TÜMLEYEN VE BÜTÜNLEYEN MODÜLLERİN HOMOLOJİK ÖZELLİKLERİ

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ÖNSÖZ

Eylül 2008'de TÜBİTAK tarafından desteklenmeye başlanılan iki yıllık "Tümleyen ve Bütünleyen Modüllerin Homolojik Özellikleri" başlıklı proje, İzmir Yüksek Teknoloji Enstitüsü'nde Doç Dr. Dilek Yılmaz, Prof. Dr. Rafail Alizade, Yrd. Doç. Dr. Engin Büyükaşık ve Yrd.Doç. Dr. Engin Mermut (9 Eylül üniversitesi) tarafından yürütülmüş ve Eylül 2010'da sonlandırılmıştır.

Bu sonuç raporu, ilk üç dönemin raporlarının özetlerini ve 4. dönem çalışmalarını içermektedir.

İçindekiler

1. RAD-TÜMLEYEN ALTMODÜLLER

R birim elemanlı herhangi bir halka olsun. R-modüllerle *sol* R-modülleri kastedeceğiz. R-$\mathcal{M}od$ ile de *sol* R-modüller kategorisini göstereceğiz.

Esas olarak Rad-tümlenmiş modüllerin bazı özellikleri ve daha genel olarak τ, R-$\mathcal{M}od$ kategorisinde bir radikal olmak üzere τ-tümlenmiş modüllerin bazı özelliklerini elde ettik. Rad-tümleyenleri (=eş düzenli alt modülleri) ve daha genel olarak τ-tümleyenleri incelememizin motivasyonu *bağıl homoloji cebirinden* gelmektedir. Aşağıdaki paragraflarda açıklanacağı üzere tümleyenlerle ilgili *öz sınıfları* incelemekteyiz. Cevapladığımız ana sorulardan biri ne zaman bütün R-modüllerin Rad-tümlenmiş olduğudur. Bu problemle uğraşırken, radikal modüller, indirgenmiş modüller ve eş atomik modüller yararlı olmuştur (bakınız [38]). Bu kavramların tanımlarında, Rad yerine R-$\mathcal{M}od$ kategorisinde bir τ radikali kullandığımızda elde edilen modüller de pek tabi ki τ-tümlenmiş modüllerin özelliklerinin incelenmesinde işe yaramaktadır.

Bir M modülü için, M'nin $\operatorname{Rad} U = U$ şartını sağlayan bütün U alt modüllerinin toplamını $P(M)$ ile gösterelim. Her R halkası için, $P({}_RR)$'nin iki-taraflı bir ideal olacağını da not edelim.

Abel gruplarının düzenli alt grupları kavramı modüllere [26] ve [27]'de genelleştirilmiştir: Modüllerin bir $f : K \to L$ monomorfizmasına *düzenli* denir eğer her basit modül S, $L \to L/\operatorname{Im} f$ projeksiyonuna göre *projektif* ise, yani $\operatorname{Hom}(S, L) \to \operatorname{Hom}(S, L/\operatorname{Im} f) \to 0$ dizisi tam ise. Bunun duali olarak [3]'de eş düzenli alt modüller tanımlanmıştır: Modüllerin bir $f : K \to L$ monomorfizmasına eş düzenli denir eğer $\operatorname{Rad} M = 0$ şartını sağlayan her M modülü bu f monomorfizmasına göre *injektif* ise, yani $\operatorname{Hom}(L, M) \to \operatorname{Hom}(K, M) \to 0$ dizisi tamsa. [27]'de belirtildiği üzere tümleyen alt modüller kısa tam dizilerin bir öz sınıfının tanımlanmasını sağlar (bakınız ayrıca [12]). Eş düzenli monomorfizmaların tanımladığı kısa tam dizilerin sınıfı da radikali sıfır olan modüller tarafından injektif olarak üretilen öz sınıftır ve bu öz sınıf tümleyen alt modüllerle tanımlanan öz sınıfı içerir. Eş düzenli alt modül tanımında, Rad yerine R-$\mathcal{M}od$ kategorisinde bir τ radikali de alabiliriz. Bu durumda eş düzenli alt modüllerin Rad-tümleyen olarak karakterizasyonu şu teoremdeki $\tau = \operatorname{Rad}$ özel durumudur:

Teorem 1.1. *(bakınız [2, 1.11] veya [6, 10.11]) R-$\mathcal{M}od$ kategorisinde bir τ radikali alalım. M bir R-modül ve $V \leq M$ de bir alt modül olsun. Şunlar denktir:*

(1) $\tau(N) = 0$ şartını sağlayan her N modülü $V \hookrightarrow M$ içermesini göre injektiftir;
(2) öyle bir $U \leq M$ alt modülü vardır ki $U + V = M$ ve $U \cap V = \tau(V)$;
(3) öyle bir $U \leq M$ alt modülü vardır ki $U + V = M$ ve $U \cap V \leq \tau(V)$.

Eğer bu denk koşullardan herhangibiri sağlanırsa, V'ye M'de bir τ-tümleyen *denir.*

Yukardaki önermedeki en son koşulun tümleyen olma tanımına ne kadar benzediğine dikkat ediniz: Tümleyen tanımındaki $U \cap V \ll V$ ($U \cap V$, V'de küçük alt modül) yerine $U \cap V \leq \tau(V)$ gelmiştir. Tümleyenlerdekine benzer tanımlar da şu şekilde verilir: Bir M modülünün U ve V alt modülleri için, V, M'de U'nun bir *τ-tümleyenidir* denir eğer $U + V = M$ and $U \cap V \leq \tau(V)$ sağlanırsa. M modülüne *τ-tümlenmiştir* denir eğer M'nin her alt modülünün M'de bir τ-tümleyeni varsa. τ-tümleyenler ve τ-tümlenmiş modüllerle ilgili bazı özellikler için bakınız [2] ve [6]. Özel olarak $\tau = \text{Rad}$ aldığımızda, bir alt modülün *Rad-tümleyeni* ve *Rad-tümlenmiş* modül tanımlarını elde ederiz. Ayrıca bakınız [33]; bu makalede Rad-tümlenmiş modüllere *genelleştirilmiş tümlenmiş* denmektedir. Rad-tümlenmiş modüllerle ilgili sonuçların taranması için bakınız [24, Ch.6].

Elde ettiğimiz esas sonucumuz şudur:

Teorem 1.2.

(1) Her sol R-modülü Rad-tümlenmiştir ancak ve ancak $R/P(R)$ sol mükemmel bir halka ise.

(2) Eğer R bir sol duo halkası ise, yani, bütün sol idealleri çift taraflı idealler ise, bu durumda R (sol R-modül olarak) Rad-tümlenmiştir ancak ve ancak $R/P(R)$ yarı-mükemmel ise.

Dedekind bölgeleri üzerinde, tümlenmiş modüllerin yapısı [38]'da tam olarak belirlenmiştir. Bu yapıyı kullanarak da şu sonucu da elde ettik:

Teorem 1.3. *Eğer R bir Dedekind bölgesi ise, bir R-modül M Rad-tümlenmiştir ancak ve ancak M'nin bölünebilir kısmı olan $D = P(M)$ için M/D tümlenmiş ise.*

2. C-İNJEKTİF MODÜLLER: KAPALI ALTMODÜLLERE GÖRE BAĞIL İNJEKTİFLİK

Bütün halkaların birleşme özelliğine sahip ve birim elemanlı halkalar olduğunu ve bütün modüllerin birimli sol modüller olduğunu kabul edelim.

Bir R-modülünün bir K alt modülüne M'de *kapalı* denir eğer K'nın M'de hiç öz büyük genişlemesi yok ise. Tabii ki, M'nin her dik toplam terimi M'de kapalıdır. M modülünün bir L alt modülünü alalım. Zorn'un Önsav'ından M'nin öyle bir K alt modülünün varlığı elde edilir ki bu K alt modülü M'nin L'yi içeren alt modülleri arasında L'nin K'da büyük alt modül olması özelliğine göre maksimal bir alt modüldür; bu durumda K alt modülü M'nin kapalı bir alt modülü olur. Bir M modülüne CS-modül denir eğer her kapalı alt modülü, M'nin bir dik toplam terimi ise. Bu durumda, M'nin her alt modülü M'deki bir dik toplam teriminin içinde büyük bir alt modül olur. Kapalı alt modüller ve CS-modüllerle ilgili olarak [7] ve [23] kaynaklarına bakınız.

M bir R-modül olsun. [30]'da bir R-modül X'e, *M-c-injektif* modül denir eğer M'nin her kapalı alt modülü K için, her $\varphi : K \to X$ homomorfizması bir $\theta : M \to X$ homomorfizmasına genişletilebilirse. X'e *c-injektif* denir eğer X her R-modülü M için c-injektif ise. Eğer M bir CS-modül ise bu durumda her R-modülü M-c-injektif olur. [31, Theorem 6]'da şu gösterilmektedir: Eğer R bir Dedekind bölgesi ve M de basit modüllerin dik çarpımı ise, bu durumda M, M-c-injektif olur ama M bir CS modül olmak durumunda değildir (bakınız [31, Proposition 2]).

Elde ettiğimiz ana sonuçlardan biri şudur: Eğer R bir Dedekind bölgesi ise, bir R-modül X c-injektiftir ancak ve ancak öyle bir Y R-modülü varsa ki Y basit R-modüllerin ve injektif bir modülün dik çarpımıdır ve X modülü de Y'nin bir dik toplam terimine izomorftur.

Böyle bir dik toplam teriminin de aslında homojen yarı-basit R-modüllerin ve injektif R-modüllerin dik çarpımına izomorf olmak zorunda olduğunu da gösterdik. İlgili özellikler için [4] ve [32] makalelerine de bakınız.

R bir halka olsun ve $\mathcal{E}$ de R'nin boş olmayan bir idealler topluluğu olsun. [20]'i takip ederek, bir R-modülü M'nin bir L alt modülüne, M'de *$\mathcal{E}$-saf* diyeceğiz eğer $L \cap IM = IL$ koşulu $\mathcal{E}$'deki her I ideali için sağlanıyorsa (ayrıca [22] kaynağına da bakınız). Bir R-modül X'e *$\mathcal{E}$-saf-injektif* diyeceğiz eğer her R-modül M ve M'nin her $\mathcal{E}$-saf alt modülü L için, her $\varphi : L \to X$ homomorfizması, M'ye genişletilebiliyorsa. Özel olarak, bu kavramlarla $\mathcal{E}$'nin R'nin tüm sol primitif ideallerinden oluştuğu durumla ilgilendik. R'nin tüm sol primitif ideallerinden oluşan kümeyi $\mathcal{P}$ ile gösterelim.

Honda [15, pp. 42-43] tarafından bir A abel grubunun bir B alt grubuna, A'da düzenli denir eğer her p asal sayısı için $pB = B \cap pA$ sağlanırsa (ayrıca bakınız [10]). Bizim terminolojimize göre, bir $\mathbb{Z}$-modül A'nın B alt modülü A'da düzenlidir ancak ve ancak B alt modülü A'nın $\mathcal{P}$-saf alt modülü ise.

R Dedekind bölgesi üzerindeki c-injektif modülleri karakterize ettiğimiz ana sonuca ulaşırken daha önce $\mathcal{P}$-saf-injektif modülleri daha geniş bir halkalar sınıfında karakterize ettik. Sonra da R Dedekind bölgesi üzerinde c-injektif modüllerin, $\mathcal{P}$-saf-injektif modüller ile aynı olduğunu elde ettik. Daha da ötesi Dedekind bölgelerini Noether tamlık bölgeleri arasında şu özellikle karakterize edebildik: R bir Noether tamlık bölgesi ise, R'nin Dedekind bölgesi olması için gerek ve yeter şart her basit R-modülün c-injektif olmasıdır.

Elde ettiğimiz ana sonuçları aşağıdaki teoremlerle özetleyeceğiz.

Yukarda bahsettiğimiz daha geniş halkalar sınıfı olarak esasta şu tür halkalarla ilgileneceğiz: öyle R halkaları ki her P sol primitif ideali için R/P Artin halkası olsun. Değişmeli halkalar tabi ki bu özelliğe sahiptir. Genel olarak bir polinom özdeşliği sağlayan halkalar da Kaplansky'nin bir teoreminden dolayı bu özelliğe sahiptir (örneğin bakınız [21, Theorem 13.3.8]). Bir R halkasına soldan tamamıyle sınırlı denir eğer R'nin her asal homomorfik görüntüsündeki her büyük left ideal sıfırdan farklı iki taraflı bir ideal içerirse. Bir R halkasına sol FBN halkası denir eğer R soldan tamamıyle sınırlı ve soldan Noether bir halka ise. Bilinmektedir ki eğer R bir sol FBN halkası ise, bu durumda her P sol primitif ideali için R/P Artin halkası olur (örneğin bakınız [13, Proposition 8.4]). Eğer R, yarı-mükemmel bir halka ise yine her P sol primitif ideali için R/P Artin halkası olur. Bu özelliği sağlayan halkalara son bir örnek olarak da şunu verebiliriz: Roseblade [29, Corollary A] şunu göstermektedir: eğer $J = \mathbb{Z}$ veya J sonlu bir cisim, G sonluyla-polisiklik bir grup ve R de $J[G]$ grup halkası ise yine bu durumda her P sol primitif ideali için R/P Artin halkası olur. Dolaysıyla bütün bu halkalar için şu sonuçlarımız var:

Teorem 2.1. *R öyle bir halka olsun ki her P sol primitif ideali için R/P Artin halkası olsun. Bu durumda bir R-modül X $\mathcal{P}$-saf-injektiftir ancak ve ancak öyle bir Y R-modülü varsa ki Y basit R-modüllerin ve injektif modüllerin dik çarpımıdır ve X modülü de Y'nin bir dik toplam terimine izomorftur.*

R öyle bir halka olsun ki her P sol primitif ideali için R/P Artin halkası olsun. Bu durumda, bir R-modül M'de $\mathcal{P}$-saf olan M'nin önemli bir alt modüller sınıfı vardır: tümleyen alt modüller. Hatırlatacak olursak, herhangi bir R-halkası için bir R-modülünün herhangi bir L alt modülüne M'de bir *tümleyen* denir eğer M'nin öyle bir N alt modülü varsa ki $M = N + L$ ve L bu özelliğe göre minimal bir alt modüldür (denk olarak $M = N + L$ ve $N \cap L$, L'de küçük bir alt modüldür).

Teorem 2.2. *R öyle bir halka olsun ki her P sol primitif ideali için R/P Artin halkası olsun. M de herhangi bir R-modül olsun. Bu durumda M'deki her tümleyen alt modül M'nin $\mathcal{P}$-saf bir alt modülüdür.*

Teorem 2.3. *R mükemmel bir halka olsun. Bir R-modül M'nin bir L alt modülü M'de bir tümleyen alt modüldür ancak ve ancak L alt modülü M'nin $\mathcal{P}$-saf bir alt modülü ise.*

Teorem 2.4. *R bir Dedekind bölgesi olsun. Bir R-modül M'nin bir K alt modülü M'de kapalıdır ancak ve ancak K, M'nin $\mathcal{P}$-saf bir alt modülü ise.*

Lemma 2.5. *R bir Dedekind bölgesi olsun. Bu durumda bir R-modül X için aşağıdakiler birbirine denktir:*

(i) *X c-injektiftir.*
(ii) *X $\mathcal{P}$-saf-injektiftir.*
(iii) *X, basit R-modüllerin ve injektif modüllerin dik çarpınımın bir dik toplam terimine izomorftur.*

R bir halka olsun. Bir (sol) R-modül M'nin bir K alt modülüne M'de *saf* denir eğer her (sonlu sunulan) sağ R-modül U için abel gruplarının

$$U \otimes_R K \xrightarrow{1_U \otimes i} U \otimes_R M$$

homomorfizması bir monomorfizma ise (burada $i : K \to M$ içerme homomorfizması ve $1_U : U \to U$ birim homomorfizmadır). Eğer R bir Dedekind bölgesi ise (daha genel olarak bir Prüfer tamlık bölgesi ise) bir (sol) R-modül M'nin bir K alt modülü M'de saftır ancak ve ancak her $a \in R$ için $K \cap aM = aK$ koşulu sağlanırsa. Bir X R-modülüne *saf-injektif* denir eğer her R-modül M ve M'nin her saf alt modülü K için her $\varphi : K \to X$ homomorfizması M'ye genişletilebilirse.

