SEMIGROUP THEORY AND SOME APPLICATIONS

A Thesis Submitted to
the Graduate School of Engineering and Sciences of
İzmir Institute of Technology
in Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

in Mathematics

by
Tuğba YAZAN

July 2020
İZMİR

ACKNOWLEDGMENTS

Foremost I would like to express my deepest gratitude to my supervisor Asst. Prof. Dr. Ahmet BATAL, for his academic guidance, motivating talks, help and endless patience during my studies with him.

I would like to thank Asst. Prof. Dr. Haydar GÖRAL and Assoc. Prof. Dr. Türker ÖZSARI for being members of my thesis committee and valuable comments on my studies.

I would like to thank TÜBİTAK for graduate students scholarship, namely "TÜBİTAK-BİDEB 2210-A Yurtiçi Yüksek Lisans Burs Programı" supporting my master program studies. I am grateful to the council for their kind support.

I also want to thank my close friend Res. Asst. Ece Hazal KORKMAZ for motivating me during this process.

Lastly, I would like to thank my parents and my sister Saliha Sultan YAZAN for their love, support, and understanding throughout my education. I am grateful to them for their confidence.

ABSTRACT

SEMIGROUP THEORY AND SOME APPLICATIONS

In the present thesis, we consider the evolution equation (Cauchy problem) which is the basis for our study. We show how various linear partial differential equations can be transformed into the Cauchy problem form. Solving the Cauchy problem is equivalent to find a family of evolution operators $T(t)$ which sends the initial state of the system to the solution state at a later time t. It turns out that this family of operators $T(t)$ must satisfy some properties which we call semigroup properties. We state the Hille-Yosida and Lumer-Phillips theorems to characterize contraction semigroups. Moreover, we apply these theorems to the heat and wave equations as examples. We also consider strongly continuous operator groups and Stone's theorem. Finally, we give some essential conditions to obtain wellposed evaluation equation and introduce an inhomogeneous Cauchy problem.

Keywords: Strongly Continuous Operator Semigroup, Contraction Semigroup, Cauchy Problem, Hille-Yosida Theorem, Lumer-Phillips Theorem

ÖZET

SEMİGRUP TEORİSİ VE BAZI UYGULAMALARI

Bu tezde, çalışmamızın temelini oluşturan ilerleme denklemi (Cauchy problemi) ele alındı. Çeşitli lineer kısmi diferansiyel denklemlerin Cauchy problem formuna nasıl dönüştürülebildiğini gösterdik. Cauchy problemini çözmek, sistemin başlangıç konumunu t zaman sonraki çözüm konumuna götüren $T(t)$ ilerleme operatör ailesi bulmaya eşdeğerdir. Bu $T(t)$ operatörleri ailesinin semigrup özellikleri olarak adlandırdığımız bazı özellikleri karşılaması gerektiği ortaya çıktı. Daralan semigrupları karakterize etmek için Hille-Yosida ve Lumer-Phillips teoremlerini açıkladık. Dahası bu teoremleri örnek olarak ısı ve dalga denklemlerine uyguladık. Ayrıca güçlü sürekli operatör gruplarını ve Stone teoremini de inceledik. Son olarak, iyi tanımlanmış ilerleme denklemini elde etmek ve homojen olmayan Cauchy problemini tanıtmak için bazı temel koşullar sunduk.

Anahtar Kelimeler: Güçlü Sürekli Operatör Semigrupları, Daralan Semigruplar, Cauchy Problemi, Hille-Yosida Teorem, Lumer-Phillips Teorem

TABLE OF CONTENTS

LIST OF SYMBOLS

$B(X,Y)$	Space of all bounded linear operators from Banach space X to another Banach space Y
$B(X)$	For $B(X,X)$
$C(S,X)$	Vector space of continuous functions for $S \subseteq \mathbb{R}^n$
$C_0(S,X)$	Space of continuous functions that vanish at infinity
$C_c(S,X)$	Space of compactly support functions
$D(A)$	Domain of an operator A
I	Identity operator
$R(A)$	Range of an operator A
$supp\, f$	Set of points in X where f is non-zero
X^*	Dual space of X
$(x \mid y)$	Scalar product on a Hilbert space H for $x, y \in H$
$\langle x, y \rangle$	Action of y on x as a functional
$\mathbb{R}_+$	$=[0,\infty)$
$\mathbb{R}_-$	$=(-\infty,0]$
A'	Hilbert adjoint of an operator A
A^*	Adjoint operator of an operator A

CHAPTER 1

INTRODUCTION

Let us consider the following Cauchy problem

$$u'(t) = Au(t), \quad t \geq 0, \tag{1.1}$$
$$u(0) = u_0$$

where u belongs to the state space X and A is an operator on X with domain $D(A)$. Finding the solution of the above problem is equivalent to find an evolution rule which describes how the next state of the system follows from the current state. Mathematically such a rule can be described by a one-parameter family of operators $T(t)$ which send the initial state u_0 at $t = 0$ of the system to $T(t)u_0$ at a later time t. For example if $X = \mathbb{C}^n$ and A is an $n \times n$ matrix, corresponding family of operators is of the form $T(t) = e^{tA}$, hence the solution $u(t)$ at any time $t > 0$ can be computed as $u(t) = e^{tA}u_0$. However if X is an infinite-dimensional Banach space and A is an unbounded operator, existence and computation of such family of evolution operators $T(t)$ are not trivial. Investigation of such family of evolution operators $T(t)$ under this general setting leads to the development of the area which we now call semigroup theory.

The theory of semigroups on Banach spaces was developed by the Hille-Yosida theorem in 1948 with valuable works of E. Hille and K. Yosida. This theorem states some conditions on an operator A to generate strongly continuous contraction semigroup. W. Felder, I. Miyedera, and R. Phillips generalize the Hille-Yosida theorem to semigroups which are not contractions in (Feller, 1953). By the Lumer-Phillips theorem, some conditions are replaced by a more suitable one (Lumer, 1961).

Semigroup theory has many fields of application, for instance, functional differential equations, integro-differential equations, quantum mechanics, infinite dimensional control theory. So far, a huge number of connections to other disciplines of mathematics have been explored such as ergodic theory, numerical analysis, partial differential equations, stochastic processes.

The thesis is organized as follows.

In chapter 2, we collect some essential tools from functional analysis, operator theory and spectral theory.

In chapter 3, we introduce strongly continuous operator semigroups $T(\cdot)$. More clearly, we start with the definition of C_0-semigroup and establish a relation between each semigroup and its generator A. It is also shown that the C_0-semigroup gives the unique solution of the Cauchy problem.

In chapter 4, we are interested in answering the question of how to check that a given operator generates a strongly continuous semigroup. For this purpose, we construct several essential conditions and study the Hille-Yosida theorem which characterizes the generators of contraction semigroups using the resolvent estimate. We prove this main theorem by means of Yosida's idea explained in (Engel and Nagel, 1999).

Chapter 5 is devoted to the study of the Lumer-Phillips theorem which gives a necessary and sufficient condition for a given operator A to generate a contraction semigroup. To apply this theorem we need two new notions such as dissipativity and range condition. We introduce the concept of C_0-group and proceed with the Stone's theorem (Stone, 1932). It states that skew-adjoint operators generate the unitary C_0-group on a Hilbert space.

In chapter 6, we also present examples to show the application of general results given in Hille-Yosida theorem and Lumer-Phillips theorem (Lumer, 1961) such as heat and wave equations.

In chapter 7, we show that the existence of a strongly continuous operator semigroup of the Cauchy problem being wellposed. To that purpose, we introduce the notion of wellposedness. Then we proceed with the study of an inhomogeneous Cauchy problem.

In conclusion, we summarize the main results obtained in this thesis.

CHAPTER 2

PRELIMINARIES

This chapter consists of some basic definitions and facts. We will use the results of the closed graph theorem and state the adjoint operators. For functional analysis tools and more details, we refer the reader to (Kreyszig, 1978), (Schnaubelt, 2012) and (Hundertmark et al., 2013). Also one may find the source about Sobolev spaces, weak derivatives as well as Gauss' and Green's formula and more details in (Schnaubelt, 2012) and (Hundertmark et al., 2013).

Notation: For a given operator A, $D(A)$ denotes its domain.

Definition 2.1 *Let X and Y be normed spaces and let $A : D(A) \subseteq X \to Y$ be a linear operator. A is called closed if its graph*

$$Gr(A) = \{(x, y) \mid x \in D(A) \text{ and } y = Ax \}$$

is closed in the Cartesian product $X \times Y$. The graph norm is defined by $\|x\|_A := \|x\|_X + \|Ax\|_Y$. We will denote $(D(A), \|\cdot\|_A)$ by $[D(A)]$.

Lemma 2.1 *Let X and Y be Banach spaces and $A : D(A) \subseteq X \to Y$ be a closed operator with $D(A)$. The closed graph theorem states that if $D(A)$ is closed then A is continuous.*

Property 2.1 *Let A be a closed operator on a normed space X and g be a continuous function on $[a, b]$ with $g(t)$ in $D(A)$ for every $t \in [a, b]$ such that Ag is continous on $[a, b]$. We thus get,*

$$\int_a^b g(t)\, \mathrm{d}t \in D(A) \quad \text{and} \quad A \int_a^b g(t)\, \mathrm{d}t = \int_a^b Ag(t)\, \mathrm{d}t.$$

Property 2.2 *Let f be a continuous function from an interval $[a, b]$ to normed space X. By the Fundamental Theorem of Calculus, the map*

$$t \to \int_a^t f(\xi)\, \mathrm{d}\xi$$

is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(\xi)\,\mathrm{d}\xi = f(t), \quad \textit{for all } t \in [a,b]. \tag{2.1}$$

Let g be a continuously differentiable function from $[a,b]$ to X and $t \in [a,b]$. We have

$$\int_a^t g'(\xi)\,\mathrm{d}\xi = g(t) - g(a). \tag{2.2}$$

Property 2.3 *Let f be a continuous function from an interval $[a,b)$ to a normed space X and $t \in [a,b)$. Then we have*

$$\frac{1}{h}\int_t^{t+h} f(\tau)\,\mathrm{d}\tau \to f(t) \quad \textit{as } h \to 0^+.$$

Property 2.4 *$C_c(\mathbb{R}) = \{g \in C(\mathbb{R}) \mid supp\, g$ is compact$\}$ is dense in $C_0(\mathbb{R}) = \{g \in C(\mathbb{R}) \mid g(s) \to 0$ as $|s| \to \infty\}$.*

Proof For all $n \in \mathbb{N}$ we take a function $\varphi_n \in C(\mathbb{R})$ with

$$\varphi_n(s) = \begin{cases} 1, & s \in [-n,n] \\ s+n+1, & s \in [-n-1,-n] \\ -s+n+1, & s \in [n,n+1] \\ 0, & \text{otherwise} \end{cases}$$

and $supp\,\varphi_n \subseteq (-n-1, n+1)$. For each $h \in C_0(\mathbb{R})$ then we have $\varphi_n h \in C_c(\mathbb{R})$ and

$$\|h - \varphi_n h\|_\infty = \sup_{\|s\| \geq n} |(1-\varphi_n(s))h(s)| \leq \sup_{\|s\| \geq n} |h(s)| \to 0 \quad \text{as}\, n \to \infty.$$

□

Property 2.5 *A is closed if the resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$ exists and is bounded for* *at least one value of parameter $\lambda \in \mathbb{C}$.*

Proof Suppose that $(\lambda I - A)^{-1}$ exists and it is bounded for some $\lambda \in \mathbb{C}$ then $\rho(A) \neq \emptyset$. Let $x_n \in D(A)$, $x_n \to x$ and $Ax_n \to y$. Then

$$x = \lim_{n\to\infty}(\lambda I - A)^{-1}(\lambda I - A)x_n = (\lambda I - A)^{-1}\lim_{n\to\infty}(\lambda I - A)x_n = (\lambda I - A)^{-1}(\lambda x - y). \quad (2.3)$$

Therefore, $x \in (\lambda I - A)^{-1}X = D(A)$. From equality (2.3),

$$(\lambda I - A)x = (\lambda I - A)(\lambda I - A)^{-1}(\lambda x - y) = \lambda x - y.$$

We obtain $Ax = y$ and thus A is closed. □

2.1. Adjoint Operator

Definition 2.2 *Let X and Y be topological vector spaces and let $A : D(A) \subseteq X \to Y$ be a linear operator. A is called **densely defined** if $D(A)$ is dense in X.*

Definition 2.3 *Let X be a Banach space and let A be a linear densely defined operator on X. The adjoint A^* is given by $A^*x^* := z^*$ for each $x^* \in D(A^*)$, where*

$$D(A^*) := \{x^* \in X^* \,|\, \exists z^* \in X^* \forall x \in D(A) \,:\, \langle Ax, x^*\rangle = \langle x, z^*\rangle\}.$$

Definition 2.4 *Let X be a Hilbert space with an inner product $(\cdot|\cdot)$ and let A be a linear densely defined operator on X. The Hilbert adjoint A' of A is given by $A'y := z$ for each $y \in D(A')$, where*

$$D(A') := \{y \in X \,|\, \exists z \in X \;\forall x \in D(A) \,:\, (Ax|y) = (x|z)\}.$$

Definition 2.5 *Let X be a Hilbert space and $A : D(A) \to X$ be a linear and densely* *defined operator on X. Then A is called **symmetric** if for each $x, z \in D(A)$,*

$$(Ax|z) = (x|Az).$$

Definition 2.6 *Let A be a linear densely defined operator on a Hilbert space. A is called **self-adjoint** if $A = A'$ and **skew-adjoint** if $-A = A'$.*

Definition 2.7 *A map T on a complex vector space X is called **antilinear** if*

$$T(u + v) = T(u) + T(v) \quad u, v \in X,$$
$$T(\alpha u) = \overline{\alpha} T(u) \quad \alpha \in \mathbb{C},\ u \in X.$$

Property 2.6 *((Schnaubelt, 2012), Theorem 4.7) Let A be a closed and symmetric operator. Then the following assetions are satisfied.*

- ***a)*** *If $\rho(A) \cap \mathbb{R} \neq \emptyset$, then $\sigma(A) \subseteq \mathbb{R}$.*
- ***b)*** *$\sigma(A) \subseteq \mathbb{R}$ if and only if A is self-adjoint.*

Property 2.7 *(Hundertmark et al., 2013) Let A and C be linear operators. If $A \subseteq C$ and $\rho(A) \bigcap \rho(C) \neq \emptyset$ then we have $A = C$.*

Proof Suppose that $\lambda \in \rho(A) \bigcap \rho(C)$ then $\lambda I - A$ and $\lambda I - C$ are bijective. By the assumption $A \subseteq C$, we have $\lambda I - A \subseteq \lambda I - C$ such that $\lambda I - A$ is surjective and $\lambda I - C$ is injective. Then we need to show that $D(\lambda I - C) \subseteq D(\lambda I - A)$. Take $x \in D(\lambda I - C)$. By the surjectivity of $\lambda I - A$ there exist $y \in D(\lambda I - A)$ such that $(\lambda I - C)x = (\lambda I - A)y = (\lambda I - C)y$. The injectivity of $\lambda I - C$ gives $x = y$ and so $x \in D(\lambda I - A)$. Hence $\lambda I - A = \lambda I - C$ which also imply $A = C$. □

Definition 2.8 *Let A be a linear and bounded operator on a Hilbert space X. A is called **unitary** if it has inverse with $A^{-1} = A'$.*

2.2. Weak Derivative and Sobolev Space

Definition 2.9 *Let $U \subseteq \mathbb{R}^n$ be open, $i \in \{1, ..., n\}$ and $p \in [1, \infty]$. Let u be a function in $L^p(U)$. We say u has a* ***weak derivative*** *υ in $L^p(U)$ if there exists a function $\upsilon \in L^p(U)$ such that*

$$\int_U u(x)\partial^i \varphi(x)\,\mathrm{d}x = (-1)^{|i|} \int_U \upsilon(x)\varphi(x)\,\mathrm{d}x \tag{2.4}$$

for all $\varphi \in C_c^\infty(U)$ and we set $\upsilon := \partial^i u$ where $\partial^i := \partial_1^{i_1} ... \partial_n^{i_n}$ and $|i| := i_1 + i_2 + ... + i_n$.

