

QUANTUM COIN FLIPPING, QUBIT MEASUREMENT, AND GENERALIZED FIBONACCI NUMBERS

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The problem of Hadamard quantum coin measurement in n trials, with an arbitrary number of repeated consecutive last states, is formulated in terms of Fibonacci sequences for duplicated states, Tribonacci numbers for triplicated states, and N -Bonacci numbers for arbitrary N -plicated states. The probability formulas for arbitrary positions of repeated states are derived in terms of the Lucas and Fibonacci numbers. For a generic qubit coin, the formulas are expressed by the Fibonacci and more general, N -Bonacci polynomials in qubit probabilities. The generating function for probabilities, the Golden Ratio limit of these probabilities, and the Shannon entropy for corresponding states are determined. Using a generalized Born rule and the universality of the n -qubit measurement gate, we formulate the problem in terms of generic n -qubit states and construct projection operators in a Hilbert space, constrained on the Fibonacci tree of the states. The results are generalized to qutrit and qudit coins described by generalized Fibonacci- N -Bonacci sequences.

Keywords: Fibonacci numbers, quantum coin, qubit, qutrit, qudit, quantum measurement, Tribonacci numbers, N -Bonacci numbers

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1. Introduction

The Fibonacci numbers have been known from ancient times as “nature’s numbering system,” and have applications to the growth of every living thing, from natural plants (branches of trees, arrangement of leaves) to human proportions and architecture (the Golden Section $\varphi = (1 + \sqrt{5})/2 \approx 1.6$). Quantum calculus of Fibonacci numbers and Fibonacci divisors regarded as q -numbers with Golden Ratio bases was developed in our papers [1], [2]. Some applications of this calculus, including the description of n -qubit states based on the Binet formula for coherent qubit states, were derived. Normalization constants and the probability of measurement in these states are determined by Fibonacci numbers and Fibonacci divisors. In particular, it was shown that for a two-qubit state with $n = 2$ and an arbitrary odd Fibonacci divisor $F_n^{(k)}$, the level of entanglement in terms of concurrence is expressible in terms of the Lucas numbers L_k only, as a decreasing function of k [2].

In this paper, we describe the n -measurement problem for qubit states with repeated identical states, which is naturally related to the Fibonacci numbers and their generalizations to Tribonacci and N -Bonacci

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cases. For a maximally random qubit state, such as the Hadamard quantum coin state, this problem is a quantum analogue of the classical coin tossing. The last one is an 18th century problem formulated by A. De Moivre in his book on probability theory “The doctrine of chances” [3]. Some useful references on the coin flipping problem and the corresponding classical dynamical system can be found in [4].

Here, we work with a quantum coin as a two-level quantum system, representing the qubit unit of quantum information. This quantum coin can be represented by spin one-half, such that the measurement of this spin in the z -component plays the role of quantum coin flipping [5]. The quantum coin flipping has applications as a protocol for encrypting messages for secure quantum communication. A single-qubit coin determines one-dimensional [5] and alternate two-dimensional [6] quantum walks. The Hadamard gate represents the simplest fair quantum coin and the corresponding Hadamard quantum walk [7]. The interesting question is a relation between classical and quantum coins. On one hand, in [8], quantum states are mapped to the classical coin probabilities by replacing Einstein’s sentence “God does not play dice” with the statement “God plays coins.” On the other hand, according to observation by Albrecht and Phillips, the outcome of coin flip is a truly quantum measurement, and it is actually a Schrödinger cat, and hence the 50 : 50 outcome of a coin toss can be derived from quantum physics [9].

In this paper, we formulate a quantum version of the classical coin flipping problem, as a quantum measurement problem for the quantum coin. In Sec. 2, we introduce the maximally random qubit state as a quantum coin. In Sec. 3, we describe measurement of this qubit state as an Hadamard coin. By a generalized Born rule and the universality of the n -qubit measurement gate, we formulate the problem in terms of generic n -qubit states. We show that for duplicated states, the Fibonacci sequence of numbers determines allowed configurations of states and the corresponding probabilities. It turns out that the Fibonacci tree for computational quantum states becomes identical to the classical Fibonacci rabbit problem and the corresponding tree. For triplicated states and generic N -plicated states, these configurations are determined by sequences of Tribonacci and N -Bonacci numbers. Generic qubit coin measurement is described in Sec. 4. In Sec. 5, we consider an arbitrary position of duplicated, triplicated, and N -plicated states and the corresponding probabilities. Section 6 is devoted to generating functions for probabilities, the Golden Ratio in measurements, and the Shannon entropy for the states. Projection operators on quantum tree states and corresponding probabilities are the subject of Sec. 7. In Sec. 8, we treat a quantum qutrit coin with three output states of measurement and generalized Fibonacci and Tribonacci numbers. In Sec. 9, we consider the most general case of a qudit coin and the corresponding generalized Fibonacci type sequences.

2. Maximally random qubit state as quantum coin

For the one-qubit state in the computational basis

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle, \tag{1}$$

where $|c_0|^2 + |c_1|^2 = 1$, probabilities of measuring the respective states $|0\rangle$ and $|1\rangle$ are $p_0 = |\langle 0|\psi\rangle|^2$ and $p_1 = |\langle 1|\psi\rangle|^2$. From this standpoint, the qubit can be regarded as a random variable state with two outcome states and corresponding probabilities p_0 and p_1 , $p_0 + p_1 = 1$. The Shannon entropy for these probabilities

$$S = -p_0 \log_2 p_0 - p_1 \log_2 p_1 = -|c_0|^2 \log_2 |c_0|^2 - |c_1|^2 \log_2 |c_1|^2, \tag{2}$$

rewritten in the form

$$S = -|\langle 0|\psi\rangle|^2 \log_2 |\langle 0|\psi\rangle|^2 - |\langle 1|\psi\rangle|^2 \log_2 |\langle 1|\psi\rangle|^2 \tag{3}$$

is a natural measure of the uncertainty in the result of measurement [10], quantifying the deficiency in the information that the outcome state $|\psi\rangle$ gives about further measurements. The entropy depends on

the basis states and changes as the basis is transformed. It takes the maximal value $S = 1$ for $p_0 = p_1 = 1/2$ or $|c_0| = |c_1| = 1/\sqrt{2}$, giving the Hadamard-type qubit states

$$|\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\varphi}}{\sqrt{2}}|1\rangle. \quad (4)$$

The Hadamard states

$$|\pm\rangle = \frac{1}{\sqrt{2}}|0\rangle \pm \frac{1}{\sqrt{2}}|1\rangle \quad (5)$$

are generated by the Hadamard gate, $|+\rangle = H|0\rangle$, $|-\rangle = H|1\rangle$, and respectively correspond to $\varphi = 0$ and $\varphi = \pi$. Since the probabilities to measure states $|0\rangle$ and $|1\rangle$ are equal $p_0 = p_1 = 1/2$, qubit state (4) can naturally be called the quantum coin. The states of this quantum coin, as maximally random states, belong to the unit circle on the equator of the Bloch sphere, the midway between the $|0\rangle$ and $|1\rangle$ states,

$$\left|\frac{\pi}{2}, \varphi\right\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\varphi}}{\sqrt{2}}|1\rangle. \quad (6)$$