Teorem 2.6. *R bir Dedekind bölgesi olsun. Bir R-modül X c-injektiftir ancak ve ancak X homojen yarı-basit modüller ve injektif modüllerin dik çarpımı ise.*

Teorem 2.7. *R değişmeli bir Noether tamlık bölgesi ve P de R'nin bir maksimal ideali olsun. Bu durumda aşağıdakiler birbirine denktir:*

(i) *R/P modülü c-injektiftir.*
(ii) *$M = R \oplus R$ serbest modülü için R/P modülü c-M-injektiftir.*
(iii) *P tersi olan bir idealdir.*

Sonuç 2.8. *Değişmeli bir Noether tamlık bölgesi R Dedekind bölgesi olur ancak ve ancak her basit R-modül c-injektif ise.*

3. $\mathcal{W}supp$ SINIFI VE İLGİLİ ÖZSINIFLARA GÖRE İNJEKTİF VE PROJEKTİF MODÜLLER

$\mathcal{S}mall$ sınıfı Im(α)' nın B' de küçük alt modül olduğu tüm

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$$

kısa tam dizilerinden oluşur. Zayıf tümleyenlerden yola çıkarak tanımlanan $\mathcal{W}supp$ sınıfı Im(α)' nın B' de bir zayıf tümleyeni olduğu tüm

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$$

kısa tam dizilerinden oluşur. $\mathcal{S}$ sınıfı Im(α)' nın B' de bir tümleyeni bulunduğu ve Zöschinger' in κ-eleman diye tanımladığı tüm

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$$

kısa tam dizilerinden oluşuyor. $\mathcal{SB}$ sınıfı da Im(α)' nın B' de Im(α)$\cap V$ sınırlı modül olacak şekilde bir V tümleyeninin bulunduğu ve Zöschinger' in β-eleman diye tanımladığı tüm

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$$

kısa tam dizilerinden oluşuyor. $\mathcal{S}mall$, $\mathcal{W}supp$, $\mathcal{S}$ ve $\mathcal{SB}$ sınıfları genelde öz sınıf oluşturmayabilir ve birbirinden farklı sınıflardır. Bu sınıfları içeren en küçük öz sınıflarla (bir $\mathcal{A}$ sınıfını içeren en küçük öz sınıf $\langle\, \mathcal{A} \,\rangle$ şeklinde gösterilmiştir) ilgili aşağıdaki sonuca ulaşılmıştır.

Teorem 3.1. *$\langle \mathcal{W}supp \rangle = \langle \mathcal{S}mall \rangle = \langle \mathcal{S} \rangle$.*

Özel olarak R'yi tamlık bölgesi olarak aldığımızda $\mathcal{S}mall$ sınıfıyla ilgili aşağıdaki sonuca ulaşılmıştır.

Teorem 3.2. *Bir R tamlık bölgesi üzerinde her sınırlı modül $\langle \mathcal{S}mall \rangle$-eşinjektiftir.*

Sınırlı modüller sınıfını $\mathcal{B}$ ve bu sınıf tarafından eşinjektif üretilmiş öz sınıfı $\underline{k}(\mathcal{B})$ ile gösterelim.

Sonuç 3.3. *Bir R tamlık bölgesi üzerinde $\underline{k}(\mathcal{B}) \subseteq \langle \mathcal{S}mall \rangle$.*

R halkasına ek koşullar koyarak $\mathcal{SB}$ sınıfı için aşağıdaki sonuca ulaşılmıştır.

Teorem 3.4. *Bir R Noether tamlık bölgesi için, $\mathcal{SB} = \underline{k}(\mathcal{B})$, yani bu durumda $\mathcal{SB}$ bir öz sınıf oluşturur.*

Yukarıdaki sonuç bir R Dedekind bölgesi için de geçerlidir.

Teorem 3.5. *R Dedekind bölgesi üzerinde, $\mathcal{S}$-injektif modüller tam olarak injektif modüllerdir.*

Bir $\mathcal{P}$ sınıfına göre projektif olan tüm modülleri $\pi(\mathcal{P})$ ve R halkası üzerinde burulma modülleri kategorisini $\mathcal{T}_R$ ile gösterelim.

Teorem 3.6. *Bir R Dedekind bölgesi üzerinde, $\mathcal{T}_R$ burulma modülleri kategorisinde aşağıdakiler doğrudur.*

(a) $\text{Ext}(C, A)$*' nın κ-elemanları bir öz sınıf oluşturur.*
(b) $\pi(\mathcal{W}supp) = \pi(\mathcal{S}) = \pi(\mathcal{S}mall) = \{0\}$.
(c) *$\mathcal{S}$-injektif modüller tam olarak $\mathcal{T}_R$ kategorisindeki injektif modüllerdir.*

Teorem 3.7. *R bir Dedekind bölgesi olmak üzere R üzerinde burulma R-modüllerinin oluşturduğu $\mathcal{T}_R$ kategorisinde X modülünün $\mathcal{S}$-eşinjektif olması için X modülünün indirgenmiş kısmının her asal bileşeninin sınırlı olması gerek ve yeterlidir.*

Teorem 3.8. *Bir R Dedekind bölgesi üzerinde, $\mathcal{T}_R$ burulma modülleri kategorisinde aşağıdakiler doğrudur.*

(a) $\text{Ext}(C, A)$*' nın β-elemanları bir öz sınıf oluşturur.*
(b) $\pi(\mathcal{SB}) = \{0\}$.
(c) *$\mathcal{SB}$-injektif modüller tam olarak $\mathcal{T}_R$ kategorisindeki injektif modüllerdir.*

Teorem 3.9. *R bir Dedekind bölgesi olmak üzere R üzerinde burulma R-modüllerinin oluşturduğu $\mathcal{T}_R$ kategorisinde X modülünün $\mathcal{SB}$-eşinjektif olması için X modülünün indirgenmiş kısmının sınırlı olması gerek ve yeterlidir.*

4. DUAL SONLU EŞKAPALI MODÜLLER

Dual sonlu eş kapalı alt modüller, bazı temel özellikleri [6]'te verilen *eş kapalı* alt modüllerin genelleşmesi olarak tanımlanmış ve aşağıdaki gibi sınıflandırılmıştır.

Lemma 4.1. *Bir M modülü ve M'nin bir N alt modülü için aşağıdakiler denktir.*

(i) *N alt mödülü M'nin dual sonlu eş kapalı alt modülüdür.*
(ii) *N alt modülünün $N/K \ll M/K$ olacak şekilde K maximal alt modülü yoktur.*
(iii) *K alt modülü N'nin bir maximal alt modülü ise, M'nin $K = N \cap L$ olacak şekilde bir maximal L alt modülü vardır.*
(iv) *Her basit S modülü $f : N \to M$ içerme homomorfizmasına göre injektifdir, yada denk olarak,* $\text{Hom}(M, S) \to \text{Hom}(N, S) \to 0$ *dizisi tamdır.*

Abel gruplarının düzenli alt grupları kavramı modüllere [26] ve [27]'de genelleştirilmiştir: Modüllerin bir $f : K \to L$ monomorfizmasına *düzenli* denir eğer her basit modül S, $L \to L/\operatorname{Im} f$ projeksiyonuna göre *projektif* ise, yani $\text{Hom}(S, L) \to \text{Hom}(S, L/\operatorname{Im} f) \to 0$ dizisi tam ise. Monomorfizmaları düzenli monomorfizma olan tüm kısa tam diziler bir öz sınıf oluşturmaktadır ve tanımdan dolayı bu öz sınıf tüm basit sol R-modüller tarafından projektif olarak üretilmektedir.

Dual sonlu eş-kapalı alt modüller ve tüm basit sol R-modüller tarafından injektif olarak üretilen öz sınıf ile ilgili aşağıdaki sonuç elde edilmiştir.

Teorem 4.2. $0 \longrightarrow N \stackrel{\alpha}{\longrightarrow} M \longrightarrow L \longrightarrow 0$ *kısa tam dizisinde* $\text{Im}(\alpha)$*' nın M' nin dual sonlu eş-kapalı alt modülü olması için gerek ve yeter koşul her basit R-modülünün bu diziye göre injektif olmasıdır, yani her basit R-modül S için* $\text{Hom}(M, S) \to \text{Hom}(N, S) \to 0$ *dizisi tam ise.*

Eş düzenli monomorfizmaların tanımladığı kısa tam dizilerin sınıfı radikali sıfır olan modüller tarafından injektif olarak üretilen öz sınıftır ve bu öz sınıf tümleyen alt modüllerle tanımlanan öz sınıfı içerir (bakınız [19]). Her basit modülün radikali sıfır olduğundan, her eş düzenli $f : K \to L$ monomorfizması için $f(K)$, L'nin dual sonlu eş-kapalı alt modülüdür. Bu nedenle dual sonlu eş kapalı monomorfizmaların tanımladığı kısa tam diziler sınıfı, eş düzenli monoforfizmaların tanımladığı kısa tam dizilerin sınıfını içermektedir. Söz konusu öz sınıfların denk olduğu bir durum aşağıdaki gibi elde edilmiştir.

Teorem 4.3. *R yarıyerel bir halka ise, eş düzenli monomorfizmaların tanımladığı öz sınıf ile, basit moduller tarafından injektif olarak üretilen öz sınıf (ya da denk olarak, dual sonlu eş kapalı monomorfizmaların oluşturduğu öz sınıf) birbirine denktir. Eğer $R/\mathrm{Jac}(R)$ düzgün (Von Neumann regular) bir halka ise, ifadenin terside doğrudur.*

5. EŞKAPALI ALTMODÜLLERLE TANIMLANAN ÖZ SINIF

R ile birim elemanlı herhangi bir halkayı göstereceğiz. Bütün modüller sol R-modüller olacak. Sol R-modüllerin kategorisini $R\text{-}\mathcal{M}od$ ile göstereceğiz.

Her tümleyen alt modül, eş kapalı alt modüldür. Tümleyen alt modüllerle bu ilişkisinden dolayı eş kapalı alt modüllerle tanımlanan kısa tam dizilerin sınıfını düşünüyoruz. Bunun bir öz sınıf oluşturduğunu gösterdik. Bunun için eş kapalı altmodüllerin bazı özelliklerini kullandık; bu özelliklerin çoğunu, örneğin yakın zamanda çıkmış olan [6, 3.7] monografında bulabilirsiniz. Yakın zamanda çıkmış olan [39, A.4] makalesinde önemli bir özellik verilmiştir.

Eş kapalı alt modüllerle tanımlanan kısa tam dizilerin sınıfını $\mathcal{C}o\text{-}\mathcal{C}losed_{R\text{-}\mathcal{M}od}$ ile göstereceğiz: Bu sınıf R-modüllerin öyle

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

kısa tam dizilerden oluşuyor ki $f(A)$, B'de eş kapalı olsun.

Eş kapalı alt modüllerin bahsettiğimiz özelliklerini kullanarak $\mathcal{C}o\text{-}\mathcal{C}losed_{R\text{-}\mathcal{M}od}$ sınıfının bir öz sınıf oluşturduğunu gözledik.

Öncelikle öz sınıf tanımını da verelim:

$\mathcal{P}$ ile R-modüllerin kısa tam dizilerinden oluşan bir sınıfı gösterelim.

Eğer bir

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

kısa tam dizisi $\mathcal{P}$ sınıfında ise f monomorfizmasına *$\mathcal{P}$-monomorfizma* ve g epimorfizmasına da *$\mathcal{P}$-epimorfizma* diyelim.

$\mathcal{P}$ sınıfına bir *öz sınıf* denir eğer şu özellikleri sağlarsa (bakınız [5], [18, Ch.12, §4], [26, §2] and [25, Introduction]):

- P1. Eğer $\mathbb{E}$, $\mathcal{P}$ sınıfında ise, o zaman $\mathbb{E}$'ye izomorf olan her kısa tam dizi de $\mathcal{P}$ sınıfındadır.
- P2. Parçalanan kısa tam dizilerin hepsi $\mathcal{P}$ sınfındadır.
- P3. İki tane $\mathcal{P}$-monomorfizmanın bileşkesi de $\mathcal{P}$-monomorfizmadır, tabi ki bileşke tanımlı ise. İki tane $\mathcal{P}$-epimorfizmanın bileşkesi de $\mathcal{P}$-epimorfizmadır, tabi ki bileşke tanımlı ise.

P4. Eğer g ve f monomorfizma ve $g \circ f$ $\mathcal{P}$-monomorfizma ise, o zaman f de $\mathcal{P}$-monomorfizmadır. Eğer g ve f epimorfizma ve $g \circ f$ $\mathcal{P}$-epimorfizma ise, o zaman g de $\mathcal{P}$-epimorfizmadır.

Eş kapalı alt modül tanımını da verelim.

Tanım 5.1. $K \subseteq L \subseteq M$ alt modülleri için, $K \subseteq L$ içermesine M'de *eşküçük* denir ve bu $K \underset{M}{\overset{cs}{\hookrightarrow}} L$ ile gösterilir eğer $L/K \ll M/K$ (yani L/K, M/K'nın küçük alt modülü ise).

Tanım 5.2. Bir M modülünün bir L alt modülüne, M'de *eşkapalı* denir ve bu $L \overset{cc}{\hookrightarrow} M$ ile gösterilir eğer L'nin hiçbir öz alt modülü K yoksa ki $K \underset{M}{\overset{cs}{\hookrightarrow}} L$ olsun.

Eş kapalı alt modüllerle ilgili kullandığımız özellikler şunlardır:

Önerme 5.3. *[6, 3.7] $K \subseteq L \subseteq M$ alt modüllerini alalım.*

(i) *Eğer $L \overset{cc}{\hookrightarrow} M$ ise, o zaman $L/K \overset{cc}{\hookrightarrow} M/K$ olur.*

(ii) *Eğer $K \overset{cc}{\hookrightarrow} M$ ise, o zaman $K \overset{cc}{\hookrightarrow} L$ olur. Tersi de doğrudur eğer $L \overset{cc}{\hookrightarrow} M$ olursa.*

Lemma 5.4. *[39, Lemma A.4] $K \subseteq L \subseteq M$ altmodüllerini alalım. Eğer $K \overset{cc}{\hookrightarrow} M$ and $L/K \overset{cc}{\hookrightarrow} M/K$ ise, o zaman $L \overset{cc}{\hookrightarrow} M$ olur..*

Bu özellikleri kullanarak öz sınıf tanımında istenen P1, P2, P3 ve P4 özelliklerinin sağlandığını göstererek şunu elde ederiz:

Teorem 5.5. *$\mathcal{C}o$-$\mathcal{C}losed_{R\text{-}\mathcal{M}od}$ sınıfı, bir* **öz sınıf** *oluşturur.*

6. $\mathcal{N}eat_{R-\mathcal{M}od}$-EŞINJEKTİF VE BASİT PROJEKTİF MODÜLLER

R ile birim elemanlı herhangi bir halkayı göstereceğiz. Bütün modüller sol R-modüller olacak. Sol R-modüllerin kategorisini R-$\mathcal{M}od$ ile göstereceğiz.

[34] makakesinde tanımlanıp incelenen maks-injektif modüllerin aslında $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ öz sınıfına göre eşinjektif modüller olduğunu gözledik. Bu sayede ilgili bazı sonuçları genelleme ve ilgili başka sorular ortaya koymamız mümkün oldu. Buna dual olarak basit-projektif modülleri de tanımladık ve aslında bunlar da dual sonlu eş kapalı alt modüllerle tanımlanan öz sınıfa göre eşprojektif modüller oluyorlar. Aşağıda gerekli tanımları verip elde ettiğimiz sonuçları özetleyeceğiz.

