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Dedekind harmonic numbers

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Abstract. For any number field, we define Dedekind harmonic numbers with respect to this number field. First, we show that they are not integers except finitely many of them. Then, we present a uniform and an explicit version of this result for quadratic number fields. Moreover, by assuming the Riemann hypothesis for Dedekind zeta functions, we prove that the difference of two Dedekind harmonic numbers are not integers after a while if we have enough terms, and we prove the non-integrality of Dedekind harmonic numbers for quadratic number fields in another uniform way together with an asymptotic result.

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1. Introduction

Let K be a number field, the set of primes be $\mathbb{P}$ and p always denote a prime number. An element of K which is a root of a monic polynomial with integer coefficients is called an algebraic integer. The set of algebraic integers of K is called the ring of integers of K and is denoted by $\mathcal{O}_K$. It is well-known that $\mathcal{O}_K$ is a Dedekind domain, in other words, it is Noetherian, integrally closed and its prime ideals are maximal. Thus, its non-zero proper ideals factor into prime ideals uniquely. Moreover, for any non-zero ideal I of $\mathcal{O}_K$, the norm of I is defined as $N_{K/\mathbb{Q}}(I) = |\mathcal{O}_K/I|$ which is always finite. We will use $N(I)$ in short.

Given a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$, one sees that $\mathfrak{p} \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. Therefore, $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$ for some $p \in \mathbb{P}$ and we say that $\mathfrak{p}$ lies above p. To add, $\mathcal{O}_K/\mathfrak{p}$ is a field extension of $\mathbb{F}_p$ and since it is finite, $N(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}|$ is a power of p. In particular, $N(\mathfrak{p}) = p^{f_\mathfrak{p}}$ where $f_\mathfrak{p}$ is the *inertial degree* of $\mathfrak{p}$ and defined as the dimension of the $\mathbb{F}_p$ vector space $\mathcal{O}_K/\mathfrak{p}$.

Now, take any prime p and let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ that lies above p. The exact power of $\mathfrak{p}$ dividing $p\mathcal{O}_K$ is called the *ramification index* of $\mathfrak{p}$ and is denoted by $e_\mathfrak{p}$. If we

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write

$$p\mathcal{O}_K = \mathfrak{p}_1^{e_{\mathfrak{p}_1}} \dots \mathfrak{p}_m^{e_{\mathfrak{p}_m}}$$

for some prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ and a positive integer m, we say that p is *ramified* if $e_{\mathfrak{p}_i} > 1$ for some i and *unramified* otherwise. Also, since the norm is multiplicative, we have the following identity

$$d = \sum_{i=1}^{m} e_{\mathfrak{p}_i} f_{\mathfrak{p}_i}, \tag{1}$$

where d is the degree of the number field K. Moreover, if $e_{\mathfrak{p}_i} = f_{\mathfrak{p}_i} = 1$ for every i, then we say that p *splits completely*. Finally, if $m = 1$ and $e_{\mathfrak{p}_1} = 1$, then we say that p is *inert* in K. It is known that for any $n > 1$, the n-th harmonic number h_n which is defined as

$$\sum_{i=1}^{n} \frac{1}{i}$$

is not an integer, see [8]. Moreover, if $n > m \geq 1$, the difference $h_n - h_m$ is never an integer by [4].

Extending the definition of harmonic numbers, we define the n-th Dedekind harmonic number as follows.

DEFINITION 1.1

The n-th Dedekind harmonic number $h_K(n)$ is defined as

$$\sum_{\substack{0 \neq I \subseteq \mathcal{O}_K \\ N(I) \leq n}} \frac{1}{N(I)}, \tag{2}$$

where the sum ranges over all non-zero ideals of $\mathcal{O}_K$ with norm less than or equal to n.

Note that the sum in Equation (2) is finite as for any $n \geq 1$, the set

$$\{0 \neq I \subseteq \mathcal{O}_K : N(I) \leq n\}$$

is finite by (1). The idea of this analogue of harmonic numbers comes from the Dedekind zeta function of a number field. The Dedekind zeta function of K is defined as

$$\zeta_K(s) = \sum_{0 \neq I \subseteq \mathcal{O}_K} \frac{1}{N(I)^s}$$

for any complex number s with $\mathrm{Re}(s) > 1$. Notice that when $K = \mathbb{Q}$, we have

$$\zeta_{\mathbb{Q}}(s) = \zeta(s) \quad \text{and} \quad h_{\mathbb{Q}}(n) = h_n.$$

As $s \to 1^+$, $\zeta_K(s)$ diverges to infinity so that the integrality of $h_K(n)$ rises a reasonable question. Also, note that for any positive integer n, the rational number $h_K(n)$ can be written as

$$h_K(n) = \sum_{i=1}^{n} \frac{a_i}{i}$$

where a_i is the number of ideals $0 \neq I \subseteq \mathcal{O}_K$ of norm exactly i.

1.1 *Order of growth of Dedekind harmonic numbers*

We know that the order of growth of the n-th harmonic number is $\log n$. The aim of this section is to show that the order of growth of the n-th Dedekind harmonic number is $c_K \log n$ for some constant c_K depending on the number field K. After showing this fact, we will express c_K explicitly. Then, we state our results. We start by setting $A(x)$ to be

$$\sum_{n\le x} a_n.$$

It is known that (see [6])

$$A(x) = c_K x + O_K\left(x^{1-\frac{1}{d}}\right)$$

where c_K is a constant depending on K and d is the degree of the number field K. To write c_K explicitly, first, let us say that $A(x) = c_K x + R(x)$ where $R(x) = O_K(x^{1-\frac{1}{d}})$. Now, the partial summation gives us the following equality

$$\sum_{n\le x} \frac{a_n}{n^s} = \frac{A(x)}{x^s} + s\int_1^x \frac{A(t)}{t^{s+1}}\mathrm{d}t = \frac{A(x)}{x^s} + s\int_1^x \frac{c_K t + R(t)}{t^{s+1}}\mathrm{d}t.$$

As $s \to 1^+$, we have that

$$\begin{aligned}\sum_{n\le x} \frac{a_n}{n} &= \frac{A(x)}{x} + c_K\int_1^x \frac{1}{t}\mathrm{d}t + \int_1^x \frac{R(t)}{t^2}\mathrm{d}t\\ &= c_K \log x + c_K + \frac{R(x)}{x} + \int_1^\infty \frac{R(t)}{t^2}\mathrm{d}t - \int_x^\infty \frac{R(t)}{t^2}\mathrm{d}t\\ &= c_K \log x + c_K + O_K(x^{-\frac{1}{d}}) + \int_1^\infty \frac{R(t)}{t^2}\mathrm{d}t - \int_x^\infty \frac{R(t)}{t^2}\mathrm{d}t.\end{aligned}$$