The whole set of states $|\varphi\rangle$ in (4) can be generated by the phase gate $R(\varphi)$ applied to the Hadamard state $|+\rangle$, producing rotation of the unit circle

$$|+\rangle \text{ ————— } \boxed{R(\varphi)} \text{ ————— } |\varphi\rangle. \quad (7)$$

The classical coin could exist in two states, but not in a superposition of these states. A generic superposition of two quantum coins (maximally random qubits) is not a maximally random state and is not a fair quantum coin anymore. Indeed, for the state

$$\alpha|+\rangle + \beta|-\rangle = \frac{\alpha + \beta}{\sqrt{2}}|0\rangle + \frac{\alpha - \beta}{\sqrt{2}}|1\rangle, \quad (8)$$

where $|\alpha|^2 + |\beta|^2 = 1$, the probabilities are not equal,

$$p_0 = \frac{1}{2} + \text{Re}(\alpha\bar{\beta}), \quad p_1 = \frac{1}{2} - \text{Re}(\alpha\bar{\beta}), \quad (9)$$

and the state is not maximally random. We say that states of this type (8) are generic qubit coin states. To make the superposition maximally random, we have to choose $p_0 = p_1 = 1/2$ in (9). This implies that $\text{Re}(\alpha\bar{\beta}) = 0$ and hence $\arg \alpha = \arg \beta + \pi/2$. Solving the normalization condition, we find $\alpha = \cos \gamma e^{i \arg \alpha}$, $\beta = i \sin \gamma e^{i \arg \alpha}$. Then superposition state (8) becomes

$$\alpha|+\rangle + \beta|-\rangle = e^{i \arg \alpha} \left[\frac{e^{-i\gamma}|0\rangle + e^{i\gamma}|1\rangle}{\sqrt{2}} \right] = e^{i \arg \alpha - i\gamma} \left[\frac{|0\rangle + e^{2i\gamma}|1\rangle}{\sqrt{2}} \right].$$

This state, up to the global phase, is in the form of quantum coin (4), corresponding to rotation of Hadamard states along the equatorial circle.

3. Quantum coin flipping and measurement

The flipping of a quantum coin is an application of the X gate to the “heads” state $|0\rangle$ and the “tails” state $|1\rangle$. If the coin is initialized in the “heads” state $|0\rangle$, then by applying the Hadamard gate the quantum computer produces the state $|+\rangle = H|0\rangle$. A more general quantum coin state $|\varphi\rangle = R(\varphi)|+\rangle = R(\varphi)H|0\rangle$ in (4) can be obtained by applying the phase gate $R(\varphi)$ to state $|+\rangle$ as in (7).

The measurement M of a quantum coin state,

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \text{ ————— } \boxed{M} \text{ ————— } |i\rangle, \quad i = 0, 1,$$

gives states $|0\rangle$ or $|1\rangle$ with equal probabilities $p_0 = p_1 = 1/2$. This is equivalent to the fair coin tossing.

3.1. Universality of n -qubit measurement. The measurement gate M is a universal one-qubit measurement gate. This means that the measurement gate for an arbitrary n -qubit state can be realized by applying the one-qubit measurement gate M to each of the n qubits [11]. This becomes clear from the generalized Born rule applied to an arbitrary $(n + 1)$ -qubit state, which can be represented in the form

$$|\psi\rangle_{n+1} = c_0|0\rangle|\phi_0\rangle_n + c_1|1\rangle|\phi_1\rangle_n, \quad (10)$$

where states $|\phi_0\rangle_n$ and $|\phi_1\rangle_n$ are normalized, but not necessarily orthogonal. If, by applying the one-qubit M gate, one measures only a single qubit (obtaining $|0\rangle$ with probability p_0 and $|1\rangle$ with probability p_1), then the $(n + 1)$ -qubit state becomes the product state $|0\rangle|\phi_0\rangle_n$ or $|1\rangle|\phi_1\rangle_n$. Applying this rule n -times reduces the process of measurement to the application of multiple copies of a single elementary hardware, the one-qubit measurement gate.

3.2. Duplicated states and Fibonacci numbers. Here, we count the probability of measurement of the quantum coin states in n trials, to obtain a consecutive pattern of the $|1\rangle$ states only in last two final measurements. Based on a generalized Born rule and the universality of the n -qubit measurement gate, we formulate the problem in terms of generic n -qubit states. This can be done by applying the one-qubit measurement gate M to the Hadamard state n times. The results of these measurements can be ordered as an n -qubit computational state, and the first question is how many n -qubit states of following form exist:

$$\underbrace{|\ast\rangle \otimes \cdots \otimes |\ast\rangle}_{n-2} \otimes \underbrace{|1\rangle \otimes |1\rangle}_2 \equiv \underbrace{|\ast\rangle \otimes \cdots \otimes |\ast\rangle}_n |1\rangle |1\rangle. \quad (11)$$

Here, in the first $n - 2$ measurements, the state $|1\rangle$ can appear at most once. This is equivalent to asking how many computational basis n -qubit states of form (11) exist. We let this number be denoted by A_n . By direct computation, it is easy to obtain the first few values $A_2 = 1$, $A_3 = 1$, $A_4 = 2$, $A_5 = 3$, etc.

We note that this problem is similar to the classical Fibonacci problem with adult and young rabbits. The state $|0\rangle$ is associated with an adult rabbit, and allows the following state on the left to be either the adult state $|0\rangle$ or the young state $|1\rangle$. But the young state $|1\rangle$ cannot be followed by the young state $|1\rangle$, but only by the adult state $|0\rangle$ (the young becomes adult). The remarkable relation between the rabbit problem and computational quantum states is shown by the Fibonacci tree for n -qubit states in Fig. 1.

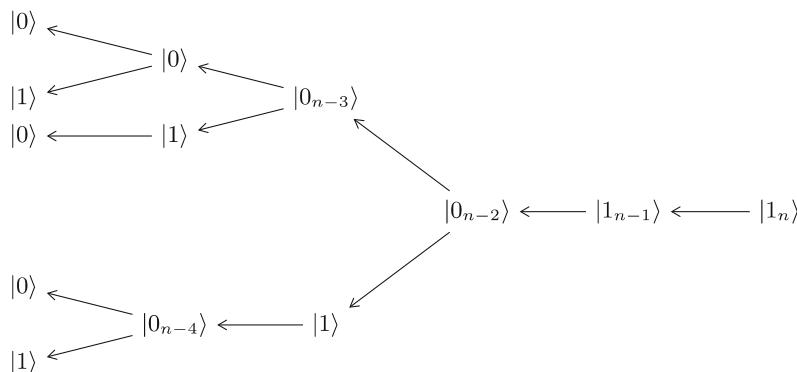


Fig. 1. Fibonacci tree of states.