Definition 2.10 *Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. The* ***Sobolev space*** *$W_p^k(U)$ consists of functions $u \in L^p(U)$ such that for each multi index i with $|i| \leq k$, the mixed weak derivative $\partial^i u \in L^p(U)$. The norm of $u \in W_p^k(U)$ is defined by*

$$||u||_{k,p} = \left(||u||_p^p + \sum_{i=1}^{k} ||\partial^i u||_p^p \right)^{1/p}, \quad \textit{if } p < \infty,$$

and

$$||u||_{k,p} = \max_{1 \leq i \leq k} \{||u||_\infty, ||\partial^i u||_\infty\}, \quad \textit{if } p = \infty.$$

The Sobolev space with $p = 2$ is denoted by $W_2^k(U) = H^k(U)$ and for $k = 0$, we set $W_p^0(U) = L^p(U)$.

Definition 2.11 *Let U be an open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$ and $p \in [1, \infty)$. The closure of $C_c^\infty(U)$ in $W_p^k(U)$ is denoted by $\mathring{W}_p^k(U)$.*

Theorem 2.1 *(Schnaubelt, 2012) Let U be an open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$ and $p \in [1, \infty)$. Then $\mathring{W}_p^k(\mathbb{R}^n) = W_p^k(\mathbb{R}^n)$ and $\mathring{H}^k(U) = \mathring{W}_2^k(U)$.*

Theorem 2.2 *(Schnaubelt, 2012) Let U be an open and bounded subset of $\mathbb{R}^n$ with $\partial U \in C^2$. Let $p \in [1, \infty]$, $F \in W_p^1(U)^n$ and $\varphi \in W_{p'}^1(U)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have* ***Gauss' formula***

$$\int_U \mathrm{div(F)}\, \varphi\, \mathrm{dx} = -\int_\mathrm{U} \mathrm{F} \cdot \nabla\varphi\, \mathrm{dx} + \int_{\partial \mathrm{U}} \mathrm{n} \cdot \mathrm{F}\, \varphi\, \mathrm{d}\sigma \tag{2.5}$$

where n is the outer unit normal of ∂U.

If $u \in W_p^2(U)$ and $v \in W_{p'}^2(U)$ with $F = \nabla u$, we obtain ***Green's formula***

$$\int_U (\Delta uv - u\Delta v)\,\mathrm{d}x = \int_{\partial U} (\partial_n uv - u\partial_n v)\,\mathrm{d}\sigma. \tag{2.6}$$

Property 2.8 *Let $U \subseteq \mathbb{R}^n$ be an open subset, $p \in (1,\infty)$, $F \in \mathring{W}^1_p(U)^n$ and $\varphi \in \mathring{W}^1_{p'}(U)$. Then*

$$\int_U \mathrm{div(F)}\,\varphi\,\mathrm{dx} = -\int_{\mathrm{U}} \mathrm{F}\cdot\nabla\varphi\,\mathrm{dx}. \tag{2.7}$$

If $U = \mathbb{R}^n$, equations (2.5) *and* (2.6) *hold without the boundary integral.*

Property 2.9 *Let U be an open bounded subset in $\mathbb{R}^n$ and $1 \leq p < \infty$. The **Poincaré's inequality** states that for a constant $\delta > 0$ such that*

$$\forall u \in \mathring{W}^1_p(U) \qquad \int_U |\nabla u|^p \,\mathrm{d}x \geq \delta\,\|u\|^p_p. \tag{2.8}$$

2.3. Fourier Transform

Definition 2.12 *For an integrable function f in $\mathbb{R}^n$, the **Fourier transform** is defined by*

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)\,e^{-i\xi.x}\,\mathrm{d}x,$$

where $\xi \in \mathbb{R}^n$ and $\xi\cdot x = \sum_{k=1}^n \xi_k x_k$.

Property 2.10 *(Hundertmark et al., 2013) The Fourier transform extends to a unitary operator $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with $(\mathcal{F}^{-1}f)(x) = (\mathcal{F}f)(-x)$. Let $k \in \mathbb{N}$ and $j \in 1,...,n$. Then the following properties hold.*

a) $\mathcal{F}(\partial^\alpha u) = i^{|\alpha|}\,\xi^\alpha \mathcal{F}u.$

b) $\partial_j u = i\mathcal{F}^{-1}(\xi_j \hat{u})$ *for* $u \in H^1(\mathbb{R}^n)$.

c) $H^k(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) \,:\, |\xi|^k_2 \hat{u} \in L^2(\mathbb{R}^n)\}$.

CHAPTER 3

STRONGLY CONTINUOUS SEMIGROUPS

In many books on semigroup theory, the definitions and properties related to strongly continuous semigroups are presented. We follow (Engel and Nagel, 1999), (Pazy, 2012) and (Hundertmark et al., 2013).

Definition 3.1 *Let X be a complex Banach space. We call a map $T(\cdot) : \mathbb{R}_+ \to B(X)$ the strongly continuous operator semigroup or C_0-semigroup if the following properties are satisfied.*

a) $T(0) = I$*, where I is the identity operator on X.*

b) $T(t+s) = T(t)T(s) \quad$ *for all $t,s \geq 0$.*

c) *For every $x \in X$ the orbit,*

$$T(\cdot)x : \mathbb{R}_+ \to X, \quad t \to T(t)x \quad \text{is continuous}.$$

Definition 3.2 *The generator A of $T(\cdot)$ is defined by*

$$Ax := \lim_{t \to 0^+} \tfrac{1}{t}(T(t)x - x)$$

where

$$D(A) := \{x \in X : \lim_{t \to 0^+} \tfrac{1}{t}(T(t)x - x) \text{ exists in } X\}.$$

If A is the generator of $T(\cdot)$, we also say that A generates $T(\cdot)$.

The conditions a) and b) in Definition 3.1 are called the semigroup laws and c) is the strong continuity.

Example 3.1 *Let A be a linear bounded operator on a Banach space X and let*

$$S_n = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots + \frac{t^n}{n!}A^n, \qquad \text{for } t \geq 0.$$

Our claim is that S_n is Cauchy. Let $\varepsilon > 0$. Since $e^s = \sum_{k=0}^{\infty} \frac{s^k}{k!}$ converges for every $s \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \frac{|t|^k \|A\|^k}{k!} < \varepsilon$. Then, for all $n, m > N$

$$\|S_n - S_m\| = \left\| \sum_{k=0}^{n} \frac{t^k}{k!} A^k - \sum_{k=0}^{m} \frac{t^k}{k!} A^k \right\| = \left\| \sum_{k=m+1}^{n} \frac{t^k}{k!} A^k \right\| \leq \sum_{k=m+1}^{n} \left\| \frac{t^k}{k!} A^k \right\| \leq \sum_{k=m+1}^{n} \frac{|t|^k \|A\|^k}{k!} < \varepsilon.$$

Since S_n is Cauchy and X is a Banach space, S_n converges. Let us denote the limit by

$$T(t) := e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

Now we check that $T(\cdot)$ satisfies the conditions of C_0-semigroup. For $t, s \geq 0$, we have

$$\begin{aligned} T(t)T(s) = e^{tA}e^{sA} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \sum_{m=0}^{\infty} \frac{s^m}{m!} A^m \\ &= (I + tA + \frac{t^2}{2!} A^2 + \cdots).(I + sA + \frac{s^2}{2!} A^2 + \cdots) \\ &= I + (t+s)A + \frac{(t+s)^2}{2!} A^2 + \frac{(t+s)^3}{3!} A^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(t+s)^n}{n!} A^n = e^{(t+s)A} = T(t+s) \end{aligned}$$

and also $T(0) = e^{0A} = I$. $T(t)$ is uniformly continuous as follows

$$\|T(t) - I\| = \left\| \sum_{n=1}^{\infty} \frac{(tA)^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{t^n \|A\|^n}{n!} - 1 = e^{t\|A\|} - 1 \to 0$$

as $t \to 0$. Uniform continuity of $T(t)$ implies strong continuity.

$T(\cdot)$ satisfies the conditions of C_0-semigroup and also $T(\cdot)$ is continuously differentiable with $\frac{d}{dt} e^{tA} = Ae^{tA}$. In addition, the solution $u : \mathbb{R}_+ \to X$ of equation (1.1) can be described as $u(t) = e^{tA} u_0$ for $u_0 \in X$. The conditions given above are satisfied for any bounded linear operator A on a Banach space X.

Note that the above example shows the C_0-semigroup $T(t) = e^{tA}$ for a given bounded operator A is exponentially bounded, i.e., $\|T(t)\| \leq e^{t\|A\|}$. This situation is not special to semigroups generated by a bounded operator only. Indeed for any C_0-semigroup, exponentially boundedness is satisfied as the following lemma states.

Lemma 3.1 *Let $T(\cdot)$ be a C_0-semigroup. There are constants $\omega \geq 0$ and $M \geq 1$ such that $||T(t)|| \leq Me^{\omega t}, \ 0 \leq t < \infty$.*

Proof Let us prove that there is an $\xi > 0$ such that $||T(t)||$ is bounded for $0 \leq t \leq \xi$. For a contradiction, suppose that the claim is false then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ which converges to 0. As $n \rightarrow \infty$ and $||T(t_n)|| \geq n$. Uniform boundedness principle implies that for some $x \in X$, $||T(t_n)x||$ is unbounded. But this contradicts to definition of strong continuity of semigroups. We conclude that $||T(t)|| \leq M$ for $0 \leq t \leq \xi$ as $||T(0)|| = 1$, M must be grater than or equal to 1. Let $\omega = \xi^{-1} \log M \geq 0$ and $t = n\xi + \delta$ where $0 \leq \delta \leq \xi$ and using semigroup properties, we get

$$||T(t)|| = ||T(\delta)T(\xi)^n|| \leq M.M^n \leq M.M^{\frac{t}{\xi}} \leq Me^{\omega t} \quad \text{for} \quad 0 \leq t < \infty.$$

□

Definition 3.3 *For a C_0-semigroup $T(\cdot)$ with a generator A, we call*

$$\omega_0(T) := \omega_0(A) := inf\{\omega \in \mathbb{R} \mid \exists M_\omega \geq 1 : ||T(t)|| \leq M_\omega e^{\omega t} \text{ for all } t \geq 0\}$$

the growth (exponential) bound of $T(\cdot)$.

Lemma 3.2 *(Engel and Nagel, 1999) Let $T(t)$ be a semigroup on a Banach space X. Then the following conditions are equivalent.*

a) *$T(t)_{t\geq 0}$ is strongly continuous.*

b) $\lim_{t\to 0^+} T(t)x = x$ *for every $x \in X$.*

c) *There exist $t_f > 0$ and a dense subspace $S \subseteq X$ such that*

- ***i)*** $sup\{||T(t)|| \mid 0 \leq t \leq t_f\} < \infty$,
- ***ii)*** $\lim_{t\to 0^+} T(t)x = x$ *for each $x \in S$.*

Proof The assertion $a) \Rightarrow ii)$ follows from the definition of strong continuity. To prove that $a) \Rightarrow i)$, we suppose that the negation of condition $i)$ is true. Assume that there is a sequence $(t_n)_{n\in\mathbb{N}}$ that converges to 0 such that $\|T(t_n)\|$ diverge to infinity as $n \to \infty$. This implies that by uniform boundedness principle, for some element $x \in X$ $\|T(t_n)x\|$ is unbounded for all $n \in \mathbb{N}$ and hence $T(t)_{t\geq 0}$ is not continuous.

To show the implication $c) \Rightarrow b)$, we assume $\|T(t)\| \leq M$ for all $0 < t < t_f$. Let $z \in X$ and $\varepsilon > 0$. Since S is dense in X, there exist $x \in S$ such that $\|x - z\| < \varepsilon$ and also condition ii) implies that there is $t_f > 0$ such that $\|T(t)x - x\| < \varepsilon$ for all $t < t_f$. Then for all $t < t_f$, we have

$$\begin{aligned}\|T(t)z - z\| &\leq \|T(t)(z - x)\| + \|T(t)x - x\| + \|x - z\| \\ &\leq \|T(t)\|\,\|z - x\| + \varepsilon + \varepsilon \leq (M + 2)\varepsilon.\end{aligned}$$

Thus $T(t)$ is strongly continuous for all $x \in X$ and $t \geq 0$. The proof is completed by showing that $b) \Rightarrow a)$. We have for every $x \in X$ and $t, h > 0$

$$\begin{aligned}\lim_{h\to 0^+} \|T(t + h)x - T(t)x\| &\leq \|T(t)\|. \lim_{h\to 0^+} \|T(h)x - x\| \\ &\leq Me^{\omega t} \lim_{h\to 0^+} \|T(h)x - x\| = 0,\end{aligned}$$

which proves the right continuity. For $t \geq h > 0$, note that $\|T(t - h)\| \leq Me^{\omega(t-h)} \leq Me^{|\omega|t}$. Hence

$$\begin{aligned}\lim_{h\to 0^+} \|T(t - h)x - T(t)x\| = \lim_{h\to 0^+} \|T(t - h)x - T(t - h)T(h)x\| &\leq \|T(t - h)\|. \lim_{h\to 0^+} \|T(h)x - x\| \\ &\leq Me^{|\omega|t} \lim_{h\to 0^+} \|T(h)x - x\| = 0,\end{aligned}$$

which proves the left continuity. □

Definition 3.4 *Let A be a linear operator on X with $D(A)$ and let $x \in D(A)$. Then a* *function* $u : \mathbb{R}_+ \to X$ *is the solution of the Cauchy problem if*

$$\begin{aligned} u'(t) &= Au(t), \quad t \geq 0, \\ u(0) &= x \end{aligned} \tag{3.1}$$

where $u \in C^1(\mathbb{R}_+, X)$ *satisfies* $u(t) \in D(A)$ *for each* $t \geq 0$.

Proposition 3.1 *For the generator A of a* C_0*-semigroup* $T(t)_{t\geq 0}$*, the following conditions hold.*

a) *If* $x \in D(A)$ *then* $T(t)x \in D(A)$ *and* $AT(t)x = T(t)Ax$ *for all* $t \geq 0$.

b) *The function* $u : \mathbb{R}_+ \to X$*,* $t \to T(t)x$ *is unique solution of* (3.1).