Here, since $R(t) = O_K(x^{1-\frac{1}{d}})$ the integral

$$\int_1^\infty \frac{R(t)}{t^2}\mathrm{d}t$$

is convergent so that it is a constant c'_K depending on K. Therefore,

$$\begin{aligned}\sum_{n\le x} \frac{a_n}{n} &= c_K \log x + c_K + c'_K - \int_x^\infty \frac{R(t)}{t^2}\mathrm{d}t + O_K(x^{-\frac{1}{d}})\\ &= c_K \log x + c_K + c'_K + O_K\left(\int_x^\infty \frac{1}{t^{1+\frac{1}{d}}}\mathrm{d}t\right) + O_K(x^{-\frac{1}{d}})\\ &= c_K \log x + b_K + O_K(x^{-\frac{1}{d}})\end{aligned}$$

where $b_K = c_K + c'_K$. To sum up,

$$h_K(n) \sim c_K \log n. \tag{3}$$

Now, we are ready to find c_K. For $s > 1$, as $x \to \infty$,

$$\frac{A(x)}{x^s} \to 0$$

and we have the following equality:

$$\lim_{x\to\infty}\sum_{n\le x}\frac{a_n}{n^s} = \zeta_K(s) = s\int_1^\infty \frac{c_K t + R(t)}{t^{s+1}}\mathrm{d}t$$
$$= c_K s\int_1^\infty \frac{1}{t^s}\mathrm{d}t + s\int_1^\infty \frac{R(t)}{t^{s+1}}\mathrm{d}t$$
$$= c_K\frac{s}{s-1} + s\int_1^\infty \frac{R(t)}{t^{s+1}}\mathrm{d}t.$$

Note that the last integral is finite for $s \ge 1$ since $R(t) = O_K\left(t^{1-\frac{1}{d}}\right)$. Therefore, multiplying both sides by $s-1$, we have that

$$(s-1)\zeta_K(s) = c_K s + s(s-1)\int_1^\infty \frac{R(t)}{t^{s+1}}\mathrm{d}t.$$

Taking limit as $s \to 1^+$, by the *analytic class number formula* (see [5, Chapter 8 Theorem 5]),

$$\lim_{s\to 1^+}(s-1)\zeta_K(s) = c_K = \frac{2^{r_1+r_2}\pi^{r_2}R_K}{|\mu(K)|\sqrt{\Delta_K}}\, h_K,$$

where r_1 is the number of real embeddings of K, r_2 is the number of non-conjugate complex embeddings, R_K is the regulator of K, h_K is the class number of K, Δ_K is the absolute value of the discriminant of K and $\mu(K)$ is the group of roots of unity in K.

In conclusion, as $n \to \infty$, $h_K(n)$ diverges by Equation (3) so that the integrality of the n-th Dedekind harmonic number seems to be an intriguing question. Here, we first prove the following theorem.

Theorem A. *Let K be a number field. Then, there exists a positive integer n_K depending only on K such that for any $n \ge n_K$, the n-th Dedekind harmonic number $h_K(n)$ is not an integer.*

The first part of our next result is a uniform and an explicit version of Theorem A for quadratic number fields.

Theorem B.

(*i*) *For any quadratic number field $K = \mathbb{Q}(\sqrt{d})$ where $d \not\equiv 1, 17 \pmod{24}$ is a square-free integer, the* n-th *Dedekind harmonic number is not an integer for any $n \ge 4$.*

(*ii*) *For any quadratic number field $K = \mathbb{Q}(\sqrt{d})$ where $d \equiv 1 \pmod{24}$ is a square-free integer, the n-th Dedekind harmonic number $h_K(n)$ is not an integer for $n \ge 4$ if*

- $n \in [2^e, 2^{e+1})$ *for some positive even integer e or,*
- $n \in [2^e, 2^{e+1})$ *for some positive integer $e \equiv 3 \pmod 4$ or,*
- $n \in [3^y, 3^{y+1})$ *for some positive integer $y \not\equiv 2 \pmod 3$.*

(*iii*) *For any quadratic number field $K = \mathbb{Q}(\sqrt{d})$ where $d \equiv 17 \pmod{24}$ is a square-free integer, the n-th Dedekind harmonic number $h_K(n)$ is not an integer for $n \ge 9$ if*

- $n \in [2^e, 2^{e+1})$ *for some positive even integer e or,*
- $n \in [2^e, 2^{e+1})$ *for some positive integer $e \equiv 3 \pmod 4$ or,*
- $n \in [3^y, 3^{y+1})$ *for some positive even integer y.*

In particular, when $K = \mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{2})$, then the corresponding Dedekind harmonic number is not an integer for any $n \geq 2$ so that the bound for n_K in Theorem B can be lowered. However, this can be seen in the first case analysis of the theorem, namely, when $d \equiv 2, 3 \pmod 4$.

It is well-known that the Dedekind zeta function $\zeta_K(s)$ can be extended to the entire complex plane, see [5, Chapter 8]. The Riemann hypothesis for $\zeta_K(s)$, DRH for short, states that if $\zeta_K(s) = 0$ and $0 < \mathrm{Re}(s) < 1$, then $\mathrm{Re}(s) = \frac{1}{2}$. Assuming DRH, we obtain the following result. The first part of the following result is in the same spirit that of [4]. The second part yields the non-integrality of Dedekind harmonic numbers for quadratic number fields in a uniform way, and this puts some light on the cases $d \equiv 1 \pmod{24}$ and $d \equiv 17 \pmod{24}$ in Theorem B. The second part of the following theorem implies its third part which states that for almost all pairs (d, n), where d is a square-free integer and $n \geq 1$ and $K = \mathbb{Q}(\sqrt{d})$, the corresponding Dedekind harmonic number $h_K(n)$ is not an integer.

Theorem C. *For any number field K, let d_K and Δ_K denote the degree and the absolute value of the discriminant of K, respectively.*

(1) *Assume DRH for the number field K. There exist constants $\beta, x_1 > 0$ such that the difference*

$$h_K(n) - h_K(m)$$

is never an integer for any positive integers $n > m \geq x_1$ whenever

$$n - m \geq \beta(d_K \log m + \log \Delta_K)\sqrt{m}.$$

(2) *Assume DRH for all quadratic number fields $K_d = \mathbb{Q}(\sqrt{d})$ where d is a square-free integer. Let $0 < c < 1$ be given. Then, there exists a constant $n_c > 0$ such that whenever $n \geq n_c$ and $|d| \leq e^{c\sqrt{n/2}}$, the n-th Dedekind harmonic number $h_{K_d}(n)$ is not an integer.*

(3) *Assume DRH for all quadratic number fields $K_d = \mathbb{Q}(\sqrt{d})$ where d is a square-free integer and let Q be the set of square-free integers in $\mathbb{Z}$. Set*

$$S(x) = |\{(d, n) \in ([-x, x] \cap Q) \times [1, x] \mid h_{K_d}(n) \notin \mathbb{Z}\}|.$$

That is, $S(x)$ counts the number of pairs $(d, n) \in Q \times \mathbb{Z}_{>0}$ inside the rectangle $[-x, x] \times [1, x]$, where the corresponding Dedekind harmonic number $h_{K_d}(n)$ is not an integer. Then,

$$S(x) = 2xQ(x) + O(x \log^2 x),$$

where $Q(x) = |Q \cap [0, x]|$. In other words, for almost all such pairs (d, n), the corresponding Dedekind harmonic number $h_{K_d}(n)$ is not an integer as

$$S(x) \sim 2xQ(x).$$

Note that in the third part of Theorem C, the error term $O(x)$ is inevitable as for $n = 1$ we have that $h_K(n) = 1$ for any number field K. Thus under DRH, we are very close to that error term. Furthermore, the third part of Theorem C yields that

$$S(x) \sim \frac{12}{\pi^2} x^2$$

as $Q(x) \sim \frac{6}{\pi^2} x$.