By analyzing this Fibonacci tree, we note that the number A_n (the number of allowed states) is equivalent to the number of different paths (of length n) in this tree. Starting from a branching point at the state $|0_{n-2}\rangle$ (see Fig. 1) with the number A_n , we see that it is the sum of paths from the branching

state $|0_{n-3}\rangle$ with the number A_{n-1} , and from the branching state $|0_{n-4}\rangle$ with the number A_{n-2} . This is why the number of paths satisfies the recursion formula

$$A_n = A_{n-1} + A_{n-2}. \quad (12)$$

This formula is the defining relation for the Fibonacci sequence of numbers. The Fibonacci numbers $1, 1, 2, 3, 5, 8, \dots$, are defined by the recursion formula

$$F_n = F_{n-1} + F_{n-2}, \quad (13)$$

with the initial values $F_0 = 0, F_1 = 1$. Comparing the two sequences, we have the formula $A_n = F_{n-1}$, $n = 2, 3, \dots$, for the number of states (11).

Hadamard quantum coin measurement. Measurements of quantum coin (4) give states $|0\rangle$ or $|1\rangle$ with equal probabilities $p_0 = p_1 = 1/2$. Therefore, the probability to have n -qubit configuration (11) is the product

$$P_n = A_n \frac{1}{2^n} = \frac{F_{n-1}}{2^n}, \quad n = 2, 4, \dots \quad (14)$$

The probabilities P_n themselves satisfy a recursion formula for the generalized Fibonacci numbers

$$P_n = \frac{1}{2}P_{n-1} + \frac{1}{2^2}P_{n-2}, \quad P_1 = 0, \quad P_2 = \frac{1}{2^2}. \quad (15)$$

The first few numbers here are $P_3 = 1/2^3$, $P_4 = 2/2^4$, $P_5 = 3/2^5$, $P_6 = 5/2^6$, etc. These probabilities can be calculated directly from the Fibonacci tree in Fig. 1, by attaching the probability $1/2$ to every state.

3.3. Triplicated states and Tribonacci numbers. We now count the probability of measurement quantum coin states in n trials so as to obtain a repeated pattern of the $|1\rangle$ states only in the last three measurements. We can order the results of these measurements as an n -qubit state and ask how many n -qubit states of following form exist:

$$\underbrace{|*\rangle \otimes \dots \otimes |*\rangle}_{n-3} \otimes \underbrace{|1\rangle \otimes |1\rangle \otimes |1\rangle}_3 \equiv \underbrace{|*\rangle \dots |*\rangle |1\rangle |1\rangle |1\rangle}_n. \quad (16)$$

Here, in the first $n - 3$ measurements, the state $|1\rangle$ can appear at most twice. This is equivalent to asking how many computational basis n -qubit states of form (16) exist. We call these states the allowed states. Denoting this number as A_n , by direct computation we obtain the first few values $A_3 = 1, A_4 = 1, A_5 = 2, A_6 = 4$, etc. The set of allowed n -qubit states is shown in Fig. 2, which we call the Tribonacci tree for computational basis states.

As in the preceding case, the number A_n is equivalent to the number of different paths (of length n) in this tree. Starting from the branching point at the state $|0_{n-3}\rangle$ (see Fig. 2), we see that the number A_n is the sum of paths at the state $|0_{n-4}\rangle$ with the number A_{n-1} , $|0_{n-5}\rangle$ with the number A_{n-2} , and $|0_{n-6}\rangle$ with the number A_{n-3} . Then the number of paths satisfies the recursion formula

$$A_n = A_{n-1} + A_{n-2} + A_{n-3}. \quad (17)$$

This formula is the defining relation for the Tribonacci sequence of numbers. The Tribonacci numbers $1, 1, 2, 4, 7, 13, \dots$, are defined by the recursion formula

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \quad T_0 = 0, \quad T_2 = 1. \quad (18)$$

Comparing the two sequences, we have the allowed number of states (16) in the form $A_n = T_{n-1}$, $n = 3, 4, \dots$

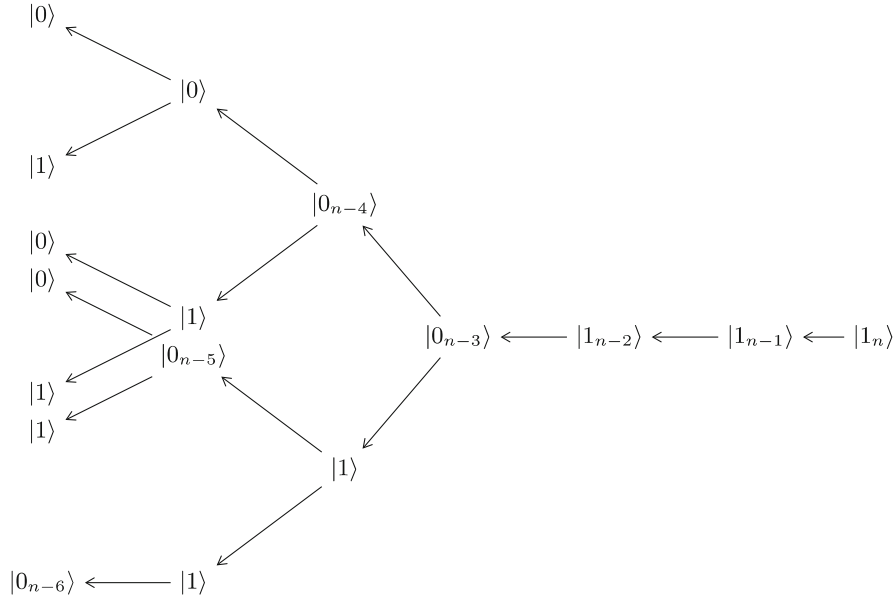


Fig. 2. Tribonacci tree of states.

Quantum coin measurement. Similarly to the foregoing, the probability to obtain n -qubit configuration (16) is the product

$$P_n = A_n \frac{1}{2^n} = \frac{T_{n-1}}{2^n}, \quad n = 3, 4, \dots \quad (19)$$

The probabilities P_n satisfy a recursion formula for the generalized Tribonacci numbers

$$P_n = \frac{1}{2}P_{n-1} + \frac{1}{2^2}P_{n-2} + \frac{1}{2^3}P_{n-3}, \quad P_1 = 0, \quad P_2 = 0, \quad P_3 = \frac{1}{2^3}. \quad (20)$$

The first few numbers of this sequence are $P_4 = 1/2^4$, $P_5 = 1/2^4$, $P_6 = 1/2^4$, $P_7 = 7/2^7$. The probabilities also result from the Tribonacci tree in 2, by attaching probabilities $1/2$ to every state.

