WHEN CERTAIN RELATIVE PROJECTIVITY AND INJECTIVITY CONDITIONS IMPLY THE GLOBAL PROJECTIVITY AND INJECTIVITY

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ABSTRACT

WHEN CERTAIN RELATIVE PROJECTIVITY AND INJECTIVITY CONDITIONS IMPLY THE GLOBAL PROJECTIVITY AND INJECTIVITY

A right R-module M is called R-projective provided that it is projective relative to the right R-module R_R. One of the parts of this thesis deals with the rings whose all nonsingular right modules are R-projective. For a right nonsingular ring R, we prove that R_R is of finite Goldie rank and all nonsingular right R-modules are R-projective if and only if R is right finitely Σ-CS and flat right R-modules are R-projective. Then, R-projectivity of the class of nonsingular injective right modules is also considered. Over right nonsingular rings of finite right Goldie rank, it is shown that R-projectivity of nonsingular injective right modules is equivalent to R-projectivity of the injective hull $\mathrm{E}(R_R)$.

As a second goal, we deal with simple-injective modules. For a right module M, we prove that M is simple-injective if and only if M is min-N-injective for every cyclic right module N. The rings whose simple-injective right modules are injective are exactly the right Artinian rings. A right Noetherian ring is right Artinian if and only if every cyclic simple-injective right module is injective. The ring is quasi-Frobenius if and only if simple-injective right modules are projective. For a commutative Noetherian ring R, we prove that every finitely generated simple-injective R-module is projective if and only if $R = A \times B$, where A is quasi-Frobenius and B is hereditary. An abelian group is simple-injective if and only if its torsion part is injective.

ÖZET

BAZI BAĞIL PROJEKTİFLİK VE İNJEKTİFLİK KOŞULLARININ GLOBAL PROJEKTİFLİĞİ VE İNJEKTİFLİĞİ GEREKTİRDİĞİ DURUMLAR

Bir sağ R-modül M, eğer sağ R-modül R_R'ye göre projektif oluyorsa bu durumda M modülüne R-projektif denir. Bu tezin ilk kısmında üzerindeki tüm tekil olmayan sağ modüllerin R-projektif olduğu halkalar ile ilgileniyoruz. Bir sağ tekil olmayan R halkası için, R_R'nin sonlu Goldie ranka sahip olması ve tüm tekil olmayan sağ R-modüllerinin R-projektif olmasının ancak ve ancak R halkası sağ sonlu Σ-CS ve düz sağ R-modülleri R-projektif olan bir halka ise sağlandığını kanıtladık. Daha sonra, tekil olmayan injektif sağ modül sınıfının R-projektifliğini ele aldık. Sonlu sağ Goldie ranka sahip sağ tekil olmayan halkalar üzerinde tekil olmayan injektif sağ modüllerin R-projektifliğinin halkanın injektif bürümü olan E(R_R)'nin R-projektif olmasına denk olduğunu gösterdik.

Bu tezde, ikinci bir amaç olarak, basit-injektif modüller ile ilgili olan ilişkileri gözlemliyoruz. İlk olarak, bir sağ R-modül M'nin basit-injektif olmasının M'nin tüm devirli sağ R-modüllere göre mininjektif olmasına denk olduğunu ispatladık. Basit-injektif sağ modülleri injektif olan halkaların tam olarak sağ Artin halkalar olduğunu gösterdik. Ayrıca, değişmeli Noether bir R halkası için her sonlu üretilmiş basit-injektif R-modülün projektif olmasının ancak ve ancak A quasi-Frobenius ve B hereditary bir halka olmak üzere, $R = A \times B$ formunda iken gerçeklendiğini kanıtladık. Bununla beraber, bir değişmeli grubun basit-injektif olmasının onun burulma kısmının injektif olmasına denk olduğunu gösterdik.

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LIST OF SYMBOLS AND ABBREVIATIONS

R	an associative ring with identity element unless otherwise stated
M	a right R-module unless otherwise stated
$\mathcal{M}od$-R	the category of all right R-modules
R-$\mathcal{M}od$	the category of all left R-modules
M/K or $\frac{M}{K}$	the factor module of M modulo K
$M \oplus N$	the direct sum of modules M and N
$\mathrm{J}(R)$	the Jacobson radical of a ring R
$\mathrm{Rad}(M)$	the Jacobson radical of a module M
$\mathrm{Soc}(M)$	the socle of a module M
$\mathrm{E}(M)$	the injective hull (envelope) of a module M
$\mathrm{cl}(M)$	the composition length of a module M
$\mathrm{Z}(M)$	the singular part of a module M
$\mathrm{T}(M)$	the torsion part of a module M
$\mathrm{Hom}(M,N)$	the set of all homomorphisms from a module M to a module N
$\mathrm{Ker}(f)$	the kernel of a homomorphism f
$\mathrm{Im}(f)$	the image of a homomorphism f
$\mathrm{Ext}^1_R(M,N)$	the set of all equivalence classes of short exact sequences starting with the module N and ending with the module M
$\mathbb{Z}$	the ring of integers
$\mathbb{Q}$	the field of rational numbers

CHAPTER 1

INTRODUCTION

In ring and module theory, investigation of rings in respect to homological properties of classes of modules over the rings has been recurrent topic in the last century. This phenomenon enables us to discover new classes of rings and modules, and to pave the way for numerous important studies in this topic. The most important homological objects in module categories are injective and projective modules. They are not only significant for ring and module theory but also have a remarkable impact on homological algebra.

Let R be a ring. Given right R-modules M and N, M is said to be N-injective (or injective relative to N) if for every monomorphism $f : K \to N$, and every homomorphism $g : K \to M$, there exists a homomorphism $h : N \to M$ such that the following diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[dl, dashed, "h"] \\ & M & \end{tikzcd}$$

commutes, that is, $hf = g$. A right R-module M is called injective if it is injective relative to every right R-module N. On the other hand, M is said to be N-projective (or projective relative to N) if for every epimorphism $f : N \to K$ and every homomorphism $g : M \to K$, there exists a homomorphism $h : M \to N$ such that $fh = g$, that is, the following diagram

$$\begin{tikzcd} & M \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\ N \arrow[r, "f"'] & K \arrow[r] & 0 \end{tikzcd}$$

commutes. A right R-module M is projective in case M is projective relative to every right R-module N.

Baer's criterion for injectivity asserts that a right R-module M is injective if and only if each homomorphism from any right ideal I of R into M extends to R, namely

M is R-injective. Dually, a right R-module M is called R-projective provided that each homomorphism $f : M \to R/I$, where I is any right ideal, factors through the canonical epimorphism $\pi : R \to R/I$. Unlike Baer's criterion, R-projective modules need not be projective. For example, the $\mathbb{Z}$-module $\mathbb{Q}$ is $\mathbb{Z}$-projective whereas it is not projective.

In 1976, Faith asked that when R-projectivity implies projectivity for all right R-modules (Faith, 1976). This problem had been open for some time. Later on, the result of (Sandomierski, 1964) which states that every R-projective module is projective over a right perfect ring were partially extended by (Ketkar & Vanaja, 1981) to semiperfect rings for modules with small radicals. Then, for many years there had been an effort to find a non-perfect ring which still satisfies this property. However, such an example could not have been found. One of the recent answers to Faith's problem was due to (Alhilali, Ibrahim, Puninski & Yousif, 2017). They gave examples of rings which are not right perfect over which some R-projective modules are not projective. Several years ago, Faith's problem was solved by Trlifaj and in fact the problem is undecidable in ZFC (see (Trlifaj, 2019), (Trlifaj, 2020)). The undecidability result of Trlifaj has increased the attention to this topic.

In addition to what has been said, characterizing rings by projectivity of some classes of their modules is a classical problem in ring and module theory. A result of Bass ((Anderson & Fuller, 1992),Theorem 28.4) states that a ring R is right perfect if and only if each flat right R-module is projective. On the other hand, Goodearl proved the following remarkable theorem which is an inspiration source for our result (Theorem 3.1): Let R be a right nonsingular ring. Then, all nonsingular right R-modules are projective if and only if R is right perfect, left semihereditary and the injective hull $\mathrm{E}(R_R)$ is flat, see ((Goodearl, 1976), Theorem 5.21).

Since characterizing rings in terms of projectivity of some classes of modules has an important role in understanding the structure of the ring, there had been a natural necessity to study the notion of R-projectivity and rings characterized by R-projectivity of some classes of their modules. With this motivation, recently, in (Amini, Ershad & Sharif, 2008) and (Amini, Amini & Ershad, 2009), the rings whose flat right R-modules are R-projective were considered and these rings are termed as right almost-perfect rings (right A-perfect, for short). As they are between perfect and semiperfect rings, this class of rings has attracted a lot of attention. Besides, the rings whose injective right R-modules are R-projective were characterized in (Alagöz & Büyükaşık, 2021).

At this point, it is natural to ask "What are the rings over which each nonsingular right R-module is R-projective?".

One of the main purposes of this thesis is to derive necessary and sufficient conditions on a right nonsingular ring R under which all nonsingular right R-modules are R-projective and to describe the structure of such rings.

Along the way, in Chapter 3, Section 1, some properties of nonsingular right modules will be investigated. We first recall the following result due to Turnidge (see ((Turnidge, 1970), Theorem 2.1)) which can also be found in ((Goodearl, 1976), Proposition 5.16): Let R be a right nonsingular ring. Then, all nonsingular right R-modules are flat if and only if R is left semihereditary and $\mathrm{E}(R_R)$ is flat. Related to this result, in Chapter 3, Section 1, we obtain that every flat right R-module is nonsingular if and only if R is right nonsingular and pure submodules of free right R-modules are closed. Over a right nonsingular ring R, we prove that pure submodules of nonsingular right R-modules are closed if and only if R_R is of finite Goldie rank. We also show that every flat right R-module is nonsingular over a right semihereditary semiperfect ring, over a right nonsingular right perfect ring, over a right semihereditary right A-perfect ring, and over a right nonsingular ring of finite right Goldie rank.

In Chapter 3, Section 2, we call a ring R *right NR* in case all nonsingular right R-modules are R-projective. We show that for a right nonsingular right *NR* ring, all nonsingular right modules are flat. If R is of finite right Goldie rank and right nonsingular, then every right *NR*-ring is right A-perfect. We prove the following theorem which characterize the rings whose nonsingular modules are R-projective (see Theorem 3.1).

Theorem: Let R be a right nonsingular ring. Then, the following statements are equivalent.

(1) R is a right *NR*-ring and R_R is of finite Goldie rank.

(2) R is semihereditary, right A-perfect, right CS and $\mathrm{E}(R_R)$ is flat.

(3) R is right finitely Σ-CS and right A-perfect.

As a consequence of this theorem, we obtain that if R is a semiprime right and left Goldie ring, then R is a right *NR*-ring if and only if R is semihereditary and right

A-perfect. In the same section, for a right nonsingular ring R of finite right Goldie rank, we also give the following equivalent conditions to the statement that every nonsingular injective right R-module is R-projective (see Theorem 3.2).

Theorem: Let R be a right nonsingular ring having finite right Goldie rank. Then, the following statements are equivalent.

(1) Every nonsingular injective right R-module is R-projective.

(2) $\mathrm{E}(R_R)$ is R-projective.

(3) $\mathrm{E}(R_R) = U_R \oplus V_R$, where U is projective and $\mathrm{Hom}(V, R/I) = 0$ for each right ideal I of R.

In particular, over a right nonsingular right Noetherian ring, nonsingular injective right R-modules are R-projective if and only if $\mathrm{E}(R_R) = U_R \oplus V_R$, where U is projective and $\mathrm{Rad}(V) = V$.

In Chapter 3, Section 3, nonsingular covers will be considered. Let $\mathcal{N}$ be the class of all nonsingular right R-modules. Following Enochs (Enochs & Jenda, 2000), an *$\mathcal{N}$-precover* (or a *nonsingular precover*) of a right R-module M is a homomorphism $\varphi : N \to M$ with $N \in \mathcal{N}$ such that for any homomorphism $\psi : N' \to M$ with $N' \in \mathcal{N}$, there exists $\lambda : N' \to N$ such that $\varphi\lambda = \psi$. An $\mathcal{N}$-precover $\varphi : N \to M$ is said to be an *$\mathcal{N}$-cover* (or a *nonsingular cover*) if every endomorphism λ of N with $\varphi\lambda = \varphi$ is an isomorphism. Works on the torsion-free covers date back to 1960s and some of the results about the existence of torsion-free covers for abstract torsion theories were given in (Teply, 1969), (Golan & Teply, 1973), (Teply, 1976). As a particular corollary, Teply proved that nonsingular covers exist for all right modules over a right nonsingular ring of finite right Goldie rank (see (Teply, 1969)). This result was further discussed and a sort the of converse of this result was given in (Cheatham, 1971). Then, in 2003, Bican extended the aforementioned result for Goldie's torsion theory: For Goldie's torsion theory $(\mathcal{T}, \mathcal{F})$, the class $\mathcal{F}$ is a covering class if and only if $(\mathcal{T}, \mathcal{F})$ is of finite type. If moreover, the ring R is right nonsingular, then these conditions are equivalent to the following condition: Every nonzero right ideal of R contains a finitely generated essential right ideal ((Bican, 2003), Theorem 2).

We will denote by

$$\mathcal{N}^{\perp} = \{X \in \mathcal{M}od\text{-}R : \operatorname{Ext}^1_R(N, X) = 0 \text{ for all } N \in \mathcal{N}\}$$

the right orthogonal class of the class $\mathcal{N}$ of all nonsingular right R-modules. In Chapter 3, Section 3, several properties of the class $\mathcal{N}^{\perp}$ of right modules are obtained. Particularly, we show that a right nonsingular ring R having finite right Goldie rank is right NR if and only if nonsingular covers of finitely generated right R-modules are (finitely generated) projective.