2. Proof of Theorem A

Let K be a number field of degree d. For any positive integer n, recall that $h_K(n)$ can be written as

$$h_K(n) = \sum_{i=1}^{n} \frac{a_i}{i},$$

where a_k is the number of ideals $0 \neq I \subseteq \mathcal{O}_K$ of norm k. We set

$$\pi_1 = \{p \in \mathbb{P} : a_p \neq 0\}$$

and

$$\pi_2 = \{p \in \mathbb{P} : a_p = 0\}.$$

Note that $\mathbb{P} = \pi_1 \cup \pi_2$. Also, define

$$\pi_1(x) = |\pi_1 \cap [1, x]|.$$

By the *prime ideal theorem* (see [2,6]), we know that

$$\pi_K(x) = |\{\mathfrak{p} \subseteq \mathcal{O}_K : N(\mathfrak{p}) \leq x\}| \sim \frac{x}{\log x}.$$

Observe that the prime ideal theorem is an extension of the prime number theorem (see [1]), which states that

$$\pi(x) \sim \frac{x}{\log x},$$

where $\pi(x) = |\{p \in \mathbb{P} : p \leq x\}|$ is the prime counting function. Notice that

$$\begin{aligned} \pi_K(x) &= \sum_{\substack{\mathfrak{p} \subseteq \mathcal{O}_K \\ N(\mathfrak{p}) = p \leq x}} 1 + \sum_{\substack{\mathfrak{p} : N(\mathfrak{p}) = p^2 \\ p \leq \sqrt{x}}} 1 + \cdots + \sum_{\substack{\mathfrak{p} : N(\mathfrak{p}) = p^d \\ p \leq \sqrt[d]{x}}} 1 \\ &= \sum_{p \leq x} a_p + \sum_{p \leq \sqrt{x}} a_{p^2} + \cdots + \sum_{p \leq \sqrt[d]{x}} a_{p^d}. \end{aligned}$$

Moreover, by Equation (1), we know that

$$a_{p^i} \leq \frac{d}{i},$$

for any $i \geq 1$. Therefore, for any $i \geq 1$, one has that

$$\sum_{p \leq \sqrt{x}} a_{p^i} \leq \frac{d}{i} \pi(\sqrt{x}). \tag{4}$$

Thus

$$\sum_{p \leq \sqrt{x}} a_{p^2} + \cdots + \sum_{p \leq \sqrt[d]{x}} a_{p^d} = O_d(\sqrt{x}).$$

This in turn yields that

$$|\{\mathfrak{p} \subseteq \mathcal{O}_K : N(\mathfrak{p}) \in \mathbb{P}, \ \ N(\mathfrak{p}) \leq x\}| \sim \frac{x}{\log x}$$

so that

$$q(x) = \sum_{p \le x} a_p \sim \frac{x}{\log x}.$$

To add, we have

$$\lim_{x\to\infty} \frac{q(2x)}{q(x)} = \lim_{x\to\infty} \frac{\frac{2x}{\log 2x}}{\frac{x}{\log x}} = 2.$$

This gives that $\lim_{x\to\infty} q(2x) - q(x) = \infty$. Therefore,

$$\sum_{\substack{x<p\le 2x \\ p\in\pi_1}} a_p \to \infty \text{ as } x \to \infty.$$

On the other hand,

$$\pi_1(2x) - \pi_1(x) = \sum_{\substack{x<p\le 2x \\ p\in\pi_1}} 1 \ge \frac{1}{d} \sum_{\substack{x<p\le 2x \\ p\in\pi_1}} a_p = \frac{1}{d}(q(2x) - q(x))$$

since $a_p \le d$ for any prime $p \in \mathbb{P}$. Hence, we also have that

$$\lim_{x\to\infty} \pi_1(2x) - \pi_1(x) = \infty.$$

Therefore, if x is sufficiently large, then there is always a prime number p in $\pi_1 \cap (x, 2x]$. Thus, there exists a positive integer n_K greater than $2d$ such that if we take any $n \ge n_K$ and choose a prime $p \in \pi_1 \cap (\frac{n}{2}, n]$, then p does not divide a_p since $1 \le a_p \le d < p$. As

$$h_K(n) = 1 + \frac{a_2}{2} + \cdots + \frac{a_p}{p} + \cdots + \frac{a_n}{n}$$

and $2p > n$, this yields that the only multiple of p lying in $[1, n]$ is just p itself. Hence, we obtain that the p-adic order of $h_K(n)$ is -1. This completes the proof. □

3. Proof of Theorem B

Suppose that K is a quadratic number field, namely,

$$K = \mathbb{Q}(\sqrt{d}),$$

where d is a square-free integer. Now, our goal is to compute n_K's explicitly and then show that it is at most 4 uniformly in K except for the cases that $d \equiv 1, 17 \pmod{24}$. For these cases, a uniform bound for n_K in d may not be possible as one may observe from the concluding remark at the end. Let us denote the *discriminant* of K by D. It is known that if $d \equiv 1 \pmod 4$, then $D = d$ and if $d \equiv 2, 3 \pmod 4$, then $D = 4d$. The *Dirichlet L-function* associated to a given Dirichlet character χ modulo q is given by

$$L(s, \chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s}.$$

Now, for any prime number p, let us define

$$\chi_D(p) = \begin{cases} 1 & \text{if } p \text{ splits,} \\ -1 & \text{if } p \text{ is inert,} \\ 0 & \text{if } p \text{ ramifies.} \end{cases}$$

Then, χ_D yields a Dirichlet character modulo $|D|$ (see [1,6]). It can be extended to all integers. Moreover, we have the following identity

$$\zeta_K(s) = \zeta(s)L(s, \chi_D).$$

Note that these Dirichlet series converge absolutely in the half plane $Re(s) > 1$. In this half plane, if we write the Dirichlet series $\zeta_K(s)$ as

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

then, since it is a multiplication of two Dirichlet series $\zeta(s)$ and $L(s, \chi_D)$, we have that

$$a_n = (1 * \chi_D)(n),$$

where $1 * \chi_D$ is the Dirichlet convolution of the unit function 1 and χ_D. As a result,

$$a_n = \sum_{b|n} \chi_D(b).$$

Here, $\chi_D(n)$ is actually the Kronecker symbol $\left(\frac{D}{n}\right)_K$ where it is defined as follows:

(I) $\left(\frac{D}{p}\right)_K = 0$ when $p \mid d$,

(II)

$$\left(\frac{D}{2}\right)_K = \begin{cases} 1 & \text{when } D \equiv 1 \pmod 8, \\ -1 & \text{when } D \equiv 5 \pmod 8, \end{cases}$$

(III) For any odd prime, $\left(\frac{D}{p}\right)_K$ is the usual Legendre symbol modulo p,

(IV)

$$\left(\frac{D}{-1}\right)_K = \begin{cases} 1 & \text{when } D > 0, \\ -1 & \text{when } D < 0, \end{cases}$$

(V) $\left(\frac{D}{n}\right)_K$ is totally multiplicative.

