Uniform Asymptotic Stability by Indefinite Lyapunov Functions

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***Abstract*—In this work, we consider Uniform Asymptotic Stability (UAS) of nonlinear time-varying systems. We utilize an indefinite signed polynomial of Lyapunov Function (LF) for the upper bound of the derivative of LF. This special bound is especially useful for perturbation problems. Compared to the ones in the literature we improve the upper bound of the LF and its related properties. Since UAS is the first step for input to state stability (ISS) and integral ISS, it should be thought that these improvements will give rise to new advances in real-world applications as well.**

***Index Terms*—nonlinear time varying systems, uniform asymptotic stability, input-to-state stability, Lyapunov second method, indefinite Lyapunov function.**

I. Introduction

Consider the nonlinear system

$$\dot{x} = f(t,x), \quad t_0 \geq 0 \tag{1}$$

and its controlled version

$$\dot{x} = h(t,x,u), \quad t_0 \geq 0 \tag{2}$$

where $f \in C[J \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n]$ is locally Lipschitz in x and u and piecewise continuous in t, assuming the input signal $u(t)$ as piecewise continuous and bounded function of $t \geq 0$.

The nonlinear system structures (1) and (2) are the most common equations encountered in real world applications. When it comes to understand its long time behaviour, Lyapunov Function method is quite common as well, [7], [11], [24] and [25]. But negativity assumption for its derivative so the difficulty in finding a suitable Lyapunov Function tailored with system equation leads research to relax its conditions [16] and [18].

One of the main attempts to this end is bounding the derivative $\dot{V}(t,x)$ with a linear form of $V(t,x)$. Especially in recent decade, many research has been done using this form,

$$\dot{V}(t,x) \leq g(t)V(t,x). \tag{3}$$

Authors put some conditions on $g(t)$ and so tried to give some conclusions on different kinds of stability and for different kinds of system structures, [2], [3], [4], [8], [9], [10], [13], [14], [19], [22], [26] and [27]. Since this approach is quite different from the ones which have been used so far, authors generalized many of the given stability conditions in the literature. These approaches can actually be summarized as an application of Gronwall inequality - [15] and classical Comparison Lemma - [11] and their extensions.

One of the research direction to this end is UAS which is a conservative form of Asymptotic Stability (AS) of the system, [1]. UAS has great importance since it's the main requirement for ISS and iISS and also since it takes part in converse theorems so in perturbation problems, [11]. To receive UAS & ISS via indefinite LF (ILF), authors of [19] used the upper bound (3) with some special conditions on the function $V(t,x)$. [4] and [27] improved this condition and match it with the stability behaviour of a Linear Time-Varying (LTV) system.

In this work, we use a different kind of upper bound structure for the derivation of the LF comparing with the ones in the ILF literature. Actually the bound that we use is a generalized form of many of them. Here, the ILF that we used is especially effective for the following reasons:

1) To relax the conditions given for $g(t)$ so to remove the complete dependency on $g(t)$. This gives some of this load to the other coefficient that take part in the upper bound of $\dot{V}(t,x)$.
2) For perturbation problem of a system [23].
Perturbation of a UAS or uniformly bounded (UB) linear & nonlinear system is a well studied issue in the literature, [2], [9], [8], [11] and [21]. The approach that we have in this work is a different point of view to this problem and improve many of the conditions given in them.

Nomenclature: Throughout the paper we use the following abbreviations and definitions. By the negative powers of V, we mean the multiplicative inverse of it, not functional inverse; $\mathbb{R}$ is the set of real numbers, $J := [0,\infty), J^+ := (0,\infty), J^- := (-\infty, 0)$, by $C^n[A,B]$ we mean n times differentiable functions from A to B, *PC* represents piecewise continuous functions; $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ are the families of

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class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions, [11]. $||.||$ is the standard Euclidean norm. $H_i(t)[f(t), g(t)]$ (or simply $H_i(t)$ if f and g are clear) is a series generated by $f(t)$ and $g(t)$ in the sense of Lemma 2 of [22], LF is a Lyapunov Function V.