In Chapter 4, we turn our attention from the concept of relative projectivity to certain relative injectivity conditions.

Quasi-Frobenius rings (QF-rings, for short) were introduced by Nakayama in the study of representations of algebras ((Nakayama, 1939) and (Nakayama, 1941)). Afterwards, QF-rings played a central role in ring theory, and numerous characterizations were given by various authors, see for instance (Ikeda, 1952), (Ikeda & Nakayama, 1954), (Eilenberg & Nakayama, 1955), (Osofsky, 1966), (Faith, 1966), (Faith & Walker, 1967), (Nicholson & Yousif, 1997 - II). In particular, Ikeda characterized these rings as the left (right) self-injective, left and right Artinian rings ((Ikeda, 1952), Theorem 1). There has been a great deal of research devoted to improve Ikeda's previously mentioned result by weakening the Artinian condition or the injectivity or both. In the same paper of (Ikeda, 1952), the concept of mininjectivity for rings (in the Artinian case) appeared as a property for characterizing QF-rings and it was shown that a ring R is QF if and only if it is right and left Artinian and right and left mininjective ((Nicholson & Yousif, 2003), Theorem 2.30). In the early 1980s, Harada (Harada, 1982) introduced the notions of mininjective modules and rings as follows: Let M and N be right R-modules. M is said to be min-N-injective if for every simple submodule K of N, and every homomorphism $f : K \to M$, there exists a homomorphism $h : N \to M$ such that $h|_K = f$. If we take the right R-module R_R for N, then M is called mininjective, that is, $\operatorname{Ext}^1_R(R/S, M) = 0$ for every simple right ideal S of R. On the other hand, according to Harada (Harada, 1992), M is said to be simple-N-injective if for every submodule K of N, and every homomorphism $f : K \to M$ with $f(K)$ simple extends to N. If $N = R_R$, in this case M is called simple-injective. In (Nicholson & Yousif, 1997 - II), Nicholson and Yousif proved that R

is QF if and only if it is left perfect, right and left simple-injective, see also ((Nicholson & Yousif, 2003), Theorem 6.39).

The concepts of mininjectivity and simple-injectivity of rings and modules aroused interest and many papers and results on them and their generalizations have appeared in the literature, see (Amin, Fathi & Yousif, 2008), (Amin, Yousif & Zeyada, 2005), (Mao, 2008), (Mao, 2007), (Nicholson & Yousif, 1997 - I), (Nicholson, Park & Yousif, 2000), (Yousif & Zhou, 2004), (Mao, 2009).

The second goal of this thesis is to address some questions about the aforementioned modules that have not been considered so far. Namely, in this part of the thesis, mainly we are interested in characterizing the rings whose simple-injective right modules are injective (respectively, projective), and determine the structure of simple-injective modules over certain rings including the ring of integers.

This chapter is organized as follows.

Section 4.1 is mainly an extension of some results on strongly simple-injective modules obtained in (Amin, Fathi & Yousif, 2008). Some results are shown to hold for simple-injective modules. For example, we prove that a right R-module M is simple-injective if and only if M is min-N-injective for every cyclic right R-module N. For a class C of right R-modules which is closed under submodules, we show that every module in C is simple-injective if and only if every simple module in C is injective.

In section 4.2, we consider the rings whose simple-injective right modules are injective (respectively, projective). We characterize the rings whose simple-injective modules are injective (see Theorem 4.1).

Theorem: R is right Artinian if and only if every simple-injective right R-module is injective.

From this, we deduce that quotients of simple-injective right R-modules are injective if and only if R is right Artinian and right hereditary. For a right Noetherian ring R, we prove that every cyclic simple-injective right R-module is injective if and only if R is right Artinian. A ring R is QF if and only if every simple-injective right R-module is projective. For a commutative Noetherian ring R, we give the the structure of the rings whose finitely generated simple-injective R-modules are projective (Theorem 4.3).

Theorem: Let R be a commutative Noetherian ring. The following statements are equivalent.

(1) Every finitely generated simple-injective R-module is projective.

(2) $R = A \times B$, where A is QF and B is hereditary.

In the last section of this chapter, for a commutative domain R and an R-module M, we prove that M is simple-injective if and only if the torsion part $\mathrm{T}(M)$ of M is simple-injective. For a commutative hereditary Noetherian ring, we prove that M is simple-injective if and only if $\mathrm{Z}(M)$ is simple-injective. In particular, an abelian group is simple-injective if and only if its torsion part is injective. We show that the notions of simple-injective, strongly simple-injective, soc-injective and strongly soc-injective coincide over the ring of integers.

CHAPTER 2

PRELIMINARIES

Throughout the thesis, the symbol R, will be used to imply an associative ring with identity $1 \neq 0$, and all modules are unital R-modules unless otherwise stated. As usual, we denote by $\mathcal{M}od$-R and R-$\mathcal{M}od$ the category of all right R-modules and all left R-modules, respectively. Also, for a module M, the notions $\mathrm{Rad}(M)$ and $\mathrm{Soc}(M)$ stand for the Jacobson radical and the socle of M, respectively.

This chapter is prepared with the aim of collecting the definitions and results which are frequently used throughout the thesis. We shall usually state the definitions and results for right modules which have obvious left versions. We do not deal with every term in ring and module theory. Actually, we accept the fundamentals of ring and module theory, and homological algebra. For further and deeper results, and detailed proofs, see for example (Anderson & Fuller, 1992), (Lam, 1999), (Lam, 2001) and (Enochs & Jenda, 2000).

In the first section of this chapter, our concern is the main subjects of this work, namely, the notions of relative projectivity and injectivity, and their properties. In the second section, using the book (Goodearl, 1976), we summarize the concepts of nonsingular modules and nonsingular rings, since they are important especially in our study of investigation of the rings whose nonsingular modules are R-projective that are presented in Chapter 3. In the last section, we will recall some notions from homological algebra.

2.1. Relative projectivity and injectivity

In the first place, the concepts of relative injectivity and relative projectivity was handled by Sandomierski in the study of (Sandomierski, 1964), and with this work the author provided an important contribution to the development of this theory.

Definition 2.1 Given right R-modules M and N, M is said to be ***N*-injective** (or **injective relative to** N) if for every monomorphism $f : K \to N$, and every homomorphism g :

$K \to M$, there exists a homomorphism $h : N \to M$ such that $hf = g$. Equivalently, M is said to be N-injective if for every submodule S of N, and every homomorphism $g : S \to M$ extends to a homomorphism $h : N \to M$. A right R-module M is called **injective** if it is injective relative to every right R-module N.

Definition 2.2 Let M and N be right R-modules. M is said to be **N-projective** (or **projective relative to N**) if for every epimorphism $f : N \to K$ and every homomorphism $g : M \to K$, there exists a homomorphism $h : M \to N$ such that $fh = g$. Equivalently, M is said to be N-projective if for each submodule S of N, every homomorphism $g : M \to N/S$ factors through the canonical epimorphism $\pi_S : N \to N/S$, that is, there exists a homomorphism $h : M \to N$ such that $\pi h = g$. A right R-module M is **projective** in case M is projective relative to every right R-module N.

At this point, we state a characterization of projective modules which we will need in the proof of Proposition 4.5.

Proposition 2.1 *((Cartan & Eilenberg, 1956), Proposition 5.1) In order that a module P be projective, it is necessary and sufficient that every diagram*

$$\begin{tikzcd} & P \arrow[d, "g"] & \\ Q \arrow[r, "f"'] & K \arrow[r] & 0 \end{tikzcd}$$

in which the row is exact and Q is injective, can be embedded in a commutative diagram

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dashed, "h"'] & \\ Q \arrow[r, "f"'] & K \arrow[r] & 0 \end{tikzcd}$$

that is, there exists a homomorphism $h : P \to Q$ such that $fh = g$.

Proposition 2.2 (Baer's criterion) *A right R-module M is injective if and only if for any right ideal I of R, any homomorphism $f : I \to M$ can be extended to $g : R \to M$.*

Contrary to the Baer's criterion, dual Baer criterion does not hold, that is, there exists R-projective modules which are not projective. For example, the $\mathbb{Z}$-module $\mathbb{Q}$ is $\mathbb{Z}$-projective since $\mathbb{Q}_{\mathbb{Z}}$ is divisible, but it is not projective.

We shall state some properties of relative projective modules as they will be used in the following chapters.

Proposition 2.3 *((Anderson & Fuller, 1992), Proposition 16.10) Let N be a right R-module and let $(M_\alpha)_{\alpha\in A}$ be an indexed set of right R-modules. Then, $\bigoplus_{\alpha\in A} M_\alpha$ is N-projective if and only if each M_α is N-projective.*

Proposition 2.4 *((Anderson & Fuller, 1992), Proposition 16.12) Let M be a right R-module.*

(1) If

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

is a short exact sequence in $\mathcal{M}od$-R and M is N-projective, then M is projective to both N' and N''.

(2) If M is projective relative to each $N_1, \ldots, N_n$, then M is $\bigoplus_{i=1}^n N_i$-projective.

We end this section by summarizing the concept of Morita equivalence and some properties of relative projective modules related to Morita equivalence. For more on the theory, the reader might consult for example (Anderson & Fuller, 1992) and (Lam, 1999).

Recall that a **category** C consists of a class of objects, $\mathrm{Obj}\, C$, and morphism sets $\mathrm{Mor}_C(K, M)$ for every $K, M \in \mathrm{Obj}\, C$ (an element f of $\mathrm{Mor}_C(K, M)$ is denoted by $f : K \mapsto M$) together with a pairing, called composition

$$\begin{aligned} \mathrm{Mor}_C(M, N) \times \mathrm{Mor}_C(K, M) \to & Mor_C(K, N) \\ (g, f) \mapsto & gf, \end{aligned}$$

and this composition satisfies the following axioms:

(1) Composition is associative, that is, if $f \in \mathrm{Mor}_C(N, L)$, $g \in \mathrm{Mor}_C(M, N)$ and $h \in \mathrm{Mor}_C(K, M)$, then $(fg)h = f(gh)$.

(2) Each $\mathrm{Mor}_C(K, K)$ contains a distinguished element 1_K, and each 1_K is an identity, that is, if $f \in \mathrm{Mor}_C(K, M)$, then $f = f1_K = 1_M f$.

A **covariant functor** $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are categories, is a function which assigns each object K of $\mathrm{Obj}\,\mathcal{C}$ to the object $F(K)$ of $\mathrm{Obj}\,\mathcal{D}$ as well as each morphism $f : K \to M$ in $\mathrm{Mor}_{\mathcal{C}}(K, M)$ to the morphism $F(f) : F(K) \to F(M)$ in $\mathrm{Mor}_{\mathcal{D}}(F(K), F(M))$ such that $F(gf) = F(g)F(f)$ for all morphisms $f, g \in \mathrm{Mor}_{\mathcal{C}}(K, M)$ whenever the composite is defined, and $F(1_K) = 1_{F(K)}$ for all $K \in \mathrm{Obj}\,\mathcal{C}$. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is a **category equivalence** in case there is a functor (necessarily covariant) $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $GF \cong 1_C$ and $FG \cong 1_D$. Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be **equivalent**, denoted by $\mathcal{C} \approx \mathcal{D}$, in case there exists a category equivalence from one to the other. At this point, we shall note that for module categories, more generally for additive categories to be equivalent, there must be an additive equivalence. Moreover, a property $\mathcal{P}$ on objects (respectively, morphisms) in a module category $\mathcal{M}od\text{-}R$ is said to be a **categorical property** if, for any category equivalence $F : \mathcal{M}od\text{-}R \to \mathcal{M}od\text{-}S$, whenever $M \in \mathcal{M}od\text{-}R$ (respectively, $g \in \mathrm{Mor}_{\mathcal{M}od\text{-}R}(M, N)$) satisfies $\mathcal{P}$, so does $F(M)$ (respectively, $F(g)$).

Definition 2.3 Two rings R and S are said to be **Morita equivalent** (abbreviated $R \approx S$) in case $\mathcal{M}od\text{-}R \approx \mathcal{M}od\text{-}S$, that is, there exists an additive category equivalence between these categories of modules. A ring theoretic property $\mathcal{P}$ is said to be **Morita invariant** if, whenever R has the property $\mathcal{P}$, so does every $S \approx R$.

Proposition 2.5 *(Anderson & Fuller, 1992) Let R and S be Morita equivalent rings via an equivalence $F : \mathcal{M}od\text{-}R \to \mathcal{M}od\text{-}S$. Let M and N be right R-modules. Then*

(1) M is N-projective if and only if $F(M)$ is $F(N)$-projective.

(2) M is projective if and only if $F(M)$ is projective.

(3) M is finitely generated if and only if $F(M)$ is finitely generated.

2.2. Nonsingular modules and rings

The notion of right and left singular ideals of a ring was first introduced by Johnson in ((Johnson, 1951), p. 894), and later the singular submodule of a module was defined in ((Johnson, 1957), p. 537). In this section, our objective is to state some definitions and results relevant to the concept of nonsingular modules and nonsingular rings on the ground that we use them in the next chapters commonly. For the details of proofs and more about the information in this section, see (Goodearl, 1976).

First of all, it will be advantageous to remind the notions of essential and small submodule of an R-module. A submodule K of a right R-module M is said to be a **small submodule** of M, denoted by $K \ll M$, if for any proper submodule L of M, we have that $K + L \neq M$. On the other hand, a submodule N of a right R-module M is called an **essential submodule** of M, written $N \trianglelefteq M$, in case $N \cap S \neq 0$ for any nonzero submodule S of M.