Tanım 6.1. $\mathcal{P}$ bir öz sınıf olsun. Bir C modülüne $\mathcal{P}$*-eşprojektif* denir eğer

$$0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 0$$

şeklindeki C ile *biten* modüllerin bütün kısa tam dizileri $\mathcal{P}$ öz sınıfında ise. Bir A modülüne $\mathcal{P}$*-eşinjektif* denir eğer

$$0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 0$$

şeklindeki A ile *başlayan* modüllerin bütün kısa tam dizileri $\mathcal{P}$ öz sınıfında ise.

Tanım 6.2. $\mathcal{M}$, modüllerin bir sınıfı olsun.

$$\pi^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} |\, \mathrm{Hom}(M, \mathbb{E}) \text{ is exact for all } M \in \mathcal{M}\}$$

sınıfı $\mathcal{M}$ tarafından *projektif olarak üretilen* öz sınıftır.

$$\iota^{-1}(\mathcal{M}) = \{\mathbb{E} \in \mathcal{A}bs_{R\text{-}\mathcal{M}od} |\, \mathrm{Hom}(\mathbb{E}, M) \text{ is exact for all } M \in \mathcal{M}\}.$$

sınıfı $\mathcal{M}$ tarafından *injektif olarak üretilen* öz sınıftır.

Maks-injektif modül tanımı injektif modül tanımının bir genellemesidir:

Tanım 6.3. [34] M bir R-modül olsun. M modülüne *maksimallere göre injektif* veya kısaca *maks-injektif* denir eğer R halkasının her maksimal sol ideali P için her $f : P \to M$ homomorfizması bir $g : R \to M$ homomorfizmasına genişletilebilirse:

$$\begin{array}{ccc} P & \xhookrightarrow[max.]{} & R \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\ M & & \end{array}$$

Dikkat edilirse bu injektif modüller için olan Baer kriterinde halkanın her sol ideali alınması yerine her sol maksimal ideali alınması ile elde edilmiş bir tanımdır.

$\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ öz sınıfı basit R-modüller tarafından projektif olarak üretilen öz sınıftır.
Özel olarak gözlediğimiz şu sonuç aslında projektif olarak üretilen öz sınıflar için genel olarak verilebilen bir sonuçtur (bakınız [25, Proposition 9.5]):

Önerme 6.4. *Bir M modülü $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-eşinjektif bir modüldür ancak ve ancak her basit S modülü için* $\mathrm{Ext}^1_R(S, M) = 0$ *ise.*

Bu koşul da max-injektif olmaya denktir (bakınız [34, Proposition 2.2]); şu da kolayca gösterilir:

Önerme 6.5. *Bir M modülü için aşağıdakiler denktir:*

(i) *M maks-injektif bir modüldür.*
(ii) *Her basit S modülü için* $\mathrm{Ext}^1_R(S, M) = 0$.
(iii) *M modülü, basit modüllerle biten bütün kısa tam dizilere göre injektiftir: S basit modül ise şu diagram değişmeli olarak tamamlanabilmektedir:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & S & \longrightarrow & 0 \\ & & {\scriptstyle \alpha}\downarrow & \swarrow {\scriptstyle \tilde{\alpha}} & & & & & \\ & & M & & & & & & \end{array}$$

Sonuç 6.6. *Bir M modülü maks-injektiftir ancak ve ancak M modülü $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-eşinjektif bir modül ise.*

Bir R halkasına *sol C-halkası* denir eğer her B (sol) R-modülü ve B'nin her büyük öz alt modülü A için $\mathrm{Soc}(B/A) \neq 0$ ise, yani B/A modülünün basit bir alt modülü varsa (bakınız [28]).

Örneğin, bir Dedekind halkası bir C-halkasıdır. Dolayısı ile esas tamlık bölgeleri de C-halkalarıdır.

Önerme 6.7. *[28, Proposition 1.2] Bir R halkası için şunlar denktir:*

(1) R bir C*-halkasıdır.*

(2) R halkasının her büyük öz sol ideali için $\mathrm{Soc}(R/I) \neq 0$.

Bir R halkasına *sol yarı-Artin halka* denir eğer R halkasının her öz sol ideali için $\mathrm{Soc}(R/I) \neq 0$. Tabi bu durumda sol yarı-Artin halkalar aynı zamanda C-halkalarıdır.

Önerme 6.8. *[34, Theorem 3.1] Eğer R bir sol yarı-Artin halka ise, bu durumda her sol maks-injektif R-modül aslında inkektif modüldür.*

$\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ öz sınıfı bütünleyen alt modüllerle tanımlanan öz sınıftır. $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ öz sınıfı her zaman için $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ öz sınıfını içerir ve [12]'de gösterildiği üzere bu iki öz sınıf birbirine eşittir ancak ve ancak R bir C-halkası ise. $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ öz sınıfına göre eşinjektif olan modüller sadece injektif modüllerdir. Dolayısıyla şu sonucu daha önce biliyorduk zaten:

Önerme 6.9. *[19, Proposition 3.7.4] R bir* C*-halkası ise, bir R-modülü M injektiftir ancak ve ancak bütün basit S modülleri için* $\mathrm{Ext}^1_R(S, M) = 0$ *ise, yani M maks-injektif bir modül ise (Önerme 6.5'den dolayı).*

[9, Theorem 4.4.1]'de , şu gösterilmiştir: bir tamlık bölesi R için şunlar denktir:

(i) Sıfırdan farklı burulmalı her R-modülünün basit alt modülü vardır.

(ii) Bir M R-modülü injektiftir ancak ve ancak her basit S modülü için $\mathrm{Ext}^1_R(S, M) = 0$ ise.

Bir tamlık bölgesi C halkasıdır ancak ve ancak sıfırdan farklı burulmalı her R-modülünün basit alt modülü varsa (bakınız orengin [19, Proposition 3.3.9]). Dolayısı ile [9, Theorem 4.4.1] sonucunu şu şekilde ifade edebiliriz.

Sonuç 6.10. *R bir tamlık bölgesi olsun. R bir* C*-halkasıdır ancak ve ancak bütün $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-eşinjektif modüller (yani maks-injektif modüller) sadece injektif modüllerden oluşuyorsa.*

Bu problem de üzerinde çalıştığımız genel problemin özel halidir. Sorumuz bütün $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-eşinjektif modüllerin (yani maks-injektif modüllerin) sadece injektif modüllerden oluştuğu R halkalarını karakterize etmek. Yukardaki sonuç, tamlık bölgeleri arasında cevabı C-halkaları olarak karakterize ediyor. Amacımız bunun daha geniş bir halkalar sınıfında da halen doğru olup olmayacağını anlamaktır. Daha önce belirttiğimiz üzer eğer R bir C-halkası ise $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-eşinjektif modüllerin (yani maks-injektif modüllerin) sadece injektif modüllerden oluştuğunu biliyoruz. Yani tersinin doğru olup olmadığını veya hangi halkalar sınıfında doğru olacağını bulmayı amaçlıyoruz.

Maks-injektif kavramına dual olarak *basit-projektif modülleri* tanımlayıp bunlarla ilgili özellikleri incelemeyi planlamaktayız:

Tanım 6.11. Bir M modülüne *basit-projektif* diyelim eğer M modülü, çekirdeği basit modül olan bütün modül epimorfizmalarına göre projektifse, yani S'nin basit modül olduğu şu şekildeki diagramlar değişmeli olarak tamamlanabiliyorsa:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & M & & \end{array}$$

Tabi kolayca şunu gösterebiliriz:

Önerme 6.12. *Bir M modülü için şunlar denktir:*

(i) *M basit-projektif bir modüldür.*
(ii) *Her basit S modülü için $\mathrm{Ext}^1_R(M,S)=0$ sağlanır.*

Öz sınıflarla ilgili şu sonucu kullanacağız:

Önerme 6.13. *(bakınız örneğin [19, Proposition 2.6.5]) Modüllerin bir $\mathcal{M}$ sınıfı tarafından injektif olarak üretilen $\mathcal{P}=\iota^{-1}(\mathcal{M})$ öz sınıfını alalım. Bir R-modülü C için şunlar denktir:*

(1) C modülü $\mathcal{P}$-eşprojektiftir.
(2) $\mathcal{M}$ sınıfındaki bütün M modülleri için $\mathrm{Ext}^1_R(C,M)=0$ sağlanır.

Maks-injektif modüllerdekine benzer biçimde şu sonucu elde ederiz:

Önerme 6.14. *Basit modüller tarafından injektif olarak üretilen*

$$\mathcal{P}=\iota^{-1}(\{\textit{bütün basit } R\textit{-modüller}\})$$

öz sınıfı için, bir M modülü basit-projektiftir ancak ve ancak M modülü $\mathcal{P}$-eşprojektif ise.

Basit-projektif modülleri de aslında bu sonuç nedeni ile tanımlayıp incelemek istemekteyiz çünkü basit modüller tarafından injektif olarak üretilen $\mathcal{P}=\iota^{-1}(\{$bütün basit R-modüller$\})$ öz sınıfı ilgilendiğimiz bir öz sınıftır ve bu proje raporunun eklerinde açıkladığımız dual sonlu eşkapalı alt modüllerle tanımlanan öz sınıftır. Bu özsınıfın eşprojektif modüllerinin ne zaman sadece projektif modüllerden oluştuğunu anlamak istemekteyiz. Maks-injektif modüllerde olan bazı sonuçların basit-projektif modüller için karşılığı olup olmadığına bakacağız. Basit-projektif modüllerin sadece projektif modüllerden oluştuğu halkaları en azından bazı halka sınıfları arasında karakterize etmeye çalışacağız.

7. $\overline{\mathcal{W}supp}$ ÖZ SINIFI VE $\overline{\mathcal{W}supp}$-EŞİNJEKTİF MODÜLLER

Daha önceki dönemlerde incelediğimiz, $\mathrm{Im}(\alpha)$' nın B' de küçük alt modül olduğu tüm $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ kısa tam dizilerinden oluşan $\mathcal{S}mall$ sınıfı, $\mathrm{Im}(\alpha)$' nın B' de bir zayıf tümleyeni olduğu tüm kısa tam dizilerden oluşan $\mathcal{W}supp$ sınıfı ve $\mathrm{Im}(\alpha)$' nın B' de bir tümleyeni bulunduğu ve Zöschinger' in κ-eleman diye tanımladığı tüm kısa tam dizilerden oluşan $\mathcal{S}$ sınıfı genelde öz sınıf oluşturmayabilir ve birbirinden farklı sınıflardır. Öte yandan bunların ürettikleri, yani bunları içeren en küçük öz sınıflar eşittir: $\langle\mathcal{S}mall\rangle=\langle\mathcal{S}\rangle=\langle\mathcal{W}supp\rangle$ (burada bir $\mathcal{A}$ sınıfını içeren en küçük öz sınıf $\langle\,\mathcal{A}\,\rangle$ şeklinde gösterilmiştir). Bu dönemki çalışmalarımız esasen bu öz sınıf üzerine yoğunlaşmıştır.

Bir $f : A \longrightarrow A'$ homomorfizması için, $f^* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C, A')$ homomorfizması $\mathrm{Ext}(C, A)$'nın $\mathcal{W}supp$ sınıfına ait elemanlarını $\mathrm{Ext}(C, A')$'nün $\mathcal{W}supp$ sınıfına ait elemanlarına götürmektedir. Fakat aynı sonuç bir $g : C' \longrightarrow C$ homomorfizması yardımıyla tanımlanan $g_* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C', A)$ homomorfizması için geçerli değildir. $\mathcal{W}supp$ sınıfına, bu sınıftaki elemanların birinci değişkene göre alınan görüntülerini de ekleyerek oluşturulan sınıfı $\overline{\mathcal{W}supp}$ ile gösterelim. Bu dönem elde ettiğimiz esas sonuç $\overline{\mathcal{W}supp}$ sınıfının $\mathcal{S}mall$, $\mathcal{W}supp$ ve $\mathcal{S}$ sınıflarını içeren en küçük öz sınıf olduğunu kanıtlamaktır.

Teorem 7.1. *Bir R kalıtsal halkası üzerinde $\overline{\mathcal{W}supp}$ sınıfı bir öz sınıftır.*

Bu teoremi kanıtlamak için aşağıdaki lemmaların doğru olduğunu gösterdik.

Lemma 7.2. *Bir $f : A \longrightarrow A'$ homomorfizması için, $f^* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C, A')$ homomorfizması $\mathrm{Ext}(C, A)$'nın $\overline{\mathcal{W}supp}$ sınıfına ait elemanlarını $\mathrm{Ext}(C, A')$'nün $\overline{\mathcal{W}supp}$ sınıfına ait elemanlarına götürmektedir.*

Lemma 7.3. *Bir $g : C' \longrightarrow C$ homomorfizması için, $g_* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C', A)$ homomorfizması $\mathrm{Ext}(C, A)$'nın $\overline{\mathcal{W}supp}$ sınıfına ait elemanlarını $\mathrm{Ext}(C', A)$'nın $\overline{\mathcal{W}supp}$ sınıfına ait elemanlarına götürmektedir.*

Lemma 7.4. *$\mathrm{Ext}(C, A)$'nın $\overline{\mathcal{W}supp}$ sınıfına ait elemanları bir alt grup oluşturur.*

Lemma 7.5. *İki $\overline{\mathcal{W}supp}$-monomorfizmanın bileşkesi $\overline{\mathcal{W}supp}$-monomorfizmadır.*

$\langle \mathcal{W}supp \rangle$ sınıfı $\mathcal{W}supp$'ı içeren bir öz sınıf olduğundan $\overline{\mathcal{W}supp}$ sınıfını da içermektedir. Öte yandan teoremden dolayı $\overline{\mathcal{W}supp}$ bir öz sınıf olduğundan $\langle \mathcal{W}supp \rangle \subseteq \overline{\mathcal{W}supp}$'dır, dolayısıyla aşağıdaki sonucu elde ederiz.

Sonuç 7.6. $\langle \mathcal{S}mall \rangle = \langle \mathcal{S} \rangle = \langle \mathcal{W}supp \rangle = \overline{\mathcal{W}supp}$.

Bir sınıfın injektif modülleri bu sınıfın ürettiği öz sınıfın injektif ve projektif modülleri ile aynı olduğundan aşağıdaki sonucu elde ederiz.

Sonuç 7.7. *R Dedekind bölgesi üzerinde, $\overline{\mathcal{W}supp}$-injektif modüller tam olarak injektif modüllerdir.*

R Dedekind bölgesi üzerinde $\overline{\mathcal{W}supp}$-eşinjektif modüller ile ilgili aşağıdaki sonuçlar elde edilmişir.

Teorem 7.8. *R Dedekind bölgesi üzerinde bir M modülü için aşağıdaki koşullar denktir.*
- (a) *M modülü $\overline{\mathcal{W}supp}$-eşinjektiftir.*
- (b) *M modülü $\mathcal{W}supp$-eşinjektiftir.*
- (c) *M/N injektif ve M'nin injektif bürümünde küçük olacak şekilde M'nin bir N altmodülü vardır.*

$\overline{\mathcal{W}supp}$-eşinjektif modüller için bu kriteri kullanarak Dedekind bölgesi üzerinde bazı özel modüllerin $\overline{\mathcal{W}supp}$-eşinjektif olup olmadığı söylenebilir.

Sonuç 7.9. *Bir A modulünün herhangi bir monomorfizma altındaki görüntüsü küçük ise A modülü $\overline{\mathcal{W}supp}$-eşinjektiftir.*

Sonuç 7.10. *Dedekind halkası üzerinde her eş atomik modül* $\overline{\mathcal{W}supp}$*-eşinjektiftir.*

Sonuç 7.11. *Dedekind bölgesi üzerinde her sonlu üretilmiş modül* $\overline{\mathcal{W}supp}$*-eşinjektiftir.*

Gözlem 7.12. İndirgenmiş (yani bölünebilir alt grubu olmayan) $\overline{\mathcal{W}supp}$-eşinjektif modüller eş atomik olmayabilir. Örneğin Abel grupları kategorisinde J_p, p-sel sayılar grubu $\overline{\mathcal{W}supp}$-eşinjektiftir, fakat eş atomik değildir. Başka bir örnek: Rasyonel sayılar grubunda paydaları 1'den büyük sayıların karesine bölünmeyen sayıların oluşturduğu alt grup (p asal sayı olmak üzere $1/p$ şeklindeki sayıların oluşturduğu alt grup) da $\overline{\mathcal{W}supp}$-eşinjektiftir, fakat eş atomik değildir.