At this point, we refer the reader to check [6]. To add, since a_n is the Dirichlet convolution of multiplicative functions, it is also multiplicative. Now, we are ready to prove the first part of the theorem. We present n_K's explicitly so that $h_K(n)$ is not an integer for any $n \geq 4$ and for $K = \mathbb{Q}(\sqrt{d})$ where d is a square-free integer with

$$d \not\equiv 1, 17 \pmod{24}.$$

From now on, χ will represent χ_D.

Case 1: $d \equiv 2, 3 \pmod 4$. For any positive integer k, we have the following coefficients:

$$a_{2^k} = \sum_{b|2^k} \chi(b) = \chi(1) + \chi(2) + \cdots + \chi(2^k) = 1$$

since when $d \equiv 2, 3 \pmod 4$, we have $D = 4d$ so that $2 \mid D$. Therefore,

$$\chi(2^k) = \chi(2) = 0.$$

For $n \geq 2$, we have $2^m \leq n < 2^{m+1}$ for some positive integer m. Then

$$\begin{aligned} h_K(n) &= \frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_n}{n} \\ &= 1 + \left(\frac{a_2}{2} + \frac{a_3}{3}\right) + \left(\frac{a_4}{4} + \frac{a_5}{5} + \frac{a_6}{6} + \frac{a_7}{7}\right) \\ &\quad + \cdots + \left(\frac{a_{2^m}}{2^m} + \cdots + \frac{a_n}{n}\right) \\ &= 1 + \underbrace{\left(\frac{1}{2} + \frac{a_3}{3}\right)}_{\text{of 2-adic val:}-1} + \underbrace{\left(\frac{1}{4} + \frac{a_5}{5} + \frac{a_6}{6} + \frac{a_7}{7}\right)}_{\text{of 2-adic val:}-2} \\ &\quad + \cdots + \underbrace{\left(\frac{1}{2^m} + \cdots + \frac{a_n}{n}\right)}_{\text{of 2-adic val:}-m}. \end{aligned}$$

Thus, whenever $n > 1$, we have $\nu_2(h_K(n)) < 0$ so that $h_K(n)$ is not an integer. In other words, n_K can be chosen to be 2.

Case 2: $d \equiv 1 \pmod 4$. Recall that when $d \equiv 1 \pmod 4$, we have $D = d$. Also,

$$\left(\frac{D}{2}\right)_K = \begin{cases} 1 & \text{when } D \equiv 1 \pmod 8, \\ -1 & \text{when } D \equiv 5 \pmod 8, \end{cases}$$

$$\left(\frac{D}{3}\right)_K = \begin{cases} -1 & \text{when } D \equiv 2 \pmod 3, \\ 0 & \text{when } D \equiv 0 \pmod 3, \\ 1 & \text{when } D \equiv 1 \pmod 3. \end{cases}$$

Subcase 1. First, suppose that $d \equiv 5 \pmod 8$. Then, $\chi(2) = -1$.

Subcase 1.1. Assume that $\chi(3) = -1$. We have

$$a_{2^{2m}} = 1, \;\; a_{2^{2m+1}} = 0, \;\; a_3 = 0 \quad \text{and} \quad a_{3.2^{2m}} = 0.$$

Now, take any positive integer $n \geq 4$ so that $2^{2e} \leq n < 2^{2e+2}$ for some integer $e \geq 1$. Then, we can write

$$h_K(n) = (1 + 0 + 0)$$

plus blocks of the form

$$\left(\frac{1}{2^{2m}} + \frac{a_{2^{2m}+1}}{2^{2m}+1} + \cdots + \frac{a_{2^{2m+1}}}{2^{2m+1}} + \cdots + \frac{a_{3.2^{2m}}}{3 \cdot 2^{2m}} + \cdots + \frac{a_{2^{2m+2}-1}}{2^{2m+2}-1}\right) \tag{5}$$

and plus the last block

$$\left(\frac{1}{2^{2e}} + \cdots + \frac{a_n}{n}\right).$$

However, since $a_{2^{2m+1}}$ and $a_{3.2^{2m}}$ vanish, the block in (5) has 2-adic valuation $-2m$. Thus,

$$h_K(n) \;= \left(1 + \frac{0}{2} + \frac{0}{3}\right) + \left(\frac{1}{4} + \cdots + \frac{0}{8} + \cdots + \frac{0}{12} + \cdots + \frac{a_{15}}{15}\right)$$

$$+\left(\frac{1}{16}+\cdots+\frac{a_{63}}{63}\right)+\cdots+\left(\frac{1}{2^{2e}}+\cdots+\frac{a_n}{n}\right)$$

is not an integer for any $n \geq 4$. In other words, n_K can be chosen to be 4 in this case.

Subcase 1.2. When $\chi(2) = -1$ and $\chi(3) = 0$, we have that

$$a_{2^{2m}} = 1,\ a_{2^{2m+1}} = 0, a_{3^m} = 1 \quad \text{and} \quad a_{2\cdot 3^m} = 0.$$

Therefore, given $n \geq 3$ where $3^m \leq n < 3^{m+1}$ and $m \geq 1$, we can write

$$\begin{aligned} h_K(n) &= \left(1+\frac{2}{2}\right)+\left(\frac{1}{3}+\cdots+\frac{0}{2\cdot 3}+\cdots+\frac{a_{3^2-1}}{3^2-1}\right) \\ &\quad +\cdots+\left(\frac{1}{3^m}+\cdots+\frac{a_n}{n}\right). \end{aligned}$$

Note that $h_K(n)$ consists of blocks of the form (or some part of it possibly for the last block)

$$\left(\frac{1}{3^m}+\cdots+\frac{0}{2\cdot 3^m}+\cdots+\frac{a_{3^{m+1}-1}}{3^{m+1}-1}\right)$$

which has 3-adic valuation $-m$. Thus, n_K can be chosen 3 in this case.