Theorem 2.1 *((Goodearl, 1976), Theorem 1.10) Given any module M, there exists a module E, containing M, such that*

(1) E is a maximal essential extension of M in the sense that $M \trianglelefteq E$, and whenever $M \trianglelefteq A$, the inclusion map $M \to E$ extends to a monomorphism $A \to E$.

(2) E is a minimal injective extension of M in the sense that E is injective, and any monomorphism $M \to E'$ with E' injective extends to a monomorphism $E \to E'$.

Any module E satisfying the conditions of Theorem 2.1 is called an **injective hull** of M (or an injective envelope of M). We use the notation $\mathrm{E}(M)$ to stand for an injective hull of M. As an important remark, $\mathrm{E}(M)$ is unique up to isomorphism.

Definition 2.4 Given any right R-module M, the **singular submodule** of M is defined as the following set

$$\mathrm{Z}(M) = \{m \in M : mI = 0 \text{ for some essential right ideal } I \text{ of } R\}.$$

Definition 2.5 A right R-module M is called **singular** if $\mathrm{Z}(M) = M$, and called **nonsingular** if $\mathrm{Z}(M) = 0$. A ring R is said to be a **right nonsingular ring** if R is nonsingular as a right R-module over itself, that is, if $\mathrm{Z}(R_R) = 0$. Similarly, we say that R is a **left nonsingular ring** provided that $\mathrm{Z}({}_RR) = 0$.

Note that right and left nonsingularity are not equivalent which can be seen from ((Goodearl, 1976), p. 36, Exercise 1).

Furthermore, for a given right R-module M, the **submodule** $\mathrm{Z}_2(M)$ is defined by $\mathrm{Z}_2(M)/\mathrm{Z}(M) = \mathrm{Z}(M/\mathrm{Z}(M))$.

As an important remark, the class of all nonsingular right R-modules is closed under submodules, direct products, direct sums, essential extensions and module extensions, whereas the class of all singular right R-modules is closed under submodules, factor modules, and direct sums (see (Goodearl, 1976), Proposition 1.22).

Proposition 2.6 *((Goodearl, 1976), Proposition 1.20(b)) An R-module M is singular if and only if there exists a short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} M \longrightarrow 0$$

such that f is an essential monomorphism, that is, $f(A)$ is an essential submodule of B.

By the above proposition, for an R-module M, whenever N is an essential submodule of M, then the quotient module M/N is singular. The converse of this is not true in general. For instance, if $M = \mathbb{Z}/2\mathbb{Z}$ and $N = 0$, in that case, M/N is a singular $\mathbb{Z}$-module whereas N is not an essential submodule of M. The next proposition enables us to observe when the converse holds.

Proposition 2.7 *((Goodearl, 1976), Proposition 1.21) Let M be nonsingular, and let N be a submodule of M. Then, M/N is singular if and only if N is an essential submodule of M.*

Another class of modules that we shall recall are closed submodules. A submodule N of a module M is said to be a **closed submodule**, if N has no proper essential extension inside M. For example, every direct summand of a module M is a closed submodule of M. As a crucial observation, we shall stress that any closed submodule of an injective module is injective ((Goodearl, 1976), Corollary 1.9).

At this point, it is beneficial to state the following lemma that we use frequently in Chapter 3.

Lemma 2.1 *((Sandomierski, 1968), Lemma 2.3) Let N be a submodule of a right module M.*

(1) If $\mathrm{Z}(M/N) = 0$, then N is closed in M.

(2) If N is closed in M and $\mathrm{Z}(M) = 0$, then $\mathrm{Z}(M/N) = 0$.

Before we give the necessary and important properties of nonsingular rings, we will mention about the concept of finite Goldie rank. A right R-module M is said to be **of finite Goldie rank** provided that M contains no infinite independent families of nonzero submodules. For example, all Noetherian modules and Artinian modules are of finite Goldie rank. A ring R is said to be **of finite right Goldie rank** if the right R-module R_R is of finite Goldie rank.

The next proposition serves as a useful tool while determining whether a module is of finite Goldie rank or not.

Proposition 2.8 *((Goodearl, 1976), Proposition 3.13(a)) A right R-module M is of finite Goldie rank if and only if every submodule of M has a finitely generated essential submodule.*

Recall that a right R-module M is said to be **flat** if given any monomorphism $f : A \to B$ of left R-modules, the tensored homomorphism

$$M \otimes_R A \xrightarrow{1_M \otimes f} M \otimes_R B$$

is also a monomorphism.

For a right nonsingular ring, we have the following properties.

Proposition 2.9 *(Goodearl, 1976) Let R be a right nonsingular ring. Then, the following statements hold.*

(1) R is of finite right Goldie rank if and only if all direct sums of nonsingular injective right R-modules are injective.

(2) R is of finite right Goldie rank if and only if every nonsingular injective right R-module is a direct sum of indecomposable modules.

(3) Let R be of finite right Goldie rank. Then, the injective hulls $\mathrm{E}(R_R)$ and $\mathrm{E}({}_RR)$ coincide if and only if $\mathrm{E}(R_R)$ is a flat right R-module.

(4) If R is of finite right Goldie rank, then all flat right R-modules are nonsingular.

We proceed by giving a few definitions of some classes of rings. A ring R is called **semilocal** if $R/\,\mathrm{J}(R)$ is semisimple Artinian where $\mathrm{J}(R)$ is the Jacobson radical of R. A ring R is called **semiperfect**, if R is semilocal and idempotents of $R/\operatorname{J}(R)$ can be lifted to R. There is no distinction between being a right or left semiperfect ring.

Theorem 2.2 *((Lam, 2001), Theorem 23.6) A ring R is semiperfect if and only if the identity element* 1 *can be decomposed into* $e_1 + e_2 + \cdots + e_n$*, where* e_i*'s are mutually orthogonal local idempotents.*

Before we give the definition of a right perfect ring, we need the notion of a right T-nilpotent ideal: A one sided ideal J of a ring R is called **right *T*-nilpotent** (respectively, **left *T*-nilpotent**) if for any sequence $(a_k)_{k=1}^{\infty}$ of elements in J, there exists an integer $n \geq 1$ such that $a_n \ldots a_2 a_1 = 0$ (respectively, $a_1 a_2 \ldots a_n = 0$). A ring R is called a **right perfect ring** (respectively, **left perfect ring**), if R is semilocal and the Jacobson radical $\operatorname{J}(R)$ of R is right T-nilpotent (respectively, left T-nilpotent). If R is both right and left perfect, we call R a perfect ring. By the famous theorem called Bass' Theorem P, we know that a ring R is right perfect if and only if every flat right R-module is projective, see ((Anderson & Fuller, 1992), Theorem 28.4). Right perfect rings do not have to be left perfect. For instance, see the example given by Bass in ((Lam, 2001), p. 245).

A ring R is said to be **right (left) hereditary** if every right (left) ideal of R is projective. Similarly, R is called **right (left) semihereditary** if every finitely generated right (left) ideal of R is projective. A ring R which is both right and left hereditary (respectively, semihereditary) is called a hereditary (semihereditary) ring. Clearly, right hereditary rings are right semihereditary.

Theorem 2.3 *((Lam, 1999), Theorem 4.67) Let R be a right semihereditary ring. Then, every submodule of a flat right or left R-module is flat.*

Now, we return our attention to the main concept of nonsingular rings. We shall state the following remarkable theorems due to the fact that they play a key role in our study introduced in Chapter 3.

Theorem 2.4 *((Goodearl, 1976), Proposition 5.16) Let R be a right nonsingular ring. Then, all nonsingular right R-modules are flat if and only if R is left semihereditary and* $\operatorname{E}(R_R)$ *is flat.*

Theorem 2.5 *((Goodearl, 1976), Theorem 5.21) If R is a right nonsingular ring, then the following conditions are equivalent.*

(1) All nonsingular right R-modules are projective.

(2) R is right perfect, left semihereditary and $\mathrm{E}(R_R)$ *is flat.*

2.3. Homological algebra aspect

In this section, we first aim to introduce the concept of torsion theories and specifically remind Goldie's torsion theory. Afterwards, we shall state the general definitions of covers and envelopes, as well as collect some of their properties by following (Xu, 1996) and (Bican, 2003). The reason why we collect these tools is that we will need them in the study related to nonsingular covers in Section 3.3. For the unexplained terms and concepts of homological algebra, we direct the reader to (Enochs & Jenda, 2000).

2.3.1. Torsion theories

The concept of torsion theory for abelian categories had been introduced by (Dickson, 1966) formally, even though the concept was in the work of (Gabriel, 1962) and (Maranda, 1964) earlier. On the ground that it is a wide theory, one can need more definitions or details than we collect here. For this reason, see for example ((Stenström, 1975), Chapter 6) or alternatively ((Benli, 2015), Chapter 2.2).

A **torsion theory** $(\mathcal{T}, \mathcal{F})$ for the category of $\mathcal{M}od\text{-}R$ consists of two classes of right R-modules $\mathcal{T}$ and $\mathcal{F}$ satisfying the properties

$$\mathcal{T} = \{M \in \mathcal{M}od\text{-}R : \mathrm{Hom}(M, F) = 0 \text{ for every } F \in \mathcal{F}\}$$

and

$$\mathcal{F} = \{M \in \mathcal{M}od\text{-}R : \mathrm{Hom}(T, M) = 0 \text{ for every } T \in \mathcal{T}\}.$$

If we take the classes

$$\mathcal{T} = \{M \in \mathcal{M}od\text{-}R : \mathrm{Z}(M) \trianglelefteq M\} = \{M \in \mathcal{M}od\text{-}R : \mathrm{Z}_2(M) = M\}$$

and

$$\mathcal{F} = \{M \in \mathcal{M}od\text{-}R : \mathrm{Z}(M) = 0\},$$

then the pair $(\mathcal{T}, \mathcal{F})$ becomes the torsion theory known as Goldie's torsion theory (for the details we refer the reader to ((Stenström, 1975), p. 139 and p. 148)). It should be pointed out that in Goldie's torsion theory when $\mathrm{Z}(R_R) = 0$, the class $\mathcal{T}$ will be exactly the class of singular modules.

Finally, a torsion theory $(\mathcal{T}, \mathcal{F})$ is said to be **of finite type** if each right ideal I for which R/I is in $\mathcal{T}$ contains a finitely generated right ideal J for which R/J is in $\mathcal{T}$ (see ((Bican, 2003), p. 396)), and said to be **hereditary** if the class $\mathcal{T}$ is closed under submodules (see ((Stenström, 1975), p.141)).

2.3.2. Covers and envelopes

The theory of covers and envelopes goes back to 1950's. Since its beginnings, the main concern about them has been showing their existence according to some classes of modules. The first problem that has been considered related to covers and envelopes is defining covers and envelopes in a general setting. Enochs first made a general definition of covers and envelopes via diagrams for a given class of modules (Enochs, 1981).

In the following definitions for the rest of this section, all classes of modules are assumed to be closed under isomorphisms, under taking finite direct sums and direct summands.

Definition 2.6 Let $\mathcal{X}$ be a class of right R-modules. For an R-module M, an **$\mathcal{X}$-cover** is a module homomorphism $\varphi : X \to M$ with $X \in \mathcal{X}$ satisfying the following conditions:

(1) For every homomorphism $\varphi' : X' \to M$ with $X' \in \mathcal{X}$, there exists a homomorphism

$f : X' \to X$ such that $\varphi f = \varphi'$, that is, f completes the following diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] & M \\ & X' \arrow[u, "\varphi'"'] \arrow[ul, dashed, "f"] \end{tikzcd}$$

commutatively.

(2) For every endomorphism $f : X \to X$, if $\varphi f = \varphi$, then f must be an automorphism, that is, the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] & M \\ & X \arrow[u, "\varphi"'] \arrow[ul, dashed, "f"] \end{tikzcd}$$

can be completed only by automorphisms of X.

If the first condition holds (and perhaps not the second condition), $\varphi : X \to M$ is called an **$\mathcal{X}$-precover**.

Note that an $\mathcal{X}$-cover need not be epic. Also, whenever it exists, it is unique up to isomorphism, that is, if $\varphi_i : X_i \to M$, $i = 1, 2$, are two different $\mathcal{X}$-covers for a module M, then $X_1 \cong X_2$ (see ((Xu, 1996), Proposition 3.1)).

In addition, a class $\mathcal{X}$ of right modules over any ring R is said to be **covering** (respectively, **precovering**), if every right R-module has an $\mathcal{X}$-cover (respectively, $\mathcal{X}$-precover).

Definition 2.7 Let $\mathcal{X}$ be a class of right R-modules. For an R-module M, an **$\mathcal{X}$-envelope** of M is a homomorphism $\varphi : M \to X$ such that the following conditions hold:

(1) For every $\varphi' : M \to X'$ with $X' \in \mathcal{X}$, there exists a homomorphism $f : X \to X'$ such that $\varphi' = f\varphi$, that is f completes the following diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\varphi'"'] & X \arrow[dl, dashed, "f"] \\ X' & \end{tikzcd}$$

commutatively.

(2) If f is an endomorphism of X with $\varphi = f\varphi$, then f must be an automorphism, that is, the following diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & X \arrow[dl, dashed, "f"] \\ X & \end{tikzcd}$$

can be completed only by automorphisms of X.

Similarly, if the first one holds (and perhaps not the second), $\varphi : M \to X$ is called an $\mathcal{X}$**-preenvelope**, and also envelopes, if exist, are unique up to isomorphism.

In the same manner, a class $\mathcal{X}$ of right modules over any ring R is said to be **enveloping** (respectively, **preenveloping**), if every right R-module has an $\mathcal{X}$-envelope (respectively, $\mathcal{X}$-preenvelope).