Öte yandan $\overline{\mathcal{W}supp}$-eşinjektif burulma grupları tam olarak betimlenmiştir.

Önerme 7.13. *R Dedekind bölgesi üzerinde bir A indirgenmiş ve burulma modülü için aşağıdaki koşullar denktir.*

(a) *A is coatomic.*
(b) *Her p asal sayısı için A'nın p-bileşeni sınırlıdır.*
(c) *A is* $\overline{\mathcal{W}supp}$*-eşinjektiftir.*

$\overline{\mathcal{W}supp}$-eşinjektif modüllerle ilgili aşağıdaki sonuç da elde edilmiştir.

Önerme 7.14. *M modülü* $\overline{\mathcal{W}supp}$*-eşinjektif ise, M'nin burulma kısmının radikalinin M'de bir tümleyeni vardır.*

$\overline{\mathcal{W}supp}$-eşprojektif modüllerle ilgili aşağıdaki sonuç elde edilmiştir.

Önerme 7.15. *Sonlu gösterilen her modül* $\overline{\mathcal{W}supp}$*-eşprojektiftir.*

Öte yandan örneğin Abel grupları kategorisinde sınırlı gruplar $\overline{\mathcal{W}supp}$-eşprojektif olmayabilir. Dolayısıyla $\overline{\mathcal{W}supp}$ sınıfı $\mathrm{Ext}(C, A)$ grubunun burulma altgrubuna karşılık gelen $Text$ sınıfını içermeyebilir, bu da $\mathrm{Ext}(C, A)$ grubunun $\overline{\mathcal{W}supp}$ sınıfına karşılık gelen altgrubunun ve bu sınıfın global boyutunun incelenmesini zorlaştırıyor.

8. EŞ ATOMİK TÜMLEYEN ALTMODÜLLER

R bir kalıtsal halka, M bir R-modül ve U bir altmodül olsun. $M = U + V$ ve $U \cap V$ eş atomik olacak şekilde bir V altmodülü varsa U ya V nin *eş atomik tümleyeni* denir. $\mathrm{Ext}_R(C, A)$ da bir $\mathbb{E} : 0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ kısa tam dizisine σ-tam denir eğer $\mathrm{Im}\,\alpha$ B de eş atomik tümleyen ise.

Bu bölümde, Σ ile göstereceğimiz tüm σ-tam dizilerin sınıfını inceleyeceğiz.

Lemma 8.1. (i) *$f : A \longrightarrow A'$ bir homomorfizma ise, $f_* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C, A')$ altında σ-elemanlar korunur.*

(ii) *$g : C' \longrightarrow C$ bir homomorfizma ise, $g^* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C', A)$ altında σ-elemanlar korunur.*

Kanıt. (i) Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, $\mathrm{Ext}(C, A)$ da bir kısa tam dizi ve $f : A \longrightarrow A'$ keyfi bir homomorfizm olsun. $f_*(E) = E_1$ olmak üzere aşağıdaki diagram değişmelidir ve satırları tamdır.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow[\alpha]{} & B & \longrightarrow & C & \longrightarrow & 0 : E \\ && {\scriptstyle f}\downarrow && {\scriptstyle f'}\downarrow && \| && \\ 0 & \longrightarrow & A' & \xrightarrow[\alpha']{} & B' & \longrightarrow & C & \longrightarrow & 0 : E_1 \end{array}$$

Eğer V, $\operatorname{Im}\alpha$ nın B de bir eş atomik tümleyeni ise, o zaman $\operatorname{Im}\alpha + V = B$ ve $V \cap \operatorname{Im}\alpha$ eş atomiktir. Böylece, pushout diagramından, $f'(V)+\operatorname{Im}\alpha' = B'$ elde ederiz ve $f'(V)\cap\operatorname{Im}\alpha' = f'(V\cap\operatorname{Im}\alpha)$ eş atomiktir, çünkü $V\cap\operatorname{Im}\alpha$ eş atomiktir ve eş atomik bir modülün homomorfik görüntüsü de eş atomiktir.

(ii) $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, $\operatorname{Ext}(C, A)$ da bir kısa tam dizi ve $g : C' \longrightarrow C$ keyfi bir homomorfizma olsun. $g^*(E) = E_1$ olmak üzere aşağıdaki diagram değişmelidir ve satırları tamdır.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow[\alpha']{} & B' & \xrightarrow[\beta']{} & C' & \longrightarrow & 0 : E_1 \\ && \| && {\scriptstyle g'}\downarrow && {\scriptstyle g}\downarrow && \\ 0 & \longrightarrow & A & \xrightarrow[\alpha]{} & B & \xrightarrow[\beta]{} & C & \longrightarrow & 0 : E \end{array}$$

V, $\operatorname{Ker}\beta$ nın B de bir eş atomik tümleyeni olsun. Yani, $\operatorname{Ker}\beta + V = B$ ve $V \cap \operatorname{Ker}\beta$ eş atomik olsun. O zaman, pullback diagramından, $g'^{-1}(V) + \operatorname{Ker}\beta' = B'$ elde ederiz. Çünkü g' bize $D' = g'^{-1}(V) \cap \operatorname{Ker}\beta'$ ve $D = V \cap \operatorname{Ker}\beta$ arasında bir izomorfizma verir ve eş atomik bir modülün epimorfik görüntüsü de eş atomiktir. Böylece, D' eş atomiktir. □

Sonuç 8.2. $\operatorname{Ext}(C, A)$ *nın bir σ-elemanın katı da yine bir σ-elemandır.*

Teorem 8.3. *σ-elemanların sınıfı Σ ile $\overline{\mathcal{W}supp}$-elemanların sınıfı $\overline{\mathcal{W}supp}$ çakışıktır.*

Kanıt. Kabul edelim ki A'nın B de bir eş atomik tümleyeni olsun. O zaman, B nin bir altmodülü V vardır öyleki $B = A + V$ ve $A \cap V$ eş atomiktir. Böylece, aşağıdaki diagram değişmelidir ve satırları tamdır:

$$\begin{array}{ccccccccc} && 0 && 0 &&&& \\ && \downarrow && \downarrow &&&& \\ && A\cap V & = & A\cap V &&&& \\ && \downarrow && \downarrow &&&& \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 : E \\ && \downarrow && {\scriptstyle \gamma}\downarrow && \| && \\ 0 & \longrightarrow & A/A\cap V & \longrightarrow & B/A\cap V & \xrightarrow{\alpha} & C & \longrightarrow & 0 : E_1 \\ && \downarrow && \downarrow &&&& \\ && 0 && 0 &&&& \end{array}$$

Açıkça, α bir $\mathcal{S}plit$-epimorfizmadır ve eş atomik moduller $\overline{\mathcal{W}supp}$-eşinjectif olduğundan, γ bir $\overline{\mathcal{W}supp}$-epimorfizmadır. Böylece, $\alpha \circ \gamma$ bir $\overline{\mathcal{W}supp}$-epimorfizmadır, ve E bir $\overline{\mathcal{W}supp}$-elemandır. Tersini kanıtlamak için, kabul edelim ki $E \in \overline{\mathcal{W}supp}$ olsun. O zaman $\mathcal{W}supp$ sınıfında bir E_1 vardır öyle ki aşağıdaki diagram değişmelidir ve satırları tamdır:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & C & \longrightarrow & 0 : E \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha'} & B' & \longrightarrow & C' & \longrightarrow & 0 : E_1 \end{array}$$

Eğer V, $\operatorname{Im}\alpha'$ nın B''de bir zayıf tümleyeni ise, o zaman $\operatorname{Im}\alpha' + V = B'$ ve $\operatorname{Im}\alpha' \cap V \ll B'$ dır. Böylece $\operatorname{Im}\alpha' \cap V$ eş atomiktir [37, Lemma 3.3] ve Lemma 8.1(ii) den dolayı, E bir σ-elemandır. □

R bir ayrık değer halkası, $K \neq R$ R nin kesirler cismi ve (p) de maksimal ideali olsun. Eğer A, B'nin eş atomik altmodülü ise B'de küçük olmak zorunda değildir, fakat $B/\operatorname{Rad}(B)$ yarıbasit olduğundan, ve

$$X/\operatorname{Rad}(B) \oplus (A + \operatorname{Rad}(B))/\operatorname{Rad}(B) = B/\operatorname{Rad}(B)$$

olduğundan, $X \cap A \ll B$ olmak üzere $X + A = B$ elde ederiz . Böylece, her eş atomik altmodülün bir zayıf tümleyeni vardır.

Lemma 8.4. *Bir ayrık değer halkası üzerinde $\mathcal{W}supp$ bir özsınıf oluşturur.*

Kanıt. A'nın B'de eş atomik tümleyeni olduğunu varsayalım. O zaman B'nin bir V altmodülü vardır öyleki $B = A + V$ ve $A \cap V$ eş atomiktir. O halde aşağıdaki diagram değişmelidir ve satırları tamdır:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & A \cap V & = & A \cap V & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 : E \\ & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \| & & \\ 0 & \longrightarrow & A/A \cap V & \longrightarrow & B/A \cap V & \xrightarrow{\alpha} & C & \longrightarrow & 0 : E_1 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

$A \cap V$ eş atomik olduğundan, γ bir $\mathcal{W}supp$-epimorfizmadır. O zaman, $\alpha \circ \gamma$ bileşkesi $\mathcal{W}supp$-epimorfizmadır. Böylece, E bir $\mathcal{W}supp$-elemandır. □

9. MAKSİMAL ALTMODÜLLERİ TÜMLEYEN OLAN MODÜLLER

[1] de maksimal altmodüllerinin tümleyenleri olan altmodüller incelenmiştir. [1] deki sonuçlardan yola çıkarak maksial altmodülleri tümleyen olan ve ayrıca, maksimal altmodülleri dik toplam olan modülleri inceledik.

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MODULES WHOSE MAXIMAL SUBMODULES ARE SUPPLEMENTS

ENGIN BÜYÜKAŞIK AND DILEK PUSAT-YILMAZ

Abstract. We study modules whose maximal submodules are supplements (direct summands). For a locally projective module, we prove that every maximal submodule is a direct summand if and only if it is semisimple and projective. Over a commutative domain, every maximal submodule of a torsion module is a direct summand if and only if every maximal ideal is idempotent and every nonzero proper ideal is an intersection of finitely many maximal ideals.

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1. Introduction

Let R be a unitary ring and M be a left R-module. A submodule N of M is called a *supplement* if there exists another submodule L such that N is minimal with respect to the property that $N + L = M$. This is equivalent to $N + L = M$ and $N \cap L \ll N$. A module M is called *supplemented* if every submodule has a supplement. Several authors have been recently attracted by different generalizations of supplemented modules. An interesting example of this situation has been studied in [1], where modules M in which the kernel of any epimorphism from M to a finitely generated module has a supplement are studied. These modules are characterized as modules whose maximal submodules have supplements, (see [1, Theorem 2.8]). Motivated by these results, we study in this paper some dual notions. Namely, modules in which any maximal submodule is a supplement, and modules in which any maximal submodule is a direct summand. For the sake of brevity, we call them *ms-modules* and *md-modules*, respectively.

We begin by studying some basic properties of md-modules. In particular, we show that homomorphic images and that a module M containing an md-module L is also md provided that L is not contained in any maximal submodule of M (Proposition 2.2). In general, md-modules need not be closed under extensions. But we show that M is an md-module provided that L and M/L are md-modules where L is a closed submodule of M. These basic results allow us to characterize semilocal rings as those rings in which any module with zero Jacobson radical is an md-module.

In Section 3, we study locally projective md-modules. Locally projective modules were introduced by Huisgen-Zimmermann in [20] and they coincide with the flat strict Mittag-Leffler modules in the sense of Raynaud and Gruson (see [10]). These modules are closely

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related to pure submodules of direct products of free modules (see [20]). And it has been recently observed by several authors that there exists a strong connection between the existence of nontrivial locally projective modules in the functor category of a ring (in the sense that they are not projective) and the construction of separable modules and the pure semisimplicity of certain subcategories of modules over the ring (see e.g. [8, 9, 11, 12, 21]). In particular, it is proved in [21] that any ring R which is not left perfect has locally projective left modules which are not projective. Motivated by these relations, we show in Section 3 that any locally projective md-module is semisimple projective. In particular, we deduce that any projective md-module is semisimple.

In Section 4, we characterize the coatomic modules whose maximal submodules are supplement (Theorem 4.3). As a consequence for a module M over a left perfect ring, we prove that every maximal submodule of M is a supplement if and only if $\operatorname{Rad} K = \operatorname{Rad} M$ for every maximal submodule K of M.

In Section 5, we prove that the class of ms-modules is strictly larger than class of md-modules. We close this paper by studying md-modules over commutative domains. We show that any (cyclic) torsion module over a commutative domain is an md-module if and only if any maximal ideal is idempotent and any ideal is a finite intersection of finitely many maximal ideals. Zöschinger proved that over a Dedekind domain, a submodule of a module is closed if and only it is coclosed. Using this result we obtain that ms-modules and md-modules coincide over Dedekind domains. This allows us to determine completely the structure of md-modules over Dedekind domains.

Throughout this paper, R will be an associative ring with identity and all modules are unital left R-modules. By $N \subseteq M$ we shall mean that N is a submodule of M. Let $L \subseteq M$, L is said to be *small* in M, denoted by $L \ll M$, if $L + K \neq M$ for every proper submodule $K \subseteq M$. Dually, a submodule $L \subseteq M$ is called *essential* in M, denoted by $L \trianglelefteq M$, if $L \cap K \neq 0$ for every nonzero $K \subseteq M$. By $\operatorname{Rad} M$ and $\operatorname{Soc}(M)$, we denote the Jacobson radical and the socle of M, respectively. A submodule L of M is called *closed* in M if $L \trianglelefteq K$ for some $K \subseteq M$, implies $L = K$. Dually, a submodule N of M is called *coclosed* in M if $N/K \ll M/K$ implies $K = N$ for every submodule K of N.

It is easy to see that a maximal submodule of a module is either essential or a direct summand. Therefore a module is an md-module if and only if every maximal submodule is a closed submodule.

2. Modules whose maximal submodules are direct summands

Let M and N be R-modules. N is said to be an *M-generated* module if there is an epimorphism $f : M^{(\Lambda)} \to N$ for some index set Λ.

Some properties of md-modules are given in the following proposition.

Proposition 2.1. *Let M be an md-module. Then*

(1) *every homomorphic image of M is an md-module,*
(2) *every direct summand of M is an md-module,*
(3) *an arbitrary sum of md-modules is an md-module,*
(4) *every M-generated module is an md-module.*

Proof. (1) follows directly, and (2) is a consequence of (1).

(3) Let $M = \sum_{i\in I} M_i$ where M_i is an md-module for each $i \in I$. Let K be a maximal submodule of M. Then $M_i \nsubseteq K$ for some $i \in I$. Then $M = M_i + K$, and so $M_i \cap K$ is a maximal submodule of M_i. Since M_i is an md-module, there is a submodule $L \subseteq M_i$ such that $M_i = L \oplus M_i \cap K$ for some $L \subseteq M$. Then it is straightforward to see that the sum $M = K + L$ is direct. Hence M is an md-module.