3.4. N -plicated states and N -Bonacci numbers. We can generalize the previous results and ask how many A_n of the n qubit states ($N < n$) of the following form exist:

$$\underbrace{[*] \otimes \dots \otimes [*]}_{n-N} \otimes \underbrace{|1\rangle \otimes \dots \otimes |1\rangle}_N \equiv \underbrace{[*] \dots [*]|1\rangle \dots |1\rangle}_n. \quad (21)$$

For the first few states, we have $A_N = 1$, $A_{N+1} = 1$, $A_{N+2} = 2$, $A_{N+3} = 4$, $A_{N+4} = 8$. In Fig. 3, we show the corresponding N -Bonacci tree of n -qubit states.

From Fig. 3 we can infer the recursion formula

$$A_{n-N} = A_{n-N-1} + A_{n-N-2} + \dots + A_{n-2N}, \quad (22)$$

showing that these numbers are expressible by the N -Bonacci numbers. The N -Bonacci numbers B_k are defined by the recursion formula and initial values

$$\begin{aligned} B_k &= B_{k-1} + B_{k-2} + \dots + B_{k-N}, \\ B_0 &= 0, \quad B_1 = 0, \quad \dots, \quad B_{N-2} = 0, \quad B_{N-1} = 1. \end{aligned} \quad (23)$$

Then the first few N -Bonacci numbers are $B_N = 1$, $B_{N+1} = 2$, $B_{N+2} = 4$. Comparing (22) and (23), we conclude that the number of allowed n -qubit states is $A_n = B_{n-1}$, $n = N, N+1, N+2, \dots$

4.1. Duplicated qubit measurement. The measurement of arbitrary qubit state (1) give states $|0\rangle$ or $|1\rangle$ with the respective probabilities $p_0 = |c_0|^2$ and $p_1 = |c_1|^2$. We let P_n denote the probability of measuring these quantum coin states in n trials so as to obtain a repeated pattern of the states $|1\rangle$ only in the last two final measurements. If we order the results of the measurements in the form of an n -qubit state, then the set of allowed states is described by the Fibonacci tree. This Fibonacci tree is shown in Fig. 1, but in this case it includes the probability p_0 associated with each $|0\rangle$ state and p_1 with each $|1\rangle$ state. The first few probabilities are $P_2 = p_1^2$, $P_3 = P_4 = p_0 p_1^2$. The probabilities P_n are given by Fibonacci polynomials in the variables $p_0 = 1 - p_1$ and p_1 , satisfying the recursion formula

$$P_n = p_0(P_{n-1} + p_1 P_{n-2}), \quad P_1 = 0, \quad P_2 = p_1^2. \quad (28)$$

For a one-qubit state on the Bloch sphere $|\theta, \varphi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\varphi}|1\rangle$, the probabilities are $p_0 = \cos^2(\theta/2)$ and $p_1 = \sin^2(\theta/2)$. Then the probabilities P_n become polynomials in $\cos\theta$. The first few polynomials are

$$P_2 = \frac{1}{4}(1 - \cos\theta)^2, \quad P_3 = P_4 = \frac{1}{8}(1 - \cos\theta)(1 - \cos^2\theta).$$

4.2. Triplicated qubit measurement. We let P_n denote the probability of measuring qubit coin states in n trials with a triplicated pattern of the $|1\rangle$ states appearing only in the three final measurements. If we order the results of the measurements in the form of an n -qubit state, then the set of allowed states is described by the Tribonacci tree shown in Fig. 2, where the probability p_0 is associated with every state $|0\rangle$ and the probability p_1 with every $|1\rangle$ state. The first few probabilities are $P_3 = p_1^3$, $P_4 = P_5 = P_6 = p_1^3 p_0$, $P_7 = p_1^3 p_0(1 - p_1^3)$, etc. It is not difficult to establish that these probabilities are given by Tribonacci polynomials in terms of the probabilities $p_0 = 1 - p_1$, p_1 , with the recursion formula

$$P_n = p_0(P_{n-1} + p_1 P_{n-2} + p_1^2 P_{n-3}), \quad P_1 = P_2 = 0, \quad P_3 = p_1^3. \quad (29)$$

For a one-qubit state on Bloch sphere, the probabilities $P_n(\cos\theta)$ are polynomials and the first few are

$$P_3 = \frac{1}{8}(1 - \cos\theta)^3, \quad P_4 = P_5 = P_6 = \frac{1}{16}(1 - \cos\theta)^3(1 + \cos\theta).$$

4.3. N -plicated qubit measurement. The recursion formula and initial values for the probabilities are

$$P_n = p_0(P_{n-1} + p_1 P_{n-2} + \cdots + p_1^{N-1} P_{n-N}), \quad (30)$$

$$P_1 = P_2 = \cdots = P_{N-1} = 0, \quad P_N = p_1^N.$$

The first few polynomials

$$P_N = p_1^N, \quad P_{N+1} = P_{N+2} = \cdots = P_{2N} = p_0 p_1^N, \quad (31)$$

$$P_{2N+1} = p_0^2 p_1^N (1 + p_1 + p_1^2 + \cdots + p_1^{N-1}) = p_0 p_1^N (1 - p_1^N);$$

are polynomials on the Bloch sphere

$$P_N = \frac{1}{2^N}(1 - \cos\theta)^N, \quad P_{N+1} = \cdots = P_{2N} = \frac{1}{2^{N+1}}(1 - \cos\theta)^N(1 + \cos\theta).$$

The polynomials correspond to the addition of different paths on the N -Bonacci tree (see Fig. 3), by attaching the probability p_0 to every $|0\rangle$ state and p_1 to every $|1\rangle$ state.

5. Arbitrary position of repeated states

In the preceding section, we described the situation with consecutive states $|1\rangle$ in the last positions of an n -qubit state. Now we are interested in the probabilities to have consecutive states $|1\rangle$ in arbitrary position.

5.1. Arbitrary position of duplicated states. Here, we treat the generic case of an arbitrary position of duplicated states in Hadamard coin states. If the duplicated $|1\rangle|1\rangle$ states appear in n -qubit states only at the last positions $n-1$ and n ,

$$\leftarrow \underbrace{|\ast\rangle \otimes \cdots \otimes |\ast\rangle}_{n-2} \otimes \underbrace{|1\rangle \otimes |1\rangle}_2 \equiv \leftarrow \underbrace{|\ast\rangle \cdots |\ast\rangle |1\rangle |1\rangle}_n, \quad (32)$$

then the Fibonacci tree grows to the left and the number of allowed states is $A_n = F_{n-1}$. If these states appear only in the first two positions $n=1$ and $n=2$,

$$\rightarrow \underbrace{|1\rangle \otimes |1\rangle}_2 \otimes \underbrace{|\ast\rangle \otimes \cdots \otimes |\ast\rangle}_{n-2} \equiv \rightarrow \underbrace{|1\rangle \cdots |\ast\rangle}_n, \quad (33)$$

then the Fibonacci tree grows to the right and the number of allowed states is the same, $A_n = F_{n-1}$.