Subcase 1.3. When $\chi(2) = -1$ and $\chi(3) = 1$, we have

$$a_{2^{2m}} = 1,\ a_{2^{2m+1}} = 0 \quad \text{and} \quad a_3 = 2.$$

For any $n \geq 3$, we can write

$$h_K(n) = \left(1+0+\frac{2}{3}\right)$$

plus blocks of the form (or some part of it possibly for the last block)

$$\left(\frac{1}{2^{2m}}+\cdots+\frac{0}{2^{2m+1}}+\cdots+\frac{2}{3\cdot 2^{2m}}+\cdots+\frac{a_{2^{2m+2}-1}}{2^{2m+2}-1}\right)$$

which has 2-adic valuation $-2m$ for each block. Therefore, by the same argument as in the previous case, n_K can be chosen 3.

Subcase 2. Suppose that $d \equiv 1 \pmod 8$ so that $\chi(2) = 1$. If $d \not\equiv 1, 17 \pmod{24}$, then $\chi(3) = 0$.

When $\chi(2) = 1$ and $\chi(3) = 0$, we have

$$a_{2^m} = m+1, a_{3^m} = 1 \quad \text{and} \quad a_{2\cdot 3^m} = 2.$$

Thus

$$\begin{aligned} h_K(n) &= \left(1+\frac{2}{2}\right)+\left(\frac{1}{3}+\cdots+\frac{2}{2\cdot 3}+\cdots+\frac{a_{3^2-1}}{3^2-1}\right) \\ &\quad +\cdots+\left(\frac{1}{3^m}+\cdots+\frac{a_n}{n}\right). \end{aligned}$$

Hence $h_K(n)$ consists of blocks of the form (or some part of it possibly for the last block)

$$\left(\frac{1}{3^m}+\cdots+\frac{2}{2\cdot 3^m}+\cdots+\frac{a_{3^{m+1}-1}}{3^{m+1}-1}\right)$$

which has 3-adic valuation $-m$. Thus, n_K can be chosen 3 in this case.

Hence, we proved the first part of Theorem B.

Now, we prove the second part of the theorem by analysing the case $d \equiv 1 \pmod{24}$. In this case, $\chi(2) = 1$ and $\chi(3) = 1$ so that

$$a_{2^m} = m + 1, a_{3^m} = m + 1 \quad \text{and} \quad a_{2^{m_1} . 3^{m_2}} = (m_1 + 1)(m_2 + 1).$$

We investigate the 2-adic and the 3-adic valuation of $h_K(n)$. Let us start with the 2-adic valuation of it. Given $n \geq 4$, we can write $2^e \leq n < 2^{e+1}$ for some positive integer e. Therefore,

$$h_K(n) = \left(1 + \frac{2}{2} + \frac{2}{3}\right)$$

plus blocks of the form (or some part of it possibly for the last block)

$$\left(\frac{m+1}{2^m} + \cdots + \frac{a_{2^{m+1}-1}}{2^{m+1} - 1}\right).$$

Consequently, the last block will be

$$\left(\frac{e+1}{2^e} + \cdots + \frac{a_n}{n}\right)$$

such that if e is even, then $e + 1$ is odd and the 2-adic valuation of $h_K(n)$ will be $-e < 0$. Thus, $h_K(n)$ is not an integer in this case.

Now, suppose that e is odd. We have $2^e \leq n < 3 \cdot 2^{e-1}$ or $3 \cdot 2^{e-1} \leq n < 2^{e+1}$, provided that $e \geq 3$ as $n \geq 4$ is assumed.

Subcase 1. $2^e \leq n < 3 \cdot 2^{e-1}$. If we write $h_K(n)$ as above, for the last block we obtain that

$$\left(\frac{e+1}{2^e} + \cdots + \frac{a_n}{n}\right)$$

such that $e + 1$ is divisible by 2. Thus, let us write

$$h_K(n) = \left(\frac{e}{2^{e-1}} + \frac{e+1}{2 \cdot 2^{e-1}} + q\right) = \left(\frac{\frac{3e+1}{2}}{2^{e-1}} + q\right)$$

for some rational number q with $\nu_2(q) > -e + 1$. Now, if $3e + 1$ is divisible by 2 only once, then $h_K(n)$ is not an integer as $\nu_2(h_K(n)) = -e + 1 < 0$. That is, if $e \equiv 3 \pmod 4$, then $h_K(n)$ is not an integer.

Subcase 2. $3 \cdot 2^{e-1} \leq n < 2^{e+1}$. In this case, the last block of $h_K(n)$ can be written as

$$\left(\frac{e+1}{2^e} + \cdots + \frac{2 \cdot e}{3 \cdot 2^{e-1}} + \cdots + \frac{a_n}{n}\right).$$

Similar to the previous case, the 2-adic valuation of $h_K(n)$ is determined by the terms having multiples of 2^{e-1} in their denominators. Moreover, we can omit the term $\frac{2 \cdot e}{3 \cdot 2^{e-1}}$ and write $h_K(n)$ as follows:

$$h_K(n) = \left(\frac{e}{2^{e-1}} + \frac{e+1}{2 \cdot 2^{e-1}} + q\right) = \left(\frac{\frac{3e+1}{2}}{2^{e-1}} + q\right)$$

for some rational number q with $\nu_2(q) > -e + 1$. Thus, again if $3e + 1$ is divisible by 2 only once, then $h_K(n)$ is not an integer, namely when $e \equiv 3 \pmod 4$.

Now, we continue our observation with the 3-adic valuation of $h_K(n)$. For any $n \geq 3$, we can write $3^y \leq n < 3^{y+1}$ for some positive integer y. Then

$$h_K(n) = \left(1 + 1 + \frac{2}{3}\right)$$

plus blocks of the form (or some part of it possibly for the last block)

$$\left(\frac{m+1}{3^m} + \cdots + \frac{2(m+1)}{2 \cdot 3^m} + \cdots + \frac{a_{3^{m+1}-1}}{3^{m+1}-1}\right).$$

We have 2 cases as follows and since we look for any increase in the 3-adic valuation of $h_K(n)$, we only consider the last block.

Subcase 1. $3^y \leq n < 2 \cdot 3^y$. We have

$$h_K(n) = \left(1 + 1 + \frac{2}{3}\right) + \cdots + \left(\frac{y+1}{3^y} + \cdots + \frac{a_n}{n}\right).$$

Hence, if $y \not\equiv 2 \pmod 3$, then $\nu_3(h_K(n)) = -y < 0$ so that $h_K(n)$ is not an integer. However, if $y \equiv 2 \pmod 3$, then the 3-adic valuation of $h_K(n)$ might increase.

Subcase 2. $2 \cdot 3^y \leq n < 3^{y+1}$. In this case, the last block of $h_K(n)$ will be

$$\left(\frac{y+1}{3^y} + \cdots + \frac{2(y+1)}{2 \cdot 3^y} + \cdots + \frac{a_n}{n}\right).$$

Considering the highest exponents of 3 in the denominators, the last block can be rewritten as

$$\frac{2(y+1)}{3^y} + q,$$

where $\nu_3(q) > -y$. Similarly, if $y \not\equiv 2 \pmod 3$, then $h_K(n)$ is not an integer but if $y \equiv 2 \pmod 3$, then the 3-adic valuation of $h_K(n)$ may be non-negative. Hence, we proved the second part of the theorem.