(4) follows from (1) and (3). □

Proposition 2.2. *Let M be an R-module and $N \subseteq M$. Suppose N is an md-module and M/N has no maximal submodules. Then M is an md-module.*

Proof. Let K be a maximal submodule of M. If $N \subseteq K$, then K/N would be a maximal submodule of M/N which is impossible, so we must have $M = N + K$. Since $M/K \cong N/(N \cap K)$ is simple, $N \cap K$ is a maximal submodule of N. Since N is an md-module, $N \cap K \oplus L = N$ for some simple submodule $L \subseteq N$. Then $M = K + N = K + K \cap N + L = K + L$. Since L is simple, $K \cap L = 0$. That is, K is a direct summand of M, and so M is an md-module. □

Let M be a module with no maximal submodules, i.e. if $\operatorname{Rad} M = M$, then M is an md-module (take $N = 0$ in the above Proposition).

In general, a submodule of an md-module need not be an md-module. For example, the $\mathbb{Z}$-module ${}_{\mathbb{Z}}\mathbb{Q}$ is an md-module, because it has no maximal submodules. On the other hand, ${}_{\mathbb{Z}}\mathbb{Q}$ does not contain any nonzero proper md-submodule, because every submodule of ${}_{\mathbb{Z}}\mathbb{Q}$ is essential in ${}_{\mathbb{Z}}\mathbb{Q}$. We have the following result for particular submodules.

Proposition 2.3. *Let M be an md-module. Then any coclosed submodule N of M with $\operatorname{Soc}(M) \subseteq N$ is an md-module.*

Proof. Let K be a maximal submodule of N. Since N is coclosed, we have $N/K + T/K = M/K$ for some proper submodule $T/K \subseteq M/K$. Then $(N/K) \cap (T/K) = 0$ because N/K is a simple module. Now we get $M/K = N/K \oplus T/K$ and so $N \cap T = K$. Then $N/K \cong M/T$ is also simple, hence T is a maximal submodule of M. Since M is an md-module, $M = T \oplus S$ for some simple submodule S of M. Then $S \subseteq \operatorname{Soc}(M) \subseteq N$. By modular law, we get $N = N \cap T \oplus S = K \oplus S$. That is, K is a direct summand of N. Hence N is an md-module. □

Let M be an R-module. If U and M/U are md-modules for some $U \subset M$, then M need not be an md-module. To see this, let p be a prime integer and $M = \mathbb{Z}/p^2\mathbb{Z}$ and let $U = pM$. Then U and M/U are both simple modules, hence md-modules. Clearly, U is a maximal submodule of M and U is not a direct summand of M. Hence M is not an md-module.

Proposition 2.4. *Let M be an R-module and L be a closed submodule of M. If L and M/L are md-modules, then M is an md-module.*

Proof. Let K be a maximal submodule of M. If $K + L = M$, then $M/K \cong L/(L \cap K)$ is simple, so $L \cap K$ is a maximal submodule of L. Since L is an md-module, $L = L \cap K \oplus S$

for some simple submodule $S \subseteq L$. Then $M = K + L = K + L \cap K + S = K + S$ and $K \cap S = 0$. So that K is a direct summand of M. If $L \subseteq K$, then K/L is a maximal submodule of M/L, so K/L is a direct summand of M/L. That is, $M/L = K/L \oplus N/L$ for some submodule N/L of M/L. Since N/L is simple, L is a maximal submodule of N. As L is closed in M, $L \cap S = 0$ for some nonzero $S \subseteq N$. So $L \oplus S = N$ with S a simple submodule of M. We get $M = K + N = K + L + S = K + S$ and $K \cap S = 0$. So K is a direct summand of M. Hence M is an md-module. □

For a module M let $s(M)$ be the sum of all simple submodules of M that are direct summands of M.

Theorem 2.5. *For an R-module M, the following are equivalent.*

(1) *M is an md-module,*
(2) *$M/s(M)$ has no maximal submodules,*
(3) *$M/\operatorname{Soc}(M)$ has no maximal submodules.*

Proof. (1)⇒(2) Let M be an md-module and suppose K is a maximal submodule of M such that $s(M) \subseteq K$. Then $M = K \oplus S$ for some simple submodule $S \subseteq M$. Hence $S \subseteq s(M) \subseteq K$, a contradiction. Therefore $M/s(M)$ has no maximal submodules.

(2)⇒(3) Clear, because any submodule of M containing $\operatorname{Soc}(M)$ also contains $s(M)$.

(3)⇒(1) Clearly $\operatorname{Soc}(M)$ is an md-module. Then (3) and Proposition 2.2 implies that M is an md-module. □

Note that, if M is a finitely generated module, then every submodule is contained in a maximal submodule. In this case, M is an md-module if and only if it is semisimple by Theorem 2.5. In particular, R is a semisimple (artinian) ring if and only if ${}_RR$ is an md-module.

Proposition 2.6. *Let M be a module such that $s(M)$ is finitely generated. Then M is an md-module if and only if $M = s(M) \oplus N$ where $N \subseteq M$ with $N = \operatorname{Rad} N = \operatorname{Rad} M$.*

Proof. First note that the (composition) length $l(s(M))$ of $s(M)$ is finite. The proof is by induction on the length $l(s(M))$ of $s(M)$. First suppose $l(s(M)) = 0$. Then clearly $s(M) = 0$. So that M has no maximal submodules, because M is an md-module. Then $\operatorname{Rad} M = M$, and so we are done. Suppose $l(s(M)) = n > 0$ and each md-submodule of M with length less than n has the desired decomposition. Let K be a maximal submodule of M. Then $M = K \oplus S$ for some $S \subseteq s(M)$. Now, K is an md-module by Proposition 2.1(2) and $l(s(K)) = n - 1$. By the induction hypothesis, $K = s(K) \oplus N$ where $\operatorname{Rad} N = N$. Then $M = S \oplus K = S \oplus s(K) \oplus N = s(M) \oplus N$, and this completes the proof.

For the converse, note that a module with no maximal submodules is an md-module. Now if $M = s(M) \oplus N$ with $N = \operatorname{Rad} N$, then both $s(M)$ and N are md-modules. Hence M is an md-module by Proposition 2.1(3). □

Proposition 2.7. *The following are equivalent for any ring R.*

(1) *Every R-module with zero radical is an md-module.*
(2) *$R/J(R)$ is an md-module.*

(3) *R is semilocal.*

Proof. (1) $\Rightarrow$ (2) Since $\mathrm{Rad}(R/J(R)) = 0$, this is clear.

(2) $\Rightarrow$ (3) $R/J(R)$ is a finitely generated R-module, and so $R/J(R)$ is semisimple. Hence R is semilocal.

(3) $\Rightarrow$ (1) Let M be an R-module with $\mathrm{Rad}\, M = 0$. Since $J(R)M \subseteq \mathrm{Rad}(M)$, the module M is an $R/J(R)$-module. Then M is semisimple, and so it is an md-module. □

3. Locally projective modules

Let R be a ring and let us denote $\mathrm{Soc}(_RR)$ by S. As S is a two-sided ideal, R/S has a canonical ring structure. Moreover, for any R-module M, we have that M/SM is an R/S-module. Let us note that a module M is semisimple projective if and only if $M = SM$

The proof of the following lemma is straightforward.

Lemma 3.1. *Let M be a left R-module, X be an R/S-module and $f : M \to X$ be a homomorphism of R-modules. Then $SM \subseteq \mathrm{Ker}(f)$ where SM is the R-submodule of M generated by the products of elements of S by elements of M.*

Let F be a module. We recall that F is called *locally projective* if for any epimorphism $p : X \to Y$, any homomorphism $g : F \to Y$, and any finitely generated submodule Z of F, there exists a homomorphism $h : F \to X$ such that $p \circ h \mid_Z = g \mid_Z$ (see e.g. [20]).

Every projective module is in particular locally projective. But the converse is far from being true. It was proved in [20, Examples 2.3(1)] that any pure submodule of a projective module is locally projective. This means, for instance, that if F is a flat module and $q : R^{(I)} \to F$ is an epimorphism, then $\mathrm{Ker}(q)$ is always locally projective. But it cannot be projective if we choose a flat module having projective dimension bigger than one. In fact, a main result of [21, Theorem 10] asserts that if R is a ring which is not left perfect, then there always exists a locally projective left R-module which is not projective.

The notion of locally projective modules coincides with that of flat strict Mittag-Leffler modules in the sense of Raynaud and Gruson [10] and their existence has been shown to have a strong relation with the decomposition properties of modules into direct summands (see e.g. [11, 12]). Bearing in mind this connection, we will prove in this section that any locally projective md-module is trivial in the sense that it is a direct sum of simple projective modules.

We first need to prove the following easy lemma.

Lemma 3.2. *Let F be a locally projective module. Then any finitely generated direct summand of F is projective.*

Proof. Let N be a finitely generated direct summand of F and let $p : R^{(n)} \to N$ be an epimorphism. Let us denote by $u : N \to F$ the inclusion and let $\pi : F \to N$ be an epimorphism such that $\pi \circ u = 1_N$. As F is locally projective and N is finitely generated, there exists a homomorphism $h : F \to R^{(n)}$ such that $p \circ h \mid_N = \pi \mid_N$. But this means that N is a direct summand of $R^{(n)}$ and therefore, projective. □

We can now state the main result of this section.

Theorem 3.3. *Let F be a locally projective module. If every maximal submodule is a direct summand, then F is semisimple projective.*

Proof. We need to show that $SF = F$. Assume on the contrary that $SF \neq F$ and let us choose $0 \neq x \in F \setminus SF$. Let $p : R^{(I)} \to F$ be an epimorphism for some index set I. As F is locally projective, there exists a homomorphism $h : F \to R^{(I)}$ such that $p \circ h(x) = x$.

We claim that $\mathrm{Im}(h) \subseteq (J + S)^{(I)}$. Otherwise, if we call $\pi : R^{(I)} \to R^{(I)}/(J + S)^{(I)}$ the canonical projection, we have that $\pi \circ h \neq 0$. And, as $\mathrm{Rad}(R^{(I)}/(J + S)^{(I)}) = 0$, this means that there exists an epimorphism $q : R^{(I)}/(J + S)^{(I)} \to C$ onto a simple module C such that $q \circ \pi \circ h \neq 0$. Our hypothesis implies now that C is a direct summand of F, which must be projective by Lemma 3.2. Hence $C \subseteq SF$. But this is a contradiction, since otherwise $q \circ \pi \circ h = 0$.

Let us now choose a finite subset $K \subseteq I$ such that $h(x) \subseteq R^{(K)}$. Say that $h(x) = \sum_{i \in K} r_i e_i$ where $r_i \in R$. Again, for any $i \in K$, we may choose a finite subset $K_i \subseteq I$ such that $h \circ p(e_i) \subseteq R^{(K_i)}$. Let us call $K' = K \cup (\bigcup_{i \in K} K_i)$. Then, for any $i \in K$, we can find elements $r_{ij} \in R$ such that $h \circ p(e_i) = \sum_{j \in K'} r_{ij} e_j$. Thus we get that

$$h(x) = hph(x) = hp(\sum_{i \in K} r_i e_i) = \sum_{i \in K} r_i hp(e_i) = \sum_{i \in K} r_i (\sum_{j \in K_i} r_{ij} e_j).$$

So if we call $\phi : R^{(K')} \to R^{(K')}$ the endomorphism whose matrix with respect to the basis $\{e_j\}_{j \in K'}$ is (r_{ij}), we get that $\phi \circ h(x) = h(x)$. Let us enlarge the row vector $(r_i)_K$ to a vector in $R^{(K')}$ by setting $r_j = 0$ if $j \in K' \setminus K$. We deduce from the above equality that $(r_j)_{j \in K'} = (r_j)_{j \in K'} \cdot (r_{ij})_{i,j \in K'}$. So if we call $I_{K'}$ the identity matrix of size K', then $(r_j)_{j \in K'} \cdot (I_{K'} - (r_{ij})_{i,j \in K'}) = 0$,

On the other hand, as we know that $\mathrm{Im}(h) \subseteq (J + S)^{(I)}$, and S is a two-sided ideal of R, we deduce that all entries of the matrix $(r_{ij} + S)_{i,j \in K'}$ belong to the Jacobson radical of R/S and therefore, it is a quasi-regular matrix by [2, Corollary 17.13]. This means that the matrix $I_{K'} - (r_{ij} + S)$ is invertible in the matrix ring $M_{K'}(R/S)$ and thus, the row matrix $(r_i)_{i \in K'} = (0 + S)$ is in $M_{K'}(R/S)$, i.e. $r_i \in S$ for any $i \in K$. But this means that $h(x) \in S^{(I)}$ and, as any simple quotient of F is a direct summand, we deduce that $x = p \circ h(x) \in SF$. A contradiction, since we were assuming that $x \notin SF$ □

In particular, we get the following corollary.

Corollary 3.4. *Any projective md-module is semisimple.*

4. Maximal submodules that are supplements

In this section we shall study modules whose maximal submodules are supplements, and we call them *ms-modules* for short. Clearly any direct summand is a supplement, and hence md-modules are ms-modules. We shall prove that the converse need not be true in general.

It can be verified easily that the properties in Proposition 2.1 and Proposition 2.2 are also held for ms-modules.

Recall that a module is called *coatomic* provided that every submodule is contained in a maximal submodule. First, we shall characterize coatomic ms-modules. Then we will obtain a characterization of ms-modules over left perfect rings. We begin with following:

Lemma 4.1. *Let M be a coatomic module and N be a coclosed submodule of M. Then N is coatomic.*

Proof. Suppose $\mathrm{Rad}(N/K) = N/K$ for some $K \subseteq N$. Then $N/K \subseteq \mathrm{Rad}(M/K) \ll M/K$. Then $N/K \ll M/K$, and hence $N = K$ because N is coclosed. Therefore N is coatomic. □

Lemma 4.2. *Let M be a module with* $\mathrm{Rad}\, M = 0$. *Then M is an ms-module if and only if it is an md-module.*

Proof. The proof is straightforward. □

Theorem 4.3. *Let R be any ring and M be a coatomic R-module. Then M is an ms-module if and only if the following conditions hold:*
(i) Every maximal submodule N of M is coatomic and $\mathrm{Rad}\, N = \mathrm{Rad}\, M$,
(ii) $M/\mathrm{Rad}\, M$ *is semisimple.*

Proof. Suppose M is an ms-module and K is a maximal submodule of M. Then K is a supplement in M, so K is coatomic by Lemma 4.1, and $\mathrm{Rad}\, K = K \cap \mathrm{Rad}\, M = \mathrm{Rad}\, M$ by [19, 41.1], this proves (i). Now (ii) follows from Lemma 4.2 and the fact that coatomic md-modules are semisimple.

Conversely, let K be a maximal submodule of M. Then $K/\mathrm{Rad}\, M$ is a direct summand of $M/\mathrm{Rad}\, M$ by (ii), so $K + L = M$ and $K \cap L = \mathrm{Rad}\, M$ for some submodule $L \subsetneq M$. Since K is coatomic and $\mathrm{Rad}\, K = \mathrm{Rad}\, M$, we have $K \cap L = \mathrm{Rad}\, K \ll K$, that is K is a supplement of L in M. Hence M is an ms-module. □

A ring R is called a left *max* ring if $\mathrm{Rad}\, M \ll M$ for every left R-module M. Equivalently, R is a left max ring if and only every (nonzero) left R-module is coatomic. R is a *left perfect* ring if R is a left max ring and $R/\mathrm{Rad}\, R$ is semisimple as a left R-module (see [2]). For every module M over a left perfect ring, we have $M/\mathrm{Rad}\, M$ is semisimple.

Now, from Theorem 4.3 we obtain the following corollary.

Corollary 4.4. *Let R be a left perfect ring and M be an R-module. Then M is an ms-module if and only if* $\mathrm{Rad}\, K = \mathrm{Rad}\, M$ *for every maximal submodule K of M.*

An R-module M is called π-*projective* if for every two submodules U, V of M with $U + V = M$, there exists $f \in \mathrm{End}(M)$ with $\mathrm{Im}(f) \subseteq U$ and $\mathrm{Im}(1 - f) \subseteq V$.

A projective module P together with an epimorphism $f : P \to M$ such that $\mathrm{Ker}(f) \ll P$ is called a *projective cover* of M. A ring R is semiperfect if and only if every simple left R-module has a projective cover if and only if the left (right) R-module R is supplemented (see [19, 42.6]).