II. Fundamental Concepts

We first start by giving some fundamental definitions, [7], [11] and [24].

Definition 1: *The equilibrium $x = 0$ of (1) is said to be*:

- *stable if, for every t_0 and each $\epsilon > 0$, there is a $\delta(\epsilon, t_0) > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon,\ \forall t \geq t_0 \geq 0; \tag{4}$$

- *uniformly stable (US) if $\delta = \delta(\epsilon)$, i.e. it's independent of t_0, such that (4) is satisfied;*
- *attractive if for every t_0, there is a positive constant $c = c(t_0) > 0$ such that*

$$x(t) \to 0 \ \ as \ \ t \to \infty,\ \forall \|x(t_0)\| < c. \tag{5}$$

- *uniformly attractive (UA) if the constant c in (5) is independent from t_0 or equivalently for each $\eta > 0$, there is $T = T(\eta) > 0$ such that*

$$||x(t)|| < \eta, \quad \forall t \geq t_0 + T(\eta), \forall ||x(t_0)|| < c \tag{6}$$

- *asymptotically stable (AS) if it is stable and attractive*
- *equiasymptotically stable (EAS) if it is stable and UA*
- *uniformly asymptotically stable (UAS) if it is US and UA*
- *exponentially stable (ES) if there exist c, k and $\alpha \in \mathcal{R}^+$ such that*

$$||x(t)|| \leq k||x(t_0)||e^{-\alpha(t-t_0)}, \quad \forall ||x(t_0)|| < c$$

and globally exponentially stable (GES) if it holds for all initials.

Compared to time invariant nonlinear systems, time varying ones have a rich variety of stability types, 36 of [7]. For example, while global UAS (GUAS) and global AS (GAS) are the same concepts for linear time invariant systems, it's not for time varying ones. Thus we encounter different variants of examples when we study time varying systems. It's also worth to emphasize that a system may be US and attractive but not UAS, or it may be an EAS system which is not UAS. Thus, uniformity for both stability and attractivity is necessary for UAS. Some counter examples should be found in 5.3 of [15], 35, 36 of [7] and 4.5 of [11].

***Lemma** 1:* **(UAS by Comparison Functions- [11])**

The system (1) is UAS if and only if there exist a function $\beta \in \mathcal{KL}$ and a positive constant c, independent of t_0, such that

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0) \ \ \forall t \geq t_0 \geq 0, \quad \forall \ ||x(t_0)|| < c \tag{7}$$

We give the concept of stable function which is introduced in [26].

***Lemma** 2:* Consider the following scalar LTV system

$$\dot{y}(t) = \mu(t)y(t), \quad t \in J \tag{8}$$

where $y : J \to \mathbb{R}$ is the state function, $\mu \in PC(J, \mathbb{R})$. Then $\mu(t)$ is AS if the system (8) is AS which is equivalent to

$$\lim_{t\to\infty} \int_{t_0}^{t} \mu(\lambda)d\lambda = -\infty. \tag{9}$$

In a similar manner, we refer to the ES or uniform ES (UES) of the system (8) when we say a function is ES or UES, [27].

III. Main Results

In the work [22], we gave an indefinite signed upper bound for the derivative of the Lyapunov Function V as

$$\dot{V}(t, x) \leq \pi(t)V^m(t, x) + \mu(t)V(t, x) \tag{10}$$

which is a boundary structure often studied in the literature [6], [8], [14] and [17] where $\pi, \mu \in \mathcal{C}^\infty, m \neq 1$. Then we proved the AS of system (1) without uniformity. But it's known that uniformity is one of the main requirements in many adaptations and real world applications like input to state stability (ISS) or integral ISS (iISS). Because it's a well known fact that, if the system (2) is ISS, then necessarily the uncontrolled version ($u = 0$ case of (2)) is UAS, [11] and [12]. Now, in this section, we give UAS with some revised and improved conditions comparing with the works in the literature, [4], [19] and [27].