For a given class $\mathcal{X}$ of right R-modules, let

$$\mathcal{X}^{\perp} = \{G \in \mathcal{M}od\text{-}R : \operatorname{Ext}^1_R(X, G) = 0 \ \text{ for all } \ X \in \mathcal{X}\}$$

and

$$^{\perp}\mathcal{X} = \{F \in \mathcal{M}od\text{-}R : \operatorname{Ext}^1_R(F, X) = 0 \ \text{ for all } \ X \in \mathcal{X}\}.$$

These classes are called right and left orthogonal classes of $\mathcal{X}$, respectively. A right R-module M is said to have a **special** $\mathcal{X}$**-precover** if there is an exact sequence

$$0 \longrightarrow G \longrightarrow X \longrightarrow M \longrightarrow 0$$

with $X \in \mathcal{X}$ and $G \in \mathcal{X}^{\perp}$. On the other hand, M is said to have a **special** $\mathcal{X}$**-preenvelope** if there is an exact sequence

$$0 \longrightarrow M \longrightarrow X \longrightarrow F \longrightarrow 0$$

with $X \in \mathcal{X}$ and $F \in {}^{\perp}\mathcal{X}$.

Now, we remind Wakamatsu's Lemma.

Lemma 2.2 *((Göbel & Trlifaj, 2006), Lemma 2.1.13) Let M be a right R-module and $\mathcal{X}$ a class of right R-modules which is closed under module extensions.*

(1) Let $\varphi : X \to M$ be an epic $\mathcal{X}$-cover of M. Then, φ is special.

(2) Let $\varphi : M \to X$ be a monic $\mathcal{X}$-envelope of M. Then, φ is special.

In 2003, among other things, Bican proved the following noticeable theorem.

Theorem 2.6 *((Bican, 2003), Theorem 2) Let $(\mathcal{T}, \mathcal{F})$ be Goldie's torsion theory. The following conditions are equivalent.*

(1) $\mathcal{F}$ is a covering class.

(2) $(\mathcal{T}, \mathcal{F})$ is of finite type.

If moreover, the ring R is right nonsingular, then these conditions are equivalent to the following condition:

(3) Every nonzero right ideal of R contains a finitely generated essential right ideal.

By specializing the class of modules, all the existing covers and envelopes can be obtained. Let $\mathcal{N}$ be the class of all nonsingular right R-modules. Since the class $\mathcal{N}$ contains the class of projective modules over a right nonsingular ring, every right module has an epic nonsingular cover ($\mathcal{N}$-cover) over a right nonsingular ring of finite right Goldie rank by the above Theorem 2.6.

Observing the fact that the right orthogonal class

$$\mathcal{N}^{\perp} = \{X \in \mathcal{M}od\text{-}R : \mathrm{Ext}^1_R(N, X) = 0 \text{ for all } N \in \mathcal{N}\}$$

of the class $\mathcal{N}$ of all nonsingular right R-modules contains all injective right R-modules, we have that $\mathcal{N}^{\perp}$-envelopes are monic.

CHAPTER 3

R-PROJECTIVITY OF NONSINGULAR MODULES

In this chapter, we object to investigate the rings whose nonsingular right modules are R-projective. We call these rings as right NR-rings. After stating some useful properties in the first section, some characterizations of NR-rings will be discussed in Section 3.2 and Section 3.3. Moreover, R-projectivity of the class of nonsingular injective modules will be considered.

3.1. Results for nonsingular rings

In this section, we shall prove some properties of nonsingular rings which will be necessary for the work in Section 3.2. We begin this section by recalling the notion of pure submodules.

A submodule T of a right R-module M is said to be a **pure submodule** if

$$0 \longrightarrow T \otimes_R A \longrightarrow M \otimes_R A$$

is exact for all left R-modules A, or equivalently, if

$$\mathrm{Hom}(A, M) \longrightarrow \mathrm{Hom}(A, M/T) \longrightarrow 0$$

is exact for all finitely presented right R-modules A. An exact sequence

$$0 \longrightarrow T \hookrightarrow M \longrightarrow M/T \longrightarrow 0$$

is said to be **pure exact** if T is a pure submodule of M.

In (Durğun, 2013), it was proved that a ring R is right Noetherian if and only if pure submodules of right R-modules are closed. Considering only nonsingular right modules, we have the following corresponding result over right nonsingular rings.

Proposition 3.1 *Let R be a right nonsingular ring. Then, the following are equivalent.*

(1) R_R is of finite Goldie rank.

(2) Pure submodules of nonsingular right R-modules are closed.

Proof (1) $\Rightarrow$ (2) Let A be a pure submodule of a nonsingular right R-module M. Suppose for the contrary that A is not closed in M. In that case, there exists a proper essential extension B of A in M. For $b \in B \setminus A$, if we set $K = A + bR$, then K/A becomes singular by Proposition 2.6. Moreover, K/A is cyclic and so $K/A \cong R/I$ for some right ideal I of R. The right ideal I is essential in R by Proposition 2.7. Additionally, there exists a finitely generated essential submodule I' of I with the help of the finiteness condition on R_R. Now, consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, hook, "pure"] & K \arrow[r, "\pi"] & \frac{R}{I} \arrow[r] & 0 \\
& & & \frac{R}{I'} \arrow[u, "f\neq 0"'] \arrow[ul, dashed, "g"] &
\end{tikzcd}$$

where f is just the projection of R/I' modulo I/I'. By the fact that R/I' is finitely presented, we have $\pi g = f$ for some $g : R/I' \to K$. However, using $I' \trianglelefteq R$, we obtain that $g(Z(R/I')) = g(R/I') \leq Z(K) = 0$, that is, $g = 0$ which is a contradiction.

(2) $\Rightarrow$ (1) Take a family $\{E_\gamma\}_{\gamma\in\Gamma}$ of nonsingular injective right R-modules. Because $\bigoplus_{\gamma\in\Gamma} E_\gamma$ is pure in $\prod_{\gamma\in\Gamma} E_\gamma$, by the assumption it is closed. Considering ((Goodearl, 1976), Corollary 1.9), which states that any closed submodule of an injective module is injective, we see that $\bigoplus_{\gamma\in\Gamma} E_\gamma$ is injective. Consequently, Proposition 2.9(1) yields that R_R is of finite Goldie rank. □

Before proving Corollary 3.1, we state the following lemma of (Holm & Jørgensen, 2008).

Lemma 3.1 *((Holm & Jørgensen, 2008), Lemma 4.7(iii)) Let $(\mathcal{T},\mathcal{F})$ be a torsion theory. If $(\mathcal{T},\mathcal{F})$ is hereditary and of finite type, then $\mathcal{F}$ is closed under pure submodules and pure quotient modules.*

Corollary 3.1 *The following are equivalent for a right nonsingular ring R.*

(1) R_R is of finite Goldie rank.

(2) Pure submodules of nonsingular right R-modules are closed.

(3) The torsion theory $(\mathcal{S}, \mathcal{N})$ is of finite type, where $\mathcal{S}$ is the class of all singular right R-modules and $\mathcal{N}$ is the class of all nonsingular right R-modules.

(4) Nonsingular right R-modules are closed under pure quotients.

Proof (1) $\Leftrightarrow$ (2) is shown in Proposition 3.1. (1) $\Leftrightarrow$ (3) is proved in Theorem 2.6. (3) $\Rightarrow$ (4) follows from Lemma 3.1 and (4) $\Rightarrow$ (2) can be easily seen from Lemma 2.1(1). □

In Proposition 3.2 we use the following characterization of flat modules.

Lemma 3.2 *((Lam, 1999),Corollary 4.86(1)) Let*

$$\varepsilon : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of right R-modules. Assume that B is flat. Then, ε is pure if and only if C is flat.

Proposition 3.2 *Every flat right R-module is nonsingular if and only if R is right nonsingular and pure submodules of free right R-modules are closed.*

Proof We immediately obtain that $Z(R_R) = 0$ since R_R is flat. Now, let K be a pure submodule of a free right R-module F. Then, F/K is a flat right R-module by Lemma 3.2. By the assumption, F/K is nonsingular and Lemma 2.1(1) implies that K is closed in F. Conversely, let M be a flat right R-module and consider the short exact sequence

$$0 \longrightarrow K \hookrightarrow F \longrightarrow M \longrightarrow 0$$

where F is a free right R-module. As M is flat, K is a pure submodule of F by Lemma 3.2. Hence, K is closed in F and so $F/K \cong M$ is nonsingular by Lemma 2.1(2). □

Following (Amini, Ershad & Sharif, 2008), a ring R is called **right almost-perfect** (**right A-perfect**, for short) if every flat right R-module is R-projective. These are exactly the rings over which flat covers of finitely generated right R-modules are projective (see Theorem 3.4). It was shown in (Amini, Ershad & Sharif, 2008) that right A-perfect rings lie properly between right perfect rings and semiperfect rings.

In the following proposition, we give some examples of rings whose flat right modules are nonsingular.

Proposition 3.3 *Over the following rings R, all flat right R-modules are nonsingular.*

(1) R is right nonsingular ring of finite right Goldie rank.

(2) R is right semihereditary and semiperfect.

(3) R is right nonsingular and right perfect.

(4) R is right semihereditary and right A-perfect.

Proof (1) This holds by Proposition 2.9(4). We include the proof for completeness. Let M be a flat right R-module and $f : F \to M$ be an epimorphism where F is a free right R-module. By the assumption and Corollary 3.1, we obtain that $M \cong F/\operatorname{Ker}(f)$ is nonsingular.

(2) Let M be a flat right R-module. We show that every finitely generated submodule K of M is nonsingular which implies that M is nonsingular. For a finitely generated submodule K of M, we have that K is flat by Theorem 2.3. Due to fact that over semiperfect rings every finitely generated flat right R-module is projective (see ((Lam, 1999), p. 161, Exercise 21)), we obtain that K is projective and so nonsingular as desired.

(3) It can be obtained from the fact that flat right modules are projective over right perfect rings.

(4) This part follows from (2). □

3.2. NR-rings

The right nonsingular rings whose nonsingular right modules are projective were characterized in Theorem 2.5. These are exactly the right perfect, left semihereditary rings with $\mathrm{E}(R_R)$ is flat. On the other hand, the rings whose flat right and injective right modules are R-projective were characterized in (Amini, Ershad & Sharif, 2008) and (Alagöz & Büyükaşık, 2021), respectively. Motivated by the aforementioned rings, in this section, we investigate the rings whose nonsingular right modules are R-projective. Moreover, we consider R-projectivity of nonsingular injective right R-modules.

Definition 3.1 A ring R is called **right *NR*** if every nonsingular right R-module is R-projective. Left NR-rings are defined similarly. If R is both right and left NR, then R is called an NR-ring.

Clearly, the rings whose nonsingular right modules are projective are right NR. We shall see in Example 3.1 that the converse is not true in general.

Lemma 3.3 *((Ketkar & Vanaja, 1981), Theorem 1) Let R be a semiperfect ring. Let M be an R-projective right R-module satisfying* $\mathrm{Rad}(M)$ *is small in M. Then, M is projective.*

Proposition 3.4 *Let R be a right NR-ring having finite right Goldie rank with* $\mathrm{Z}(R_R) = 0$ *and M be a nonsingular right R-module with* $\mathrm{Rad}(M) \ll M$*. Then, M is projective.*

Proof Since R is of finite right Goldie rank and right nonsingular, flat right R-modules are nonsingular by Proposition 3.3. Then, R becomes right A-perfect as R is right NR. Thus, R is semiperfect by ((Amini, Ershad & Sharif, 2008), Remark 3.8) and so Lemma 3.3 yields that every right nonsingular R-module with small radical is projective. □

The next result indicates that being NR-ring is a Morita invariant property.

Proposition 3.5 *Let R and S be Morita equivalent rings. Then, R is a right NR-ring if and only if S is a right NR-ring.*

Proof A right R-module M is R-projective if and only if M is N-projective for any finitely generated projective right R-module N by Proposition 2.4. Now, by ((Lam, 1999), p. 501, Exercise 2), being nonsingular is a categorical property. Moreover, by Proposition 2.5, projectivity, relative projectivity and being finitely generated are preserved by Morita equivalence, hence the proof is clear. □

A right R-module M is called ***CS*** if every closed submodule of M is a direct summand of M and a ring R called **right *CS*** if the right module R_R is CS. A ring R is called **right Σ-*CS*** (respectively, **right finitely Σ-*CS***) if every (respectively, finite) direct sum of copies of R_R is CS. If R is both right and left CS (respectively, Σ-CS and finitely Σ-CS), then R is said to be a CS (respectively, Σ-CS and finitely Σ-CS) ring (the reader might consult (Dung, Huynh, Smith & Wisbauer, 1994)).

Now, we state a weaker form of ((Dung, Huynh, Smith & Wisbauer, 1994), Chapter 4, 12.17).

Proposition 3.6 *((Dung, Huynh, Smith & Wisbauer, 1994), Chapter 4, 12.17) Let R be a right nonsingular ring with injective hull* $\mathrm{E}(R_R)$. *If R is right finitely* Σ-CS, *then R is right semihereditary and* $\mathrm{E}(R_R)$ *is flat. In this case, R is also left semihereditary.*

Proposition 3.7 *Let R be a right NR-ring. Then the following properties hold.*

(1) Finitely generated nonsingular right R-modules are projective.

(2) All nonsingular right R-modules are flat.

Moreover, if R is right nonsingular, then:

(3) R is right finitely Σ-*CS.*

(4) R is right and left semihereditary.

Proof (1) Assembling the right NR-ring assumption and the fact that finitely generated R-projective right R-modules are projective which was given in ((Alhilali, Ibrahim, Puninski & Yousif, 2017), Lemma 2.1), we are done.