Proof For part a) we take $x \in D(A)$ and $h > 0$, then $T(t)x \in D(A)$ if $\lim\limits_{h\to 0^+} \frac{1}{h}(T(t+h)x - T(t)x)$ exists. Indeed

$$\begin{aligned} \lim_{h\to 0^+} \tfrac{1}{h}(T(t+h)x - T(t)x) &= \lim_{h\to 0^+} \tfrac{1}{h}(T(t)T(h)x - T(t)x), \quad \text{since } T(t) \text{ is continuous,} \\ &= T(t) \lim_{h\to 0^+} \tfrac{1}{h}(T(h)x - x) = T(t)Ax. \end{aligned}$$

Hence by definition $T(t)x \in D(A)$, $AT(t)x = T(t)Ax$ and also $T(\cdot)x$ is differentiable from right. In addition, for $0 < h < t$

$$\lim_{h\to 0^+} \tfrac{1}{h}(T(t)x - T(t-h)x) = \lim_{h\to 0^+} T(t-h)\tfrac{1}{h}(T(h)x - x) = T(t)Ax,$$

which shows that $T(\cdot)x$ is differentiable from left as well. Hence we obtain $T(\cdot)x \in C^1(\mathbb{R}_+, X)$ and $\frac{d}{dt}T(\cdot)x = AT(\cdot)x$. Thus u solves the equation (3.1). For uniqueness, suppose that ω is a solution of the equation (3.1) and $t > 0$. We define $\mu(s) = T(t-s)\omega(s)$, s in $[0, t]$. Taking derivative of both sides with respect to s, we have

$$\mu'(s) = -T(t-s)A\omega(s) + T(t-s)\omega'(s) = T(t-s)(-A\omega(s) + \omega'(s)) = 0$$

since ω is a solution of (3.1). Consequently, for each functional x^* in X^* the function $\langle \mu(\cdot), x^* \rangle$ has derivative which equals to 0 and so it is constant. Then we have

$$\langle \omega(t), x^* \rangle = \langle \mu(t), x^* \rangle = \langle \mu(0), x^* \rangle = \langle T(t)x, x^* \rangle$$

for all $x^* \in X^*$ and $t \geq 0$. We obtain $T(\cdot)x = \omega$ which shows the uniqueness of solution. □

CHAPTER 4

CHARACTERIZATION OF GENERATORS AND HILLE-YOSIDA THEOREM

Lemma 4.1 *Let $T(\cdot)$ be a C_0-semigroup with generator A. Then $S(t) := e^{\mu t}T(\alpha t)$ is also a C_0-semigroup generated by $B = \mu I + \alpha A$ with $D(A) = D(B)$ where $\mu \in \mathbb{C}$ and $\alpha > 0$.*

Proof We assume that $T(\cdot)$ is a C_0-semigroup and then we need to show that $S(t)$ satisfies the semigroup law and strong continuity. First, we have

$$S(t+s) = e^{\mu(t+s)}T(\alpha(t+s)) = e^{\mu t}T(\alpha t)e^{\mu s}T(\alpha s) = S(t)S(s), \quad \text{for all } t, s \geq 0,$$

and $S(0) = I$. From the strong continuity of $T(t)$,

$$\lim_{t\to 0^+} S(t)x = \lim_{t\to 0^+} e^{\mu t}T(\alpha t)x = e^0 T(0)x = x, \quad \text{for all } x \in X.$$

We conclude that $S(\cdot)$ is a C_0-semigroup. Let B be the generator of $S(\cdot)$. Then

$$\begin{aligned} Bx = \lim_{t\to 0^+} \tfrac{1}{t}(S(t)x - x) &= \lim_{t\to 0^+} \tfrac{1}{t}(e^{\mu t}T(\alpha t)x - x) \\ &= \lim_{t\to 0^+} \tfrac{\alpha}{\alpha t}(e^{\mu t}T(\alpha t)x - x) \\ &= \lim_{t\to 0^+} \alpha e^{\mu t}\Big(\Big(\tfrac{1}{\alpha t}(T(\alpha t)x - x)\Big) + \tfrac{1}{t}(e^{\mu t}x - x)\Big) \\ &= \alpha Ax + \mu x, \end{aligned}$$

which also shows $D(A) = D(B)$. □

Lemma 4.2 *(Hundertmark et al., 2013) Let $T(\cdot)$ be a C_0-semigroup with generator A, $t > 0$ and $x \in X$. Then $\int_0^t T(s)x\,\mathrm{d}s \in D(A)$ and*

$$T(t)x - x = A\int_0^t T(s)x\,\mathrm{d}s \quad \textit{if } x \in X \tag{4.1}$$

$$= \int_0^t T(s)Ax\,\mathrm{d}s \quad \textit{if } x \in D(A). \tag{4.2}$$

Proof Let $t > 0$ and $x \in X$, we have

$$\begin{aligned}
\lim_{h\to 0^+} \tfrac{1}{h}(T(h) - I)\int_0^t T(s)x\,\mathrm{d}s &= \lim_{h\to 0^+} \frac{1}{h}\left(\int_0^t T(h+s)x\,\mathrm{d}s - \int_0^t T(s)x\,\mathrm{d}s\right) \\
&= \lim_{h\to 0^+} \frac{1}{h}\left(\int_h^{t+h} T(s)x\,\mathrm{d}s - \int_0^t T(s)x\,\mathrm{d}s\right) \\
&= \lim_{h\to 0^+} \frac{1}{h}\left(\int_h^{t} T(s)x\,\mathrm{d}s + \int_t^{t+h} T(s)x\,\mathrm{d}s - \int_0^t T(s)x\,\mathrm{d}s\right) \\
&= \lim_{h\to 0^+} \left(\frac{1}{h}\int_t^{t+h} T(s)x\,\mathrm{d}s - \frac{1}{h}\int_0^h T(s)x\,\mathrm{d}s\right) \quad \text{by Property 2.3,} \\
&= T(t)x - x.
\end{aligned}$$

We conclude that $\int_0^t T(s)x\,\mathrm{d}s$ belongs to $D(A)$ and also (4.1) is satisfied. If $x \in D(A)$, first note that in part a) of the proof of Proposition 3.1 we showed $\frac{\mathrm{d}}{\mathrm{d}t}T(\cdot)x = AT(\cdot)x$. Hence

$$\begin{aligned}
\int_0^t T(s)Ax\,\mathrm{d}s = \int_0^t AT(s)x\,\mathrm{d}s &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}T(s)x\,\mathrm{d}s \\
&= \int_0^t (T(s)x)'\,\mathrm{d}s \quad \text{by Property 2.2,} \\
&= T(t)x - x.
\end{aligned}$$

□

The next proposition yields some essential properties of a generator of a C_0-semigroup .

Proposition 4.1 *If A is a generator of a C_0-semigroup then A is a closed and densely defined operator.*

Proof We begin by proving the closedness of A. Let x_n be a sequence in $D(A)$ which converges to $x \in X$ and let Ax_n converge to $y \in X$. From equation (4.2),

$$\begin{aligned}\frac{1}{t}(T(t)x - x) &= \lim_{n\to\infty} \frac{1}{t}(T(t)x_n - x_n)\\ &= \lim_{n\to\infty} \frac{1}{t}\int_0^t T(s)Ax_n\,\mathrm{d}s \quad \text{since } T(t) \text{ is continuous,}\\ &= \frac{1}{t}\int_0^t T(s)y\,\mathrm{d}s.\end{aligned}$$

Then Property 2.3 gives that

$$Ax := \lim_{t\to 0^+} \tfrac{1}{t}(T(t)x - x) = y,$$

which shows $x \in D(A)$ and $Ax = y$. Hence A is closed. To prove density property, let $x \in X$ and for $h > 0$ we set $x_h = \frac{1}{h}\int_0^h T(s)x\,\mathrm{d}s$. From Lemma 4.2, we know $x_h \in D(A)$. Moreover by Property 2.3, x_h converges to x as $h \to 0$. □

Definition 4.1 *Let A be a given linear close operator on a Banach space X. The* ***resolvent set*** *is defined to be $\rho(A) := \{\lambda \in \mathbb{C} \mid \lambda I - A : D(A) \to X \text{ is bijective}\}$. For $\lambda \in \rho(A)$ the family of bounded linear operators $R(\lambda, A) = (\lambda I - A)^{-1}$ is called the* ***resolvent*** *of A.*

Proposition 4.2 *Let $T(\cdot)$ be a C_0-semigroup generated by A. If for some $\lambda \in \mathbb{C}$*

$$R(\lambda)x := \int_0^\infty e^{-\lambda s}T(s)x\,\mathrm{d}s$$

exists for each $x \in X$ then $\lambda \in \rho(A)$ and $R(\lambda, A) = R(\lambda)$.

Proof By Lemma 4.1, we see that $T_\lambda(s) = e^{-\lambda s}T(s)$ is a C_0-semigroup with generator $A - \lambda I$. For all $x \in X$, we have

$$
\begin{aligned}
\lim_{h\to 0^+}\frac{1}{h}(T_\lambda(h)-I)R(\lambda)x &= \lim_{h\to 0^+}\left(\lim_{t\to\infty}\frac{1}{h}(T_\lambda(h)-I)\int_0^t T_\lambda(s)x\,\mathrm{d}s\right) \quad \text{by the continuity of } T_\lambda,\\
&= \lim_{h\to 0^+}\left(\lim_{t\to\infty}\frac{1}{h}\int_0^t T_\lambda(h+s)x\,\mathrm{d}s - \lim_{t\to\infty}\frac{1}{h}\int_0^t T_\lambda(s)x\,\mathrm{d}s\right)\\
&= \lim_{h\to 0^+}\left(\lim_{t\to\infty}\frac{1}{h}\int_h^{t+h} T_\lambda(s)x\,\mathrm{d}s - \lim_{t\to\infty}\frac{1}{h}\int_0^t T_\lambda(s)x\,\mathrm{d}s\right)\\
&= \lim_{h\to 0^+}\left(\lim_{t\to\infty}\frac{1}{h}\int_h^{t} T_\lambda(s)x\,\mathrm{d}s + \lim_{t\to\infty}\frac{1}{h}\int_t^{t+h} T_\lambda(s)x\,\mathrm{d}s - \lim_{t\to\infty}\frac{1}{h}\int_0^t T_\lambda(s)x\,\mathrm{d}s\right)\\
&= \lim_{h\to 0^+}\left(\lim_{t\to\infty}\frac{1}{h}\int_t^{t+h} T_\lambda(s)x\,\mathrm{d}s - \frac{1}{h}\int_0^h T_\lambda(s)x\,\mathrm{d}s\right)
\end{aligned}
$$

Since $\int_0^\infty e^{-\lambda s}T(s)x\,\mathrm{d}s$ exists

$$
\begin{aligned}
&= -\frac{1}{h}\int_0^h T_\lambda(s)x\,\mathrm{d}s\\
&= -e^{-\lambda 0}T(0)x\\
&= -x.
\end{aligned}
$$

Here we find that $R(\lambda)x \in D(A-\lambda I) = D(A)$ and $(A-\lambda I)R(\lambda)x = -x$. For $x \in D(A)$, we get

$$
\begin{aligned}
R(\lambda)(\lambda I - A)x &= \lim_{t\to\infty}\int_0^t e^{-\lambda s}T(s)(\lambda I - A)x\,\mathrm{d}s \quad \text{since } T(s)Ax = AT(s)x,\\
&= \lim_{t\to\infty}\int_0^t (\lambda I - A)e^{-\lambda s}T(s)x\,\mathrm{d}s \quad \text{by Property 2.1,}\\
&= (\lambda I - A)\lim_{t\to\infty}\int_0^t e^{-\lambda s}T(s)x\,\mathrm{d}s = (\lambda I - A)R(\lambda)x = x.
\end{aligned}
$$

By Proposition 4.1, A is a closed operator. We thus have $\lambda \in \rho(A)$ and $R(\lambda, A) = R(\lambda) = (\lambda I - A)^{-1}$. □

Corollary 4.1 *For all $n \in \mathbb{N}, \omega \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, if $\omega_0(A) < \omega < \operatorname{Re}\lambda$, then*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n} \tag{4.3}$$

where $M \geq 1$ satisfying $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

Proof By Theorem 1.13 in (Schnaubelt, 2012), the resolvent map is analytic with

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n n!\ R(\lambda, A)^{n+1}. \tag{4.4}$$

Then, we have for all $x \in X$

$$R(\lambda, A)^n x = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, A)x = \frac{(-1)^{n-1}}{(n-1)!} \int_0^\infty \frac{d^{n-1}}{d\lambda^{n-1}} e^{-\lambda s} T(s)x \, \mathrm{d}s. \tag{4.5}$$

On the other hand, if $\operatorname{Re}\lambda > \omega$,

$$\|e^{-\lambda s} T(s)x\| \leq Me^{-\operatorname{Re}\lambda s} e^{\omega s} x = Me^{(\omega - \operatorname{Re}\lambda)}$$

which is integrable. Hence $\int_0^\infty e^{-\lambda s} T(s)x \, \mathrm{d}s$ exists. By Proposition 4.2

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda s} T(s)x \, \mathrm{d}s.$$

When we apply induction to the following identity

$$\frac{d}{d\lambda} R(\lambda, A)x = \frac{d}{d\lambda} \int_0^\infty e^{-\lambda s} T(s)x \, \mathrm{d}s = \int_0^\infty (-s) e^{-\lambda s} T(s)x \, \mathrm{d}s \tag{4.6}$$

we get

$$\frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, A)x = \int_0^\infty (-1)^{n-1} s^{n-1} e^{-\lambda s} T(s)x \, \mathrm{d}s. \tag{4.7}$$

Substituting (4.7) into (4.5), we have

$$R(\lambda, A)^n x = \frac{1}{(n-1)!}\int_0^\infty s^{n-1}e^{-\lambda s}T(s)x\,\mathrm{d}s.$$

Taking norm of both sides,

$$\begin{aligned}
\|R(\lambda, A)^n x\| &= \frac{1}{(n-1)!}\left\|\int_0^\infty s^{n-1}e^{-\lambda s}T(s)x\,\mathrm{d}s\right\| \\
&\leq \frac{1}{(n-1)!}\int_0^\infty \|s^{n-1}e^{-\lambda s}T(s)x\|\,\mathrm{d}s \quad \text{since } \|\,T(s)\| \leq Me^{\omega s}, \\
&\leq \frac{M}{(n-1)!}\int_0^\infty s^{n-1}e^{-\lambda s}e^{\omega s}\|x\|\,\mathrm{d}s \\
&= \frac{M}{(n-1)!}\int_0^\infty s^{n-1}e^{(\omega-(\mathrm{Re}\,\lambda+i\,Im\lambda))s}\|x\|\mathrm{d}s \quad \text{since } |e^{-i(Im\lambda)s}| = 1, \\
&\leq \frac{M}{(n-1)!}\int_0^\infty s^{n-1}e^{(\omega-\mathrm{Re}\,\lambda)s}\|x\|\,\mathrm{d}s \\
&\leq \frac{M}{(n-1)!}\frac{(n-1)!}{(\mathrm{Re}\,\lambda-\omega)^n}\|x\| = \frac{M}{(\mathrm{Re}\,\lambda-\omega)^n}\|x\|, \quad \text{for all } x \in X.
\end{aligned}$$

□

Lemma 4.3 *Let A be a closed, densely defined operator and $M \geq 1$, $\omega \geq 0$ such that $[\omega, \infty) \subseteq \rho(A)$ and $\|R(\lambda, A)\| \leq \frac{M}{\lambda}$ for each $\lambda \geq \omega$. The following assertions are satisfied as $\lambda \to \infty$.*

a) *$\lambda R(\lambda, A)x$ converges to x for each $x \in X$.*

b) *$\lambda AR(\lambda, A)z = \lambda R(\lambda, A)Az$ converges to Az for each $z \in D(A)$.*

Proof

a) From the definition of resolvent of A, $R(\lambda, A)(\lambda I - A) = I$, which implies

$$\lambda R(\lambda, A)x - R(\lambda, A)Ax = x, \quad \text{for all } x \in D(A)$$

and thus $\|\lambda R(\lambda, A)x - x\| = \|R(\lambda, A)Ax\| \leq \frac{M}{\lambda}\|Ax\| \to 0$ as $\lambda \to \infty$. Let $x \in X$ and $\varepsilon > 0$. Since $D(A)$ is dense in X, there exists $a \in D(A)$ such that $\|x - a\| < \varepsilon$. On the other hand, since $\lambda R(\lambda, A)a - a \to 0$ as $\lambda \to \infty$, there exists $\lambda_s > 0$ such that for all $\lambda > \lambda_s$, $\|\lambda R(\lambda, A)a - a\| < \varepsilon$. Hence,

$$\begin{aligned}
\|\lambda R(\lambda, A)x - x\| &= \|\lambda R(\lambda, A)x - \lambda R(\lambda, A)a + \lambda R(\lambda, A)a - a + a - x\| \\
&\leq \|\lambda R(\lambda, A)(x - a)\| + \|\lambda R(\lambda, A)a - a\| + \|a - x\| \\
&= \|\lambda R(\lambda, A)\|\varepsilon + \varepsilon + \varepsilon \\
&= (M + 2)\varepsilon,
\end{aligned}$$

which shows $\lambda R(\lambda, A)x \to x$.

b) Since A and $\lambda I - A$ commute, we obtain

$$\begin{aligned}
AR(\lambda, A) = A(\lambda I - A)^{-1} &= (\lambda I - A)^{-1}(\lambda I - A)A(\lambda I - A)^{-1} \\
&= (\lambda I - A)^{-1}A(\lambda I - A)(\lambda I - A)^{-1} \\
&= (\lambda I - A)^{-1}A = R(\lambda, A)A.
\end{aligned}$$