Consider (10) and multiply each sides by $(1 - m)V^{-m}(t, x)\overline{\mu}(t)$ where $\overline{\mu}(t) = (m - 1)\int_{t_0}^{t} \mu(\lambda)d\lambda$. Then we have

$$m < 1 \Rightarrow \frac{d}{dt}\left[V^{1-m}exp(\overline{\mu}(t))\right] \leq (1 - m)\pi(t)exp(\overline{\mu}(t)) \tag{11}$$

$$m > 1 \Rightarrow \frac{d}{dt}\left[V^{1-m}exp(\overline{\mu}(t))\right] \geq (1 - m)\pi(t)exp(\overline{\mu}(t)) \tag{12}$$

In [22], we also gave a restriction for the sign of the function $\pi(t)$:

$$\pi(t) \in \mathcal{C}^\infty(J, I),\ I = \begin{cases} J^+, & m < 1 \\ J^-, & m > 1 \end{cases} \tag{13}$$

Here, we gave the sign assumptions of $\pi(t)$ to obtain a positive right hand side for (11) and (12). The sign assumption for $m > 1$ is taken so that it's possible to have negative powers of each sides for the next step. On the other hand, the sign assumption for $m < 1$ should be regarded redundant. Because the integration process preserves the sign and after that the resulting left hand side is positive so the right hand side is.

Let us consider again the integration of the right hand side of (11) and (12). Integrating both sides, we obtain

$$m < 1 \Rightarrow V^{1-m}(t, x(t)) \leq (1 - m)\frac{\int_{t_0}^{t} \pi(\tau)exp(\overline{\mu}(\tau))d\tau}{exp(\overline{\mu}(t))}$$

$$+\frac{V_0^{1-m}}{exp(\overline{\mu}(t))} \tag{14}$$

where $V_0 := V(t_0, x_0)$ and inequality is opposite direction for $m > 1$.

When we consider the right hand side of above inequality, it's hard to remove the dependency on t_0 so to choose a $\delta = \delta(\epsilon)$ as we emphasized in the work [22]. This makes more difficult to receive uniform asymptotic stability (UAS).

Now, instead of using signed assumptions (13), let us assume

$$\int_{t_0}^{t} \pi(\tau)exp(\overline{\mu}(\tau))d\tau < 0 \tag{15}$$

where $t \geq t_0 \geq 0$.

We have two reasons to put this assumption. First one is to be able to take negative powers for the case $m > 1$. Second one is to be able to increase right hand side so to receive UAS for $m < 1$. These kind of assumptions, that is, to assume the resulting integral to have a fixed signed instead of the integrand that we considered is common in the literature, Theorem 5 of [4], Theorem 1 of [5]. Even this negativity seems a restriction, it will enable us to relax some properties of the coefficient $\mu(t)$ by taking into account the effect of the other coefficient $\pi(t)$.

Now assume further that $V(t,x)$ is bounded by $\alpha_i \in \mathcal{K}_\infty,\ i = 1, 2$, that is,

$$\alpha_1(||x(t)||) \leq V(t,x) \leq \alpha_2(||x(t)||) \tag{16}$$

and (15) is satisfied. Then the integral term given in (14) can be removed and it turns to the following form for any m.

$$V(t,x(t)) \leq V_0\ e^{\int_{t_0}^{t} \mu(\lambda)d\lambda} \tag{17}$$

This leads us to the following estimate:

$$||x(t)|| \leq \alpha_1^{-1}\big[\alpha_2||x(t_0)||e^{\int_{t_0}^{t} \mu(\lambda)d\lambda}\big] \tag{18}$$

Now we have some alternatives to receive UAS, i.e. to have a class $\mathcal{KL}$ function on the right hand side. The right hand side of (18) is clearly a *class* $\mathcal{K}$ function by Lemma 4.2 of [11]. At this stage, to make it also a *class* $\mathcal{L}$ function, while [27] assumes

$$\dot{V}(t,x) \leq \mu(t)V(t,x) \tag{19}$$

and $\mu(t)$ to be a UES function, on the other hand [4] proposes some others. But we suppose that both the upper bound of $\dot{V}(t,x)$ and the conditions on $\mu(t)$ should be relaxed and revised with some alternative conditions.