(2) Let M be a nonsingular right R-module and N be a finitely generated submodule of M. Since N is nonsingular, it is projective by (1). We conclude that M is flat by the fact that every module is a direct limit of its finitely generated submodules and the direct limit of projective modules is flat.

(3) Let K be a closed submodule of $R^{(n)}$. Then, $R^{(n)}/K$ is nonsingular by Lemma 2.1(2), and so projective by (1). Therefore, the sequence

$$0 \longrightarrow K \hookrightarrow R^{(n)} \longrightarrow R^{(n)}/K \longrightarrow 0$$

splits, that is, K is a direct summand of $R^{(n)}$ which in turn yields that R is right finitely Σ-CS.

(4) This follows from (3) and Proposition 3.6. □

Recall that a **uniform** module is a nonzero module M such that any two nonzero submodules of M have nonzero intersection. Equivalently, M is uniform if and only if $M \neq 0$ and every nonzero submodule of M is essential in M. Note that a right R-module M is of finite Goldie rank if and only if M has an essential submodule which is a direct sum of finitely many uniform submodules (see (Goodearl, 1976), Proposition 3.19(a)).

Now, we are ready to give a characterization of right NR-rings.

Theorem 3.1 *Let R be a right nonsingular ring. The following statements are equivalent.*

(1) R is a right NR-ring and R_R is of finite Goldie rank.

(2) R is semihereditary, right A-perfect, right CS and $\mathrm{E}(R_R)$ *is flat.*

(3) R is right finitely Σ-CS and right A-perfect.

If any of these statements is satisfied, then the classes of all flat right R-modules and all nonsingular right R-modules coincide.

Proof (1) $\Rightarrow$ (2) By Proposition 3.3, we have that all flat right R-modules are nonsingular. Therefore, all flat right R-modules become R-projective by the right NR-ring assumption, and so R is right A-perfect. In addition, by Proposition 3.7(4), R is both right and left semihereditary, and by Proposition 3.7(2), $\mathrm{E}(R_R)$ is flat. In able to show that R is right CS, let I be a closed right ideal of R. By Lemma 2.1(2), R/I is nonsingular. Then, R/I is projective by Proposition 3.7(1) which implies that I is a direct summand of R. Therefore, R is right CS.

(2) $\Rightarrow$ (1) Let M be a nonsingular right R-module. The assumptions $\mathrm{E}(R_R)$ is flat and R is left semihereditary imply that M is flat by Theorem 2.4. As R is right A-perfect, we see that M is R-projective, that is, R is a right NR-ring. Since every right A-perfect ring is semiperfect, there exist orthogonal idempotents $e_1, \ldots, e_n$ in R such that

$$R_R = e_1R \oplus \cdots \oplus e_nR$$

and each e_iR is an indecomposable right R-module (see Theorem 2.2). From this, by applying the same arguments as in ((Dung, 1990), Theorem 3.1), we conclude that R is of finite right Goldie rank. For completeness, we give this deduction. Let C be a nonzero closed submodule of e_iR. Then, C_R is closed in R_R too. Hence, $R_R = C_R \oplus C'$ for some submodule C' of R_R. From this, we see that C_R is also a direct summand of e_iR. It follows that $C = e_iR$ is a uniform right ideal of R. Hence, R_R has finite Goldie rank.

(1) $\Rightarrow$ (3) By Proposition 3.7(3), R is right finitely Σ-CS. Moreover, since flat right R-modules are nonsingular (see Proposition 3.3), we obtain that all flat right R-modules are R-projective, that is, R is right A-perfect.

(3) $\Rightarrow$ (1) Note that over right finitely Σ-CS rings, finitely generated nonsingular right R-modules are projective (see ((Dung, Huynh, Smith & Wisbauer, 1994), Corollary 11.4). By this fact, we obtain that nonsingular right R-modules are flat. Now, by the A-perfectness assumption, we have that R is right NR. For the remaining part, recall that right finitely Σ-CS rings are also right CS. Therefore, combining this with being A-perfect, as in the proof of (2) $\Rightarrow$ (1), we conclude that R is of finite right Goldie rank.

Now, for the last statement, suppose one of these conditions holds. Then, nonsingular right R-modules are flat follows from Theorem 2.4, and the converse holds by Proposition 3.3. □

Remark 3.1 For the sake of simplicity, call a ring R right G-ring if all nonsingular right R-modules are flat. Clearly, if R is any right G-ring which is also right A-perfect, then R is a right NR-ring. By Theorem 3.1, the converse implication holds in the case when R is right nonsingular and of finite right Goldie rank. In other words, the NR-property of the ring in that particular case is just the conjunction of two known properties: of being a G-ring, and of being A-perfect.

As a consequence of Theorem 3.1, we have the following corollaries.

Corollary 3.2 *Let R be a right nonsingular ring. If R is a right A-perfect left semihereditary ring with* $\mathrm{E}(R_R)$ *is flat, then R is a right NR-ring.*

Recall that a **right Goldie ring** is a ring R such that R is of finite right Goldie rank and such that the right annihilator ideals in R satisfy the ascending chain condition (ACC). Left Goldie rings are defined similarly.

Lemma 3.4 *((Goodearl, 1976), Corollary 3.32) Let R be semiprime. Then, R is a right Goldie ring if and only if R is right nonsingular and R is of finite right Goldie rank.*

Proposition 3.8 *((Dung, Huynh, Smith & Wisbauer, 1994), Corollary 12.18) Let R be a semiprime right and left Goldie ring. Then, the following statements are equivalent.*

(1) R is right finitely Σ-CS.

(2) R is left finitely Σ-CS.

(3) R is right semihereditary.

(4) R is left semihereditary.

The fact that $E(R_R)$ is flat over semiprime right and left Goldie rings (see Proposition 2.9 and Lemma 3.4) together with Proposition 3.8 give rise to the following corollary.

Corollary 3.3 *Let R be a semiprime right and left Goldie ring. Then, the following statements are equivalent.*

(1) R is a right NR-ring.

(2) R is semihereditary and right A-perfect.

(3) R is finitely Σ-CS and right A-perfect.

Semihereditary commutative local domains are valuation domains. Since A-perfect rings are semiperfect, and semiperfect domains are local, we have the following corollary.

Corollary 3.4 *Let R be a commutative domain. Then, the following statements are equivalent.*

(1) R is NR.

(2) R is an A-perfect valuation domain.

By the following example, we show that there are right NR-rings which are not right perfect.

Example 3.1 Let F be a field and $R = F[[x]]$ be the ring of formal power series in one indeterminate x. Then, R is a valuation domain and also, R is an A-perfect ring which is not perfect by ((Amini, Ershad & Sharif, 2008), Example 3.11). Thus, R is an NR-ring by Corollary 3.4.

In (Alagöz & Büyükaşık, 2021), the authors studied the rings whose injective right modules are R-projective. In the following proposition, we characterize when every nonsingular injective right R-module is R-projective over a right nonsingular ring of finite right Goldie rank. Before we prove the characterization, we shall state a lemma.

Lemma 3.5 *((Faith, 1973), Lemma 4)) Let R be a right nonsingular ring and I be a right ideal of R. Then, the following statements are equivalent.*

(1) I is a closed right ideal of R.

(2) R/I embeds in $\mathrm{E}(R_R)$.

Theorem 3.2 *Let R be a right nonsingular ring having finite right Goldie rank. Then, the following statements are equivalent.*

(1) Every nonsingular injective right R-module is R-projective.

(2) $\mathrm{E}(R_R)$ *is R-projective.*

(3) $\mathrm{E}(R_R) = U_R \oplus V_R$*, where U is projective and* $\mathrm{Hom}(V, R/I) = 0$ *for each right ideal I of R.*

Proof (1) $\Rightarrow$ (2) is clear since $\mathrm{E}(R_R)$ is nonsingular and injective.

(2) $\Rightarrow$ (1) Let M be a nonsingular injective right R-module. Then, by Proposition 2.9, M can be written as a direct sum of indecomposable injective right R-modules N_γ, that is, $M = \bigoplus_{\gamma\in\Gamma} N_\gamma$. Now, let K_γ be a nonzero cyclic submodule of N_γ. Since K_γ is nonsingular, for each $\gamma \in \Gamma$, we see that K_γ is isomorphic to a submodule of $\mathrm{E}(R_R)$ by Lemma 3.5. However, N_γ's are uniform. This implies that N_γ's can be embedded in $\mathrm{E}(R_R)$, too. So, N_γ's are direct summands of $\mathrm{E}(R_R)$. Therefore, N_γ's are R-projective by the assumption, and then, using Proposition 2.3 we obtain that M is R-projective.

(2) $\Rightarrow$ (3) $\mathrm{E}(R_R)$ is of finite Goldie rank since R_R is of finite Goldie rank. Therefore,

$$\mathrm{E}(R_R) = U_1 \oplus \cdots \oplus U_n$$

where U_i's are indecomposable and injective right R-modules for $i = 1, \ldots, n$. Clearly, every U_i is R-projective. Now, we divide the proof into two cases:

Case 1: Let

$$\mathcal{U} = \{i \in \{1, \ldots, n\} : \mathrm{Hom}(U_i, R/I) \neq 0 \text{ for some right ideal } I \text{ of } R\}.$$

Assume $i \in \mathcal{U}$. Then, there exists a nonzero homomorphism $f : U_i \to R/I$. By the R-projectivity property of U_i, we have a nonzero homomorphism $g : U_i \to R$. As $\mathrm{Ker}(g)$

is a closed submodule of the injective module U_i, $\mathrm{Ker}(g)$ becomes a direct summand, and so $U_i = \mathrm{Ker}(g) \oplus S$ for some submodule S of U_i. However, U_i is indecomposable and g is nonzero. Thus, we conclude that g is monic which means $U_i \cong g(U_i)$ is a direct summand of R_R. Therefore, U_i is projective, and so is $\bigoplus_{i \in \mathcal{U}} U_i$.

Case 2: Let

$$\mathcal{V} = \{i \in \{1, \ldots, n\} : \mathrm{Hom}(U_i, R/I) = 0 \text{ for every right ideal } I \text{ of } R\}.$$

This gives that $\mathrm{Hom}(\bigoplus_{i \in \mathcal{V}} U_i, R/I) = 0$ for each cyclic right R-module R/I.

(3) $\Rightarrow$ (2) Clearly, such U_R and V_R are R-projective. The rest follows from Proposition 2.3. □

We deduce the following corollary by the fact that for a right R-module M over a right Noetherian ring R, $\mathrm{Rad}(M) = M$ if and only if $\mathrm{Hom}(M, R/I) = 0$ for each right ideal I of R. To see this fact, first assume that $\mathrm{Hom}(M, R/I) \neq 0$ for some right ideal I of R. Let $f : M \to R/I$ be a nonzero homomorphism. Then, by the Noetherianity assumption on R, the factor module $M/\mathrm{Ker}(f)$ contains a maximal submodule. This contradicts with $\mathrm{Rad}(M) = M$. Conversely, assume that $\mathrm{Rad}(M) \neq M$. Then, there exists a maximal submodule K of M. So, $M/K \cong R/I$ for some maximal right ideal I of R. Now, consider the nonzero homomorphism $f : M \to R/I$ which implies that $\mathrm{Hom}(M, R/I) \neq 0$ for some right ideal I of R.

Corollary 3.5 *Let R be a right nonsingular right Noetherian ring. Then, the following statements are equivalent.*

(1) Nonsingular injective right R-modules are R-projective.

(2) $\mathrm{E}(R_R)$ is R-projective.

(3) $\mathrm{E}(R_R) = U_R \oplus V_R$, where U is projective and $\mathrm{Rad}(V) = V$.

3.3. Right orthogonal class of nonsingular modules

In this section, $\mathcal{N}$ will denote the class of all nonsingular right R-modules. The class

$$\mathcal{N}^{\perp} = \{X \in \mathcal{M}od\text{-}R : \mathrm{Ext}_R^1(N, X) = 0 \ \text{ for all } \ N \in \mathcal{N}\}$$

will represent the right orthogonal class of $\mathcal{N}$. We aim to characterize right NR-rings via nonsingular covers.

Note that a right R-module C is said to be **cotorsion** (in the sense of Enochs) if $\mathrm{Ext}_R^1(F, C) = 0$ for every flat right R-module F.

Example 3.2 (1) Any injective right R-module M is contained in $\mathcal{N}^{\perp}$.

(2) Nonsingular right R-modules need not be flat in general. If R is right nonsingular, left semihereditary and $\mathrm{E}(R_R)$ is flat or if R right NR, then nonsingular right R-modules are flat (see respectively Theorem 2.4 and Proposition 3.7). So, in these cases, every cotorsion right R-module is contained in $\mathcal{N}^{\perp}$.

(3) Let R be a ring which is mentioned in Proposition 3.3. Then, every right R-module $M \in \mathcal{N}^{\perp}$ is cotorsion.

Corollary 3.6 *Let R be a right nonsingular ring. Then, the following are equivalent.*

(1) R is left semihereditary and $\mathrm{E}(R_R)$ *is flat.*

(2) All nonsingular right R-modules are flat.

(3) All cotorsion right R-modules are contained in $\mathcal{N}^{\perp}$.

Proof (1) $\Leftrightarrow$ (2) comes from Theorem 2.4 and (2) $\Rightarrow$ (3) follows from Example 3.2(2). For (3) $\Rightarrow$ (2), let M be a nonsingular right R-module. Then, $\mathrm{Ext}_R^1(M, C) = 0$ for every cotorsion right R-module C, which means that M is flat. □

Remark 3.2 Having reminded that the class $\mathcal{N}$ of all nonsingular right R-modules is closed under submodules, direct products, direct sums, essential extensions and module extensions in Section 2.3, we now conclude that over a right nonsingular ring of finite right Goldie rank, it is also closed under pure quotients by Corollary 3.1.