Hence, if we take $Az = x$ in the first assertion of lemma, we get

$$\lambda AR(\lambda, A)z = \lambda R(\lambda, A)Az \to Az, \quad \text{for all } z \in D(A).$$

□

Definition 4.2 *For all $\lambda \in \mathbb{C}$, $(\lambda I - A)R(\lambda, A) = \lambda R(\lambda, A) - AR(\lambda, A) = I$. Multiplying both sides with λ, we get $\lambda^2 R(\lambda, A) - \lambda AR(\lambda, A) = \lambda$. Letting $A_\lambda := \lambda AR(\lambda, A)$, we have*

$$A_\lambda = \lambda AR(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda I.$$

The operator A_λ is called the ***Yosida Approximation*** *of a generator A for a given λ.*

Definition 4.3 *Let $T(t)_{t\geq 0}$ be a C_0-semigroup. By Lemma 3.1, we already know that, there exists some $M \geq 1$ and $\omega \geq 0$ such that for all $t \geq 0$, $\|T(t)\| \leq Me^{\omega t}$. If it happens to be* *the case that M can be chosen as 1 and ω as 0, then we have $\|T(t)\| \leq 1$ and in this case $T(t)$ is called a* ***contraction*** *C_0-semigroup .*

Theorem 4.1 *(Hille-Yosida) Let X be a Banach space and $T(\cdot)$ be a family of operators on X. $T(\cdot)$ is a contraction C_0-semigroup generated by a linear operator A if and only if A is closed, densely defined, $(0, \infty) \subseteq \rho(A)$, and for every $\lambda > 0$, we have $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$. Moreover if $T(\cdot)$ is a contraction C_0-semigroup, then $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \subseteq \rho(A)$ and we have $\|R(\lambda, A)^n\| \leq \frac{1}{(\operatorname{Re}\lambda)^n}$, $\forall n \in \mathbb{N}$ and $\lambda \in \mathbb{C}_+$.*

Proof ($\Rightarrow$) By Proposition 4.1, if A generates a C_0-semigroup $T(\cdot)$ then A must be closed and densely defined. Then for each $x \in X$ and $\operatorname{Re}\lambda > 0$, one has

$$\begin{aligned}\|R(\lambda)x\| = \left\|\int_0^\infty e^{-\lambda s}T(s)x\,\mathrm{d}s\right\| &\leq \int_0^\infty e^{-\operatorname{Re}\lambda s}\|T(s)\|\,\|x\|\,\mathrm{d}s \quad \text{since } \|T(s)\| \leq 1,\\ &\leq \int_0^\infty e^{-\operatorname{Re}\lambda s}\|x\|\,\mathrm{d}s = \frac{1}{\operatorname{Re}\lambda}\|x\|,\end{aligned}$$

which shows $R(\lambda) = \int_0^\infty e^{-\lambda s}T(s)\,\mathrm{d}s$ is absolutely integrable and $\|R(\lambda)\| \leq \frac{1}{\operatorname{Re}\lambda}$. By Proposition 4.2 and Corollary 4.1 the result follows.

($\Leftarrow$) Let $\lambda > 0$ and A_λ be the Yosida approximation of A. Note that by definition, A_λ's are bounded operators. Hence we can safely define the family of operators e^{tA_λ} for $t \geq 0$. Indeed, by Example 3.1 we know that e^{tA_λ} forms a uniformly continuous semigroup. We proceed by showing that e^{tA_λ} is a contraction semigroup.

$$\begin{aligned}\|e^{tA_\lambda}\| = \|e^{-t\lambda}e^{t\lambda^2 R(\lambda,A)}\| &\leq e^{-t\lambda}\sum_{j=0}^\infty \frac{(t\lambda^2)^j}{j!}\|R(\lambda, A)\|^j\\ &= e^{-t\lambda}e^{t\lambda^2\|R(\lambda,A)\|} \quad \text{since } R(\lambda, A) \text{ is bounded with } \frac{1}{\lambda}, \qquad (4.8)\\ &\leq e^{-t\lambda}e^{t\lambda} = 1.\end{aligned}$$

Taking $\lambda, \mu \in \mathbb{N}$, one has

$$A_\lambda e^{tA_\mu} = A_\lambda \sum_{j=0}^\infty \frac{t^j}{j!}(A_\mu)^j = \sum_{j=0}^\infty \frac{t^j}{j!}(A_\mu)^j A_\lambda = e^{tA_\mu}A_\lambda. \qquad (4.9)$$

For $t_f > 0$ and $t \in [0, t_f]$, using (2.2) leads to

$$
\begin{aligned}
\|e^{tA_\lambda}y - e^{tA_\mu}y\| &= \left\| \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(e^{(t-s)A_\mu}e^{sA_\lambda}y)\,\mathrm{d}s \right\| \\
&= \left\| \int_0^t (-A_\mu e^{(t-s)A_\mu}e^{sA_\lambda}y + e^{(t-s)A_\mu}A_\lambda e^{sA_\lambda}y)\,\mathrm{d}s \right\| \quad \text{using (4.9)} \\
&= \left\| \int_0^t e^{(t-s)A_\mu}e^{sA_\lambda}(A_\lambda y - A_\mu y)\,\mathrm{d}s \right\| \\
&\le \int_0^t \|e^{(t-s)A_\mu}e^{sA_\lambda}\|\,\|A_\lambda y - A_\mu y\|\,\mathrm{d}s \quad \text{from (4.8)} \\
&\le t_f\|A_\lambda y - A_\mu y\|.
\end{aligned}
$$

By Lemma 4.3, $\lim_{\lambda\to\infty} A_\lambda y = Ay$ for all $y \in D(A)$. Hence $A_\lambda y$ is Cauchy which implies $e^{tA_\lambda}y$ is Cauchy as well. Hence it is convergent. Let denote its limit by $T(t)y$, i.e., $T(t)y := \lim_{\lambda\to\infty} e^{tA_\lambda}y$. Since e^{tA_n} is a contraction for all n, passing to the limit, we obtain $\|T(t)y\| \le \|y\|$ which shows $T(t)$ is a contraction for each $t \le t_f$ on the dense domain $D(A)$. By bounded extension property of bounded operators to the closure of their domain, we can extend $T(t)$ to the whole space X by preserving its norm. Hence $T(t)$ is a contraction on X.

Let $x \in X$ and let $\varepsilon > 0$. Since $D(A)$ is dense in X, $\exists y \in D(A)$ such that $\|x-y\| < \varepsilon$. Since $e^{tA_n}y \to T(t)y$, there exists $N \in \mathbb{N}$ such that $\|e^{tA_N}y - T(t)y\| < \varepsilon/2$ and also the strong continuity of e^{tA_N} implies that there exists $t^* > 0$ such that $\|e^{tA_N}y - y\| \le \varepsilon/2$ for all $t < t^*$. Then for all $t < t^*$, we have

$$
\begin{aligned}
\|T(t)x - x\| &\le \|T(t)x - T(t)y\| + \|T(t)y - y\| + \|y - x\| \\
&\le \|T(t)\|\,\|x - y\| + \|T(t)y - e^{tA_N}y\| + \|e^{tA_N}y - y\| + \|x - y\| < 3\varepsilon.
\end{aligned}
$$

Consequently, $T(t)$ is strongly continuous for all $x \in X$ and $t \ge 0$. It is obvious that $T(0) = \lim_{\lambda\to\infty} e^{0A_\lambda} = \lim_{\lambda\to\infty} I = I$ and also $T(t+s)x = \lim_{\lambda\to\infty} e^{(t+s)A_\lambda}x = \lim_{\lambda\to\infty} e^{tA_\lambda}e^{sA_\lambda}x$ for all $t, s \ge 0$. On the other hand,

$$
\begin{aligned}
\|T(t)T(s)x - e^{tA_\lambda}e^{sA_\lambda}x\| &\le \|T(t)T(s)x - e^{tA_\lambda}T(s)x\| + \|e^{tA_\lambda}T(s)x - e^{tA_\lambda}e^{sA_\lambda}x\| \\
&\le \|(T(t) - e^{tA_\lambda})T(s)x\| + \|T(s)x - e^{sA_\lambda}x\| \to 0
\end{aligned}
$$

as $\lambda \to \infty$ and $s \to 0$. So, we have $T(t+s)x = T(t)T(s)x$. By the uniqueness of the limit $T(\cdot)$ is a contraction semigroup.

Let B be a generator of C_0-semigroup $T(\cdot)$. By Property 2.7, it is enough to show that $A \subseteq B$ and $\rho(A) \cap \rho(B) \neq \emptyset$. We see that $\mathbb{C}_+ \subseteq \rho(A) \cap \rho(B)$ by the first part of the proof hence $\rho(A) \cap \rho(B) \neq \emptyset$. On the other hand, for $y \in D(A)$ and $t > 0$, from (2.2) it follows that

$$\begin{aligned} \frac{1}{t}(T(t)y - y) = \lim_{\lambda\to\infty} \frac{1}{t}(e^{tA_\lambda}y - y) &= \lim_{\lambda\to\infty} \frac{1}{t} \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} e^{sA_\lambda} y \,\mathrm{d}s \\ &= \lim_{\lambda\to\infty} \frac{1}{t} \int_0^t A_\lambda e^{sA_\lambda} y \,\mathrm{d}s \quad \text{since } e^{sA_\lambda} \text{ is continuous,} \\ &= \frac{1}{t} \int_0^t e^{sA} Ay \,\mathrm{d}s, \end{aligned}$$

as $t \to 0$, Property 2.3 gives $y \in D(B)$ and $Ay = By$ ie. $A \subseteq B$. □

Definition 4.4 *Let $T(\cdot)$ be a C_0-semigroup with generator A and $\lambda \in \mathbb{C}$. We define the spectral bound of generator A by*

$$s(A) := sup\{ \operatorname{Re} \lambda : \lambda \in \sigma(A)\} \tag{4.10}$$

which is less then or equal to $\omega_0(A) < \infty$.

Example 4.1 *Let $X = C_0(\mathbb{R}_-) = \{f \in C(\mathbb{R}_-) \,|\, f(s) \to 0$ as $s \to -\infty\}$ and $A = -\frac{d}{ds}$ with $D(A) = C_0^1(\mathbb{R}_-) = \{f \in C^1(\mathbb{R}_-) \,|\, f, f' \in X\}$. We will show that A generates the right translation semigroup $T(\cdot)$, which is defined by*

$$(T(t)f)(s) := f(s-t) \text{ for } f \in X \text{ and } t, s \in \mathbb{R}_-.$$

We check that A satisfies the assumptions of the Hille-Yosida theorem. To show closedness of A we take u_n belongs to $D(A)$ such that u_n converges uniformly to a function $u \in X$ and

Au_n converges uniformly to $f \in X$. Then

$$\left|u_n(x) - u_n(0) - \int_0^x f(r)\,\mathrm{d}r\right| = \left|\int_0^x (u_n'(r) - f(r))\,\mathrm{d}r\right| \leq \int_0^x |u_n' - f|\,\mathrm{d}r \leq x\,\|u_n' - f\|_\infty \to 0$$

as $n \to \infty$. We deduce that $u_n(x) - u_n(0)$ converges pointwise to $\int_0^x f(r)\,\mathrm{d}r$. By the assumption $u_n \to u$ uniformly, in particular $u_n \to u$ pointwise. By the uniqueness of the limit we get $u(x) = u(0) + \int_0^x f(r)\,\mathrm{d}r \in C^1(\mathbb{R}_-)$ and $u' = f \in X$ so that $u \in D(A)$ and $Au = f$.

Obviously $C_c^1(\mathbb{R}_-) \subseteq D(A)$ since every function with compact support vanishes at infinity. Our next claim is $\overline{C_c^1(\mathbb{R}_-)} = X$, which gives the density of $D(A)$ in X. By Property 2.4, we have that $C_c(\mathbb{R}_-)$ is dense in $C_0(\mathbb{R}_-)$. We choose $f \in C_c(\mathbb{R}_-)$ with supp$f \subseteq [a, 0]$. There exists a sequence of polynomials p_n converging to f uniformly on $[a-1, 0]$ by the Weierstrass approximation theorem. Taking a function $\varphi \in C_c^1(\mathbb{R}_-)$ with $\varphi = 1$ on $[a, 0]$ and supp$\varphi \subseteq (a-1, 0]$ we define $h_n = \varphi p_n \in C_c^1(\mathbb{R}_-)$ and moreover $h_n = p_n$ on $[a, 0]$. Note that

$$\|h_n - f\|_\infty = \|\varphi p_n - f\|_\infty \leq \sup_{a-1\leq t\leq a} \|\varphi(t)p_n(t) - 0\| + \sup_{a\leq t\leq 0} \|p_n(t) - f(t)\| \to 0$$

as $n \to \infty$, and thus $C_c^1(\mathbb{R}_-)$ is dense in $C_c(\mathbb{R}_-)$ and $C_c(\mathbb{R}_-)$ is dense in $C_0(\mathbb{R}_-)$. Hence $C_c^1(\mathbb{R}_-)$ is dense in $C_0(\mathbb{R}_-)$. Since $C_c^1(\mathbb{R}_-) \subseteq D(A)$. This shows that $\overline{D(A)} = X$.

Let $f \in X$ and $\lambda > 0$. In order to show the invertibility of $\lambda I - A$, one note that $u \in D(A)$ and satisfies $\lambda u - Au = f$ if and only if $u' + \lambda u = f$, $u \in C^1(\mathbb{R}_-)$, and also $u \in X$. Let $R(\lambda)f(s) := u(s) = \int_{-\infty}^s e^{-\lambda(s-\eta)} f(\eta\,\mathrm{d}\eta$ for each $s \leq 0$. Then $u \in C^1(\mathbb{R}_-) \bigcap X$ satisfies $u' + \lambda u = f$. We will now show that $R(\lambda)f$ belongs to X. Let $\varepsilon > 0$. Then there exists n_ε such that $|f(\eta)| \leq \varepsilon$ for each $\eta \leq n_\varepsilon$. For $s \leq n_\varepsilon$, we have

$$\begin{aligned}|R(\lambda)f| = \left|\int_{-\infty}^s e^{-\lambda(s-\eta)} f(\eta)\,\mathrm{d}\eta\right| &\leq \int_{-\infty}^s e^{-\lambda(s-\eta)} |f(\eta)|\,\mathrm{d}\eta \quad \textit{substituting } \; s - \eta = \omega \\ &\leq \varepsilon \int_0^\infty e^{-\lambda\omega}\,\mathrm{d}\omega = \frac{\varepsilon}{\lambda}.\end{aligned}$$

Therefore, $u(s) = R(\lambda)f(s) \to 0$ as $s \to -\infty$. Hence $u \in D(A)$ and $\lambda u - Au = f$. So

$\lambda \in \rho(A)$ *and* $R(\lambda) = R(\lambda, A)$. *If we employ the above formula for the resolvent operator, we thus get*

$$\|R(\lambda, A)f\|_\infty \leq \sup_{s\leq 0} \int_{-\infty}^{s} e^{-\lambda(s-\eta)}\|f\|_\infty \, \mathrm{d}\eta = \|f\|_\infty \int_0^\infty e^{-\lambda\omega} \, \mathrm{d}\omega = \frac{\|f\|_\infty}{\lambda} \quad \textit{where } \; s - \eta = \omega$$

for each $f \in X$ *and* $\lambda > 0$, *namely,* $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$. *Consequently Theorem 4.1 yields that* A *generates a contraction semigroup* $T(\cdot)$.