To prove the last part of the theorem, suppose that $d \equiv 17 \pmod{24}$. In this case, we have $\chi(2) = 1$ and $\chi(3) = -1$. Therefore,

$$a_{2^m} = m + 1, \;\; a_{3^{2m}} = 1, \;\; a_{3^{2m+1}} = 0 \quad \text{and} \quad a_{2 \cdot 3^m} = 2 \cdot a_{3^m}.$$

First of all, we investigate the 2-adic valuation of $h_K(n)$. For any $n \geq 4$, we have $2^e \leq n < 2^{e+1}$ for some positive integer e. Then, we can write

$$h_K(n) = \left(1 + \frac{2}{2} + 0\right)$$

plus blocks of the form (or some part of it possibly for the last block)

$$\left(\frac{a_{2^m}}{2^m} + \cdots + \frac{a_{2^{m+1}-1}}{2^{m+1}-1}\right).$$

If e is even, then $a_{2^e} = e + 1$ such that the last block has 2-adic valuation $-e < 0$. Thus, $h_K(n)$ is not an integer for any $n \geq 4$ satisfying $2^e \leq n < 2^{e+1}$ for some positive even integer e.

Now, assume that $n \geq 4$ and $2^e \leq n < 2^{e+1}$ for some positive odd integer e. We have $2^e \leq n < 3 \cdot 2^{e-1}$ or $3 \cdot 2^{e-1} \leq n < 2^{e+1}$. Then, writing $h_K(n)$ as above, the last block will be

$$\left(\frac{a_{2^e}}{2^e} + \cdots + \frac{a_n}{n}\right) \quad \text{or} \quad \left(\frac{a_{2^e}}{2^e} + \cdots + \frac{a_{3 \cdot 2^{e-1}}}{3 \cdot 2^{e-1}} + \cdots + \frac{a_n}{n}\right)$$

which is

$$\left(\frac{e+1}{2^e} + \cdots + \frac{a_n}{n}\right)$$

as $a_{3.2^{e-1}} = a_3 \cdot a_{2^{e-1}} = 0$. Since $e+1$ is divisible by 2, the 2-adic valuation of $h_K(n)$ will be determined by the terms having 2^{e-1} in their denominators. Therefore, we can write

$$h_K(n) = \left(\frac{e}{2^{e-1}} + \frac{e+1}{2 \cdot 2^{e-1}} + q\right) = \left(\frac{\frac{3e+1}{2}}{2^{e-1}} + q\right)$$

for some rational number q with $v_2(q) > -e+1$. Thus, if $3e+1$ is divisible by 2 only once, then $h_K(n)$ is not an integer. That is, if $e \equiv 3 \pmod 4$, then $h_K(n)$ is not an integer.

Now, let us continue our investigation with the 3-adic valuation of $h_K(n)$. For a given $n \geq 3$, if $3^y \leq n < 3^{y+1}$ for some positive integer y, note that

$$h_K(n) = \left(1 + \frac{2}{2}\right)$$

plus blocks of the form (or some part of it possibly for the last block)

$$\left(\frac{a_{3^m}}{3^m} + \cdots + \frac{a_{2\cdot 3^m}}{2 \cdot 3^m} + \cdots + \frac{a_{3^{m+1}-1}}{3^{m+1}-1}\right).$$

The block has 3-adic valuation $-m$ if m is even. Therefore, if $3^y \leq n < 3^{y+1}$ holds for some even integer y then $h_K(n)$ is not an integer and n_K can be chosen 9 in this case as first block with negative 3-adic valuation starts when $n = 9$. This completes the last part of the proof. □

Remark 3.1.

(i) In the second part of the previous proof, note that if $y \equiv 2 \pmod 3$, then the highest exponent of 3 that occurs in the denominators of $h_K(n)$ will be 3^{y-1}. However,

$$3^y \leq n < 3^{y+1}$$

implies that

$$3 \cdot 3^{y-1} \leq n < 9 \cdot 3^{y-1}.$$

Consequently, the fractions

$$\frac{a_{5.3^{y-1}}}{5 \cdot 3^{y-1}} \quad \text{and} \quad \frac{a_{7.3^{y-1}}}{7 \cdot 3^{y-1}}$$

may appear inside $h_K(n)$. Unfortunately, $\chi(5)$ and $\chi(7)$ must be known to find the values of

$$a_{5.3^{y-1}} \quad \text{and} \quad a_{7.3^{y-1}}.$$

Considering the possible values of $\chi(5)$ and $\chi(7)$ brings another set of subcases, which we will not elaborate further in this note.

(ii) In the last part of the previous proof, note that for a given n, if $2^e \leq n < 2^{e+1}$ is satisfied for some positive integer $e \equiv 1 \pmod 4$, then the 2-adic valuation of $h_K(n)$ will be determined by the terms $\frac{a_k}{k}$ for k a multiple of 2^{e-1}. Even though $a_{3.2^{k-1}} = 0$ holds, if $3e + 1$ is divisible by 2^k for some $k \geq 2$ then the terms $\frac{a_k}{k}$ with $k = c \cdot 2^{e-2}$ correlate with each other where c is a positive integer. As a result, the 2-adic valuation of $h_K(n)$ may increase and one has to consider the possible values $\chi(5), \chi(7), \ldots$ and so on.

Moreover, if $3^y \leq n < 3^{y+1}$ holds for some positive odd integer y, then for the last block in the proof, we may have

$$\left(\frac{a_{3^y}}{3^y} + \cdots + \frac{a_{2\cdot 3^y}}{2\cdot 3^y} + \cdots + \frac{a_n}{n}\right).$$

Since y is odd, $a_{3^y} = 0$ and $a_{2\cdot 3^y} = a_2 \cdot a_{3^y} = 0$. Consequently, to investigate the 3-adic valuation of $h_K(n)$, one has to consider the terms $\frac{a_k}{k}$ for k a multiple of 3^{y-1}. However, this will lead to some other subcases such that the possible values for $\chi(5)$ need to be considered to begin with. Therefore, by considering only the values of $\chi(2), \chi(3)$ the bound n_K may not be given explicitly.

Next, we continue with the following remark which may shed some more light on the cases $d \equiv 1 \pmod{24}$ and $d \equiv 17 \pmod{24}$ in Theorem B.