We consider the general case of allowed states with $|1\rangle|1\rangle$ states occurring only at positions k and $k+1$, where $k = 1, 2, \dots, n-1$,

$$\leftarrow \underbrace{|\ast\rangle \otimes \cdots \otimes |\ast\rangle}_{k-1} \otimes \underbrace{|1\rangle \otimes |1\rangle}_{k \quad k+1} \otimes \underbrace{|\ast\rangle \otimes \cdots \otimes |\ast\rangle}_{n-k-1} \rightarrow. \quad (34)$$

The number of states is now determined by two Fibonacci trees, one is a $(k+1)$ -qubit tree growing to the left, with the number of allowed states F_k , and the other is the tree of $n-k+1$ qubits, growing to the right, with the number of states F_{n-k} . Then the total number of allowed states is $A_n = F_k \cdot F_{n-k}$. We hence conclude that for an n -qubit state, the probability to have a duplicated $|1\rangle|1\rangle$ state only at positions k and $k+1$ is

$$P_{n,k} = \frac{F_k \cdot F_{n-k}}{2^n} = \frac{L_n - (-1)^k L_{n-2k}}{5 \cdot 2^n}, \quad k = 1, 2, \dots, n, \quad (35)$$

where L_n are the Lucas numbers and we use the identity

$$F_m \cdot F_n = \frac{L_{m+n} - (-1)^n L_{m-n}}{5}. \quad (36)$$

The number of n -qubit states in which $|1\rangle|1\rangle$ states appear just once, independently of the position, is

$$\sum_{k=1}^{n-1} F_k \cdot F_{n-k} = \frac{nL_n - F_n}{5}. \quad (37)$$

The corresponding probability to have this states occurring only once, but anywhere is

$$\sum_{k=1}^{n-1} P_{n,k} = \sum_{k=1}^{n-1} \frac{F_k \cdot F_{n-k}}{2^n} = \frac{nL_n - F_n}{5 \cdot 2^n}. \quad (38)$$

5.2. Triplicated states at an arbitrary position. For n trials with a triplicated state $|1\rangle|1\rangle|1\rangle$ at positions $n - 2, n - 1, n$, the Tribonacci tree grows to the left and the number of allowed states is $A_n = T_{n-1}$, $n = 3, 4, \dots$. The same number of states is for the triplicated state at positions 1, 2, 3, with the Tribonacci-state tree growing to the right. Now, if the triplicated state is located at arbitrary positions $k, k + 1, k + 2$, then the number of allowed states is the number T_{k+1} of states for the left Tribonacci tree times T_{n-k} for the right Tribonacci tree. This gives the number of allowed states at an arbitrary position k as $A_{n,k} = T_{k+1}T_{n-k}$ where $k = 1, 2, \dots, n - 2$. The probability of this configuration is

$$P_{n,k} = \frac{T_{k+1}T_{n-k}}{2^n}. \quad (39)$$

If the position of triplicated states is not fixed, then the number of states and the corresponding probabilities are

$$A_{n,k} = \sum_{k=1}^{n-2} T_{k+1}T_{n-k}, \quad P_{n,k} = \sum_{k=1}^{n-2} \frac{T_{k+1}T_{n-k}}{2^n}. \quad (40)$$

5.3. The N -plicated states in an arbitrary position. For the N -plicated states $|1\rangle \dots |1\rangle$ at an arbitrary position k , we have the number of allowed states and the corresponding probabilities in terms of the N -Bonacci numbers

$$A_{n,k} = B_{k+N-2}B_{n-k}, \quad P_{n,k} = \frac{B_{k+N-2}B_{n-k}}{2^n}, \quad k = 1, 2, \dots, n - N + 1. \quad (41)$$

6. Generating function and Golden Ratio

6.1. Generating function for duplicated probability. For duplicated states of the Hadamard quantum coin, the generating function of probabilities, Eq. (14), is

$$g(x) = \sum_{n=2}^{\infty} P_n x^{n-2} = \sum_{n=0}^{\infty} P_{n+2} x^n = \sum_{n=0}^{\infty} \frac{F_{n+1}}{2^{n+2}} x^n = \frac{1}{4 - 2x - x^2}. \quad (42)$$

This can be shown by using recursion formula (13) or, alternatively, by using the Binet representation for the Fibonacci numbers and geometric series. For $x = 1$, Eq. (42) implies the identity

$$\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} \frac{F_{n+1}}{2^{n+2}} = 1. \quad (43)$$

The identity is the completeness relation for duplicated states (11) and shows that a duplicated state appears once in the set of all numbers in n trials. Relation (43) allows introducing the information characterization of states by Shannon entropy (48) and the corresponding qubit states, realized by random walk in the Fock space of computational states.

6.1.1. Golden Ratio in computational states. For duplicated n -qubit states (11), the number of states is $A_n = F_{n-1}$, and for duplicated $(n + 1)$ -qubit states, it is $A_{n+1} = F_n$. The ratio of these numbers in the limit $n \rightarrow \infty$ is the Golden Ratio

$$\lim_{n \rightarrow \infty} \frac{A_{n+1}}{A_n} = \frac{F_n}{F_{n-1}} = \varphi. \quad (44)$$

6.1.2. Golden Ratio in Hadamard coin measurements. The probability to obtain duplicated states (11) in n measurements of the Hadamard coin states is $P_n = F_{n-1}/2^n$; for $n + 1$ measurements, it is $P_{n+1} = F_n/2^{n+1}$. The ratio of these probabilities in the limit $n \rightarrow \infty$ is half the Golden Ratio:

$$\lim_{n \rightarrow \infty} \frac{P_{n+1}}{P_n} = \lim_{n \rightarrow \infty} \frac{F_n}{2F_{n-1}} = \frac{1}{2}\varphi. \quad (45)$$

6.1.3. Golden Ratio for arbitrary position. We take an n -qubit state with the allowed number of states A_n and $n + 1$ qubit states with the number A_{n+1} from (37). The ratio of these numbers in the limit $n \rightarrow \infty$ is the Golden Ratio

$$\lim_{n \rightarrow \infty} \frac{A_{n+1}}{A_n} = \lim_{n \rightarrow \infty} \frac{(n+1)L_{n+1} - F_{n+1}}{nL_n - F_n} = \varphi. \quad (46)$$

The ratio of corresponding probabilities (38) to find the $|1\rangle|1\rangle$ pair in arbitrary position, but only once, in the set of n and $n + 1$ trials in the limit $n \rightarrow \infty$ is half the Golden Ratio

$$\lim_{n \rightarrow \infty} \frac{P_{n+1}}{P_n} = \lim_{n \rightarrow \infty} \frac{1}{2} \frac{(n+1)L_{n+1} - F_{n+1}}{nL_n - F_n} = \frac{1}{2}\varphi. \quad (47)$$

The same limit is valid for the ratio of probabilities $P_{n,k}$ in (35) when the pair $|1\rangle|1\rangle$ is in position $k, k + 1$, where $k = 1, 2, \dots, n - 1$.