We take $f \in D(A)$ *and define* $u(t) = T(t)f$ *for* $t \geq 0$ *to evaluate* $T(\cdot)$. *By Proposition 3.1, the unique function* $u \in C^1(\mathbb{R}_-, X)$ *and* $u(t) \in D(A)$ *for all* $t \geq 0$ *satisfies the following equation*

$$u'(t) = Au(t) = \frac{-d}{ds}u(t), \quad t \geq 0 \tag{4.11}$$
$$u(0) = f.$$

Consider $v(t) = f(\cdot - t)$ *for* $t \geq 0$. *It is obvious that* $v(t) \in X$ *and thus by the uniqueness of the solution* $v(0) = f$. *Let us show that* v *is a solution of* (4.11) *and thus* $u = v$. *For* $t_\alpha, t \geq 0$ *and* $t_\alpha \neq t$ *using Property 2.3*

$$\begin{aligned} \left\| \frac{v(t_\alpha) - v(t)}{t_\alpha - t} + f'(\cdot - t) \right\|_\infty &= \sup_{s\in\mathbb{R}_-} \left| \frac{1}{t_\alpha - t} \int_t^{t_\alpha} v'(\eta) \, \mathrm{d}\eta + f'(s-t) \right| \quad \textit{substituting } \; -f'(\cdot - t) \textit{ into } v'(t) \\ &= \sup_{s\in\mathbb{R}_-} \left| \frac{-1}{t_\alpha - t} \int_t^{t_\alpha} f'(s-\eta) \, \mathrm{d}\eta + f'(s-t) \right| \\ &\leq \sup_{s\in\mathbb{R}_-} \sup_{|\eta - t| \leq t_\alpha - t} |f'(s-t) - f'(s-\eta)| \to 0 \end{aligned}$$

as $t_\alpha \to t$ *since* $f' = Af \in C_0(\mathbb{R}_-)$ *and so* f' *is uniformly continuous. We conclude that* $\frac{d}{dt}v(t) = -f'(\cdot - t)$ *for* $t \geq 0$. *The map* $t \to f'(\cdot - t) \in X$ *is continuous and so* $v \in C_0^1(\mathbb{R}_+, X)$. *By a similar reason,* $v(t) \in C^1(\mathbb{R}_-)$ *and* $\frac{d}{ds}v(t) = f'(\cdot - t) \in X$ *so that* $v(t) \in D(A)$ *for all* $t \geq 0$ *as well as* v *holds* (4.11). *Consequently,* $T(t)f = v(t) = f(\cdot - t)$ *for each* $f \in D(A)$. *Since* $\overline{D(A)} = X$, *the equation* (4.11) *is satisfied for each* $f \in X$.

Lastly, we need to show that $\sigma(A) = \{z \in \mathbb{C} \mid \operatorname{Re} z \leq 0\}$, *namely* $(0, \infty) \subseteq \rho(A)$. *If* $\operatorname{Re}\lambda < 0$, *then* $e^{-\lambda t} \in D(A)$ *and satisfies* $Ae^{-\lambda t} = -(e^{-\lambda t})' = \lambda e^{-\lambda t}$ *so that* $\lambda \in \sigma(A)$. *Since*

$\|T\| \leq 1$, $s(A) = \omega_0(A) = 0$, *the claim follows from the closedness of* $\sigma(A)$.

Theorem 4.2 *(Feller-Miyadera-Phillips) Let A be a linear operator on a Banach space X and let* $M \geq 1$ *and* $\omega \in \mathbb{R}$. *A generates a* C_0*-semigroup* $T(\cdot)$ *satisfying* $\|T(t)\| \leq Me^{\omega t}$ *for each* $t \geq 0$ *if and only if A is closed, densely defined,* $(\omega, \infty) \subseteq \rho(A)$ *and for every* $\lambda \in (\omega, \infty)$, *we have* $\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda-\omega)^n}$ *for all* $n \in \mathbb{N}$.
In addition, if $T(\cdot)$ *is a* C_0*-semigroup, then* $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \omega\} \subseteq \rho(A)$ *and we have* $\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re}\lambda-\omega)^n}$ *for each* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega$ *and each* $n \in \mathbb{N}$.

CHAPTER 5

CHARACTERIZATION OF GENERATORS AND LUMER-PHILLIPS THEOREM

The resolvent estimate assumption of the Hille-Yosida theorem contains the usually unknown resolvent operator and thus it is sometimes difficult to apply in examples. Therefore it is important to be able to replace the resolvent estimate in Hille-Yosida theorem by some other conditions which are easier to check. For this purpose in this chapter, we introduce the Lumer-Phillips theorem where the resolvent estimate is replaced by dissipativity and some range condition. This result is extremely useful for a large number of applications. We first introduce the concept of dissipativity that is essential for applications of the Lumer-Phillips theorem. For more details, we refer to (Pazy, 2012) and (Hundertmark et al., 2013).

Definition 5.1 *Let X be Banach space and let X^* be the dual space of X. The value of $x^* \in X^*$ at $x \in X$ denoted by $\langle x, x^* \rangle$. For all $x \in X$ the duality set $J(x) \subseteq X^*$ is defined as follows*

$$J(x) := \{\, x^* \in X^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2 \,\}. \tag{5.1}$$

Note that if X is a Hilbert space with an inner product $(\cdot|\cdot)$ then the duality set $J(x)$ consists of only one element, namely $(\cdot|x)$.

Definition 5.2 *A linear operator A is dissipative if for all $x \in D(A)$ there exist $x^* \in J(x)$ such that $Re\langle Ax, x^* \rangle \leq 0$.*

Proposition 5.1 *(Schnaubelt, 2011) A linear operator A is dissipative if and only if $||\lambda x - Ax|| \geq \lambda ||x||$ is satisfied for all $\lambda > 0$ and $x \in D(A)$.*

Proof Let A be dissipative and $x \in D(A)$. So there exists $x^* \in J(x)$ such that $Re\langle Ax, x^* \rangle \leq 0$. For all $\lambda > 0$, we have

$$
\begin{aligned}
\|\lambda x - Ax\|\,\|x^*\| \geq |\langle \lambda x - Ax, x^*\rangle| &\geq Re\langle \lambda x - Ax, x^*\rangle \\
&\geq Re\langle \lambda x, x^*\rangle - Re\langle Ax, x^*\rangle \quad \text{since} \;\; Re\langle Ax, x^*\rangle \leq 0, \\
&\geq \lambda \|x\|^2.
\end{aligned}
$$

Consequently, $\|\lambda x - Ax\| \geq \lambda\|x\|$ as $\|x\| = \|x^*\|$.

Conversely suppose that $x \in D(A)$ and $\|\lambda x - Ax\| \geq \lambda\|x\|$ for every $\lambda > 0$. Let us first prove that A is dissipative if X is a Hilbert space with an inner product $(\cdot|\cdot)$. Then

$$
\begin{aligned}
\lambda^2\|x\|^2 \leq \|\lambda x - Ax\|^2 = (\lambda x - Ax \,|\, \lambda x - Ax) &= \lambda^2\|x\|^2 - 2\lambda(Ax|x) + \|Ax\|^2 \\
&\leq \lambda^2\|x\|^2 - 2\lambda Re(Ax|x) + \|Ax\|^2.
\end{aligned}
$$

It follows that $Re(Ax|x) \leq \frac{1}{2\lambda}\|Ax\|^2$ and since this is satisfied for all λ, we have $Re(Ax|x) \leq 0$. For the general case we assume that X is a Banach space and without loss of generality we take $\|x\| = 1$. If we choose $z^*_\lambda \in J(\lambda x - Ax)$ such that $\|z^*_\lambda\| = \|\lambda x - Ax\| \geq \lambda\|x\| = \lambda > 0$ and thus $\|z^*_\lambda\| \neq 0$. Setting $x^*_\lambda = \frac{z^*_\lambda}{\|z^*_\lambda\|}$ for $\lambda > 0$, we have $\|x^*_\lambda\| = 1$. Moreover

$$
\begin{aligned}
\lambda = \|\lambda x\| \leq \|\lambda x - Ax\| &= \left\langle \lambda x - Ax, \frac{z^*_\lambda}{\|z^*_\lambda\|} \right\rangle \\
&= Re\langle \lambda x - Ax, x^*_\lambda\rangle \\
&= \lambda Re\langle x, x^*_\lambda\rangle - Re\langle Ax, x^*_\lambda\rangle \\
&\leq min\{\lambda - Re\langle Ax, x^*_\lambda\rangle,\ \lambda Re\langle x, x^*_\lambda\rangle + \|Ax\|\}.
\end{aligned}
$$

So $\lambda \leq \lambda - Re\langle Ax, x^*_\lambda\rangle$ which implies $Re\langle Ax, x^*_\lambda\rangle \leq 0$ and $Re\langle x, x^*_\lambda\rangle \geq 1 - \frac{1}{\lambda}\|Ax\|$ as follows. We consider x^*_λ as a map on the two dimensional linearly independent subspace $S = span\{Ax, x\}$ of X. Since x^*_λ is bounded with $\|x^*_\lambda\| = 1$, there exists a functional z^* in S^* and a sequence λ_i such that $\lambda_i \to \infty$ and $x^*_{\lambda_i} \to z^* \in S^*$ as $i \to \infty$ since unit ball of S^* is compact. So $Re\langle Ax, z^*\rangle \leq 0$ and $Re\langle x, z^*\rangle \geq 1$. From the Hahn-Banach theorem, there is a bounded linear functional x^* on X^* which is extension of z^* such that $\|x^*\| = 1$,

$Re\langle Ax, x^* \rangle \leq 0$ and $Re\langle x, x^* \rangle \geq 1$. Then

$$1 \leq Re\langle x, x^* \rangle \leq |\langle x, x^* \rangle| \leq \|x^*\| \,.\, \|x\| = \|x^*\| = 1.$$

Consequently, $1 = \|x\| = \|x^*\| = \langle x, x^* \rangle$ and $x^* \in J(x)$. □

Definition 5.3 *For a linear operator A, we call A is* ***closable*** *if it has a closed extension. If A is closable, we define $\overline{A}$ as its closure as follows*

$$D(\overline{A}) := \{x \in X \,|\, \exists x_n \subseteq D(A), \exists y \in X : \lim_{n\to\infty} x_n = x \, and \, \lim_{n\to\infty} Ax_n = y\} \tag{5.2}$$

and set $\overline{A}x := y$ where y as in the definition of $D(\overline{A})$.

Definition 5.4 *Let* ***D*** *be a linear subspace of $D(A)$ of a linear operator A.* ***D*** *is called* ***core*** *for A if* ***D*** *is dense in $D(A)$ with respect to the graph norm*

$$\|x\|_A := \|x\| + \|Ax\|.$$

Proposition 5.2 *(Hundertmark et al., 2013) Let A generate a C_0-semigroup $T(\cdot)$ on a Banach space X. Let $\boldsymbol{D} \subseteq D(A)$ be a linear which is dense in X and $T(t)\boldsymbol{D} \subseteq \boldsymbol{D}$ for all $t \geq 0$. Then* ***D*** *is a core for A.*

Proof Take $x \in D(A)$. By Proposition 3.1, $T(t)x \in D(A)$. $T(\cdot)x : \mathbb{R}_+ \to [D(A)]$ is continuous since for each $t, s \geq 0$

$$\|T(t)x - T(s)x\|_A = \|T(t)x - T(s)x\| + \|AT(t)x - AT(s)x\|$$

converges to 0 as $t \to s$. For all $\varepsilon > 0$ there exists $t_f \in (0, 1]$ with $\|T(t)x - x\|_A < \varepsilon$ for each $t \in [0, t_f]$. Using Property 2.3, we have

$$\left\| \frac{1}{t_f} \int_0^{t_f} T(t)x \, \mathrm{d}t - x \right\|_A \leq \frac{1}{t_f} \int_0^{t_f} \|T(t)x - x\|_A \, \mathrm{d}t < \varepsilon.$$

Since $\mathbf{D}$ is dense in $D(A)$, there is a vector $y \in \mathbf{D}$ such that $\|x - y\| < \varepsilon$. Let $\overline{\mathbf{D}}$ be the closure of $\mathbf{D}$ in $[D(A)]$. We define

$$\omega = \frac{1}{t_f}\int_0^{t_f} T(t)y\,\mathrm{d}t \in \overline{\mathbf{D}},$$

which is close to x for the graph norm of A. By the given assumption $\frac{1}{t_f}\int_0^{t_f} T(t)y\,\mathrm{d}t \in \mathbf{D} \subseteq D(A)$ and so $\omega \in \overline{\mathbf{D}}$. We set $m = \sup_{t\in[0,1]} \|T(t)\|$. Then

$$\begin{aligned}
\|x-\omega\|_A &\leq \left\| x - \frac{1}{t_f}\int_0^{t_f} T(t)x\,\mathrm{d}t + \frac{1}{t_f}\int_0^{t_f} T(t)x - \frac{1}{t_f}\int_0^{t_f} T(t)y\,\mathrm{d}t \right\|_A \\
&\leq \left\| x - \frac{1}{t_f}\int_0^{t_f} T(t)x\,\mathrm{d}t \right\|_A + \left\| \frac{1}{t_f}\int_0^{t_f} T(t)x - \frac{1}{t_f}\int_0^{t_f} T(t)y\,\mathrm{d}t \right\|_A \quad \text{by Lemma 4.2,} \\
&\leq \varepsilon + \frac{1}{t_f}\left\| \int_0^{t_f} T(t)(x-y)\,\mathrm{d}t \right\| + \frac{1}{t_f}\left\| A\int_0^{t_f} T(t)(x-y)\,\mathrm{d}t \right\| \\
&\leq \varepsilon + m\|x-y\| + \frac{1}{t_f}\left\| (T(t_f)-I)(x-y) \right\| \\
&\leq \varepsilon + \left(m + \frac{m-1}{t_f} \right)\|x-y\| \leq K\varepsilon.
\end{aligned}$$

Finally, since $\omega \in \overline{\mathbf{D}}$ in $[D(A)]$, we take a vector $s \in \mathbf{D}$ with $\|\omega - s\| \leq \varepsilon$ and so $\|x - s\|_A \leq \|x - \omega\|_A + \|\omega - s\|_A \leq n\varepsilon$. □

Proposition 5.3 *For a dissipative operator A, the following properties are satisfied.*

- ***i)*** *The operator $\lambda I - A$ is injective for each $\lambda > 0$ and for $y \in R(\lambda I - A)$ we obtain $\|(\lambda I - A)^{-1}y\| \leq \frac{1}{\lambda}\|y\|$.*
- ***ii)*** *If $\lambda_0 I - A$ is surjective for some $\lambda_0 > 0$, then A is closed, $(0,\infty) \subseteq \rho(A)$ and also $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$ for each $\lambda > 0$.*
- ***iii)*** *Let A be densely defined. Then A is closable and $\overline{A}$ is also dissipative.*

Proof

- **i)** If a linear operator A is dissipative then for $x \in D(A)$, we have $\|(\lambda I - A)x\| \geq \lambda\|x\|$. Assume that $(\lambda I - A)x = 0$ we then have $\lambda\|x\| \leq \|(\lambda I - A)x\| = 0$, which implies $x = 0$ for $\lambda > 0$ and so, $A - \lambda I$ is injective. Moreover, letting $y = (\lambda I - A)x$ in $\|(\lambda I - A)x\| \geq \lambda\|x\|$ we thus get $\|y\| \geq \lambda\,\|(\lambda I - A)^{-1}y\|$.

ii) To show ii) we assume that $\lambda_0 I - A$ is surjective, then the assumption i) gives that $\lambda_0 I - A$ is invertible for some $\lambda_0 > 0$ with

$$\|(\lambda_0 I - A)^{-1} y\| \leq \frac{1}{\lambda_0}\|y\|.$$

Since $(\lambda_0 I - A)^{-1}$ exists and bounded for some $\lambda_0 \in \mathbb{C}$, we have $\rho(A) \neq \emptyset$ and thus Property 2.5 shows that A is closed. We choose an arbitrary $\lambda \in (0, 2\lambda_0)$. It satisfies $|\lambda - \lambda_0| < \lambda_0 < \frac{1}{\|R(\lambda_0, A)\|}$ and thus $\lambda \in \rho(A)$. Assertion i) also satisfies $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$ for $\lambda \in (0, 2\lambda_0)$. From the above it follows that $(0, 2\lambda_0) \subseteq \rho(A)$ and so $\left(0, \frac{3}{2}\lambda_0\right] \subseteq \rho(A)$ as $\lambda \in \rho(A)$. Proceeding by induction $\left(0, \left(\frac{3}{2}\right)^n \lambda_0\right] \subseteq \rho(A)$ for each $n \in \mathbb{N}$ and thus $(0, \infty) \subseteq \rho(A)$.

iii) Let us suppose that $\overline{D(A)} = X$. In order to show the closability of A, we take $x_n \in D(A)$ such that $x_n \to 0$ and $Ax_n \to y$ in X as $n \to \infty$. From the density assumption, there exists another sequence $y_k \in D(A)$ such that $y_k \to y$ in X as $k \to \infty$. For $\lambda > 0$ and $n, k \in \mathbb{N}$, Proposition 5.1 yields

$$\|\lambda^2 x_n + \lambda y_k - \lambda A x_n - A y_k\| = \|(\lambda I - A)(\lambda x_n + y_k)\| \geq \lambda \|\lambda x_n + y_k\|.$$

Taking the limit as $n \to \infty$, we have

$$\|(\lambda I - A) y_k - \lambda y\| \geq \lambda \|y_k\|.$$

This inequality is equivalent to $\|y_k - \frac{1}{\lambda} A y_k - y\| \geq \|y_k\|$ for $\lambda > 0$. Letting $\lambda \to \infty$, $\|y_k - y\| \geq \|y_k\|$ and it also follows that $0 \geq \|y\|$ as $k \to \infty$ and thus $y = 0$. Consequently, A is closable. To show the dissipativity of $\overline{A}$, we take $x \in D(\overline{A})$. By means of the definition of a closable operator, there is a sequence $w_n \subseteq D(A)$ which satisfies $w_n \to x$ and $A w_n \to \overline{A} x$ in X as $n \to \infty$. Since A is dissipative, it follows that

$$\|\lambda x - \overline{A} x\| = \lim_{n\to\infty} \|\lambda w_n - A w_n\| \geq \lambda \lim_{n\to\infty} \|w_n\| = \lambda \|x\|,$$

and so $\overline{A}$ is dissipative.