Remark 3.2. For the case $d \equiv 1 \pmod{24}$, we see that by [7]:

- For $K = \mathbb{Q}(\sqrt{73})$, the 5-adic valuation $\nu_5(h_K(514 + j)) = 2$ for $j \in \{0, 1, 2, 3, 4\}$.
- For $K = \mathbb{Q}(\sqrt{73})$, the 7-adic valuation $\nu_7(h_K(311 + j)) = 2$ for $j \in \{0, 1, 2, 3, 4\}$.
- For $K = \mathbb{Q}(\sqrt{97})$, the 3-adic valuation $\nu_3(h_K(681)) = 4$.
- For $K = \mathbb{Q}(\sqrt{145})$, the 2-adic valuation $\nu_2(h_K(960)) = 1$.
- For $K = \mathbb{Q}(\sqrt{217})$, the 2-adic valuations $\nu_2(h_K(807 + j)) = 6$ for $j \in \{0, 1, 2, 3, 4, 5\}$.
- For $K = \mathbb{Q}(\sqrt{265})$, the 2-adic valuation $\nu_2(h_K(9264+j)) = 1$ for $j \in \{0, 1, 2, 3, 4, 5\}$ and $\nu_2(h_K(9270)) = 3$.
- For $K = \mathbb{Q}(\sqrt{313})$, the 2-adic valuation $\nu_2(h_K(8624+j)) \geq 1$ for $j \in \{0, 1, \ldots, 24\}$ with $\nu_2(h_K(8627 + j)) = 4$ for $j \in \{0, 1, 2, 3, 4\}$.
- For $K = \mathbb{Q}(\sqrt{385})$, the 2-adic valuation $\nu_2(h_K(817)) = 4$.
- For $K = \mathbb{Q}(\sqrt{505})$, the 2-adic valuation $\nu_2(h_K(852 + j)) = 5$ for $j \in \{0, 1, 2\}$.
- For $K = \mathbb{Q}(\sqrt{-623})$, the 2-adic valuation $\nu_2(h_K(968 + j)) = 4$ for $j \in \{0, 1, 2, 3\}$.
- For $K = \mathbb{Q}(\sqrt{-695})$, the 2-adic valuation $\nu_2(h_K(864 + j)) = 4$ for $j \in \{0, 1, 2\}$ and the 3-adic valuation $\nu_3(h_K(375 + j)) = 1$ for $j \in \{0, 1, 2, 3, 4, 5, 6\}$.
- For $K = \mathbb{Q}(\sqrt{1153})$, the 71-adic valuation $\nu_{71}(h_K(928 + j)) = 2$ for $j \in \{0, 1, 2, 3, 4, 5, 6, 7\}$.

For the case $d \equiv 17 \pmod{24}$, we see via [7] that:

- For $K = \mathbb{Q}(\sqrt{-223})$, the 2-adic valuation $\nu_2(h_K(36 + i)) \geq 1$ for $i \in \{0, 1, \ldots, 10\}$ and $\nu_2(h_K(47 + i)) = 6$ for $i \in \{0, 1\}$.
- For $K = \mathbb{Q}(\sqrt{-199})$, the 3-adic valuation $\nu_3(h_K(424 + i)) = 1$ for $i \in \{0, 1, 2\}$, $\nu_3(h_K(430)) = 3$ and $\nu_3(h_K(433)) = 5$.
- For $K = \mathbb{Q}(\sqrt{209})$, the 3-adic valuation $\nu_3(h_K(423 + i)) = 2$ for $i \in \{0, 1, 2, 3, 4\}$ and $\nu_3(h_K(428 + i)) = 1$ for $i \in \{0, 1, 2\}$.

- For $K = \mathbb{Q}(\sqrt{689})$, the 2-adic valuation $\nu_2(h_K(51+i)) = 7$ for $i \in \{0,1\}$ and the 3-adic valuation $\nu_3(h_K(51+i)) \geq 1$ for $i \in \{0,1\}$.

The remark above shows some results obtained with the computer algebra system SageMath [7]. We check the p-adic valuation of Dedekind harmonic numbers for their non-integrality naturally, and it seems that there may not be a uniform bound in d for n_K via a specific p-adic valuation (for instance, $p = 2$) when $K = \mathbb{Q}(\sqrt{d})$ and $d \equiv 1 \pmod{24}$ or $d \equiv 17 \pmod{24}$ is a square-free integer.

Moreover, again by [7], we constructed a suitable list of square-free integers $d \equiv 1 \pmod{24}$, where the χ values for the primes 5, 7, 11, 13, 17 for these d values are either 0 or 1. For each choice of $\chi(5), \chi(7), \ldots, \chi(17)$, we chose positive and negative d's. In total, we obtained 57585 such d. Then, we checked the first 1000 Dedekind harmonic numbers for each number field $\mathbb{Q}(\sqrt{d})$ whether they have non-negative 2-adic and 3-adic valuations or not. However, the program could not find any example which has both non-negative 2-adic and 3-adic valuations among all these numbers. On the other hand, when $d \equiv 17 \pmod{17}$ (for instance, $d = 689$) the remark above indicates such an example.

Before proving our next result, we finish this part of our note by exhibiting the first ten values of $h_K(n)$ for various quadratic number fields as below:

n_K	$\mathbb{Q}(i)$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{17})$	$\mathbb{Q}(\sqrt{-23})$	$\mathbb{Q}(\sqrt{73})$	$\mathbb{Q}(\sqrt{97})$
1	1	1	1	1	1	1	1	1
2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	1	2	2	2	2
3	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{11}{6}$	1	2	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{8}{3}$
4	$\frac{7}{4}$	$\frac{7}{4}$	$\frac{25}{12}$	$\frac{5}{4}$	$\frac{11}{4}$	$\frac{41}{12}$	$\frac{41}{12}$	$\frac{41}{12}$
5	$\frac{43}{20}$	$\frac{7}{4}$	$\frac{25}{12}$	$\frac{29}{20}$	$\frac{11}{4}$	$\frac{41}{12}$	$\frac{41}{12}$	$\frac{41}{12}$
6	$\frac{43}{20}$	$\frac{7}{4}$	$\frac{9}{4}$	$\frac{29}{20}$	$\frac{11}{4}$	$\frac{49}{12}$	$\frac{49}{12}$	$\frac{49}{12}$
7	$\frac{43}{20}$	$\frac{57}{28}$	$\frac{9}{4}$	$\frac{29}{20}$	$\frac{11}{4}$	$\frac{49}{12}$	$\frac{49}{12}$	$\frac{49}{12}$
8	$\frac{91}{40}$	$\frac{121}{56}$	$\frac{19}{8}$	$\frac{29}{20}$	$\frac{13}{4}$	$\frac{55}{12}$	$\frac{55}{12}$	$\frac{55}{12}$
9	$\frac{859}{360}$	$\frac{1145}{504}$	$\frac{179}{72}$	$\frac{281}{180}$	$\frac{121}{36}$	$\frac{59}{12}$	$\frac{59}{12}$	$\frac{59}{12}$
10	$\frac{931}{360}$	$\frac{1145}{504}$	$\frac{179}{72}$	$\frac{281}{180}$	$\frac{121}{36}$	$\frac{59}{12}$	$\frac{59}{12}$	$\frac{59}{12}$

4. Proof of Theorem C

We first recall the following fact from [3, Theorem 2].