6.1.4. Shannon entropy. The Hadamard quantum tree in Fig. 1, at each level n , determines the probability $P_n = F_{n-1}/2^n$, and as we have seen, the sum of probabilities (43) is one, $\sum_{n=2}^{\infty} P_n = 1$. The level of randomness in this distribution can be characterized by the Shannon entropy

$$S = - \sum_{n=2}^{\infty} P_n \log_2 P_n = - \sum_{n=2}^{\infty} \frac{F_{n-1}}{2^n} \log_2 \frac{F_{n-1}}{2^n}. \quad (48)$$

The ratio test for this series gives half the Golden Ratio

$$\lim_{n \rightarrow \infty} \frac{s_{n+1}}{s_n} = \frac{1}{2}\varphi < 1$$

and the convergence of this sum. For the n th term, asymptotically, we have

$$s_n \approx n \left(\frac{\varphi}{2} \right)^n \frac{\log_2(2/\varphi)}{\sqrt{5}}. \quad (49)$$

6.2. Generating function for triplicated probability. The generating function of the Tribonacci numbers is defined as

$$g_T(x) = \sum_{n=0}^{\infty} T_n x^n = x^2 + x^3 + 2x^4 + 4x^5 + \dots. \quad (50)$$

By substituting the recursion formula for Tribonacci numbers (18) and using the values $T_{-1} = 1, T_{-2} = -1, T_{-3} = 0$, we obtain

$$g_T(x) = \frac{x^2}{1 - x - x^2 - x^3}. \quad (51)$$

For Hadamard quantum coin (4), we can define a generating function of the Tribonacci probabilities P_n (19) as

$$g(x) = \sum_{n=0}^{\infty} P_{n+3} x^n = \sum_{n=0}^{\infty} \frac{T_{n+2}}{2^{n+3}} x^n. \quad (52)$$

By using the recursion formula for Tribonacci numbers, we find

$$g(x) = \frac{1}{2^3 - 2^2x - 2x^2 - x^3}. \quad (53)$$

If we set $x = 1$ here, then we find the completeness relation for probabilities $\sum_{n=3}^{\infty} P_n = 1$. This relation allows introducing the Shannon entropy for triplicated states in the Fock space in a way similar to (48).

7. Projection operators and n -qubit states

7.1. Maximally random n -qubit state. The n -qubit state

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i_{n-1}, \dots, i_1 i_0=0}^1 |i_{n-1} \dots i_1 i_0\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle \quad (54)$$

is a maximally random state, according to the computational basis. In this case, all probabilities are equal and the Shannon entropy is maximal, $S = n$. In the Fibonacci-state tree shown in Fig. 1, with every path we can associate the projection operator on the corresponding computational state. Summing these projectors gives the projection operator onto the subspace of n -qubit states determined by the Fibonacci tree. For the first five qubit states, we have the projection operators

$$\begin{aligned} \hat{P}_2 &= |11\rangle\langle 11|, & \hat{P}_3 &= |011\rangle\langle 011|, & \hat{P}_4 &= |0011\rangle\langle 0011| + |1011\rangle\langle 1011|, \\ \hat{P}_5 &= |00011\rangle\langle 00011| + |01011\rangle\langle 01011| + |10011\rangle\langle 10011|. \end{aligned}$$

For an arbitrary n -qubit state, the projection operator to the Fibonacci tree is

$$\hat{P}_n = \sum_{i_{n-1}, \dots, i_3=0,1}^{\neq 11} |i_{n-1} \dots i_3 011\rangle\langle i_{n-1} \dots i_3 011|, \quad (55)$$

where duplicated states $|1\rangle|1\rangle$ are not included into the summation. The dimension of the Hilbert space associated with the n -qubit Fibonacci tree in Fig. 1 is given by the Fibonacci number $\dim_F H = F_{n-1}$. For the first five qubits in the above examples, we have the respective dimensions 1, 1, 2, 3. Acting by projection operator (55) on state (54), we obtain the state $|\phi\rangle = \hat{P}_n|\psi\rangle$ connected with the Fibonacci tree in Fig. 1. The inner product of this state

$$\langle\phi|\phi\rangle = \frac{F_{n-1}}{2^n}$$

gives the corresponding normalized state

$$|\Phi\rangle = \frac{|\phi\rangle}{\sqrt{\langle\phi|\phi\rangle}} = \frac{1}{\sqrt{F_{n-1}}} \sum_{i_{n-1}, \dots, i_3=0,1}^{\neq 11} |i_{n-1} \dots i_3 011\rangle. \quad (56)$$

Then the average of the projection operator describes the probability of collapse the state

$$P_n = \langle\psi|\hat{P}_n|\psi\rangle = \frac{F_{n-1}}{2^n}. \quad (57)$$

This formula coincides with probability (14) of Hadamard coin measurement in n trials.

Similar computations can be done for the triplicated and N -plicated states associated with the corresponding trees. For triplicated states, we have the projection operator

$$\hat{P}_n = \sum_{i_{n-1}, \dots, i_4=0,1}^{\neq 111} |i_{n-1} \dots i_4 0111\rangle\langle i_{n-1} \dots i_4 0111| \quad (58)$$

associated with the Tribonacci tree in Fig. 2. The dimension of the corresponding Hilbert space is determined by the Tribonacci number $\dim_T H = T_{n-1}$. The action of this operator on state (54), $\hat{P}_n|\psi\rangle$, gives the normalized state

$$|\Phi\rangle = \frac{1}{\sqrt{T_{n-1}}} \sum_{i_{n-1}, \dots, i_4=0,1}^{\neq 111} |i_{n-1} \dots i_4 0111\rangle. \quad (59)$$

Then the probability of state (14) collapse to the Tribonacci tree subspace is

$$P_n = \langle \psi | \hat{P}_n | \psi \rangle = \frac{T_{n-1}}{2^n}. \quad (60)$$

In the general case of N duplicated states, the projection operator associated with the N -Bonacci tree in Fig. 3 is

$$\hat{P}_n = \sum_{i_{n-1}, \dots, i_{N+1}=0,1}^{\neq 11\dots 1} |i_{n-1} \dots i_{N+1} 011 \dots 1\rangle \langle i_{n-1} \dots i_{N+1} 011 \dots 1|. \quad (61)$$

The dimension of the corresponding Hilbert subspace $\dim_B H = B_{n-1}$ and the probability

$$P_n = \langle \psi | \hat{P}_n | \psi \rangle = \frac{B_{n-1}}{2^n}, \quad (62)$$

are determined by the N -Bonacci numbers B_n .