□

Theorem 5.1 *(Lumer-Phillips)(Lumer, 1961) For a linear, densely defined operator A on a Banach space X the following assumptions hold.*

i) *Let A be dissipative and $R(\lambda_0 I - A)$ be dense in X for some $\lambda_0 > 0$. Then A is closable and $\overline{A}$ generates a contraction semigroup.*

ii) *Let A be dissipative and $\lambda_0 I - A$ be surjective for some $\lambda_0 > 0$. Then A generates a contraction semigroup.*

iii) *Let A generates a contraction semigroup. Then A is dissipative, $||R(\lambda, A)|| \leq \frac{1}{Re\lambda}$ for all $\lambda \in \mathbb{C}_+$ and also $\mathbb{C}_+ \subseteq \rho(A)$.*

Proof

i) Suppose that A is densely defined and dissipative, from Proposition 5.3 we deduce that $\overline{A}$ is dissipative. Since $R(\lambda_0 I - A) \subseteq R(\lambda_0 I - \overline{A})$, $\lambda_0 I - A$ has a dense range. For $y \in X$, we take a sequence $x_n \in D(\overline{A})$ such that $(\lambda_0 - \overline{A})x_n := y_n \to y$ in X as $n \to \infty$. Using the dissipativity of $\overline{A}$, we thus get

$$\begin{aligned}\lambda_0 ||x_n - x_m|| &\leq ||(\lambda_0 I - \overline{A})(x_n - x_m)|| \\ &= ||(\lambda_0 x_n - \overline{A}x_n) - (\lambda_0 x_m - \overline{A}x_m)|| \\ &= ||y_n - y_m||\end{aligned}$$

for each $n, m \in \mathbb{N}$. Accordingly, $x_n \to x$ in X and so $\overline{A}x_n = \lambda_0 x_n - y_n \to \lambda_0 x - y$ as $n \to \infty$. From the closedness of $\overline{A}$, $x \in D(\overline{A})$ and $\overline{A}x = \lambda_0 x - y$. This implies that $R(\lambda_0 I - \overline{A}) = X$ and consequently $\lambda_0 I - \overline{A}$ is surjective. By part ii) of Proposition 5.3, $\overline{A}$ is closed, $(0, \infty) \subseteq \rho(\overline{A})$ and $||R(\lambda, \overline{A})|| \leq \frac{1}{\lambda}$. Therefore $\overline{A}$ generates a contraction semigroup which follows from Hille-Yosida generation theorem.

ii) Proposition 5.3 implies that A is closed under the surjectivity of $\lambda_0 I - A$ assumption. By part i), $A = \overline{A}$ generates a contraction semigroup.

iii) Assume that A generates a contraction semigroup. We now take $x \in D(A)$ and $x^* \in J(x)$.

$$\begin{aligned} Re\langle Ax, x^*\rangle &= \lim_{t\to 0^+} Re\langle \tfrac{1}{t}(T(t)x - x), x^*\rangle \\ &= \lim_{t\to 0^+} \tfrac{1}{t}(Re\langle T(t)x, x^*\rangle - \langle x, x^*\rangle) \\ &\leq \lim_{t\to 0^+} \sup \tfrac{1}{t}(\|T\|\,\|x\|\,\|x^* - \|x\|^2) \end{aligned}$$

as $\|x^*\| = \|x\|$ and $\|T\| \leq 1$ we obtain $Re\langle Ax, x^*\rangle \leq 0$, which gives A is dissipative and also Theorem 4.1 satisfies the other assumptions. $\square$

Let us first replace the range condition of the Lumer-Phillips theorem by the injectivity of $\lambda I - A^{'}$ for some $\lambda > 0$ by means of the next corollary. Since the injectivity of $\lambda I - A^{'}$ is much easier to check then the range conditions in parts a) and b) of the Lumer-Phillips theorem .

Corollary 5.1 *Let A be a densely defined operator on Hilbert space X. If A is dissipative and $\lambda I - A^{'}$ is injective for some $\lambda > 0$ then $\overline{A}$ generates a contraction semigroup.*

Proof By means of the Lumer-Phillips theorem, we only need to show that $R(\lambda I - A)$ is dense in X. Suppose that $y \in R(\lambda I - A)^{\perp}$, then for all $x \in X$ such that $((\lambda I - A)x|y) = 0 = (x|(\lambda I - A^{'})y)$. This implies that for all $x \in X$, $(\lambda I - A^{'})y = 0$. From the injectivity of $\lambda I - A^{'}$, we get $y = 0$ and thus $R(\lambda I - A)^{\perp} = \{0\}$. $\square$

Lemma 5.1 *(Schnaubelt, 2011) Let $T(\cdot)$ be a C_0-semigroup on a Banach space X. If there is a $t_0 > 0$ such that $T(t_0)$ is invertible, then $T(\cdot)$ can be extented to a C_0-group $T(t)_{t\in\mathbb{R}}$ on X.*

Proof Let us first show that $T(t)$ is invertible for each $t \geq 0$. There exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for each $t \geq 0$. We now set $K = \|T(t_0)^{-1}\|$. It follows for $0 \leq t \leq t_0$

$$T(t_0) = T(t_0 - t)T(t) = T(t)T(t_0 - t).$$

Since $T(t_0)$ is invertible,

$$I = T(t_0)^{-1}T(t_0 - t)T(t) = T(t)T(t_0)^{-1}T(t_0 - t)$$

and thus $T(t)$ has the inverse $T(t_0)^{-1}T(t_0 - t)$ with

$$\|T(t_0)^{-1}T(t_0 - t)\| \leq \|T(t_0)^{-1}\| \, \|T(t_0 - t)\| \leq K.Me^{\omega(t_0 - t)} := K_1.$$

Thus $T^{-1}(t)$ is bounded. Moreover, let $t = mt_0 + \xi$ for $m \in \mathbb{N}$ and $\xi \in [0, t_0)$. Then

$$T(t) = T(mt_0 + \xi) = T(t_0)^m T(\xi),$$

which has the inverse $T(t_0)^{-m}T(\xi)^{-1}$. Therefore $T(t)$ is invertible and we can extend $T(\cdot)$ to $\mathbb{R}$ defining $T(t) := T(-t)^{-1}$ for $t \leq 0$. For $t, s \geq 0$,

$$T(-t)T(-s) = T(t)^{-1}T(s)^{-1} = (T(s)T(t))^{-1} = T(t + s)^{-1} = T(-t - s),$$

$$T(-t)T(s) = (T(s)T(t - s))^{-1}T(s) = T(t - s)^{-1}T(s)^{-1}T(s) = T(t - s), \quad t \geq s,$$

$$T(-t)T(s) = T(t)^{-t}T(t)T(s - t) = T(s - t), \quad s \geq t.$$

The above definition satisfies the semigroup laws. To show the strong continuity of $T(-t)_{t\leq 0}$, we take $x \in X$ and $t \in [0, t_0]$,

$$\|T(-t)x - x\| = \|T(-t)(x - T(t)x)\| \leq K_1\|T(t)x - x\| \to 0$$

as $t \to 0$. Consequently, $T(t)_{t\in\mathbb{R}}$ is a C_0-group . □

Definition 5.5 *The generator A of a strongly continuous operator group or C_0-group $T(t)_{t\in\mathbb{R}}$ is defined by*

$$Ax := \lim_{t\to 0} \tfrac{1}{t}(T(t)x - x)$$

where

$$D(A) := \{x \in X : \lim_{t\to 0} \tfrac{1}{t}(T(t)x - x) \text{ exists in } X\}.$$

From the given definition, we denote $T_+(t) := T(t)$ and $T_-(t) := T(-t)$ for $t \geq 0$ which are C_0-semigroups generated by A and $-A$, respectively.

Theorem 5.2 *(Stone, 1932) Let A be a linear, densely defined operator on a Hilbert space X. Then A generates a C_0-group $T(\cdot)$ of unitary operators if and only if A is skew-adjoint.*

Proof ($\Leftarrow$) Assume that $A' = -A$. For an element $x \in D(A) = D(A')$, we have a duality set $J(x) = \{\psi_x\}$ with $\psi_x := (\cdot|x)$. Then we evaluate

$$\langle Ax, \psi_x \rangle = (Ax|x) = (x|A'x) = -(x|Ax) = -\overline{(Ax|x)}$$

and thus $Re\,(Ax|x) = 0$. Consequently, A and A' are dissipative since $A' = -A$ and $Re\,(A'x|x) = 0$. Since $A = (-A)' = A''$, A and A' are closed as well $\overline{A} = A$ and $\overline{A'} = A'$. If $(\lambda I - A)x = 0$, then $\lambda x = Ax$ and from the dissipativity of A, we have

$$Re\,(Ax|x) = Re\,(\lambda x|x) = \lambda Re\,(x|x) = \lambda ||x||^2 = 0$$

$\lambda > 0$. Thus $\lambda I - A$ is injective and so $\lambda I - A'$. By Corollary 5.1, A and $A' = -A$ generates contraction semigroups $T_+(t)$ and $T_-(t)$ respectively. By Definition 5.5, $T(t)$ is a C_0-group and so $T(t)$ is surjective. Since $||T_+(t)|| \leq 1$ and $||T^{-1}(t)|| = ||T(-t)|| \leq 1$, $T(t)$ is bounded. The proof is completed by showing that $T(t)$ is also isometric as follows:

$$||T(t)x|| \leq ||x|| = ||T(t)T(-t)x|| \leq ||T(-t)||\,||T(t)x|| \leq ||T(t)x||$$

for each $x \in X$ and $t \in \mathbb{R}$. Therefore each $T(t)$ is unitary since T is surjective and isometric. ($\Rightarrow$) Let $T(t)$ be a unitary C_0-group with generator A. The family $T'(t)$ is a contraction semigroup generated by $-A$ since $T'(t) = T(t)^{-1} = T(-t)$ for $t \geq 0$. For each $x, y \in D(A)$,

$$(Ax|y) = \left(\lim_{t\to 0} \frac{1}{t}(T(t)x - x)|y\right) = \lim_{t\to 0}\left(x|\frac{1}{t}(T'(t)y - y)\right) = (x| - Ay),$$

which implies $-A \subseteq A'$. Since $||T(t)|| = 1$ from the isometry property, Theorem 4.2 implies $|Re\lambda| > 0$ for C_0-groups and $\sigma(A) \subseteq i\mathbb{R}$. For a closed and densely defined operator A, $\sigma(A') = \{\overline{\lambda}\,|\,\lambda \in \sigma(A)\} \subseteq i\mathbb{R}$ and also $\rho(A) \bigcap \rho(A') \neq \emptyset$. By Property 2.7, $-A = A'$. □

CHAPTER 6

EXAMPLES INVOLVING LAPLACIAN OPERATOR

Example 6.1 *Consider the heat equation*

$$\frac{\partial}{\partial t}u = \Delta u, \quad x \in \mathbb{R}^n, \ t > 0,$$
$$u(0,x) = u_0, \quad x \in \mathbb{R}^n. \tag{6.1}$$

We choose $X = L^2(\mathbb{R}^n)$ *for* $\xi \in \mathbb{R}^n$ *and define its domain*

$$D(A) := \{u \in X \mid |\xi|_2^2(\hat{u}) \in X\} = H^2(\mathbb{R}^n).$$

Using part c) of Property 2.10 and applying the inverse Fourier transform, we then obtain $Au := -\mathcal{F}^{-1}(|\xi|_2^2\hat{u}) = \Delta u$*. Note that* $\Delta u = \sum_{i=1}^{n} \partial_i^2 u = div(\nabla u)$*.*

Our aim is to show that the operator A is dissipative, self-adjoint and $\sigma(A) \subseteq \mathbb{R}_-$*. Remember that* $H^2(\mathbb{R}^n) = \mathring{H}^2(\mathbb{R}^n)$ *by Theorem 2.1, so that Theorem 2.2 and equality* (2.7) *lead to for all* $u, \upsilon \in D(A)$

$$(Au|\upsilon) = \int_{\mathbb{R}^n} \Delta u\overline{\upsilon}\,\mathrm{d}x = \int_{\mathbb{R}^n} \mathrm{div}(\nabla \mathrm{u})\overline{\upsilon}\,\mathrm{dx} = -\int_{\mathbb{R}^n} \nabla \mathrm{u}.\nabla\overline{\upsilon}\,\mathrm{dx} = \int_{\mathbb{R}^n} \mathrm{u}\Delta\overline{\upsilon}\,\mathrm{dx} = (\mathrm{u}|\mathrm{A}\upsilon),$$

which implies that A is symmetric. Moreover,

$$\int_{\mathbb{R}^n} \Delta u\overline{u}\,\mathrm{d}x = -\int_{\mathbb{R}^n} |\nabla u|^2\,\mathrm{d}x \leq 0,$$

which shows $Re(Au|u) = (Au|u) \leq 0$*, hence A is dissipative.*

To show that the range condition for the Laplacian operator on $\mathbb{R}^n$ *let* $\lambda > 0$ *and* *take $f \in X$.*

$$\hat{f} = \lambda\hat{u} - \sum_{m=1}^{n} i^2\xi_m^2\hat{u} = (\lambda + |\xi|_2^2)\hat{u}.$$

Then $\hat{u} = \frac{\hat{f}}{\lambda+|\xi|_2^2}$ which is element of X and so let $u = \mathcal{F}^{-1}(\frac{1}{\lambda+|\xi|_2^2}\hat{f})$ in X. From the inverse Fourier transform,

$$\lambda u - \Delta u = \mathcal{F}^{-1}\left(\frac{\lambda}{\lambda + |\xi|_2^2}\hat{f} - i^2\frac{|\xi|_2^2}{\lambda + |\xi|_2^2}\hat{f}\right) = f.$$

Therefore, $u \in H^2(\mathbb{R}^n)$ and $\lambda I - A$ is surjective. Moreover $H^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and thus A is densely defined. Lumer-Phillips theorem yields that A generates a contraction semigroup, and it follows that A is closed and $(0, \infty) \subseteq \rho(A)$ by the Hille-Yosida theorem. Since A is symmetric and $\rho(A) \bigcap \mathbb{R} \neq \emptyset$, Property 2.6 satisfies $\sigma(A) \subseteq \mathbb{R}$ and thus A is self-adjoint. Finally, $(0, \infty) \subseteq \rho(A)$ implies $\sigma(A) \subseteq \mathbb{R}_-$.