We also need the following fact of (Holm & Jørgensen, 2008) which is needed for the deduction of Lemma 3.6.

Theorem 3.3 *((Holm & Jørgensen, 2008), Theorem 3.4) If a class $\mathcal{F}$ contains the ground ring R and is closed under extensions, direct sums, pure submodules, and pure quotient modules, then $\mathcal{F}$ is covering and $\mathcal{F}^{\perp}$ is enveloping.*

Lemma 3.6 *If R is a right nonsingular ring of finite right Goldie rank, then all right R-modules have an $\mathcal{N}$-cover and an $\mathcal{N}^{\perp}$-envelope. Besides, all right R-modules have a special $\mathcal{N}$-precover and a special $\mathcal{N}^{\perp}$-preenvelope.*

Proof Since over a right nonsingular ring R of finite right Goldie rank, all conditions of Theorem 3.3 are satisfied by Remark 3.2, we conclude that every right R-module has an $\mathcal{N}$-cover and $\mathcal{N}^{\perp}$-envelope. Observing the facts that the class $\mathcal{N}$ contains all projective right R-modules and the class $\mathcal{N}^{\perp}$ contains all injective right R-modules, we have that $\mathcal{N}$-covers are epic and $\mathcal{N}^{\perp}$-envelopes are monic. Also, it is clear that the class $\mathcal{N}^{\perp}$ is closed under module extensions. Thus, the remaining part follows from Wakamatsu's Lemma, see Lemma 2.2. □

At this point, we emphasize that Lemma 3.6 does not extend to right nonsingular rings of infinite right Goldie rank which may be seen from the following example.

Example 3.3 Let R be the endomorphism ring of an infinite dimensional right vector space over a division ring. Then, R is von Neumann regular, right self-injective, but not semisimple (see (Goodearl, 1976), Proposition 2.23). Note that nonsingular right R-modules coincide with the flat Mittag-Leffler right R-modules by ((Herbera & Trlifaj, 2012), Corollary 2.10(i) and Example 6.8). However, the class of all flat Mittag-Leffler right R-modules is not precovering by ((Šaroch, 2018), Theorem 3.3), and so is not covering.

Before proving our next result, we will need the following theorem.

Theorem 3.4 *((Amini, Ershad & Sharif, 2008), Theorem 3.7) For a ring R the following are equivalent.*

(1) R is right A-perfect.

(2) R is semiperfect and flat covers of finitely generated right R-modules are finitely generated.

(3) Finitely generated flat right R-modules are projective and flat covers of finitely generated right R-modules are finitely generated.

(4) Flat covers of finitely generated right R-modules are projective.

(5) Flat covers of cyclic right R-modules are projective.

Proposition 3.9 *Let R be a right nonsingular ring of finite right Goldie rank. Then, the following are equivalent.*

(1) R is a right NR-ring.

(2) Nonsingular covers of finitely generated right R-modules are (finitely generated) projective.

(3) Nonsingular covers of cyclic right R-modules are (finitely generated) projective.

Proof (1) ⇒ (2) Let R be a right NR-ring. Since R is a right nonsingular ring of finite right Goldie rank, R is right A-perfect, and flat right R-modules and nonsingular right R-modules coincide by Theorem 3.1. Now, (2) follows from Theorem 3.4.

(2) ⇒ (3) is clear.

(3) ⇒ (1) This part of the proof is an analog of the proof of Theorem 3.4. □

Proposition 3.10 *Let R be a right nonsingular ring of finite right Goldie rank. Then, the following are equivalent.*

(1) Every right R-module has an $\mathcal{N}^{\perp}$-envelope which is nonsingular.

(2) Every $M \in \mathcal{N}^{\perp}$ is injective.

(3) Every $M \in \mathcal{N}^{\perp}$ is nonsingular.

(4) R is semisimple.

Proof (4) ⇒ (2), (4) ⇒ (3) and (3) ⇒ (1) are clear.

(1) ⇒ (4) Let M be a right R-module and $f : M \to L$ be its monic $\mathcal{N}^{\perp}$-envelope. Since L is nonsingular, M is also nonsingular. Hence, R is semisimple.

(2) ⇒ (4) Let A be any right R-module. By Lemma 3.6, special $\mathcal{N}$-precovers exist and so there is a short exact sequence

$$0 \longrightarrow M \longrightarrow F \longrightarrow A \longrightarrow 0$$

with $M \in \mathcal{N}^{\perp}$ and $F \in \mathcal{N}$. Then, by (2), M is injective, whence $A \in \mathcal{N}$. Therefore, R is semisimple. □

Theorem 3.5 *((Amini, Amini & Ershad, 2009), Theorem 2.8) A ring R is right A-perfect if and only if for every flat right R-module F, if $F = P + U$, where P is a finitely generated projective summand of F and U is a submodule of F, then $F = P \oplus V$ for some V in U.*

In ((Nicholson, 1976), Lemma 1.16), it was shown that for a projective right R-module M, if $M = P + K$, where P is a direct summand of M and K is a submodule of M, then there exists a submodule Q of K with $M = P \oplus Q$. Using the same method as in the proof of Theorem 3.5, one can prove the following result.

Proposition 3.11 *A ring R is right NR if and only if for every nonsingular right R-module N, if $N = P + L$, where P is a finitely generated projective direct summand of N and L is a submodule of N, then $N = P \oplus K$ for some K in L.*

CHAPTER 4

SIMPLE-INJECTIVE MODULES

In this chapter, our main goal is to state and prove characterizations of rings whose simple-injective modules are injective as well as, whose finitely generated simple-injective modules are projective. We divide this chapter into three parts. First, in attempt to declare these characterizations, we prove some useful properties for simple-injective modules. In the second part, we state and prove our main theorems, and finally we focus on simple-injective modules over commutative rings.

4.1. Certain relative injectivity conditions

In this section, first, we will remind certain relative injectivity conditions. Subsequently, we shall examine some of their properties.

Let M and N be right R-modules. According to Harada (Harada, 1992), M is said to be **simple-N-injective** if for every submodule K of N, and every homomorphism $f : K \to M$ with $f(K)$ simple extends to N, that is, the following diagram is commutative.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, hook, "i"] \arrow[d, "f"'] & N \arrow[dl, dashed, "h"] \\ & M & \end{tikzcd}$$

M is called **simple-injective** provided that M is simple-R_R-injective.

In (Harada, 1982), M is said to be **min-N-injective** if for every simple submodule K of N, and every homomorphism $f : K \to M$, there exists a homomorphism $h : N \to M$ such that the following diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, hook, "i"] \arrow[d, "f"'] & N \arrow[dl, dashed, "h"] \\ & M & \end{tikzcd}$$

commutes, that is, $hi = f$ where $i : K \to N$ is the inclusion map. If we take the right R-module R_R for N, then M is called **mininjective**.

Following (Amin, Fathi & Yousif, 2008), a right R-module M is said to be **strongly simple-injective** if M is simple-N-injective for each right R-module N.

We begin this section with the following result, which is crucial for investigating the rings whose simple-injective right modules are injective.

Proposition 4.1 *A right R-module M is simple-injective if and only if it is mininjective relative to each cyclic right R-module.*

Proof Suppose M is a simple-injective right R-module. Let I/K be a simple submodule of R/K where K is a right ideal of R and let $f : I/K \to M$ be a homomorphism. Then, $\mathrm{Im}(f\pi) = f(I/K)$ is a simple submodule of M, where $\pi : I \to I/K$ is the canonical epimorphism. Since M is simple-injective, there exists a homomorphism $h : R \to M$ such that $f\pi = hi$, where $i : I \to R$ is the inclusion. By the observation $h(K) = (f\pi)(K) = f(\pi(K)) = f(0) = 0$, we obtain that $K \subseteq \mathrm{Ker}(h)$. This yields that there exists a homomorphism $\varphi : R/K \to M$ such that $\varphi\pi' = h$, where $\pi' : R \to R/K$ is the canonical epimorphism. Now, for $\bar{a} \in I/K$, we have

$$f(\bar{a}) = f(\pi(a)) = h(i(a)) = h(a) = \varphi(\pi'(a)) = \varphi(\bar{a}) = \varphi(i'(\bar{a})),$$

where $i' : I/K \to R/K$ is the inclusion. Therefore, M is min-R/K-injective for each right ideal K of R, namely M is mininjective relative to each cyclic right R-module.

Conversely, suppose that M is mininjective relative to each cyclic right R-module. Let I be a right ideal of R and $\gamma : I \to M$ be a homomorphism such that $\gamma(I)$ a simple submodule of M. Then, $\mathrm{Ker}(\gamma)$ is a maximal submodule of I. If we set $K = \mathrm{Ker}(\gamma)$, then by the factor theorem, there is a homomorphism $\bar{\gamma} : I/K \to M$ such that $\gamma = \bar{\gamma}\pi$, where $\pi : I \to I/K$ is the canonical epimorphism. Since M is min-R/K-injective for every right ideal K of R, there is a homomorphism $h : R/K \to M$ such that $\bar{\gamma} = hi$, where $i : I/K \to R/K$ is the inclusion. Then, $h\pi'$ is the desired map, where $\pi' : R \to R/K$ is the canonical epimorphism. Indeed,

$$\gamma(a) = \bar{\gamma}\pi(a) = \bar{\gamma}(\bar{a}) = hi(\bar{a}) = h(\bar{a}) = h\pi'(a)$$

for every $a \in I$, which means that M is simple-injective. □

Proposition 4.2 *The following statements are equivalent for a right module M.*

(1) M is simple-injective.

(2) M is min-N-injective for every cyclic submodule N of $\mathrm{E}(M)$.

(3) For every simple right module S, and cyclic submodule N of $\mathrm{E}(S)$, the module M is min-N-injective.

Proof (1) ⇒ (2) follows from Proposition 4.1.

(2) ⇒ (3) Let S be a simple right module, and N be a cyclic submodule of $\mathrm{E}(S)$. Clearly, S is the unique simple submodule of N. Let $f : S \to M$ be a nonzero homomorphism. Then $\mathrm{E}(M)$ contains a copy of $\mathrm{E}(S)$. Thus, N is isomorphic to a cyclic submodule of $\mathrm{E}(M)$. Hence, (2) implies that M is min-N-injective.

(3) ⇒ (1) Let N be a cyclic right module, and S be a simple submodule of N. Let $f : S \to M$ be a homomorphism. There is a homomorphism $g : N \to \mathrm{E}(S)$ such that g is the identity on S. Let $N' = g(N)$. Apparently, N' is a cyclic submodule of $\mathrm{E}(S)$. Then by (3), there is a homomorphism $h : N' \to M$ that extends f. Now, it is straightforward to check that the map $hg : N \to M$ extends f. Thus, M is simple-injective by Proposition 4.1. □

Lemma 4.1 *A simple right module is simple-injective if and only if it is injective.*

Proof Let S be simple right module which is simple-injective. Assume that S is not injective. Let $\mathrm{E}(S)$ be the injective hull of S. Then there is a cyclic submodule X of $\mathrm{E}(S)$ which properly contains S. Moreover, S is essential in X. By Proposition 4.1, S is min-X-injective. This implies that the inclusion map $i : S \to X$ splits, that is $X = S \oplus K$ for some $K \subseteq X$. This contradicts the fact that S is essential in X. Therefore, S must be injective. This proves the necessity. The sufficiency is obvious. □

It is natural to ask "What are the rings all of whose right modules are simple-injective?". It is possible to handle this question in a more general frame as follows.

Proposition 4.3 *Let C be a class of right R-modules which is closed under submodules. The following statements are equivalent.*

(1) Every module in C is simple-injective.

(2) Every finitely generated module in $\mathcal{C}$ is simple-injective.

(3) Every cyclic module in $\mathcal{C}$ is simple-injective.

(4) Every simple module in $\mathcal{C}$ is simple-injective.

(5) Every simple module in $\mathcal{C}$ is injective.

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clear and (4) $\Rightarrow$ (5) holds by Lemma 4.1.

(5) $\Rightarrow$ (1) Let $M \in \mathcal{C}$. Let I be a right ideal of R, and $f : I \to M$ be a homomorphism with $f(I)$ is a simple submodule of M. Since $\mathcal{C}$ is closed under submodules, $f(I) \in \mathcal{C}$. Then $f(I)$ is injective by (5), and so there is a homomorphism $g : R \to f(I)$ that extends f. Therefore, M is simple-injective. $\square$

Recall that a ring R is called a **right V-ring** if every simple right R-module is injective and R is said to be a **right GV-ring** if each singular simple right R-module is injective.

Corollary 4.1 *The following statements are equivalent for a ring R.*

(1) Every right R-module is simple-injective.

(2) Every finitely generated right R-module is simple-injective.

(3) Every cyclic right R-module is simple-injective.

(4) Every simple right R-module is simple-injective.

(5) R is a right V-ring.

Using the fact that singular right modules and nonsingular right modules are closed under submodules, the following corollary can be easily obtained by Proposition 4.3. Note that nonsingular simple right modules are projective.

Corollary 4.2 *The following statements hold for a ring R.*

(1) Every nonsingular right R-module is simple-injective if and only if every projective simple right R-module is injective.

(2) Every singular right R-module is simple-injective if and only if R is a right GV-ring.

4.2. Rings whose simple-injective modules are injective (projective)

In this section, we study the rings whose (finitely generated) simple-injective right modules are injective (projective). In (Amin, Fathi & Yousif, 2008), the authors characterized the rings whose strongly simple-injective right modules are injective as follows.