***Remark** 1:* Let us give the state of the art as well and compare the given conditions. First of all, for LTV systems, UES is equivalent to UAS, Theorem 6.13 of [20]. It's further known that UAS has two parts: US and UA, Section 5.1 - 31 of [24]. (8) is a LTV system and thus US is equivalent to saying that norm of the solution $exp(\int_{t_0}^{t} \mu(\lambda)d\lambda)$ is bounded with a constant independent from t_0, Section 5.4 - 73 of [24] and Theorem 6.4 of [20]. On the other hand, UA represents convergency to zero uniformly in t_0, Section 5.1 - 29 of [24].

The works [27] and [4] consider UAS and ISS of a nonlinear time varying system. For ISS, by considering the upper bounding structure (19), while these conditions were presented with UES of $\mu(t)$ in Theorem 3-ii of [27] in a compact form, it's presented as

- $\int_{t_0}^{\infty} \mu(t)dt = -\infty$,
- $\int_{s}^{t} \mu(\lambda)d\lambda \leq M,\ \forall t \geq s \geq t_0$,

in Theorem 2 of [4]. But these two theorems actually give the same conclusion, using similar requirements because of the explanations that we gave above. This also should be seen by checking their ISS gains $\alpha_1^{-1}\big(2e^{M}\rho(||u(t)||)\big)$.

Here the convergence should be received without need of (9) but utilizing the other coefficient $\pi(t)$ in the upper bound. This of course does not only remedy the problem but also enlarge the class of $\mu(t)$ functions so the candidate Lyapunov Functions that we have. This also enriches the tools that we can use for analysis. This also probably will present a bigger ISS-gain in future works.

Now, we are ready to give our conclusion on UAS.

***Theorem** 3:* Consider the system (1). Assume that there exist such functions

1) $V : J \times \mathbb{R}^n \to J,\quad V \in \mathcal{C}^1$,
2) $\mu(t), \pi(t) : J \to \mathbb{R}$ and $\mu(t), \pi(t) \in \mathcal{C}^\infty$,
3) $\dot{V}(t,x) \leq \pi(t)V^m(t,x) + \mu(t)V(t,x) \quad \forall x \in \mathbb{R}^n - \{0\}$

that (15) and (16) hold. Then, the system (1) is UAS if one of the following conditions hold:

1) $\mu(t)$ is UAS (or equivalently UES) for any $m \neq 1$.
2) $\mu(t)$ is US, $m > 1$ and we have

$$\int_{t_0}^{t} (1-m)\pi(\tau)exp(\overline{\mu}(\tau))d\tau \geq \alpha_3(t - t_0) \tag{20}$$

or equivalently

$$(1-m)H_i(t)exp(\overline{\mu}(t)) - (1-m)H_i(t_0) \geq \alpha_3(t - t_0) \tag{21}$$

for $\alpha_3 \in \mathcal{K}_\infty$ where $H_i(t) := H_i(t)[\mu(t), \pi(t)]$.

Proof:

(1) We obtain the form (17) and so the estimation (18) easily from the inequality (10) using (16) and the assumption (15). First of all, it's clear that the right hand side of this estimation is a *class* $\mathcal{K}$ function by Lemma 4.2 of [11]. On the other hand, as the function $\mu(t)$ is UAS, it's uniformly attractive. Thus, there exists a $\sigma \in\ \mathcal{L}$ by 59 - Sec.5.1 of [24] such that the right hand side of (18) is bounded by $\alpha_1^{-1}\big[\alpha_2||x(t_0)||\sigma(t)\big]$. Consequently, as the norm of the solution is bounded by a *class* $\mathcal{KL}$ function, it's UAS by Definition 2.