As a result of the Lumer-Phillips theorem, A generates a C_0-semigroup $T(\cdot)$. Thus the function u defined by $u(t) = T(t)u_0$ for $t \geq 0$ is the unique solution of the given diffusion equation.

For the next example we will need the Lax-Milgram lemma as follows.

Theorem 6.1 *(Lax-Milgram Lemma) Let H be a Hilbert space and $\alpha : H \times H \to \mathbb{C}$ be a sesquilinear form (i.e., $u \to \alpha(u, v)$ is linear and $v \to \alpha(u, v)$ is antilinear for $u, v \in H$) which is bounded and strictly accretive, namely there exist $C, \delta > 0$*

$$||\alpha(u, v)|| \leq C\,||u||.||v|| \quad \textit{and} \quad Re\,\alpha(u, u) \geq \delta||u||^2 \tag{6.2}$$

for each $u, v \in H$. Then for all functional $\psi \in H^*$ there exists a unique vector $\omega \in H$ such that $\alpha(v, \omega) = \psi(v)$ for all $v \in H$. The map $\psi \to \omega$ is bounded and antilinear.

Proof The map $\varphi_v := \alpha(\cdot, v) \in H^*$ and $||\varphi_v(u)|| = ||\alpha(u, v)|| \leq C\,||u||.||v||$ which implies $||\varphi_v|| \leq C||v||$ for all $v \in Y$. Riesz representation theorem now yields a unique Sv satisfying

$$(u|Sv) = \varphi_v(u) = a(u, v)$$

for each $u \in H$ and $||Sv|| = ||\varphi_v|| \leq C||v||$. So S is a linear and bounded operator with constant C. In addition, the strict accretivity assumption now gives

$$\delta||v||^2 \leq Re\,\alpha(v, v) = Re\,(v|Sv) \leq |(v|Sv)| \leq C\,||v||\,||Sv||$$

and thus

$$||v|| \leq \tfrac{C}{\delta}||Sv|| \quad \text{for all } v \in H. \tag{6.3}$$

From this inequality if $||Sv|| = 0$, then we get $||v|| = 0$ so S is injective. Assuming $u \in H$ is orthogonal to the range $R(S)$, we deduce

$$0 = (u|Su) = Re\,(u|Su) = Re\,\alpha(u, u) \geq \delta||u||^2 \tag{6.4}$$

so that $u = 0$. Then $R(S)^{\perp} = \{0\}$ is equivalent to $\overline{R(S)} = H$. To show the closedness of $R(S)$, we take a sequence Su_n in $R(S)$ such that Su_n converges to x. For all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n, m > N$ then $||Su_n - Su_m|| < \epsilon$. Moreover by (6.3)

$$||u_n - u_m|| \leq \tfrac{C}{\delta}||Su_n - Su_m|| < M\epsilon.$$

Hence u_n is Cauchy and H is complete space then this sequence converges to some element $u \in H$. Since S is bounded, Su_n converges to Su. Then by uniqueness of limit $Su = x$ and $x \in R(S)$. So $R(S)$ is closed, i.e. S is surjective and hence it is invertible with $||S^{-1}|| \leq \frac{C}{\delta}$. Let $\psi \in H^*$. There exist unique $h \in H$ for all $v \in H$ such that $\psi(v) = (v|h)$ due to Riesz representation theorem. Then

$$\psi(v) = (v|h) = (v|SS^{-1}h) = \alpha(v, S^{-1}h)$$

for all $v \in H$. Setting $\omega = S^{-1}h$, we obtain $\psi(v) = \alpha(v, \omega)$. Let $K : H^* \to H$ such that

$K\psi = \omega$. Then, we have

$$\|K\psi\| = \|\omega\| = \|S^{-1}h\| \leq \|S^{-1}\|\,\|h\| \leq \|S^{-1}\|\,\|\psi\|$$

since $\|h\| = \|\psi\|$ by Riesz representation theorem and this implies that K is bounded. We set $K(\lambda\psi) = S^{-1}h^*$ such that $\lambda\psi(v) = (v|h^*)$ then $\psi(v) = (v|\frac{1}{\lambda}h^*) = (v|h)$ which implies $h^* = \bar{\lambda}h$. So

$$K(\lambda\psi) = S^{-1}h^* = S^{-1}\bar{\lambda}h = \bar{\lambda}S^{-1}h = \bar{\lambda}K\psi$$

which shows K is antilinear. If also $\tilde{\omega} \in H$ satisfies $\alpha(v, \tilde{\omega}) = \psi(v)$ for each $v \in H$, then $\delta\|\omega - \tilde{\omega}\| \leq \alpha(\omega - \tilde{\omega}, \omega - \tilde{\omega}) = 0$ as in (6.4) which gives the uniqueness of ω. □

Example 6.2 *Let U be a nonempty open bounded subset in $\mathbb{R}^n$ and $X = L^2(U)$. We the sesquilinear form as*

$$\alpha(u, v) = \int_U \nabla u.\nabla\bar{v}\,\mathrm{d}x \tag{6.5}$$

for $u, v \in \mathring{H}^1(U) =: Y$. We denote the norm $\|f\|_p = \|f\|_{L^p}$ on $L^p(U)$ and $\|f\|_{1,2} = \|f\|_2 + \|\nabla f\|_2$ on $\mathring{H}^1(U)$. Now, our aim is to construct a self-adjoint, dissipative and invertible operator A corresponding to the sesquilinear form α. Thanks to Hölder's inequality,

$$|\alpha(u, v)| = \left|\int_U \nabla u.\nabla\bar{v}\,\mathrm{d}x\right| \leq \int_U |\nabla u.\nabla\bar{v}|\,\mathrm{d}x = \|\nabla u.\nabla v\|_1 \leq \|\nabla u\|_2\|\nabla v\|_2 \leq C\|u\|_{1,2}\,\|v\|_{1,2}$$

and by Poincaré's inequality (2.8),

$$\begin{aligned} Re\,\alpha(u, u) = Re\int_U |\nabla u|^2\,\mathrm{d}x = \|\nabla u\|_2^2 &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 + \delta\frac{1}{2}\|u\|_2^2 \\ &\geq m(\|\nabla u\|_2^2 + \|u\|_2^2) \;\; \textit{where} \;\; m = min\left\{\tfrac{1}{2}, \tfrac{\delta}{2}\right\}, \\ &= m\|u\|_{1,2}^2. \end{aligned}$$

Consequently, the form α is bounded and strictly accretive and thus it satisfies the Lax-

Milgram lemma. We now introduce

$$D(A) := \{u \in Y | \ \exists f \in X \ \forall v \in Y : \alpha(u,v) = (f|v)_{L^2}\},$$
$$Au := -f, \quad \textit{where f is given by } D(A).$$

Let us first prove that f is unique. Assume that for given u, there is another $g \in X$ satisfying the condition of D(A) such that

$$\alpha(u,v) = (f|v) = (g|v).$$

Then $(f - g|v) = 0$ for all $v \in Y$. This implies that $f - g \in Y^{\perp} = \{0\}$ since Y is dense in X. Hence $f = g$. Clearly, A is linear. The map $\varphi_f : v \to (v|f)$ belongs to Y^* if $f \in L^2(U)$ and

$$||\varphi_f||_{Y^*} = \sup_{||v||_{1,2} \leq 1} |(v|f)_{L^2}| \leq \sup_{||v||_{1,2} \leq 1} ||v||_2 ||f||_2 \leq ||f||_2. \tag{6.6}$$

Lax-Milgram lemma now gives a unique $u \in Y$ such that

$$\alpha(v,u) = \varphi_f(v) = (v|f)_{L^2}, \quad \forall v \in Y,$$

which means $u \in D(A)$ and $Au = -f$. Hence A is surjective. Moreover taking $v = u$,

$$\delta ||u||_{1,2}^2 \leq Re \ \alpha(u,u) \leq |\alpha(u,u)| = |\varphi_f(u)| \leq ||\varphi_f||_{Y^*} ||u||_{1,2}$$

by using (6.6), we get

$$||u||_{1,2} \leq \frac{1}{\delta} ||\varphi_f||_{Y^*} \leq c||f||_2, \quad \textit{where} \ \ \frac{1}{\delta} = c. \tag{6.7}$$

The inequality (6.6) and (6.7) imply

$$||u||_2 \leq ||u||_{1,2} \leq c||\varphi_f||_{Y^*} \leq c||f||_2 = c||Au||_2. \tag{6.8}$$

If $Au = 0$ then inequality (6.8) gives $u = 0$. So A is injective. Thus, A is bijective. If we take

$x = Au$ in inequality (6.8), we obtain that A^{-1} is bounded since $\|u\|_2 = \|A^{-1}x\|_2 \leq c\|x\|_2$. Let $T : X \times X \to X \times X$ be a reflection map such that

$$T(x, y) = (y, x).$$

We want to showing that A is closed which means $Gr(A)$ is closed. But this is equivalent to show that A^{-1} is closed since $Gr(A) = T\,Gr(A^{-1})$. Note that $D(A^{-1}) = R(A) := X$ which is closed and A^{-1} is bounded. Hence by the closed graph theorem, A^{-1} is closed.

There is at least one value $\lambda = 0$ in $\rho(A)$ since A is bijective and has a bounded inverse. Moreover, we know that $\rho(A)$ is open. Hence there exists $r > 0$ such that $B_r(0) \subseteq \rho(A)$ in particular for $0 < \lambda_0 < r$, λ_0 in $\rho(A)$, which implies $(\lambda_0 I - A)$ is surjective.

On the other hand, for $u, v \in D(A)$ we compute

$$(Au|v)_{L^2} = (-f|v)_{L^2} = -\alpha(u, v) = -\overline{\alpha(v, u)} = \overline{(Av|u)_{L^2}} = (u|Av)_{L^2}.$$

Therefore A is symmetric. If we take $u = v$, strict accretivity property of Lax-Milgram lemma gives

$$(Au|u)_{L^2} = -\alpha(u, u) \leq -\delta\|u\|_{1,2}^2 \leq 0,$$

which implies $Re(Au|u) \leq 0$. So A is dissipative. Consequently A is densely defined, dissipative and $\lambda_0 I - A$ is surjective for some $\lambda_0 > 0$. The Lumer-Phillips theorem implies that A generates a contraction semigroup and by the Hille-Yosida theorem, we have $(0, \infty) \subseteq \rho(A)$. Finally, it is self-adjoint by means of Property 2.6.

Example 6.3 *The wave equation with Dirichlet boundary conditions on a open and bounded domain $U \subseteq \mathbb{R}^n$ is given by*

$$\begin{aligned} &\frac{\partial^2}{\partial t^2} u(t, x) = \Delta u(t, x), && x \in U,\ t \in \mathbb{R}, \\ &u(t, x) = 0, && x \in \partial U,\ t \in \mathbb{R}, \\ &u(0, x) = u_0(x), \quad \frac{\partial}{\partial t} u(0, x) = u_1(x), && x \in U. \end{aligned} \tag{6.9}$$

The domain of Dirichlet Laplacian $D(\Delta_D)$ contains $u \in \mathring{H}^1(U)$ such that there exists $f \in L^2(U)$ with

$$\forall v \in \mathring{H}^1(U) \quad \int_U \nabla u.\nabla \bar{v}\, \mathrm{d}x = \int_U f\bar{v}\, \mathrm{d}x, \tag{6.10}$$

and then we define $\Delta_D u = -f$. In order to apply the wave equation (6.9) we choose the Hilbert space $X = \mathring{H}^1(U) \times L^2(U)$ endowed with inner product structure by defining

$$\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \int_U (\nabla u_1.\nabla \bar{v}_1 + u_2 \bar{v}_2)\, \mathrm{d}x.$$

By means of Poincaré's estimate (2.8), induced norm is equivalent to the usual norm on X which is $(\|u_1\|_{1,2}^2 + \|u_2\|_2^2)^{1/2}$. On X we define the operator

$$A = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix} \textit{ with } \quad D(A) = D(\Delta_D) \times \mathring{H}^1(U).$$

The given wave equation can be transformed into the form of Cauchy problem for A in X. In this example our aim is to show the skew-adjointness of the operator A. Let $(u_1, u_2)^T, (v_1, v_2)^T \in D(A)$ and $u_1, v_1 \in D(\Delta_D)$. Then we evaluate

$$\left(A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \left(\begin{pmatrix} u_2 \\ \Delta_D u_1 \end{pmatrix} \middle| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \int_U (\nabla u_2.\nabla \bar{v}_1 + (\Delta_D u_1)\bar{v}_2)\, \mathrm{d}x$$

using by $\Delta_D u_1 = -f_1$ and equality (6.10)

$$\begin{aligned}
&= \int_U (\nabla u_2.\nabla \bar{v}_1 - f_1 \bar{v}_2)\, \mathrm{d}x \\
&= \int_U (\nabla u_2.\nabla \bar{v}_1 - \nabla u_1.\nabla \bar{v}_2)\, \mathrm{d}x \\
&= \overline{\int_U \nabla \bar{u_2} \nabla v_1\, \mathrm{d}x} - \int_U \nabla u_1 \nabla \bar{v_2}\, \mathrm{d}x
\end{aligned}$$

and also applying (6.10) and $\Delta_D v_1 = -f_2$

$$
\begin{aligned}
&= \overline{\int_U \bar{u_2} f_2 \, \mathrm{d}x} - \int_U \nabla u_1 \nabla \bar{v_2} \, \mathrm{d}x \\
&= -\overline{\int_U \bar{u_2} \Delta_D v_1 dx} - \int_U \nabla u_1 \nabla \bar{v_2} \, \mathrm{d}x \\
&= -\left(\int_U \Delta_D \bar{v_1} u_2 \, \mathrm{d}x + \int_U \nabla u_1 \nabla \bar{v_2} \, \mathrm{d}x \right) \\
&= -\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| \begin{pmatrix} v_2 \\ \Delta_D v_1 \end{pmatrix} \right) = -\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)
\end{aligned}
$$

so that A is skew-symmetric and thus iA is symmetric. Furthermore,

$$
\begin{aligned}
Re\left(A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \middle| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = Re\left(\begin{pmatrix} u_2 \\ \Delta_D u_1 \end{pmatrix} \middle| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) &= Re \int_U (\nabla u_2 . \nabla \bar{u}_1 + (\Delta_D u_1) \bar{u}_2) \, \mathrm{d}x, \\
&= Re \int_U (\nabla u_2 . \nabla \bar{u}_1 - \nabla u_1 \nabla \bar{u}_2) \, \mathrm{d}x = 0.
\end{aligned}
$$

We see that $Re(Au|u) = 0$ for all $u \in D(A)$, so that A is dissipative. Let the operator R is defined by

$$
R = \begin{pmatrix} 0 & \Delta_D^{-1} \\ I & 0 \end{pmatrix}
$$

on X where Δ_D^{-1} exists by Example 6.2. We will show that R is bounded.

$$
\| Ru \|_X = \left\| R \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\|_X = \left\| \begin{pmatrix} \Delta_D^{-1} u_2 \\ u_1 \end{pmatrix} \right\|_X = (\| \Delta_D^{-1} u_2 \|_{1,2}^2 + \| u_1 \|_2^2)^{1/2}.
$$

We will use the sesquilinear form to explain the expression $\| \Delta_D^{-1} u_2 \|_{1,2}^2$. If we take

$u = v = w$ in (6.5) and use the Cauchy-Schwarz inequality, then

$$\int \| \nabla w \|^2 \, \mathrm{d}x = \|(\Delta_D w|w)\| \leqslant \|\Delta_D w\|_2 . \|w\|_2. \tag{6.11}$$

We know that $\Delta_D w = f$ and substitute $w = \Delta_D^{-1} f$ into the sesquilinear form (6.5), *we get*

$$(f|v)_{L^2} = \int \nabla(\Delta_D^{-1} f)\nabla\bar{v} \, \mathrm{d}x, \quad \textit{for all} \;\; v \in \mathring{H}^1(U).$$

Let us take $v = \Delta_D^{-1} f$ and using the inequality (6.11),

$$\begin{aligned}\|\nabla\Delta_D^{-1} f\|_2^2 = \int \|\nabla\Delta_D^{-1} f\|^2 \, \mathrm{d}x &\le \|f\|_2 . \|\Delta_D^{-1} f\|_2, \quad \textit{by the Poincaré's Inequality} \\ &\le \frac{1}{\delta}\|f\|_2 . \|\nabla\Delta_D^{-1} f\|_2, \quad \textit{some } \delta > 0.\end{aligned}$$

It follows that

$$\|\nabla\Delta_D^{-1} f\|_2 \le \frac{1}{\delta}\|f\|_2. \tag{6.12}$$

From the inequality (6.12), it is easily seen that $\|\Delta_D^{-1} f\|_{1,2} \le c\|f\|_2$ for some $c > 0$ and $\|u_1\|_2 \le \|u_1\|_{1,2}$. Consequently,

$$(\|u_1\|_2^2 + \|\Delta_D^{-1} u_2\|_{1,2}^2)^{1/2} \le c \, (\|u_1\|_{1,2}^2 + \|u_2\|_2^2)^{1/2}.$$

Namely, $\|Ru\|_X \le c\|u\|_X$.