Proposition 4.5. *Let R be a semiperfect ring and M a π-projective R-module. Then M is an ms-module if and only if M is an md-module. In particular, ${}_RR$ is an ms-module if and only if it is semisimple.*

Proof. Necessity is clear. Now suppose M is an ms-module and let N be a maximal submodule of M. Then by hypothesis $M = N + L$ and $N \cap L \ll N$ for some $L \subseteq M$. Since R is semiperfect, the simple R-module M/N has a projective cover. So that N has a supplement L' in L by Lemma 4.40 in [16]. Then N and L' are mutual supplements. Hence N is a direct summand of M by [3, 20.9]. □

5. Example

As we have mentioned, in general an ms-module need not be an md-module. In the following two lemmas we shall prove the existence of such a module.

Lemma 5.1. *Let R be a ring and M be an R-module. Suppose M has a simple submodule U such that $U \trianglelefteq M$ and M/U is semisimple but not simple. Then M is an ms-module but not an md-module.*

Proof. It is clear from the hypothesis that $\mathrm{Soc}(M) = U$ and $U \subseteq L$ for every nonzero proper submodule L of M. In particular, U is contained in every maximal submodule of M, and hence $U \subseteq \mathrm{Rad}\, M$. Since $\mathrm{Rad}\, M/U = \mathrm{Rad}\, M/U = 0$, $\mathrm{Rad}\, M = U$. By the same argument we have $\mathrm{Rad}\, N = U$ for every submodule N of M which contains U properly. Let K be a maximal submodule of M. Then $M/U = K/U \oplus T/U$ for some $T/U \subseteq M/U$ because M/U is semisimple. We get $K + T = M$ and $K \cap T = U = \mathrm{Rad}\, K$. Clearly U is finitely generated, so $K \cap T = U \ll K$. Therefore K is a supplement of T in M. Hence M is an ms-module. Since every nonzero submodule of M contains U, K is not a direct summand of M, i.e. M is not an md-module. □

Lemma 5.2. *Let R be a complete commutative noetherian local ring with maximal ideal P. Suppose P is not principal. Then there exists an ms-module over R which is not an md-module.*

Proof. Let U be the simple R-module R/P and $E = E(U)$ be the injective hull of U. Let $V = \{e \in E \mid P^2 e = 0\}$. Then V is a submodule of E and $P(V/U) = 0$, so that V/U is a vector space over R/P. Also P/P^2 is a vector space over R/P. The dimension of these vector spaces is the respective composition length. By [18, Corollary p. 154] the composition length of V/U is the same as the composition length of P/P^2. Since P is not principal, the composition length of P/P^2 is at least two (see [17, Proposition 9.3]), so that V/U is not simple. Therefore by Lemma 5.1, V is an ms-module but not an md-module. □

Example 5.3. Let $R = \mathbb{C}[x, y]$, $P = Rx + Ry$ and $S = R/P^2$. Then S is an artinian local ring. Let $M = E_S(R/P)$ be the injective hull of the simple S-module R/P. Then $P^2 M = 0$, so M is an ms-module but not an md-module by Lemma 5.2.

Corollary 5.4. *Let M be an R-module such that $\operatorname{Rad} M$ is a simple essential submodule of M and $M/\operatorname{Rad} M \cong S_1 \oplus S_2$ for simple modules S_1 and S_2. Then M is an ms-module but not an md-module.*

Note: A concrete example satisfying the hypothesis of Corollary 5.4 can be found in [15, p. 339].

6. Modules over Commutative Rings

Throughout this section all rings are commutative. In general direct product of simple modules need not be an md-module. For instance, let F be a field and $R = F^I$ where I is an infinite index set. Then R is a direct product of simple R-modules each of which is isomorphic to F. By [13, p. 264] R is not semisimple. Hence R is not an md-module by Theorem 3.3.

In case R is commutative and noetherian, we shall prove that an arbitrary direct product of simple R-modules is an md-module. First we need the following lemma.

Lemma 6.1. *Let R be a ring and A be a finitely generated ideal of R. Let $X = \prod_{i\in I} X_i$ be the direct product of R-modules X_i. Suppose that $X_i = AX_i$ for all $i \in I$. Then $X = AX$.*

Proof. Let $A = Ra_1 + Ra_2 + \cdots + Ra_k$ for some $k \geqslant 1$, $a_i \in A\,(1 \leqslant i \leqslant k)$. For every $i \in I$, we have $X_i = AX_i = a_1X_i + \cdots + a_kX_i$. Let $x = (x_i) \in X$ where $x_i \in X_i$ for all $i \in I$. By assumption, for every $i \in I$ there exists $x_{ij} \in X_i$, $(1 \leqslant j \leqslant k)$ such that $x_i = a_1x_{i1} + \cdots + a_kx_{ik}$. Then $(x_{ij}) \in X$ $(1 \leqslant j \leqslant k)$ and $x = a_1(x_{i1}) + \cdots + a_k(x_{ik}) \in AX$. Hence $X = AX$. □

Theorem 6.2. *Let R be a noetherian ring and let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a collection of simple R-modules. Then $M = \prod_{\lambda\in\Lambda} U_\lambda$ is an md-module.*

Proof. Let $\{P_i\}_{i\in I}$ be the collection of distinct maximal ideals P_i of R such that for every $i \in I$ there exists $\lambda \in \Lambda$ with $P_iU_\lambda = 0$. For each $i \in I$ let $\Lambda_i = \{\lambda \in \Lambda \mid P_iU_\lambda = 0\}$. Let K be a maximal submodule of M and P be the maximal ideal of R such that $PM \subseteq K$. By Lemma 6.1, $P = P_j$ for some $j \in I$. Again by Lemma 6.1, if $L = \prod_{\lambda\in\Lambda'} U_\lambda$ where $\Lambda' = \bigcup\{\Lambda_i \mid i \in I \setminus \{j\}\}$, then $PL = L$. So that $L \subseteq K$. Now let $L' = \prod_{\lambda\in\Lambda_j} U_\lambda$. Then $P_jL' = 0$, so that L' is semisimple, also $M = L \oplus L'$. Then $K = L \oplus (K \cap L')$ and $K \cap L'$ is a direct summand of L'. Therefore K is a direct summand of M. Hence M is an md-module. □

Theorem 6.3. *For a domain R the following are equivalent.*

(1) *Every (cyclic) torsion R-module is an md-module.*
(2) *Every torsion R-module is semisimple.*
(3) *Every nonzero proper ideal of R is a product of finitely many maximal ideals and $P^2 = P$ for every maximal ideal P of R.*

Proof. (1) $\Rightarrow$ (2) Let M be a torsion R-module and $0 \neq m \in M$. Then $Rm \cong R/I$ for some nonzero $I \subseteq R$. By hypothesis, Rm is an ms-module, so R/I is an ms-module. Then R/I is an ms-module as an R/I-module. So R/I is a semisimple R/I-module, and so R/I is a semisimple R-module. Therefore $M = \sum_{m \in M} Rm$ is semisimple. Hence M is an md-module.

(2)$\Rightarrow$(1) Clear.

(2) $\Rightarrow$ (3) Let I be a nonzero ideal of R. Since the R-module R/I is torsion, it is semisimple. Then I is an intersection of finitely many distinct maximal ideals of R. So I is equal to the product of these ideals (see [17]). Let P be a maximal ideal of R. Since R/P^2 is a torsion module it is semisimple. So $P^2 = Q_1Q_2 \dots Q_k$ where $Q_1, Q_2, \dots, Q_k$ are distinct maximal ideals of R. Since $Q_1Q_2 \dots Q_k \subseteq P$, we have $Q_j \subseteq P$ for some j. Maximality of Q_j in R implies that $Q_j = P$. By renumbering the Q_i's, we may assume that $j = 1$. Then $R = P + Q_2 \dots Q_k$, and so $P = P^2 + PQ_2 \dots Q_k = P^2$, that is $P = P^2$.

(3)$\Rightarrow$(1) Let M be a torsion R-module and $m \in M$. Then $Rm \cong R/I$ for some nonzero ideal I of R and $I = P_1^{n_1} \dots P_k^{n_k}$ for distinct maximal ideals $P_1, \dots, P_k$ of R, so by the assumption $P_i^{n_i} = P_i$ for all $i = 1, \dots, k$. Thus $R/I \cong R/P_1 \oplus \dots \oplus R/P_k$ is semisimple, so also is Rm. Therefore $M = \sum_{m \in M} Rm$ is semisimple. □

We characterize md-modules over Dedekind domains. We begin with the following lemma which is due to Zöschinger. Using this lemma we shall prove that ms-modules and md-modules coincide over Dedekind domains.

Lemma 6.4. ([22], Lemma 3.3) *Let R be Dedekind domain, M be an R-module and $V \subseteq M$. Then V is coclosed if and only if V is closed.*

Let M be any module and $N \subseteq M$. A submodule K of M is called a *complement of* N if K is maximal in the collection of submodules L of M such that $L \cap N = 0$. A submodule T of M is called a *complement* if there is a submodule N of M such that T is a complement of N. A submodule of M is a complement if and only if it is closed (see [7, p.6]).

Proposition 6.5. *Let R be a Dedekind domain and M be an R-module. Then M is an ms-module if and only if M is an md-module.*

Proof. We only need to prove the necessity. Let N be a maximal submodule of M. Since M is an ms-module, N is a supplement in M. So N is a complement in M by Lemma 6.4, i.e. $N \cap L = 0$ for some $L \subseteq M$ and N is maximal with respect to this property. Now $L \neq 0$ because $M \cap 0 = 0$. Therefore $N + L = M$, i.e. N is a direct summand of M. □

Lemma 6.6. ([1], Lemma 4.4) *Let R be Dedekind domain. For an R-module M the following are equivalent.*

1. *M is injective.*
2. *M is divisible.*
3. *$M = PM$ for every maximal ideal P of R.*
4. *M does not contain any maximal submodule.*

Let R be a Dedekind domain and M be an R-module. For a maximal ideal P of R, the submodule $T_P(M) = \{m \in M \mid P^n m = 0$ for some positive integer $n\}$ is called the

P-primary component of M. If $M = T_P(M)$ for some maximal ideal P of R, then M is called a *P-primary module.* For a torsion module M we always have $M = \bigoplus_{P\in\Omega} T_P(M)$ where Ω is the set of all maximal ideals of R (see [4, 10.6.9]).

The divisible part of a module M is denoted by $D(M)$. By Lemma 6.6, we have $M = D(M) \oplus M'$ for some $M' \subseteq M$. If M is a divisible module, then M has no maximal submodules, and so $\operatorname{Rad} M = M$. Therefore $D(M) \subseteq \operatorname{Rad} M$ for every R-module M.

Lemma 6.7. *Let R be a Dedekind domain and M be a reduced and P-primary module for some maximal ideal $P \subseteq R$. Then M is an md-module if and only if M is semisimple.*

Proof. Suppose M is an md-module. Then $M/\operatorname{Soc}(M)$ has no maximal submodules by Proposition 2.5, so $P(M/\operatorname{Soc}(M)) = M/\operatorname{Soc}(M)$ by Lemma 6.6, that is $PM + \operatorname{Soc}(M) = M$ and this gives $P(PM+\operatorname{Soc}(M)) = P^2M = PM$. Therefore PM is divisible by Lemma 6.6, but M is reduced so that $PM = 0$. Hence M is an R/P-module, i.e. M is semisimple.

Converse is clear. ☐

Theorem 6.8. *Let R be a Dedekind domain and M be a torsion R-module. The following are equivalent.*

(1) *M is an md-module.*
(2) *$M = M_1 \oplus M_2$ where M_1 is divisible and M_2 is semisimple.*
(3) *Every submodule $U \subseteq M$ with $\operatorname{Rad} M \subseteq U$ is a direct summand of M.*

Proof. (1)⇒(2) Let D be the divisible part of M. Then $M = D \oplus N$ for some $N \subseteq M$. Since N is torsion, we have $N = \bigoplus_{P\in\Omega} T_P(N)$ and since M is an md-module $T_P(N)$ is also an md-module for every $P \in \Omega$ by Proposition 2.1(2). Then $T_P(N)$ is semisimple by Lemma 6.7. Therefore N is semisimple.

(2) ⇒ (3) We have $\operatorname{Rad} M = \operatorname{Rad}(M_1 \oplus M_2) = \operatorname{Rad} M_1 \oplus \operatorname{Rad} M_2 = \operatorname{Rad} M_1 = M_1$. Let $\operatorname{Rad} M \subseteq U \subseteq M$. Then we get $U = M_1 \oplus U \cap M_2$. Since M_2 is semisimple $M_2 = K \oplus M_2 \cap U$ for some $K \subseteq M_2$. So $M = M_1 \oplus M_2 = M_1 \oplus K \oplus M_2 \cap U = K \oplus U$.

(3)⇒(1) $\operatorname{Rad} M \subseteq P$ for every maximal submodule P of M. So, by hypothesis, every maximal submodule of M is a direct summand. Hence M is an md-module. ☐

Lemma 6.9. ([14], Example 6.34) *Let R be a domain and M be an R-module. Then the torsion submodule $T(M)$ is a closed submodule of M.*

Corollary 6.10. *Let R be domain and M be an R-module. If $T(M)$ and $M/T(M)$ are md-modules, then M is an md-module.*

If R is a Dedekind domain, then the converse also holds.

Proof. By Lemma 6.9, $T(M)$ is a closed submodule of M. Then M is an md-module by Proposition 2.4.

If R is a Dedekind domain, then $T(M)$ is a coclosed submodule of M by Lemma 6.4 and Lemma 6.9. Since every simple submodule of M is torsion, $\operatorname{Soc}(M) \subseteq T(M)$, so that $T(M)$ is an md-module by Proposition 2.3. $M/T(M)$ is an md-module by Proposition 2.1(1). ☐

Lemma 6.11. *Let R be a Dedekind domain and M be a torsion-free R-module. Then M is an md-module if and only if M is divisible.*

Proof. Suppose M is an md-module and let P be a maximal submodule of M. Then $P \oplus S = M$ for some simple submodule S of M. Thus $S \subseteq T(M) = 0$, so $P = M$, a contradiction. Hence M has no maximal submodules, and M is divisible by Lemma 6.6.

Conversely, if M is divisible, then M has no maximal submodules by Lemma 6.6. Hence M is an md-module. □

Theorem 6.12. *Let R be a Dedekind domain and M be an R-module. Then M is an md-module if and only if*
(i) $T(M) = M_1 \oplus M_2$ where M_1 is semisimple and M_2 is divisible,
(ii) $M/T(M)$ is divisible.

Proof. Suppose M is an md-module. Then $T(M)$ is an md-module by Corollary 6.10, so $T(M)$ has the desired decomposition by Theorem 6.8. Hence $M/T(M)$ is divisible by Lemma 6.11.

To prove the converse, let N be a maximal submodule of M. Then by (ii) we have $N + T(M) = M$. Since M_2 is divisible, $M_2 \subseteq \operatorname{Rad} M \subseteq N$, so $M = N + T(M) = N + M_1$. Then $N + S = M$ for some simple submodule $S \subseteq M_1$. We have $N \cap S = 0$ because S is a simple submodule. Therefore N is a direct summand of M. Hence M is an md-module. □

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RAD-SUPPLEMENTED MODULES

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Abstract. Let τ be a radical for the category of left R-modules for a ring R. If M is a τ-coatomic module, that is, if M has no nonzero τ-torsion factor module, then $\tau(M)$ is small in M. If V is a τ-supplement in M, then the intersection of V and $\tau(M)$ is $\tau(V)$. In particular, if V is a Rad-supplement in M, then the intersection of V and $\text{Rad}(M)$ is $\text{Rad}(V)$. A module M is τ-supplemented if and only if the factor module of M by $P_\tau(M)$ is τ-supplemented where $P_\tau(M)$ is the sum of all τ-torsion submodules of M. Every left R-module is Rad-supplemented if and only if the direct sum of countably many copies of R is a Rad-supplemented left R-module if and only if every reduced left R-module is supplemented if and only if $R/P(R)$ is left perfect where $P(R)$ is the sum of all left ideals I of R such that $\text{Rad}\, I = I$. For a left duo ring R, R is a Rad-supplemented left R-module if and only if $R/P(R)$ is semiperfect. For a Dedekind domain R, an R-module M is Rad-supplemented if and only if M/D is supplemented where D is the divisible part of M.