Proposition 4.4 *((Amin, Fathi & Yousif, 2008), Proposition 1.12) The following conditions are equivalent for a ring R.*

(1) R is right Artinian.

(2) Every strongly simple-injective right R-module is injective.

(3) Every strongly simple-injective right R-module is quasi-continuous.

In the following theorem, we characterize the rings whose simple-injective right modules are injective, and this generalizes Proposition 4.4.

Recall that a right R-module M is injective if and only if $\mathrm{Ext}^1_R(R/I, M) = 0$ for each right ideal I of R.

Theorem 4.1 *R is right Artinian if and only if every simple-injective right R-module is injective.*

Proof Suppose that R is right Artinian. Let M be a simple-injective right R-module and X be a right ideal of R. We shall prove that $\mathrm{Ext}^1_R(R/X, M) = 0$. Since R is right Artinian, the composition length $\mathrm{cl}(X)$ of X is finite. We proceed by induction on $\mathrm{cl}(X)$. If $\mathrm{cl}(X) = 1$, that is, X is simple, then $\mathrm{Ext}^1_R(R/X, M) = 0$, because M is simple-injective and simple-injective modules are mininjective. Suppose that $\mathrm{cl}(X) = n \geq 2$ and $\mathrm{Ext}^1_R(R/Z, M) = 0$ for each right ideal Z of R with $\mathrm{cl}(Z) = n - 1$. Let Y be a submodule of X with $\mathrm{cl}(Y) = n - 1$. Consider the short exact sequence

$$0 \to \frac{X}{Y} \to \frac{R}{Y} \to \frac{R}{X} \to 0.$$

By applying $\mathrm{Hom}(-, M)$, we obtain the following exact sequence:

$$0 \to \mathrm{Hom}\left(\frac{R}{X}, M\right) \to \mathrm{Hom}\left(\frac{R}{Y}, M\right) \to \mathrm{Hom}\left(\frac{X}{Y}, M\right) \to \mathrm{Ext}^1_R\left(\frac{R}{X}, M\right) \to \mathrm{Ext}^1_R\left(\frac{R}{Y}, M\right).$$

By the induction hypothesis, we have that $\mathrm{Ext}^1_R(R/Y, M) = 0$. Since M is simple-injective, it is mininjective relative to each cyclic right R-module by Proposition 4.1. Therefore, as X/Y is simple,

$$\mathrm{Hom}(\frac{R}{Y}, M) \to \mathrm{Hom}(\frac{X}{Y}, M)$$

is an epimorphism which implies that $\mathrm{Ext}^1_R(R/X, M) = 0$. Thus, M is injective as desired.

For the converse part, note that strongly simple-injective modules are simple-injective. Thus, the sufficiency part follows by Proposition 4.4. Also, we will give a direct proof as follows. Assume that every simple-injective right R-module is injective. Let $\{E_i\}_{i\in I}$ be an arbitrary family of injective right R-modules. Then, the image of a homomorphism $g : X \to \bigoplus_{i\in I} E_i$ where X is a simple right ideal of R is contained in $\bigoplus_{i\in F} E_i$ where F is a finite subset of the index set I. This implies that $\bigoplus_{i\in F} E_i$ is injective and so there exist a homomorphism $h : R \to \bigoplus_{i\in F} E_i$ which extends g. Namely, we have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, hook] \arrow[d, "g"'] & R \arrow[dl, dashed, "h"] \arrow[ddl, dashed, "hj"] \\
& \bigoplus\limits_{i\in F} E_i \arrow[d, "j"'] & \\
& \bigoplus\limits_{i\in I} E_i &
\end{tikzcd}$$

Hence, $\bigoplus_{i\in I} E_i$ is simple-injective and so injective by the assumption. Therefore, R is right Noetherian by Bass-Papp theorem which states that a ring R is right Noetherian if and only if every direct sum of injective right R-modules is injective (see for example ((Anderson & Fuller, 1992), Proposition 18.13)). Next, we shall show that R is right semi-Artinian. Let M be a nonzero cyclic right R-module with $\mathrm{Soc}(M) = 0$. Then every submodule of M has a zero socle, and so every submodule of M is simple-injective. Thus every submodule of M is injective by the assumption. This implies that every submodule of M is a direct summand of M, that is, M is semisimple. This contradicts the fact that M has a zero socle, that is, R is right semi-Artinian. Hence, R is right Artinian. This completes the proof. □

For a right Noetherian ring, Theorem 4.1 can be stated as follows.

Corollary 4.3 *The following statements are equivalent for a right Noetherian ring R.*

(1) Every finitely generated simple-injective right R-module is injective.

(2) Every cyclic simple-injective right R-module is injective.

(3) R is right Artinian.

Proof (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) We may show that R is right Artinian as in the proof of Theorem 4.1

(3) $\Rightarrow$ (1) follows from Theorem 4.1. $\square$

In (Amin, Yousif & Zeyada, 2005), the authors proved the following.

Theorem 4.2 *((Amin, Yousif & Zeyada, 2005), Theorem 2.8) For a projective right R-module M, the following conditions are equivalent.*

(1) Evert quotient of a soc-M-injective right R-module is soc-M-injective.

(2) Every quotient of an injective right R-module is soc-M-injective.

(3) $\mathrm{Soc}(M)$ *is projective.*

In ((Amin, Fathi & Yousif, 2008), Remark 1.5), the authors stated that quotients of simple-injective right R-modules are simple-injective if and only if $\mathrm{Soc}(R_R)$ is projective, that is, R is right PS. Although the necessity of this statement is true and its proof is similar to that of Theorem 4.2, the sufficiency is not correct in general. For example, the ring of integers $\mathbb{Z}$ is a hereditary ring, and $\mathbb{Z}_\mathbb{Z}$ is simple-injective. However, the simple factor $\mathbb{Z}/p\mathbb{Z}$ for any prime p, is not simple-injective, otherwise $\mathbb{Z}/p\mathbb{Z}$ would be injective by Lemma 4.1.

We call R **right quasi V-ring** if simple right R-modules which are not isomorphic to right ideals of R are injective. Right Kasch rings and right GV-rings are right quasi V-rings. Every ring R with $\mathrm{Soc}(R_R) = 0$ is a right quasi V-ring if and only if R is a right V-ring. We have the following for right quasi V-rings.

Proposition 4.5 *Let R be a right quasi V-ring. The following statements are equivalent.*

(1) Quotients of simple-injective right R-modules are simple-injective.

(2) R is right PS.

(3) R is a right GV-ring.

Proof (1) $\Rightarrow$ (2) Let S be a simple right ideal of R and $\pi : E \to N$ be an epimorphism with E injective. Let $f : S \to N$ be a homomorphism. Since N is simple-injective by (1), f can be extended to a homomorphism $g : R \to N$, that is, $gi = f$, where $i : S \to R$ is the inclusion map. By the projectivity of R_R, there is a homomorphism $h : R \to E$ such that $\pi h = g$. So $\pi(hi) = (\pi h)i = gi = f$, whence S is projective by Proposition 2.1.

(2) $\Rightarrow$ (1) Let M be a simple-injective right R-module and $N \subseteq M$. We shall prove that M/N is min-R/J-injective for each (proper) right ideal J of R. Let I/J be a simple submodule of R/J and $f : I/J \to M/N$ be a homomorphism. Now, if I/J is not isomorphic to a minimal right ideal of R, then I/J will be injective by the right quasi V-ring assumption. Then $f\pi : R/J \to M$ extends f, where $\pi : R/J \to I/J$ is the canonical projection. If I/J is isomorphic to a minimal right ideal of R, then I/J is projective by (2). Therefore there is a homomorphism $g : I/J \to M$ such that $eg = f$, where $e : M \to M/N$ is the canonical epimorphism. Now, simple-injectivity of M implies that $g = hi$ for some homomorphism $h : R/J \to M$, where $i : I/J \to R/J$ is the inclusion. Consequently, we obtain that $f = eg = ehi$, that is eh extends f, and so M/N is mininjective relative to each cyclic right R-module. Hence M/N is simple-injective by Proposition 4.1.

(2) $\Leftrightarrow$ (3) It is easy to see that R is a right PS and right quasi V-ring if and only if R is a right GV-ring. □

The following result is a direct consequence of Theorem 4.1.

Corollary 4.4 *R is right Artinian and right hereditary if and only if quotients of simple-injective right R-modules are injective.*

Recall that a ring R is called a **quasi-Frobenius ring** (***QF*-ring**) if R is right (or left) Artinian, right (or left) self-injective. Equivalently, R is QF if and only if every injective right R-module is projective, if and only if every projective right R-module is injective.

In ((Amin, Fathi & Yousif, 2008), Proposition 1.13), it was proved that a ring R is QF if and only if every strongly simple-injective right R-module is projective. This result is also true if one replaces strongly simple-injective by simple-injective.

Corollary 4.5 *R is QF if and only if every simple-injective right R-module is projective.*

Proof QF-rings are right Artinian, hence the necessity follows by Theorem 4.1. Sufficiency follows from the fact that injective modules are simple-injective. □

Remark 4.1 It is routine to check that if M is a right R-module with $MI = 0$ for some ideal I of R and M is simple-injective as a right R-module, then M is simple-injective as a right R/I-module. On the other hand, the converse is not true in general. For example, for every Artinian ring R, $R/\operatorname{J}(R)$ is a semisimple ring, and so $R/\operatorname{J}(R)$ is simple-injective as $R/\operatorname{J}(R)$-module. But $R/\operatorname{J}(R)$ is not simple-injective as an R-module unless the the ring R is semisimple.

Now, we shall consider the rings whose finitely generated simple-injective right modules are projective. First, we give some results which are needed for characterizing such rings. The following lemma is well known in the literature. We include its proof for completeness.

Lemma 4.2 *Let R_1 and R_2 be rings and $R = R_1 \times R_2$, and M be a right R-module. If MR_i is a finitely generated projective right R_i-module for each $i = 1, 2$, then $M = MR_1 \times MR_2$ is a finitely generated projective right R-module.*

Proof Let MR_i be a finitely generated projective right R_i-module for each $i = 1, 2$. Then, we have

$$MR_1 \oplus K \cong R_1^n$$

and

$$MR_2 \oplus L \cong R_2^t$$

for some $t, n \in \mathbb{N}$. Therefore, MR_1 and MR_2 can be seen as direct summands of R_1^{n+t} and R_2^{n+t}, respectively. Thus,

$$\begin{aligned}(MR_1 \times MR_2) \oplus (K' \times L') &\cong (MR_1 \oplus K') \times (MR_2 \oplus L') \\ &\cong R_1^{n+t} \times R_2^{n+t} \\ &\cong (R_1 \times R_2)^{n+t} = R^{n+t}\end{aligned}$$

for some $K' \subseteq R_1^{n+t}$ and $L' \subseteq R_2^{n+t}$. This makes $M = MR_1 \times MR_2$ into a direct summand of the free right R-module R^{n+t}. Hence M is a finitely generated projective right R-module. □

Proposition 4.6 *Let $R = R_1 \times R_2$ be a ring decomposition. A right R-module M is simple-injective as an R-module if and only if MR_i is simple-injective as an R_i-module for each $i = 1, 2$.*

Proof Necessity is clear by Remark 4.1. Let us prove the sufficiency. We utilize ((Lam, 1999), 3.11A). Let M be a right R-module. Then $M_R = MR_1 \oplus MR_2$, as right R-modules. Suppose MR_i is a simple-injective right R_i-module for $i = 1, 2$ and I be a right ideal of R. Then, $I = I_1 \oplus I_2$ for some right ideals $I_1 \subseteq R_1$ and $I_2 \subseteq R_2$. Let $f : I \to M$ be an R-homomorphism with $f(I)$ simple. Then $\pi_i f e_i : I_i \to MR_i$ are R_i-homomorphisms for $i = 1, 2$ with $\pi_i f e_i(I_i)$ is simple or zero, where $\pi_i : M \to MR_i$ is the projection and $e_i : I_i \to I_1 \oplus I_2$ is the injection for $i = 1, 2$. Since MR_i is a simple-injective R_i-module, there is an R_i-homomorphism $g_i : R_i \to MR_i$ that extends $\pi_i f e_i$. Then, $g = g_1 \oplus g_2$ is an R-homomorphism, and extends f. This completes the proof. □

The following results will play a crucial role in the proof of Theorem 4.3.

Proposition 4.7 *((McConnell & Robson), Theorem 5.4.6) A hereditary Noetherian ring R is a finite direct sum of Artinian hereditary rings and hereditary Noetherian prime rings.*

Proposition 4.8 *((McConnell & Robson), Proposition 5.7.18) Let R be a commutative Dedekind domain and M be a nonzero finitely generated R-module. Then, the following statements hold.*

(1) M is the direct sum of a torsion module and a torsion-free module.

(2) If M is torsion, then M is a direct sum of indecomposable cyclic modules each of which has a unique composition series.

(3) If M is torsion-free, then

(i) $M \cong I_1 \oplus \cdots \oplus I_n$ for some nonzero ideals I_i of R where $n = udim(M)$.

(ii) $M \cong R^{n-1} \oplus I$ with $I = I_1 I_2 \ldots I_n$.

(iii) if also $M \cong R^m \oplus J$ with $0 \neq J \trianglelefteq R$ then $m = n - 1$ and $J \cong I$.

Now, we prove the following result for commutative Noetherian rings.

Theorem 4.3 *Let R be a commutative Noetherian ring. The following statements are equivalent.*

(1) Every finitely generated simple-injective R-module is projective.

(2) $R = A \times B$, where A is QF and B is hereditary.