It obvious that $RX \subseteq D(A)$ and $AR = I$, as well as $RA\mu = \mu$ for all $\mu \in D(A)$. Consequently, A is invertible and by the openness of $\rho(A)$, $\lambda I - A$ is surjective for sufficiently small $\lambda > 0$. The Lumer-Phillips theorem shows that A generates a contraction semigroup, and thus A is closed.

In addition, iA is also invertible and so $0 \in \rho(iA)$. Since $\rho(iA) \cap \mathbb{R} \neq \emptyset$ then $\sigma(iA) \subseteq \mathbb{R}$, Property (2.6) yields that iA is self-adjoint and thus A is skew-adjoint. By Stone's theorem, A generates a unitary C_0-group on X.

CHAPTER 7

WELLPOSEDNESS AND INHOMOGENEOUS EVOLUTION EQUATION

We are interested in predicting behavior after t time of a given system. For this purpose, we require that many solutions for each initial value u_0 and also these solutions must be uniquely determined by u_0. Furthermore, we are only able to know the initial value approximately if very small changes in initial data result in small changes in the solution. Here we need the concept of wellposedness. For more details, we refer to (Hundertmark et al., 2013).

Definition 7.1 *Let A be a linear, closed operator on a Banach space X. The Cauchy problem* (3.1) *is called wellposed if the following conditions are true.*

i) *A is densely defined.*

ii) *For all $u_0 \in D(A)$ there exists a unique solution $u = u(\cdot\,; u_0)$ of* (3.1).

iii) *If elements $u_{0,n}, u_0 \in D(A)$ and $u_{0,n} \to u_0$ in X as $n \to \infty$, then $u(\cdot\,; u_{0,n}) \to u(\cdot\,; u_0)$ uniformly in compact subsets of $\mathbb{R}_+$, namely solutions dependence continuously on initial data.*

Theorem 7.1 *(Hundertmark et al., 2013) For a closed operator A, A generates a C_0-semigroup $T(\cdot)$ if and only if Cauchy problem* (3.1) *is wellposed. Moreover, $u = T(\cdot)u_0$ is the solution of* (3.1) *for all u_0 in $D(A)$.*

Proof Assume that A is a generator of a C_0-semigroup $T(\cdot)$, then $T(\cdot)u_0$ is the unique solution of (3.1) by means of Proposition 3.1 and A is densely defined by Proposition 4.1. Let $u_{0,n}$ converge to u_0 in X and let $\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, $\|u_{0,n} - u_0\| < \varepsilon$. Also let $T(t)$ be locally bounded namely, if we have some t_f then there exists $M_{t_f} > 0$ such that $\|T(t)\|_X \leq M_{t_f}$ for all $t \in [0, t_f]$. We then have,

$$\begin{aligned} \|u(t, u_{0,n}) - u(t, u_0)\|_X &= \|T(t)u_{0,n} - T(t)u_0\|_X, && \text{by linearity of T} \\ &= \|T(t)(u_{0,n} - u_0)\|_X, && \text{since T is bounded} \\ &\leq \|T(t)\|_X.\|u_{0,n} - u_0\|_X \\ &\leq M_{t_f}.\varepsilon \end{aligned}$$

and we conclude that the solution depends continously on initial data by using the property of locally boundedness of $T(t)$.

On the other hand, we now suppose that (3.1) is wellposed problem for closed operator A and initial condition u_0. The operator $T(t) : D(A) \to X$ is denoted by $T(t)x := u(t; x)$ for $x \in D(A)$ and $t \geq 0$ using condition ii). For $x, y \in D(A)$ and $\lambda, \mu \in \mathbb{C}$, the function v given by

$$v(t) = \lambda u(t; x) + \mu u(t; y) = \lambda T(t)x + \mu T(t)y$$

for $t \geq 0$ solves (3.1) with initial value $\lambda x + \mu y$ because A is linear. From the uniqueness of solution, we have

$$v(t) = u(t; \lambda x + \mu y) = T(t)(\lambda x + \mu y).$$

Hence $T(t)$ is linear for all $t \geq 0$.

Let us prove that for each $t_s > 0$ there exists a constant $c > 0$ such that $\|T(t)x\| \leq c\|x\|$ for each $x \in D(A)$ and each $t \in [0, t_s]$. Suppose the assertion to be false. Then there exists $t_s > 0$ and in particular for all $n \in \mathbb{N}$, there is $t_n \in [0, t_s]$ such that $\|T(t_n)\|_{X \to X} = \sup_{\|x\|=1} \|T(t_n)x\| > n$. This implies that there exists $\|x_n\| = 1$ such that $\|T(t_n)x_n\| := c_n > n$. Set $z_n := \frac{1}{c_n} x_n \in D(A)$ for every $n \in \mathbb{N}$. The initial values z_n tend to 0 as $n \to \infty$ but the norms $\|u(t_n; z_n)\| = \|T(t_n)z_n\| = \frac{1}{c_n}\|T(t_n)x_n\| = 1$ do not converge to 0. This contradicts the assumption iii) in Definition 7.1 and consequently $T(\cdot)$ is locally bounded. So we can extend each single operator $T(t)$ to a continous linear operator on $\overline{D(A)} = X$.

It is obvious that $T(0) = I$. We have $t \to T(t)x$ in X is continous on $\mathbb{R}_+$ for every $x \in D(A)$, $\overline{D(A)} = X$ and then $T(t)x$ is strongly continous at 0. Namely, for all $\varepsilon > 0$ there exist $\delta > 0$ such that $0 < t < \delta$ implies $\|T(t)x - x\| < \varepsilon$, $\forall x \in D(A)$. Let $y \in X$ then there is $x \in D(A)$ such that $\|x - y\| < \varepsilon$ and then

$$\begin{aligned}
\|T(t)y - y\| &\leq \|T(t)y - T(t)x + T(t)x - x + x - y\| \\
&\leq \|T(t)(y - x)\| + \|T(t)x - x\| + \|x - y\| \\
&\leq \|T(t)\|.\|x - y\| + \varepsilon + \varepsilon \quad \text{since T(t) is locally bounded} \\
&\leq (M_{t_f} + 2)\varepsilon.
\end{aligned}$$

We see that $T(t)x$ is strongly continous for every $x \in X$. Moreover, $t, s \geq 0$ and $x \in D(A)$. Then $u(s, x) \in D(A)$ so that $v(t) := T(t)u(s; x) = u(t; u(s; x))$ for $t \geq 0$ also corresponds to the solution of the Cauchy problem with initial condition $u(s; x)$. From the other point of view $u(t + s; x) = T(t + s)x$ for $t \geq 0$ also satisfies the problem. Since solutions are unique, we obtain $T(t)T(s)x = T(t + s)x$ which gives the semigroup law.

Let $x \in D(A)$ and B be the generator of $T(t)$. Then $Bx := \lim_{t\to 0^+} \frac{1}{t}(T(t)x - x)$ and by Definition 7.1,

$$\lim_{t\to 0^+} \frac{1}{t}(T(t)x - x) = \lim_{t\to 0^+} \frac{1}{t}(u(t, x) - x) = Ax.$$

Since $T(t)$ solves the Cauchy problem, we have $A \subseteq B$ which satisfies $D(A) \subseteq D(B)$ and $Ax = Bx$ for all $x \in D(A)$. By Definition 5.4 and Proposition 5.2, $D(A)$ is dense in $D(B)$ with respect to $\|x\|_B = \|x\| + \|Bx\|$ and $T(t)D(A) \subseteq D(A)$ for all $t \geq 0$ so that $D(A)$ is a core of B. For all $x \in D(B)$ there exist $x_n \in D(A)$ such that

$$\|x_n - x\| + \|Bx_n - Bx\| \to 0$$

$Ax_n = Bx_n \to Bx$. Consequently, the closedness of A requires $x \in D(A)$ and $A = B$. □

7.1. The Inhomogeneous Equation

In this section we introduce the inhomogeneous Cauchy problem or inhomogeneous evolution equation.

$$
\begin{aligned}
u'(t) &= Au(t) + g(t), \quad t \in (0, T) \\
u(0) &= u_0.
\end{aligned} \tag{7.1}
$$

Moreover, let the initial value $u_0 \in X$, $g : [0, T) \to X$ be a continuous function and A be a linear and closed operator.

Definition 7.2 *We call a function $u : [0, T) \to X$ is a solution of the equation (7.1) if u is continuously differentiable on $(0, T)$, $u(t) \in D(A)$ for every $t \in [0, T)$ and (7.1) is satisfied on $(0, T)$.*

From the definition $u_0 \in D(A)$. We notice that a solution of (7.1) belongs to $C([0, T), [D(A)])$ and thus u is called classical solution of (7.1).

Proposition 7.1 *(Pazy, 2012) Let $T(t)$ be a C_0-semigroup generated by A, $u_0 \in D(A)$ and $g \in C([0, T), X)$. Then the solution of (7.1) is unique and given by the following Duhamel's formula*

$$
u(t) = T(t)u_0 + \int_0^t T(t-s)g(s)\,\mathrm{d}s, \quad t \in [0, T]. \tag{7.2}
$$

Proof We assume that $(0, T) \subseteq \mathbb{R}_+$, $t \in (0, T)$ and u is the solution of (7.1). Then we set $\omega(s) = T(t-s)u(s)$, for $0 \leq s < t$ that is the solution of (7.1) at time $t - s$ for initial value $u(s)$ equals to $\omega(s)$. Then ω is continuously differentiable with derivative

$$
\begin{aligned}
\omega'(s) &= T(t-s)u'(s) - T(t-s)Au(s) \\
&= T(t-s)(Au(s) + g(s)) - T(t-s)Au(s) \\
&= T(t-s)g(s).
\end{aligned}
$$

If $g \in L^1((0,T),X)$ namely $\int_0^t \|g(s)\|_X \,\mathrm{d}s$ exists, then

$$\int_0^t \|T(t-s)g(s)\|_X \,\mathrm{d}s \le \int_0^t \|T(t-s)\|_X \|g(s)\|_X \,\mathrm{d}s \le T(t-s) \int_0^t \|g(s)\|_X \,\mathrm{d}s$$

since $T(\cdot)$ is exponentially bounded and we see that $T(t-s)g(s)$ is integrable by integrating

$$\begin{aligned}\int_0^t T(t-s)g(s)\,\mathrm{d}s = \int_0^t \omega'(s)\,\mathrm{d}s &= \omega(t) - \omega(0),\\ &= T(0)u(t) - T(t)u(0),\\ &= u(t) - T(t)u_0.\end{aligned}$$

Consequently, we obtain

$$u(t) = T(t)u_0 + \int_0^t T(t-s)g(s)\,\mathrm{d}s, \quad t \in [0,T].$$

□

Definition 7.3 *Let $T(t)$ be a C_0-semigroup generated by A, $u_0 \in X$ and $g \in C([0,T],X)$. The function $u \in C([0,T],X)$ is called mild solution if it holds*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)g(s)ds, \quad t \in [0,T].$$

We conclude that every solution of inhomogeneous Cauchy problem is mild solution from Proposition 7.1 and Definition 7.3. But the converse is not always satisfied as the following example shows.

Example 7.1 *Let $X = C_0(\mathbb{R})$, A be a derivative operator with $D(A) = C_0^1(\mathbb{R})$ and let φ be any non-differentiable function. The given A generates a C_0-group $T(\cdot)$ which is defined the left translation group $T(t)h = h(\cdot + t)$. We have $T(t)\varphi = \varphi(\cdot + t) \notin D(A)$ since φ is not differentiable.*

Let $g(s) = T(s)\varphi$ for $s \in \mathbb{R}$. The function $g : \mathbb{R} \to X$ is continuous and the mild solution *of* (7.1) *with* $u_0 = 0$ *can be find by means of Duhamel's formula. That is*

$$\begin{aligned} u(t) &= T(t)u_0 + \int_0^t T(t-s)T(s)\varphi \, \mathrm{d}s \\ &= \int_0^t T(t)\varphi \, \mathrm{d}s = tT(t)\varphi, \quad t \in \mathbb{R}. \end{aligned}$$

Hence $u(t)$ *is not solution of* (7.1) *as* $u(t) \notin D(A)$ *for* $t \neq 0$.

It follows easily that continuity of the function g is not enough to ensure the existence of solutions even though $u_0 = u(0) \in D(A)$.

CHAPTER 8

CONCLUSION

We studied the linear evolution equation (Cauchy problem). We showed that finding a solution to a Cauchy problem is equivalent to find a family of evolution operators $T(t)$ which are strongly continuous. We proved several theorems which characterize the existence of C_0-semigroup of a Cauchy problem. Then we applied these theorems to specific problems such as heat and wave equation.

REFERENCES

Bátkai, András, ., M. Kramar Fijavž, and . Rhandi, Abdelaziz (2017). *Positive operator semigroups*. Springer.

Einemann, Michael, . (2009). *Semigroup methods in finance*. Ph. D. thesis, Universität Ulm.

Engel, Klaus-Jochen, . and . Nagel, Rainer (1999). *One-parameter semigroups for linear evolution equations*, Volume 194. Springer Science & Business Media.

Engel, Klaus-Jochen, . and . Nagel, Rainer (2006). *A short course on operator semigroups*. Springer Science & Business Media.

Feller, William, . (1953). On the generation of unbounded semi-groups of bounded linear operators. *Annals of Mathematics*, 166–174.

Hille, Einar, . and . Phillips, Ralph Saul (1996). *Functional analysis and semi-groups*, Volume 31. American Mathematical Soc.

Hundertmark, Dirk, ., M. Meyries, . Machinek, Lars, and R. Schnaubelt (2013). Operator semigroups and dispersive equations. In *16th Internet Seminar on Evolution Equations*.

Kreyszig, Erwin, . (1978). *Introductory functional analysis with applications*, Volume 1. wiley New York.

Lumer, Günter, . (1961). Rs phillips-dissipative operators in a banach space. *Pacific J. of Math 11*, 679–698.

Pazy, Amnon, . (2012). *Semigroups of linear operators and applications to partial differential equations*, Volume 44. Springer Science & Business Media.

Schnaubelt, Roland, . (2011). Lecture notes on evolution equations. *Karlsruhe, April.*

Schnaubelt, Roland, . (2012). Lecture notes spectral theory. *Karlsruher Institut für Technologie 33*.

Stone, Marshall H, . (1932). On one-parameter unitary groups in hilbert space. *Annals of Mathematics*, 643–648.