Fact [3, *Theorem* 2]. Assume DRH for the number field K. There exist absolute constants $x_0, c_1, c_2 > 0$ such that for $x \geq x_0$ and $c_1(d_K \log x + \log \Delta_K)\sqrt{x} \leq h \leq x$, we have

$$\pi_K(x+h) - \pi_K(x) \geq c_2 \frac{h}{\log x},$$

where d_K is the degree of the number field K and Δ_K is the absolute value of the discriminant of K.

Assume that we have the absolute constants $x_0, c_1, c_2 > 0$ given by the previous fact. Let h be a function satisfying

$$tc_1(d_K \log x + \log \Delta_K)\sqrt{x} \le h \le x,$$

where $t = \frac{2\sqrt{2}}{c_1 c_2} + 1$. Thus there exists $x_1 \ge \max\{x_0, d_K\}$ such that for all $x \ge x_1$, the inequality $tc_1(d_K \log x + \log \Delta_K)\sqrt{x} \le x$ is preserved. By Equation (4), one can obtain that

$$\pi_K(x) = \sum_{p \le x} a_p + R(x),$$

where

$$|R(x)| \le \frac{d_K}{2} x^{1/2} + \frac{d_K}{3} x^{1/3} + \cdots + x^{1/d_K}.$$

We can also choose the above x_1 such that for all $x \ge x_1$ the inequality

$$|R(x)| \le d_K \sqrt{x}$$

holds. Therefore, by the previous fact again, we see that

$$\begin{aligned}\sum_{x<p\le x+h} a_p &\ge \frac{c_2}{\log x} h - 2d_K\sqrt{x+h} \\ &\ge \frac{c_2}{\log x} tc_1(d_K \log x + \log \Delta_K)\sqrt{x} - 2d_K\sqrt{x+h} \\ &> 2\sqrt{2} d_K \sqrt{x} - 2d_K\sqrt{2x} + \frac{tc_1c_2(\log \Delta_K)\sqrt{x}}{\log x} > 0.\end{aligned}$$

As a result, for any integer $m \ge x_1$, there is a prime $p \in \mathbb{P}$ with $a_p \ne 0$ between

$$m \quad \text{and} \quad n = m + h$$

whenever $m \ge h \ge \beta(d_K \log m + \log \Delta_K)\sqrt{m}$ for some absolute constant β. Thus, we have

$$\nu_p(h_K(n) - h_K(m)) < 0.$$

For the case when $h > m$, we can use the same argument as in Theorem A and the first part follows.

Now, we prove the second part of the theorem. Assume DRH for all quadratic number fields $K_d = \mathbb{Q}(\sqrt{d})$, where d is a square-free integer. Let $0 < c < 1$ be given. From the fact, we have $x_0, c_1, c_2 > 0$ such that for $x \ge x_0$ and $c_1(2 \log x + \log \Delta_K)\sqrt{x} \le h \le x$, the inequality

$$\pi_K(x+h) - \pi_K(x) \ge c_2 \frac{h}{\log x}$$

holds. By Theorem B, we may assume that d is congruent to 1 or 17 modulo 24, as we have a uniform bound 4 for other cases, and in this case we have that $\Delta_K = |d|$. Note that if $|d| \le e^{c\sqrt{x}}$, then $\log |d| \le c\sqrt{x}$. There exists $x_c \ge x_0$ such that whenever $x \ge x_c$, the inequalities

$$c_1(2 \log x + c\sqrt{x})\sqrt{x} \le x$$

and

$$\frac{c_2 x}{\log x} - 4\sqrt{2x} > 0$$

hold. Let us choose $h = x$. Hence, similar to the first part of the theorem, we get that

$$\sum_{x<p\leq 2x} a_p > 0 \tag{6}$$

for any quadratic field $K_d = \mathbb{Q}(\sqrt{d})$ with $|d| \leq e^{c\sqrt{x}}$. Now choose a positive integer n_c which is greater than both $2x_c$ and 4. Let $n \geq n_c$ and $|d| \leq e^{c\sqrt{n/2}}$. By (6), choose a prime $p_d \in (\frac{n}{2}, n]$ with $a_{p_d} \neq 0$. As

$$h_{K_d}(n) = 1 + \frac{a_2}{2} + \cdots + \frac{a_{p_d}}{p_d} + \cdots + \frac{a_n}{n},$$

we obtain that the p_d-adic order of $h_{K_d}(n)$ is -1. Hence, whenever $n \geq n_c$ and $|d| \leq e^{c\sqrt{n/2}}$, the n-th Dedekind harmonic number $h_{K_d}(n)$ is not an integer. This completes the proof of the second part.

Finally, we prove the third part of Theorem C. We start by taking $c = \frac{1}{2}$ in part (2) of the theorem. Hence, there exists a constant $m_0 > 0$ such that for any $n \geq m_0$ and $|d| \leq e^{\frac{1}{2}\sqrt{n/2}}$ the n-th Dedekind harmonic number $h_{K_d}(n)$ is not an integer. Now, take a sufficiently large positive real number $x > m_0$. Then, $d = e^{\frac{1}{2}\sqrt{n/2}}$ and $d = x$ intersect when $n = 8\log^2 x$ (see Figure 1).

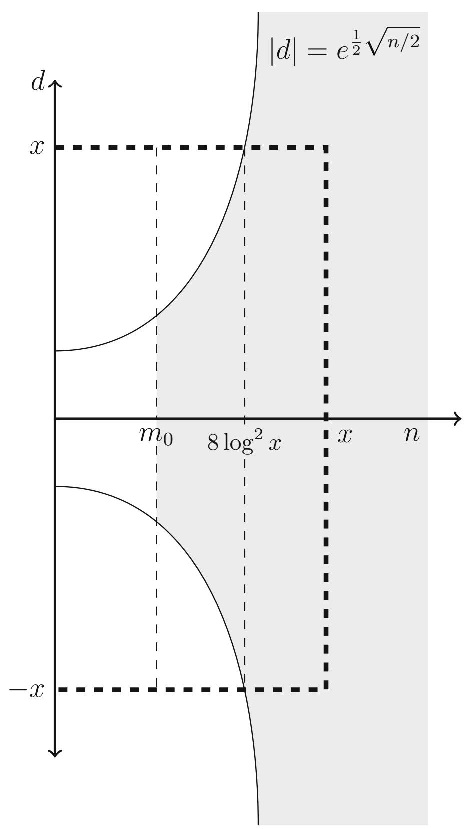

Figure 1. The graph of $|d| = e^{\frac{1}{2}\sqrt{n/2}}$. The lattice points (d, n) where the corresponding $h_{K_d}(n)$ is not an integer in the shaded area.

Now, set $Q(x) = |\{0 \le n \le x \mid n \text{ is square-free}\}|$. Therefore, we have

$$S(x) - 2Q(x)x \ll 8\log^2 x \cdot 2Q(x),$$

so that $S(x) = 2xQ(x) + O(x\log^2 x)$. Hence

$$S(x) \sim 2xQ(x)$$

as $Q(x) \sim \frac{6}{\pi^2}x$ and we obtain the result. □

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