7.2. Arbitrary n -qubit state. The above projection operators can also be applied to an arbitrary n -qubit state

$$|\Psi\rangle = \sum_{i_{n-1}, \dots, i_0=0}^1 c_{i_{n-1} \dots i_0} |i_{n-1} \dots i_0\rangle = \sum_{i=0}^{2^n-1} c_i |i\rangle, \quad \sum_{i=0}^{2^n-1} |c_i|^2 = 1. \quad (63)$$

Depending on the projector, the state is projected onto subspaces determined by the Fibonacci tree (Fig. 1), Tribonacci tree (Fig. 2), and, in general, the N -Bonacci tree (Fig. 3). Then the average of the projection operator \hat{P}_n gives the probability of collapse to an arbitrary N -Bonacci state as

$$P_n = \langle \Psi | \hat{P}_n | \Psi \rangle = \sum_{i_{n-1}, \dots, i_{N+1}=0,1}^{\neq 11\dots 11} |c_{i_{n-1} \dots i_{N+1} 011 \dots 11}|^2. \quad (64)$$

7.3. Decimal form of computational states. The second sum in (63) represents the decomposition of an n -qubit state with respect to the computational basis, counted in decimal base. By using this form, we can associate a sequence of numbers with the Fibonacci tree. To specify the same numbers but with a different number of qubits, we use the following notation. Binary numbers 11, 011, and 0011 determine the same decimal number 3, and this is why for the corresponding computational states, we use a subscript to denote the number of qubits: $|11\rangle = |3\rangle_2$, $|011\rangle = |3\rangle_3$, $|0011\rangle = |3\rangle_4$.

In Fig. 4, we show the Fibonacci tree for these states. This tree determines a sequence of numbers and corresponding states. The sequence of numbers is 3, 11, 19, 35, 43, 67, 75, 83, 131, 139, 147, 163, 171, etc.

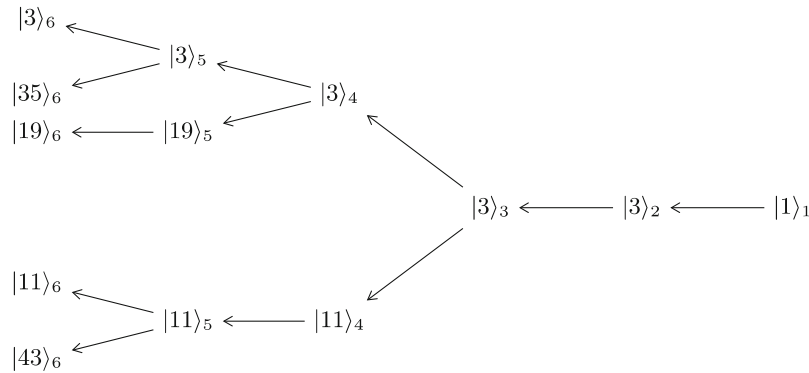


Fig. 4. Fibonacci tree of decimal states.

Every measurement of this coin produces basis states with equal probabilities $1/3$, and hence this state is the most random qutrit state in the computational basis with base 3. In the set of n trials, we count the allowed states with the last two states duplicated,

$$\underbrace{|\ast\rangle \otimes \cdots \otimes |\ast\rangle}_{n-2} \otimes \underbrace{|1\rangle \otimes |1\rangle}_2 \equiv \underbrace{|\ast\rangle \cdots |\ast\rangle |1\rangle |1\rangle}_n \quad (66)$$

where, in contrast to the qubit case, we have three states $|\ast\rangle = |0\rangle, |1\rangle, |2\rangle$. To count the number of allowed states, we can analyze the corresponding tree, similar to Fig. 1. But in this case the tree is duplicating its branches, and hence the number of states is $B_n = E_{n-2}$, where the E_n are the generalized Fibonacci numbers with the recursion formula

$$E_n = 2(E_{n-1} + E_{n-2}), \quad E_0 = 0, \quad E_1 = 1.$$

The first few numbers are 0, 1, 2, 6, 16, 44. The probability of duplicated states $|1\rangle|1\rangle$ occurring in n trials for the qutrit quantum coin is $P_n = E_{n-1}/3^n$. The recursion formula for these probabilities is

$$P_{n+1} = \frac{2}{3} \left(P_n + \frac{1}{3} P_{n-1} \right), \quad P_2 = \frac{1}{3^2}, \quad P_3 = \frac{2}{3^3}. \quad (67)$$

These probabilities are countable from the corresponding doubled Fibonacci tree with probabilities $1/3$, associated with every state.

8.2. Triplicated States.

8.2.1. Maximally Random Coin. The number $B_n = E_{n-1}$ of triplicated states

$$\underbrace{|\ast\rangle \otimes \cdots \otimes |\ast\rangle}_{n-3} \otimes \underbrace{|1\rangle \otimes |1\rangle \otimes |1\rangle}_3 \equiv \underbrace{|\ast\rangle \cdots |\ast\rangle |1\rangle |1\rangle |1\rangle}_n \quad (68)$$

where $|\ast\rangle = |0\rangle, |1\rangle, |2\rangle$, is written in terms of the generalized Tribonacci numbers, with the recursion formula

$$E_n = 2(E_{n-1} + E_{n-2} + E_{n-3}), \quad E_0 = 0, \quad E_1 = 0, \quad E_2 = 1.$$

The first five numbers are 0, 0, 1, 2, 6, 18. The probability to have a triplicated state at the end of n trials $P_n = E_{n-1}/3^n$ is controlled by the recursion formula

$$P_n = \frac{2}{3} \left(P_{n-1} + \frac{1}{3} P_{n-2} + \frac{1}{3^2} P_{n-3} \right), \quad P_1 = 0, \quad P_2 = 0, \quad P_3 = \frac{1}{3^3}. \quad (69)$$

It can also be found from the corresponding doubled Tribonacci tree by assigning probability $1/3$ to every state.

8.2.2. Arbitrary Qutrit Coin. For an arbitrary qutrit coin

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle, \quad (70)$$

the probabilities are $p_i = |c_i|^2$, $p_0 + p_1 + p_2 = 1$. For a triplicated state in n trials, we have probabilities P_n , as generalized Tribonacci polynomials in p_1 :

$$P_{n+1} = (1 - p_1)(P_n + p_1 P_{n-1} + p_1^2 P_{n-2}), \quad P_1 = P_2 = 0, \quad P_3 = p_1^3. \quad (71)$$

To count these probabilities from the doubled Tribonacci tree, we have to associate probability p_0 with the state $|0\rangle$, p_1 with the state $|1\rangle$, and p_2 with the state $|2\rangle$.