Proof (1) $\Rightarrow$ (2) Let A be the sum of Artinian submodules of R. Then $\mathrm{Soc}(R/A) = 0$, and so R/A is simple-injective. Thus R/A is projective by (1), and so $R = A \oplus B$ for some ideal B of R. Then, A is both Noetherian and semi-Artinian, and so A is Artinian. Now, let M be an injective A-module. Because of the fact that a commutative ring R is Artinian if and only if each injective R-module is a direct sum of finitely generated modules due to ((Faith & Walker, 1967), Corollary 3.2), the module M is a direct sum of finitely generated injective modules, say $M = \bigoplus_{i \in I} M_i$. Then each M_i is projective by (1), and so M is projective. Therefore A is a QF ring. On the other hand, clearly $\mathrm{Soc}(B) = 0$, and so every ideal contained in B is simple-injective. Then, every ideal contained in B is projective by (1), and so B is hereditary.

(2) $\Rightarrow$ (1) Let M be a finitely generated simple-injective R-module. Then $M = MA \oplus MB$, where MA and MB are finitely generated simple-injective A-module and B-module, respectively. Since A is a QF-ring it is Artinian, and so MA is an injective A-module by Theorem 4.1. Thus, MA is a projective A-module because A is QF. Now, set $N = MB$. We shall prove that N is projective. Since B is Noetherian and hereditary with $\mathrm{Soc}(B) = 0$,

$$B = D_1 \times \cdots \times D_n,$$

where D_i is a Dedekind domain for each $i = 1, \ldots, n$ by Proposition 4.7. By Proposition 4.6, without loss of generality, we may assume that N is a finitely generated simple-injective module over a Dedekind domain D. Since $N/T(N)$ is finitely generated and torsion-free, it is projective by Proposition 4.8(3). Then $N = \mathrm{T}(N) \oplus L$, where $\mathrm{T}(N)$ is the torsion part of N and L is projective. We will prove that $\mathrm{T}(N) = 0$. Suppose the contrary that $\mathrm{T}(N) \neq 0$. Then $\mathrm{T}(N)$ is a direct sum of indecomposable cyclic modules by Proposition 4.8(2), that is,

$$\mathrm{T}(N) = \frac{D}{P_1^{k_1}} \oplus \cdots \oplus \frac{D}{P_n^{k_n}},$$

where P_i's are maximal ideals of D and k_i's are positive integers for $i = 1, \ldots, n$. Since $\mathrm{T}(N)$ is simple-injective, $D/P_i^{k_i}$'s are simple injective as well. In particular, $D/P_1^{k_1}$ is simple-injective. For simplicity, set $P = P_1$ and $k = k_1$. Since D/P is a field and P^k/P^{k+1} is a D/P-module, P^k/P^{k+1} is a semisimple D-module. Let S/P^{k+1} be a simple submodule of P^k/P^{k+1}. Note that D/P^k contains a simple submodule isomorphic to S/P^{k+1}. Thus, there is a nonzero homomorphism $f : S/P^{k+1} \to D/P^k$. Since D/P^k is simple-injective, it is mininjective relative to each cyclic module by Proposition 4.1. Thus, there is a homomorphism $g : D/P^{k+1} \to D/P^k$ that extends the map f. Then for each $\bar{x} = x + P^k \in S/P^{k+1}$, we have

$$f(\bar{x}) = g(\bar{x}) = g(\bar{1})x.$$

Since $x \in P^k$ and $g(\bar{1}) \in D/P^k$, we have $g(\bar{1})x = 0$. Then $f = 0$, a contradiction. Therefore $\mathrm{T}(N)$ must be zero, and so $N = L$ is projective. Hence, summing up, $M = MA \oplus MB$ is a projective R-module by Lemma 4.2. This completes the proof. $\square$

Since commutative hereditary Noetherian domains are Dedekind domain, we have the following corollary.

Corollary 4.6 *Let R be a commutative Noetherian domain. The following statements are equivalent.*

(1) Every finitely generated simple-injective R-module is projective.

(2) R is a Dedekind domain.

4.3. Simple-injective modules over commutative rings

In this section, we give a characterization of simple-injective modules over the ring of integers. We prove that if R is a commutative domain, and M is an R-module, then M is simple-injective if and only if the torsion part $\mathrm{T}(M)$ of M is simple-injective. For a commutative hereditary Noetherian ring, we show that a module M is simple-injective if and only if the singular submodule $\mathrm{Z}(M)$ of M is simple-injective.

Lemma 4.3 *Let M be a right R-module and N a submodule of M. If N is simple-injective and $\mathrm{Soc}(M) \subseteq N$, then M is simple-injective.*

Proof Let I be a right ideal of R and $f : I \to M$ be a homomorphism with $f(I)$ simple submodule of M. Then $f(I) \subseteq N$ by the hypothesis. Since N is simple-injective, there is a homomorphism $g : R \to N$ such that $gi = f$, where $i : I \to R$ is the inclusion homomorphism. This shows that M is simple-injective. $\square$

Lemma 4.4 *Let R be a commutative domain, and M be an R-module. Then M is simple-injective if and only if* $\mathrm{T}(M)$ *is simple-injective.*

Proof Since $\mathrm{Soc}(M) \subseteq \mathrm{T}(M)$, sufficiency is clear by Lemma 4.3. For the necessity, suppose M is simple-injective and let I be an ideal of R and $f : I \to \mathrm{T}(M)$ be a homomorphism with $f(I)$ simple submodule of $\mathrm{T}(M)$. Since M is simple-injective, there is a homomorphism $g : R \to M$ such that, $gi = jf$, where $i : I \to R$ and $j : \mathrm{T}(M) \to M$ are the inclusion maps. As R is cyclic, $g(R)$ is a cyclic submodule M. Let $g(R) \cong R/J$ for some ideal J of R. Since $f(I) = g(I)$ is simple and $\mathrm{Soc}(R) = 0$, we have that $J \neq 0$. Thus $g(R)$ is a torsion module, and so $g(R) \subseteq \mathrm{T}(M)$. Therefore g extends f, and so $\mathrm{T}(M)$ is simple-injective. □

Proposition 4.9 *Let R be a ring and M be a right R-module. Suppose that every projective simple right module is injective. If* $\mathrm{Z}(M)$ *is simple-injective, then M is simple-injective.*

Proof Let K be a right ideal of R and I/K be a simple submodule of R/K. Assume that $f : I/K \to M$ be an R-homomorphism. If I/K is nonsingular, then it is projective as it is simple. By the hypothesis, we obtain that I/K is injective, and so I/K is a direct summand of R/K. The map $f\pi$ clearly extends f, where $\pi : R/K \to I/K$ is the natural projection. On the other hand, if I/K is singular, then $f(I/K) \subseteq \mathrm{Z}(M)$. Since $\mathrm{Z}(M)$ is simple-injective, there is a $g : R/K \to \mathrm{Z}(M)$ that extends f. Thus M is simple-injective by Proposition 4.1. □

Lemma 4.5 *((Alizade, Büyükaşık, Lopez-Permouth & Yang), Lemma 3.4) For a simple module V over a commutative Noetherian ring, the properties of injectivity, flatness and projectivity are equivalent.*

Let R be a commutative Noetherian ring. Then, every nonsingular R-module is simple-injective by Lemma 4.5 and Corollary 4.2(1). Moreover, we obtain the following corollary by Proposition 4.9.

Corollary 4.7 *Let R be a commutative Noetherian ring, and M be an R-module. If* $\mathrm{Z}(M)$ *is simple-injective, then M is simple-injective.*

Now, following (Fuchs, 1970), we remind the notion of basic subgroups for torsion groups.

Definition 4.1 *((Fuchs, 1970), p. 139)* A **basic subgroup** B of a torsion group A is a subgroup of A satisfying the following conditions:

(i) B is a direct sum of cyclic groups of prime power orders.

(ii) B is pure in A.

(iii) A/B is divisible.

Proposition 4.10 *The following are equivalent for an abelian group M.*

(1) M is simple-injective.

(2) $\mathrm{T}(M)$ is simple-injective.

(3) $\mathrm{T}(M)$ is injective

(4) $M = A \oplus B$, where A is torsion-free and B is injective.

Proof (1) $\Leftrightarrow$ (2) holds by Lemma 4.4.

(2) $\Rightarrow$ (3) By ((Fuchs, 1970),Theorem 21.3) which states that every abelian group is the direct sum of a divisible group and a reduced group, we have that $\mathrm{T}(M) = A \oplus B$ where A is a reduced and B is an injective subgroup of $\mathrm{T}(M)$. We need to prove that $A = 0$. Assume for the contrary that A is nonzero. Now, let C be a basic subgroup of A. Then, C is a direct sum of cyclic groups of prime power orders, and also C is pure in A (see Definition 4.1). Let X be a cyclic direct summand of C. Since X is a bounded and pure subgroup of A, then by ((Fuchs, 1970), Theorem 27.5), X is a direct summand of A and so a direct summand of $\mathrm{T}(M)$ which means that X is simple-injective. On the other hand, without lost of generality we may write $X \cong \mathbb{Z}_{p^n}$ where p is a prime and $n \geq 1$. By Proposition 4.1, $\mathbb{Z}_{p^n}$ is mininjective relative to each cyclic abelian group. In particular, $\mathbb{Z}_{p^n}$ is min-$\mathbb{Z}_{p^{n+1}}$-injective. Let $\gamma : p^n\mathbb{Z}_{p^{n+1}} \to \mathbb{Z}_{p^n}$ be a nonzero homomorphism. As $p^n\mathbb{Z}_{p^{n+1}}$ is simple there is a $\beta : \mathbb{Z}_{p^{n+1}} \to \mathbb{Z}_{p^n}$ such that the following diagram commutes:

$$\begin{tikzcd}
p^n\mathbb{Z}_{p^{n+1}} \arrow[r, "i"] \arrow[d, "\gamma"'] & \mathbb{Z}_{p^{n+1}} \arrow[dl, dashed, "\beta"'] \\
\mathbb{Z}_{p^n} &
\end{tikzcd}$$

Then,

$$\gamma(p^n) = \beta(1)p^n = 0$$

in $\mathbb{Z}_{p^n}$, and so $\gamma = 0$, a contradiction. Therefore, $A = 0$, and so $\mathrm{T}(M)$ is injective.

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2) are clear. □

In (Amin, Yousif & Zeyada, 2005), for right R-modules M and N, M is called **soc-N-injective** if any R-homomorphism $f : \mathrm{Soc}(N) \to M$ extends to N and, M is called **soc-injective** if it is soc-R-injective. M is said to be **strongly soc-injective** if it is soc-N-injective for all right R-modules N. Strongly simple-injective right modules and (strongly) soc-injective right modules are closed under finite direct sums by ((Amin, Fathi & Yousif, 2008), Proposition 1.4(2)) and ((Amin, Yousif & Zeyada, 2005), Corollary 2.3(1)), respectively. Now, the following corollary is clear by Proposition 4.10(4).

Corollary 4.8 *The following statements are equivalent for an abelian group M.*

(1) M is simple-injective.

(2) M is soc-injective.

(3) M is strongly soc-injective.

(4) M is strongly simple-injective.

Proposition 4.11 *Let R be a right Noetherian ring, and*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of right R-modules with C flat. If B and C are simple-injective then A is simple-injective.

Proof By Proposition 4.1, it is enough to prove that A is mininjective relative to each cyclic right R-module. Let J be a right ideal of R and I/J be a simple submodule of R/J. Consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(\frac{R}{I}, A) \arrow[r] \arrow[d] & \mathrm{Hom}(\frac{R}{I}, B) \arrow[r] \arrow[d] & \mathrm{Hom}(\frac{R}{I}, C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\frac{R}{J}, A) \arrow[r] \arrow[d] & \mathrm{Hom}(\frac{R}{J}, B) \arrow[r] \arrow[d] & \mathrm{Hom}(\frac{R}{J}, C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\frac{I}{J}, A) \arrow[r] & \mathrm{Hom}(\frac{I}{J}, B) \arrow[r] \arrow[d] & \mathrm{Hom}(\frac{I}{J}, C) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since R is right Noetherian and the given sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is pure-exact, the rows are exact as well. By the hypothesis, B and C are simple-injective. Thus, the second and the third columns are exact, too. Then, the first column is exact by 3×3 lemma. Therefore, A is simple-injective. □

Corollary 4.9 *Let R be a commutative hereditary Noetherian ring. An R-module M is simple-injective if and only if $\mathrm{Z}(M)$ is simple-injective.*

Proof Since $M/\mathrm{Z}(M)$ is nonsingular, it is simple-injective. Also $M/\mathrm{Z}(M)$ is flat by Theorem 2.4. Thus $\mathrm{Z}(M)$ is simple-injective by Proposition 4.11. This proves necessity. Sufficiency holds by Corollary 4.7. □

CHAPTER 5

CONCLUSION

In this thesis, after summarizing the concepts of certain relative projectivity and injectivity conditions and their importance in ring and module theory, we investigated both the rings whose nonsingular modules are R-projective and the rings whose simple-injective modules are injective. We proved that for a right nonsingular ring R, all nonsingular right R-modules are R-projective and R is of finite right Goldie rank if and only if R is right finitely Σ-CS and all flat right R-modules are R-projective. Another problem that we considered is that R-projectivity of the class of nonsingular injective modules. We proved that over right nonsingular rings of finite right Goldie rank, all nonsingular injective right R-modules are R-projective if and only if the injective hull $\mathrm{E}(R_R)$ of R is R-projective. Among other results, one of the main conclusions that we obtained in this study is about the characterization of the rings whose simple-injective modules are injective. We showed that these rings are exactly right Artinian rings. Furthermore, for a commutative Noetherian ring R, we proved that every finitely generated simple-injective R-module is projective if and only if $R = A \times B$, where A is QF and B is hereditary. Besides, we gave a complete characterization of simple-injective modules over the ring of integers.

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