(2) Assume that all the hypothesis hold. Then integrating both sides of (12) and using (15), we receive the following inequalities

$$V(t,x) \leq \left[\frac{1}{\frac{\int_{t_0}^{t}(1-m)\pi(\tau)exp(\overline{\mu}(\tau))d\tau}{exp(\overline{\mu}(t))} + \frac{V_0^{1-m}}{exp(\overline{\mu}(t))}}\right]^{\frac{1}{m-1}}$$

$$= \left[\frac{V_0^{m-1} exp(\overline{\mu}(t))}{V_0^{m-1} \int_{t_0}^{t} (1-m)\pi(\tau) exp(\overline{\mu}(\tau)) d\tau + 1} \right]^{\frac{1}{m-1}}.$$

By US of $\mu(t)$, we have $\int_s^t \mu(\lambda) d\lambda \leq M, \ \forall t \geq s \geq t_0$. In addition, by condition (21) and the infinite series approach that we first introduced in Lemma 2 of [22], we revise the right hand side of the above inequality as follows

$$\leq exp(M) \left(\frac{V_0^{m-1}}{V_0^{m-1} \alpha_3(t-t_0) + 1} \right)^{\frac{1}{m-1}}.$$

Here, the right hand side is an increasing function of V_0^{m-1}, so it is possible to enlarge using α_i's. Thus we have

$$||x(t)|| \leq \alpha_1^{-1} \left[\Big(\frac{\alpha_2^{m-1}(||x_0||)}{\alpha_2^{m-1}(||x_0||)\alpha_3(t-t_0) + 1} \Big)^{\frac{1}{m-1}} \right]$$

Now for simplicity, let us define $\beta(r,s) := \frac{r}{rs+K}$ where r represents $\alpha_2^{m-1}(||x_0||)$ and s represents $\alpha_3(t-t_0)$. It's known that this function $\beta(r,s) \in \mathcal{KL}$ for $K > 0$, Example 4.16 of [11]. This shows that the right hand side is a *class* $\mathcal{K}$- function of $||x_0||$ and a *class* $\mathcal{L}$ function of $t - t_0$ which shows that the system is UAS.$\square$

Example 1: Consider the system

$$\dot{x}(t) = \frac{cos(t)}{2} x - \frac{t}{2} x^3, \quad t \geq t_0 \geq 0 \tag{22}$$

and the LF $V(t,x) = x^2$. We have

$$\dot{V}(t,x) = cos(t)V - tV^2 := \mu(t)V + \pi(t)V^2.$$

Here note that $\mu(t)$ is a US function but not a UAS function. Because the equilibrium of the corresponding system $\dot{y}(t) = cos(t)y$ is even not attractive. Thus, none of the methods mentioned in the works [4], [19] and [27] can be applied.

Now, let us evaluate the problem in the view of perturbation problem of LTV systems. (22) should be regarded as perturbation of the US scalar LTV system $\dot{x}(t) = \frac{cos(t)}{2} x$. This nominal system is US, clearly the perturbation term is $g(t,x) := -\frac{t}{2}x^3$ and none of the requirements of [2] and [21] are met. But we preserve US by also adding UA.

We have without actually solving the system (22) that

$$\int_{t_0}^{t} (1-m)\pi(\tau) e^{\overline{\mu}(\tau)} d\tau = \int_{t_0}^{t} \tau e^{sin(\tau) - sin(t_0)} d\tau$$

$$\geq e^{-2} \int_{t_0}^{t} \tau d\tau = e^{-2}(t-t_0) := \alpha_3(t-t_0).$$

As the rest of the hypothesis of Theorem 2 holds, we can conclude that the nonlinear system (22) is UAS.

Consequently, we can also state as a future work that if the necessary conditions mentioned in the Theorem 2 are met UAS of a nonlinear system is preserved. This can be regarded as a corollary of our main result to this well studied problem, [21].

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