9. Quantum Qudit Coin

9.1. Duplicated States. The state of the qudit coin is a superposition of d basis states

$$|\psi\rangle = \frac{1}{\sqrt{d}}(|0\rangle + |1\rangle + \cdots + |d-1\rangle). \quad (72)$$

Measurement on this coin gives the basis states with equal probabilities $1/d$ and state (72) is a maximally random state in the computational qudit basis with base d . The measurement in n trials gives the state

$$\underbrace{|\ast\rangle \otimes \cdots \otimes |\ast\rangle}_{n-2} \otimes \underbrace{|1\rangle \otimes |1\rangle}_2 \equiv \underbrace{|\ast\rangle \cdots |\ast\rangle}_n |1\rangle |1\rangle \quad (73)$$

where $|\ast\rangle = |0\rangle, |1\rangle, \dots, |d-1\rangle$. The number of allowed states $B_n = D_{n-1}$ is expressed by the generalized Fibonacci numbers D_n , with the recursion relation

$$D_n = (d-1)(D_{n-1} + D_{n-2}), \quad D_0 = 0, \quad D_1 = 1.$$

The first few numbers are polynomials in d :

$$\begin{aligned} D_0 &= 0, & D_1 &= 1, & D_2 &= d-1, & D_3 &= (d-1)d, \\ D_4 &= (d-1)^2(d+1), & D_5 &= (d-1)^2(d^2+d-1). \end{aligned}$$

For the probability of the qudit coin in n trials, we have $P_n = D_{n-1}/d^n$. The recursion formula for these probabilities is

$$P_{n+1} = \frac{d-1}{d} \left(P_n + \frac{1}{d} P_{n-1} \right), \quad P_2 = \frac{1}{d^2}, \quad P_3 = \frac{d-1}{d^3}. \quad (74)$$

For a generic qudit coin state

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle + \cdots + c_{d-1}|d-1\rangle, \quad (75)$$

probabilities to measure basis states are $p_i = |c_i|^2$, $i = 0, 1, \dots, d-1$, and $\sum_{i=0}^{d-1} p_i = 1$. The probability P_n to measure the duplicated state $|1\rangle|1\rangle$ only at the end of n trials satisfies the recursion formula

$$P_n = (1-p_1)P_{n-1} + (1-p_1)p_1P_{n-2}, \quad P_2 = p_1^2, \quad P_3 = p_1^2(1-p_1). \quad (76)$$

In these formulas, $1-p_1 = p_0 + p_2 + \cdots + p_{d-1}$. The formula can also be generalized to the case where the duplicated state at the end of n trials is $|k\rangle|k\rangle$, where $k = 0, 1, 2, \dots, d-1$. The probability to have this state

$$\underbrace{|\ast\rangle \otimes \cdots \otimes |\ast\rangle}_{n-2} \otimes \underbrace{|k\rangle \otimes |k\rangle}_2 \equiv \underbrace{|\ast\rangle \cdots |\ast\rangle}_n |k\rangle |k\rangle \quad (77)$$

satisfies the recursion formula

$$P_n = (1-p_k)P_{n-1} + (1-p_k)p_kP_{n-2}, \quad P_2 = p_k^2, \quad P_3 = p_k^2(1-p_k), \quad (78)$$

where $1-p_k = p_0 + \cdots + p_{k-1} + p_{k+1} + \cdots + p_{d-1}$.

9.2. Triplicated States. For a maximally random coin, the number of allowed states is $B_n = E_{n-1}$, where E_n are generalized Tribonacci numbers:

$$E_n = (d-1)(E_{n-1} + E_{n-2} + E_{n-3}), \quad E_0 = 0, \quad E_1 = 0, \quad E_2 = 1.$$

The first few numbers are

$$E_3 = d-1, \quad E_4 = (d-1)d, \quad E_5 = (d-1)d^2.$$

For the probabilities of these states, this gives $P_n = E_{n-1}/d^n$ and the recursion formula

$$\begin{aligned} P_{n+1} &= \frac{d-1}{d} \left(P_n + \frac{1}{d} P_{n-1} + \frac{1}{d^2} P_{n-2} \right), \\ P_1 = P_2 &= 0, \quad P_3 = \frac{1}{d^3}, \quad P_4 = \frac{d-1}{d^4}. \end{aligned} \tag{79}$$

For general qudit coin (75), the probability P_n of a triplicated state is a polynomial in p_1 ,

$$\begin{aligned} P_{n+1} &= (1-p_1)(P_n + p_1 P_{n-1} + p_1^2 P_{n-2}), \\ P_1 = P_2 &= 0, \quad P_3 = p_1^3, \quad P_4 = p_1^3(1-p_1), \end{aligned} \tag{80}$$

where $1-p_1 = p_0 + p_2 + p_3 + \dots + p_{d-1}$.

9.3. N-plicated states. For n measurements with arbitrary qudit state (75), the probability P_n to find the N -plicated state

$$\underbrace{|\ast\rangle \otimes \dots \otimes |\ast\rangle}_{n-N} \otimes \underbrace{|1\rangle \otimes |1\rangle \dots \otimes |1\rangle}_N \equiv \underbrace{|\ast\rangle \dots |\ast\rangle |1\rangle \dots |1\rangle}_n \tag{81}$$

where $|\ast\rangle = |0\rangle, |1\rangle, \dots, |d-1\rangle$, is expressed by the generalized N -Bonacci polynomials

$$\begin{aligned} P_n &= (1-p_1)(P_{n-1} + p_1 P_{n-2} + p_1^2 P_{n-3} + \dots + p_1^{N-1} P_{n-N}), \\ P_1 = \dots = P_{N-1} &= 0, \quad P_N = p_1^N, \end{aligned} \tag{82}$$

where $1-p_1 = p_0 + p_2 + p_3 + \dots + p_{d-1}$.

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REFERENCES

1. O. K. Pashaev and S. Nalci, “Golden quantum oscillator and Binet–Fibonacci calculus,” *J. Phys A: Math. Theor.*, **45**, 015303, 23 pp. (2012), arXiv: 1107.4389.
2. O. K. Pashaev, “Quantum calculus of Fibonacci divisors and infinite hierarchy of bosonic-fermionic golden quantum oscillators,” *Internat. J. Geom. Methods Modern Phys.*, **18**, 2150075, 32 pp. (2021).
3. A. de Moivre, *The Doctrine of Chances: Or, A Method of Calculating the Probabilities of Events in Play*, Chelsea Publ., New York (1967).
4. S. F. Kennedy and M. W. Stafford, “Coin flipping, dynamical systems and the Fibonacci numbers,” *Math. Mag.*, **67**, 380–382 (1994).
5. Y. Aharonov, L. Davidovich, and N. Zagury, “Quantum random walks,” *Phys. Rev. A*, **48**, 1687–1690 (1993).
6. C. Di Franco, M. Mc Gettrick, T. Machida, and Th. Busch, “Alternate two-dimensional quantum walk with a single-qubit coin,” *Phys. Rev. A*, **84**, 042337, 8 pp. (2011).

7. S. E. Venegas-Andraca, “Quantum walks: a comprehensive review,” *Quantum Inf. Process.*, **11**, 1015–1106 (2012).
8. V. N. Chernega, O. V. Man’ko, and V. I. Man’ko, “Probability representation of quantum states as a renaissance of hidden variables – God plays coins,” *J. Russ. Laser Res.*, **40**, 107–120 (2019), arXiv:1904.09788.
9. A. Albrecht and D. Phillips, “Origin of probabilities and their application to the multiverse,” *Phys. Rev. D*, **90**, 123514, 7 pp. (2014).
10. D. Deutsch, “Uncertainty in quantum measurements,” *Phys. Rev. Lett.*, **50**, 631–633 (1983).
11. N. D. Mermin, *Quantum Computer Science*, Cambridge Univ. Press, Cambridge (2007).