1. Introduction

All rings considered in this paper will be associative with an identity element. Unless otherwise stated R denotes an arbitary ring and all modules will be *left* unitary R-modules. By $R\text{-}\mathcal{M}od$, we denote the category of left R-modules. Unless otherwise stated, τ is a radical on $R\text{-}\mathcal{M}od$. For fundamentals on module theory, see for example [17], [4] and [30]. Let R be a ring and M be an R-module. Denote by $X \leq M$ that X is a submodule of M. As usual, $\text{Rad}\, M$ denotes the radical of M and $J(R)$ denotes the Jacobson radical of the ring R. A submodule K of M is called *small* in M (denoted by $K \ll M$) if $M \neq K + T$ for every proper submodule T of M. For an index set I, $M^{(I)}$ denotes as usual the direct sum $\oplus_{i\in I} M$. The set of natural numbers is denoted by $\mathbb{N}$. See [30, §41] and the recent monograph [10] for results (and the definitions) related to (weak) supplements and (weakly) supplemented modules. Given submodules $K \leq L \leq M$, the inclusion $K \leq L$ is called *cosmall in* M if $L/K \ll M/K$ (see [10, 3.1]). A submodule $L \leq M$ is called *coclosed in* M if L has no proper submodule K for which the inclusion $K \leq L$ is cosmall in M (see [10, 3.6]).

We shall investigate some properties of Rad-supplemented modules and in general τ-supplemented modules where τ is a radical for $R\text{-}\mathcal{M}od$. The motivation for considering Rad-supplements (coneat submodules) and τ-supplements in general is given in the next section. One of the main questions we shall answer is when are all left R-modules Rad-supplemented. In the investigation of this problem, the notion of radical modules, reduced modules and coatomic modules turn out to be useful; see [32, pp. 47]. In the definitions and properties for reduced and coatomic modules, instead of Rad, we can use any (pre)radical τ on $R\text{-}\mathcal{M}od$ (see Section 3), and these will be useful in the investigation of the properties of τ-supplemented modules. For a module M, the sum of all radical submodules of M is denoted by $P(M)$, that is, $P(M)$ is the sum of all submodules U of M such that $\text{Rad}\, U = U$. For submodules U and V of a module M, the submodule V is said to be a *Rad-supplement* of U in M or U is said to *have a Rad-supplement* V in M if $U + V = M$ and $U \cap V \leq \text{Rad}\, V$. A module M is called a *Rad-supplemented module* if every submodule of M has a Rad-supplement in M. See also [29]; Rad-supplemented modules

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are called generalized supplemented modules there. In Section 6, we shall prove that every left R-module is Rad-supplemented if and only if $R/P(R)$ is left perfect. In [9], it is proved that the class of Rad-supplemented rings lies properly between those of the semiperfect and the semilocal rings. We show that a left duo ring R is Rad-supplemented as a left R-module if and only if $R/P(R)$ is semiperfect. Whenever possible the related results are given in general for a radical τ for R-$\mathcal{M}od$. See [1] and [10, §10] for some properties of τ-supplements and τ-supplemented modules. We shall investigate some further properties of τ-supplemented modules in Section 4. For some rings R, we shall also determine when all left R-modules are τ-supplemented in Section 5. We are also going to study the property $\operatorname{Rad} V = V \cap \operatorname{Rad} M$ for a submodule V of M. It is known that this holds if V is a supplement in M (see [30, 41.1]) and moreover if V is coclosed in M (see [10, 3.7]). We show that this property also holds when V is a Rad-supplement in M (Corollary 4.2); in general for a radical τ for R-$\mathcal{M}od$, we show that if V is a τ-supplement in M, then $\tau(V) = V \cap \tau(M)$. It is clear that every supplemented module is Rad-supplemented. But the converse implication fails to be true. For example, the $\mathbb{Z}$-module $\mathbb{Q}$ is Rad-supplemented but not supplemented. Since $\operatorname{Rad}\mathbb{Q} = \mathbb{Q}$ (see for example [17, 2.3.7]), $\mathbb{Q}$ is Rad-supplemented (by Proposition 4.5-(i)). But $\mathbb{Q}$ is not supplemented by example [10, 20.12]. In Section 7, we understand this example clearly and describe Rad-supplemented modules over Dedekind domains using the structure of supplemented modules over Dedekind domains which was completely determined in [32].

For definitions and elementary properties of preradicals, see [26, Ch. VI], [6] or [10, §6]. A *preradical* τ for R-$\mathcal{M}od$ is defined to be a subfunctor of the identity functor on R-$\mathcal{M}od$. Let τ be a preradical τ for R-$\mathcal{M}od$. The following module classes are defined: the *preradical* or *(pre)torsion class* of τ is

$$\mathbb{T}_\tau = \{N \in R\text{-}\mathcal{M}od \mid \tau(N) = N\}$$

and the *preradical free* or *(pre)torsion free class* of τ is

$$\mathbb{F}_\tau = \{N \in R\text{-}\mathcal{M}od \mid \tau(N) = 0\}.$$

τ is said to be *idempotent* if $\tau(\tau(N)) = \tau(N)$ for every R-module N. τ is said to be a *radical* if $\tau(N/\tau(N)) = 0$ for every R-module N. For the main elementary properties that we shall use frequently for a (pre)radical, see for example [10, pp. 55]. For R-modules $K \leq M$, we always have $(\tau(M) + K)/K \leq \tau(M/K)$. If moreover τ is a *radical* and $K \leq \tau(M)$, then $\tau(M/K) = \tau(M)/K$ [26, Ch. VI, Lemma 1.1]. When we consider a ring R as a left R-module, we already have that $A = \tau({}_RR)$ is a left ideal of R; indeed it is a two-sided ideal of R [26, Ch. VI, §1, Examples (3), pp. 139] so that we can consider the quotient ring R/A which we shall use in the results for τ-supplemented modules. For a free R-module F, the property $\tau(F) = \tau(R)F$ is easily obtained. This also holds for projective modules. See also [13] and [7] for some related concepts in torsion theories (mostly for a hereditary preradical).

2. Coneat submodules and Rad-supplements

Neat subgroups of abelian groups (introduced in [15, pp. 43-44]) have been generalized to modules in [28, 9.6] (and [27, §3]). The class of coneat submodules has been introduced in [21] and [3]: A monomorphism $f : K \to L$ is called *coneat* if each module M with $\operatorname{Rad} M = 0$ is *injective* with respect to it, that is, the Hom sequence

$$\operatorname{Hom}(L, M) \to \operatorname{Hom}(K, M) \to 0$$

is exact. See [21, Proposition 3.4.2] or [10, 10.14] or [1, 1.14] for a characterization of coneat submodules. This characterization will be the particular case $\tau = \operatorname{Rad}$ in Proposition 2.1 and this is the reason for considering Rad-supplements and in general τ-supplements given below. For more results on coneat submodules see [21], [3], [10, §10 and 20.7–8], [1] and [24].

Proper classes of monomorphisms and short exact sequences were introduced in [8] to do *relative* homological algebra. In [27, Remark after Proposition 6], it is pointed out that supplement submodules induce a proper class of short exact sequences (the term ‘low’ is used for supplements dualizing the term ‘high’ used in abelian groups). [12] uses the terminology 'cohigh' for supplements and gives more general definitions for proper classes of supplements related to another given proper class (motivated by the considerations as pure-high extensions and neat-high extensions in [14]). For the definition and properties of *proper classes*, see [25], [20, Ch. 12, §4], [28] and [22]. We shall follow the terminology and notation as in [10, §10] and [1] since we will mainly refer to these for τ-supplemented modules and Rad-supplemented modules.

Denote by $\mathbb{E}_{Suppl}$ the class of all short exact sequences induced by supplement submodules; that is $\mathbb{E}_{Suppl}$ is the class of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms such that $\mathrm{Im}(f)$ is a supplement in B. Then as mentioned above, the class $\mathbb{E}_{Suppl}$ forms a *proper* class, see for example [10, 20.7]. Every module M with $\mathrm{Rad}\, M = 0$ is $\mathbb{E}_{Suppl}$-injective that is M is injective with respect to every short exact sequence in $\mathbb{E}_{Suppl}$. Thus supplement submodules are coneat submodules by the definition of coneat submodules. In the definition of coneat submodules, using any radical τ instead of Rad, the following result is obtained. It gives us the definition of a τ-*supplement* in a module because the last condition is like the usual supplement condition except that, instead of $U \cap V \ll V$, the condition $U \cap V \leq \tau(V)$ is required.

Proposition 2.1. *(see* [10, 10.11] *or* [1, 1.11]*) Let τ be a radical for R-$\mathcal{M}od$. For a submodule $V \leq M$, the following statements are equivalent.*

(i) *Every module N with $\tau(N) = 0$ is injective with respect to the inclusion $V \hookrightarrow M$;*
(ii) *there exists a submodule $U \leq M$ such that*

$$U + V = M \text{ and } U \cap V = \tau(V);$$

(iii) *there exists a submodule $U \leq M$ such that*

$$U + V = M \text{ and } U \cap V \leq \tau(V).$$

*If these conditions are satisfied, then V is called a τ-*supplement *in M.*

The usual definitions are then given as follows. For submodules U and V of a module M, the submodule V is said to be a τ-*supplement* of U in M or U is said to *have a τ-supplement* V in M if $U + V = M$ and $U \cap V \leq \tau(V)$. A module M is called a τ-*supplemented module* if every submodule of M has a τ-supplement in M. We call M *totally τ-supplemented* if every submodule of M is τ-supplemented. A submodule N of M is said to have *ample τ-supplements in M* if for every $L \leq M$ with $N + L = M$, there is a τ-supplement L' of N with $L' \leq L$. A module M is said to be *amply τ-supplemented* if every submodule of M has ample τ-supplements in M.

For τ = Rad, the above definitions give *Rad-supplement submodules* of a module, *Rad-supplemented modules*, etc. By these definitions, a submodule V of a module M is a *coneat* submodule of M if and only if V is a *Rad-supplement* of a submodule U of M in M.

3. τ-REDUCED AND τ-COATOMIC MODULES, AND THE LARGEST τ-TORSION SUBMODULE $P_\tau(M)$

Let τ be a preradical for R-$\mathcal{M}od$ and let M be an R-module. By taking τ instead of Rad in the definitions of reduced and coatomic module definitions in [32, pp. 47], we define the following:

(i) M is said to be a τ-*torsion* module if $\tau(M) = M$, that is M is in the pretorsion class $\mathbb{T}_\tau$.
(ii) By $P_\tau(M)$ we denote the sum of *all* τ-torsion submodules of M, that is,

$$P_\tau(M) = \sum\{U \leq M \mid \tau(U) = U\}.$$

(iii) M is said to be a τ-*reduced* module if it has *no* nonzero τ-torsion submodule, that is, for every submodule U of M, $\tau(U) = U$ implies $U = 0$; equivalently, $\tau(U) \neq U$ for every nonzero submodule U of M. Clearly, M is τ-reduced if and only if M is P_τ-torsion free, that is, $P_\tau(M) = 0$.

(iv) M is said to be a τ-*coatomic* module if it has *no* nonzero τ-torsion factor module, that is, for every submodule U of M, $\tau(M/U) = M/U$ implies $U = M$; equivalently, $\tau(M/U) \neq M/U$ for every proper submodule U of M.

For $\tau = \text{Rad}$, $P_\tau(M)$ will be denoted by just $P(M)$, a Rad-torsion module is called a *radical module*, a Rad-reduced module will be called a *reduced module* and a Rad-coatomic module will be called a *coatomic module* following the terminology in [32]. Coatomic modules appear in the theory of supplemented, semiperfect, and perfect modules. See [32, Lemma 1.5] for some properties of reduced and coatomic modules. For the structure of coatomic modules over commutative Noetherian rings see [33]; the Noetherian assumption is needed to have that every submodule of a coatomic module over a commutative Noetherian ring is coatomic [33, Lemma 1.1].

For completeness note the following elementary properties of $P_\tau(M)$:

Theorem 3.1. *Let τ be a preradical for R-$\mathcal{M}od$ and let M be an R-module.*

(i) *P_τ is an* idempotent preradical.

(ii) *If $M \leq N$ for a module N, then $P_\tau(M) \leq \tau(N)$. In particular, $P_\tau(M) \leq \tau(M)$.*

(iii) *$\tau(P_\tau(M)) = P_\tau(M)$, that is, $P_\tau(M)$ is τ-torsion, and so by its definition $P_\tau(M)$ is the* largest *τ-torsion submodule of M.*

(iv) *If $P_\tau(M) \leq V$ for a submodule V of M, then $P_\tau(M) \leq \tau(V)$.*

(v) *$P_\tau(\tau(M)) = P_\tau(M)$*

(vi) *The pretorsion class of P_τ equals the pretorsion class of τ and the pretorsion free class of P_τ contains the pretorsion free class of τ:*

$$\mathbb{T}_{P_\tau} = \mathbb{T}_\tau \qquad \textit{and} \qquad \mathbb{F}_{P_\tau} \supseteq \mathbb{F}_\tau.$$

(vii) *Moreover, if τ is a radical, then the factor module $M/P_\tau(M)$ is τ-reduced, that is, $P_\tau\big(M/P_\tau(M)\big) = 0$ and so P_τ is an* idempotent radical.

Remark 3.2. In general, given any class $\mathbb{A}$ of modules, a *preradical* $\tau^{\mathbb{A}}$ is defined by setting for each module N,

$$\tau^{\mathbb{A}}(N) = \sum\{\text{Im} f \mid f : A \to N \text{ in } R\text{-}\mathcal{M}od,\ A \in \mathbb{A}\}.$$

and if $\mathbb{A}$ is a pretorsion class, then $\tau^{\mathbb{A}}$ is an idempotent preradical (see for example [10, 6.5-6]). In our case, the preradical P_τ is equal to $\tau^{\mathbb{A}}$ when the pretorsion class $\mathbb{A} = \mathbb{T}_\tau$, the torsion class of τ. See also [26, Ch. VI, §1]; P_τ is the largest idempotent preradical that is smaller than τ and see [26, Ch. VI, Exercise 4, p. 157] for the properties Theorem 3.1-(iii,v). Since P_τ is an idempotent radical when τ is a radical, it gives a torsion theory for R-$\mathcal{M}od$ with torsion class $\mathbb{T}_{P_\tau} = \mathbb{T}_\tau$ and torsion free class $\mathbb{F}_{P_\tau}$. By the results in [26, Ch. VI, §2], the properties for τ-torsion and τ-reduced modules in the following Proposition 3.4 are obtained because τ-torsion modules equate with P_τ-torsion modules and τ-reduced modules form the torsion free class $\mathbb{F}_{P_\tau}$.

Remark 3.3. See [13, pp. 29,63] for the definitions and properties of τ-dense submodules of a module and τ-cotorsionfree modules for a *hereditary* idempotent preradical τ on R-$\mathcal{M}od$: A submodule N of a module M is said to be τ-*dense* in M if M/N is τ-torsion, that is, $\tau(M/N) = M/N$, and a module M is said to be τ-*cotorsionfree* if it has *no* proper τ-dense submodules. Our definition of τ-coatomic module coincides with τ-cotorsionfree module but in our case, τ need not be idempotent or hereditary. Observe that since being τ-torsion is the same with being P_τ-torsion and P_τ is an idempotent preradical, the idempotent assumption is not a problem. But in our case τ is not assumed to be hereditary; in particular, Rad is not hereditary. The properties for τ-cotorsionfree modules given in [13] hold under this *hereditary* assumption. For example, arbitary direct sum of τ-cotorsionfree modules is τ-cotorsionfree when

τ is a *hereditary* idempotent preradical but in our case, for just an (idempotent) preradical τ, arbitrary direct sum of τ-coatomic modules need not be τ-coatomic.

Note also the following properties of τ-reduced and τ-coatomic modules which are easily proved:

Proposition 3.4. *Let τ be a preradical for R-$\mathcal{M}od$.*

(i) *The class of τ-torsion modules is closed under quotients and direct sums. Moreover, if τ is a radical, then the class of τ-torsion modules is closed under extensions.*
(ii) *The class of τ-reduced modules is closed under submodules, direct products and direct sums.*
(iii) *Every factor module of a τ-coatomic module is τ-coatomic.*
(iv) *The class of τ-reduced, respectively τ-coatomic, modules is closed under extensions, that is, if*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of modules such that A and C are τ-reduced, respectively τ-coatomic, then B is also τ-reduced, respectively τ-coatomic.

Proposition 3.5. *Let τ be a radical for R-$\mathcal{M}od$. If a module M is τ-coatomic, then $\tau(M) \ll M$.*

Proof. Suppose $\tau(M) + L = M$ for some submodule $L \leq M$. Since $M/L = (\tau(M) + L)/L \leq \tau(M/L)$, we obtain $M/L = \tau(M/L)$. This gives $L = M$ since M is τ-coatomic. Hence $\tau(M) \ll M$. □

4. τ-supplemented modules

Throughout the rest of the paper, τ denotes a radical on R-$\mathcal{M}od$ (where R is an arbitrary ring). See [1] and [10, §10] for properties of τ-supplements and τ-supplemented modules. In this section, we shall see some other properties of τ-supplemented modules. We shall frequently use the fact that any factor module of a τ-supplemented module is τ-supplemented [1, 2.2(2)].

Theorem 4.1. *If V is a τ-supplement in a module M, then $\tau(V) = V \cap \tau(M)$.*

Proof. $\tau(V) \leq V \cap \tau(M)$ always holds. To show the converse we only require to show that $(V \cap \tau(M))/\tau(V) = 0$. Since V is a τ-supplement in M, there exists a submodule $U \leq M$ such that $U + V = M$ and $U \cap V = \tau(V)$ by Proposition 2.1-(ii)). Then

$$M/(U \cap V) = (U/(U \cap V)) \oplus ((V/U \cap V)) = (U/\tau(V) \oplus (V/\tau(V)).$$

Since τ is a radical, we obtain:

$$\tau(M/\tau(V)) = \tau(U/\tau(V)) \oplus \tau(V/\tau(V)) = \tau(U/\tau(V)) \oplus 0 = \tau(U/\tau(V)).$$

By properties of a radical, since $\tau(V) \leq \tau(M)$, we have:

$$\tau(M)/\tau(V) = \tau(M/\tau(V)) = \tau(U/\tau(V)), \quad \textit{and}$$

$$\begin{aligned} (V \cap \tau(M))/\tau(V) &= (V/\tau(V)) \cap (\tau(M)/\tau(V)) = (V/\tau(V)) \cap \tau(U/\tau(V)) \\ &\leq (V/\tau(V)) \cap (U/\tau(V)) \\ &= (U \cap V)/\tau(V) = \tau(V)/\tau(V) = 0. \end{aligned}$$

□

Corollary 4.2. *If V is a Rad-supplement in a module M, then*

$$\operatorname{Rad} V = V \cap \operatorname{Rad} M.$$

Proposition 4.3. *Let K, L, M be modules such that $K \leq L \leq M$.*

(i) *If K is a τ-supplement in M, then it is a τ-supplement in L.*
(ii) *If $K \leq \tau(L)$ and L/K is a τ-supplement in M/K, then L is a τ-supplement in M.*
(iii) *If K is a τ-supplement in L and L is a τ-supplement in M, then K is a τ-supplement in M.*

Proof. (i) Since K is a τ-supplement in M, there exists a submodule $U \leq M$ such that $U + K = M$ and $U \cap K \leq \tau(K)$. So $L = L \cap M = L \cap (U + K) = L \cap U + K$ and $(L \cap U) \cap K = U \cap K \leq \tau(K)$.

(ii) Since L/K is a τ-supplement in M/K, there exists a submodule $U \leq M$ with $K \leq U$ such that $U/K + L/K = M/K$ and $(U/K) \cap (L/K) \leq \tau(L/K)$. So we obtain $U + L = M$ and

$$(U \cap L)/K = (U/K) \cap (L/K) \leq \tau(L/K) = \tau(L)/K$$

by properties of a radical since $K \leq \tau(L)$. Hence $U \cap L \leq \tau(L)$ and so L is a τ-supplement (of U) in M.

(iii) Temporarily denote by $\mathbb{E}$ the class induced by τ-supplement submodules; that is $\mathbb{E}$ is the class of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms such that $\mathrm{Im}(f)$ is a τ-supplement in B. For such a short exact sequence in the class $\mathbb{E}$, f is said to be an $\mathbb{E}$-monomorphism. By Proposition 2.1, the class $\mathbb{E}$ is the proper class injectively generated by all modules M such that $\tau(M) = 0$. By the definition of proper classes, the composition of two $\mathbb{E}$-monomorphisms is an $\mathbb{E}$-monomorphism (see [10, 10.1]). If K is a τ-supplement in L and L is a τ-supplement in M, then the inclusions $K \hookrightarrow L$ and $L \hookrightarrow M$ are $\mathbb{E}$-monomorphisms and so their composition $K \hookrightarrow M$ is also an $\mathbb{E}$-monomorphism, that is, K is a τ-supplement in M.

□

Proposition 4.4. *Let M be a module and let N, K be submodules of M such that $M = N + K$. If K is τ-supplemented, then K contains a τ-supplement of N in M.*

Proof. Since K is τ-supplemented, the submodule $N \cap K$ of K has a τ-supplement in K, that is, there exists a submodule $L \leq K$ such that $(N \cap K) + L = K$ and $(N \cap K) \cap L \leq \tau(L)$. Then $M = N + K = N + (N \cap K) + L = N + L$ and $N \cap L = (N \cap K) \cap L \leq \tau(L)$. Hence L is a τ-supplement of N in M. □

It is trivial to show that:

Proposition 4.5.
(i) *Every τ-torsion module is τ-supplemented.*
(ii) *The module $P_\tau(M)$ is τ-supplemented for every module M.*

Theorem 4.6. *If a module M is τ-reduced and τ-supplemented, then M is τ-coatomic,* $\mathrm{Rad}\, M = \tau(M)$ *and M is weakly supplemented.*

Proof. Let U be a proper submodule of M. Since M is τ-supplemented, there exists a submodule $V \leq M$ such that $U + V = M$ and $U \cap V \leq \tau(V)$. So we have $\tau(V/(U \cap V)) = \tau(V)/(U \cap V)$ by properties of a radical. We also have $\tau(V) \neq V$ since M is τ-reduced, and so $\tau(V)/(U \cap V) \neq V/(U \cap V)$. Therefore, using the fact that $M/U = (U + V)/U \cong V/(U \cap V)$ we obtain

$$\tau(M/U) \cong \tau(V/(U \cap V)) = \tau(V)/(U \cap V) \neq V/(U \cap V),$$

or equivalently, $\tau(M/U) \neq M/U$, that is, M is τ-coatomic. By Proposition 3.5, $\tau(M) \ll M$ and hence $\tau(M) \leq \mathrm{Rad}\, M$. By [1, 2.2(3)], $M/\tau(M)$ is semisimple since M is τ-supplemented. Then $\mathrm{Rad}(M/\tau(M)) = 0$ and so $\mathrm{Rad}\, M \leq \tau(M)$. Thus $\mathrm{Rad}\, M = \tau(M)$. Since $\mathrm{Rad}\, M = \tau(M) \ll M$ and M is a semilocal module (that is $M/\mathrm{Rad}\, M = M/\tau(M)$ is semisimple), we obtain that M is weakly supplemented by [19, Theorem 2.7]. □

Theorem 4.7. *If M is a τ-supplemented module, then* $\mathrm{Rad}\, M \leq \tau(M)$*, and*

$$\mathrm{Rad}(M/P_\tau(M)) = \tau(M/P_\tau(M)) = \tau(M)/P_\tau(M).$$

Proof. By [1, 2.2(3)], $M/\tau(M)$ is semisimple and so $\mathrm{Rad}(M/\tau(M)) = 0$ which gives $\mathrm{Rad}\, M \leq \tau(M)$. The module $M/P_\tau(M)$ is τ-supplemented as a factor module of the τ-supplemented module M. Since $M/P_\tau(M)$ is τ-reduced, $\mathrm{Rad}(M/P_\tau(M)) = \tau(M/P_\tau(M))$ by Theorem 4.6. By properties of a radical, $\tau(M/P_\tau(M)) = \tau(M)/P_\tau(M)$. □

Proposition 4.8. *The following are equivalent for a module M and a submodule $K \leq P_\tau(M)$:*

(i) *M is τ-supplemented;*
(ii) *M/K is τ-supplemented;*
(iii) *$M/P_\tau(M)$ is τ-supplemented.*

Proof. Since every factor module of a τ-supplemented module is τ-supplemented, $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear. To prove $(iii) \Rightarrow (i)$, take $U \leq M$. By hypothesis, there is a submodule $V \leq M$ such that $P_\tau(M) \leq V$,

$$[(U + P_\tau(M))/P_\tau(M)] + [V/P_\tau(M)] = M/P_\tau(M)$$

and

$$\begin{aligned}(U \cap V + P_\tau(M))/P_\tau(M) &= [(U + P_\tau(M))/P_\tau(M)] \cap [V/P_\tau(M)] \\ &\leq \tau(V/P_\tau(M)) = \tau(V)/P_\tau(M).\end{aligned}$$

Note that the last equality holds by Theorem 3.1-(iv). So we have $U+V = M$ and $U \cap V \leq \tau(V)$. That is V is a τ-supplement of U in M. □

Corollary 4.9. *The following are equivalent for a ring R:*

(i) *every R-module is τ-supplemented;*
(ii) *every free R-module is τ-supplemented;*
(iii) *every τ-reduced R-module is τ-supplemented.*

Proof. $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ are clear. $(ii) \Rightarrow (i)$ follows since every module is an epimorphic image of a free R-module and being τ-supplemented is preserved under passage factor modules. To prove $(iii) \Rightarrow (i)$ take an R-module M. Since $M/P_\tau(M)$ is τ-reduced, we obtain that $M/P_\tau(M)$ is τ-supplemented by the hypothesis. So M is τ-supplemented by Proposition 4.8. □

Proposition 4.10. *If V is a τ-supplement in a module M and V is τ-coatomic, then V is a supplement in M.*

Proof. Since V is a τ-supplement in M, there exists $U \leq M$ such that $U + V = M$ and $U \cap V \leq \tau(V)$. Since V is τ-coatomic, we have by Proposition 3.5 that $\tau(V) \ll V$. Then $U \cap V \leq \tau(V) \ll V$ and so V is a supplement in M. □

Proposition 4.11. *If M is a τ-reduced module that is totally τ-supplemented, then M is totally supplemented.*

Proof. Since being τ-reduced is inherited by submodules, it is enough to prove that M is supplemented. Let $U \leq M$ and V be a τ-supplement of U in M. Then $U+V = M$ and $U \cap V \leq \tau(V)$. By hypothesis, V is τ-supplemented and τ-reduced. So by Theorem 4.6, V is τ-coatomic. Then $\tau(V) \ll V$ by Proposition 3.5. Therefore $U \cap V \ll V$ and so V is a supplement of U in M. Hence M is supplemented. □

Clearly supplemented modules are Rad-supplemented and so we obtain the following:

Corollary 4.12. *If M is a reduced module, then M is totally Rad-supplemented if and only if M is totally supplemented.*

5. When are all left R-modules τ-supplemented?

In this section, we shall characterize the rings all of whose (left) modules are τ-supplemented for some particular radicals τ including Rad.

An epimorphism $f : P \to M$ is said to be a *projective cover* if P is projective and $\operatorname{Ker} f \ll P$. A property that we shall use is that if P is projective and P/U has a projective cover, then U has a supplement V in P such that V is a direct summand of P and hence projective (see [30, 42.1]). A ring R is called *left perfect* if every left R-module has a projective cover. Recall that, a subset I of a ring R is said to be *left T-nilpotent* in case for every sequence $\{a_k\}_{k=1}^{\infty}$ in I there is a positive integer n such that $a_1 \cdots a_n = 0$. A ring R is said to be a *left max ring* if every left R-module has a maximal submodule, equivalently $\operatorname{Rad}(M) \ll M$ for every left R-module M. A ring R is said to be a *semilocal* ring if $R/J(R)$ is a semisimple ring (that is a left (and right) semisimple R-module), see [18, §20]. Semilocal rings are also referred to as rings semisimple modulo their radical (see [4, §15, pp. 170-172]). For a semilocal ring R, $\operatorname{Rad} M = JM$ for every left R-module M where $J = J(R)$ (see for example [4, Corollary 15.18]). By a characterization of left perfect rings by Bass, as in for example [4, Theorem 28.4], a ring R is left perfect if and only if R is a semilocal ring and $J(R)$ is left T-nilpotent if and only if R is a semilocal left max ring. A ring R is called *left semiperfect* if every finitely generated left R-module has a projective cover. A ring R is (left or right) semiperfect if and only if the left (or right) R-module R is supplemented (see [30, 42.6]).

An epimorphism $f : N \to M$ is said to be a *τ-cover* if $\operatorname{Ker} f \leq \tau(N)$. If moreover N is projective, then f is called a *projective τ-cover*. A ring R is called left *τ-perfect* if every left R-module has a projective τ-cover. These rings are studied in [5] and [31] for the radical $\tau = \operatorname{Rad}$, and in [23] for a larger class of preradicals. A ring R is called left *τ-semiperfect* if every finitely generated left R-module has a projective τ-cover. The relation between τ-cover and τ-supplements is the following:

Proposition 5.1. [1, 2.14] *For an R-module L and $U \leq L$, the following are equivalent:*
(i) *L/U has a projective τ-cover;*
(ii) *U has a τ-supplement V which has a projective τ-cover.*

It is clear from the definitions and Proposition 5.1 that, if R is a left τ-(semi)perfect ring then every (finitely generated) left R-module is τ-supplemented. But the converse need not be true, for example when $\tau = \operatorname{Rad}$; see Example 6.2.

Lemma 5.2. *If R is a ring that is a τ-reduced left R-module and if the free left R-module $F = R^{(\mathbb{N})}$ is τ-supplemented, then $\tau(R)$ is left T-nilpotent.*

Proof. Since $P_\tau(R) = 0$ and $P_\tau(F) = (P_\tau(R))^{(\mathbb{N})} = 0$, F is τ-reduced. Then F is τ-coatomic by Theorem 4.6, and so by Proposition 3.5

$$\tau(R)F = (\tau(R))^{(\mathbb{N})} = \tau(F) \ll F.$$

Therefore $\tau(R)$ is left T-nilpotent by [4, Lemma 28.3]. □

Theorem 5.3. *If R is a ring that is a τ-reduced left R-module, then the free left R-module $F = R^{(\mathbb{N})}$ is τ-supplemented if and only if R is left perfect and $\tau(R) = J(R)$.*

Proof. Suppose $F = R^{(\mathbb{N})}$ is τ-supplemented. Then R is τ-supplemented as a direct summand of F. Since R is also τ-reduced by hypothesis, we obtain $\tau(R) = J(R)$ by Theorem 4.6. By Lemma 5.2, $J(R) = \tau(R)$ is left T-nilpotent. Since R is τ-supplemented, $R/J(R) = R/\tau(R)$ is semisimple by [1, 2.2(3)]. Hence R is left perfect by [4, Theorem 28.4]. Conversely suppose R is left perfect and $\tau(R) = J(R)$. Let $U \leq F = R^{(\mathbb{N})}$. Since R is left perfect, every left R-module, and in particular, F/U has a projective cover. Then by [30, 42.1]), U has a supplement V in the free module F such that V is a direct summand of F. Since F is free, its direct summand V is projective. So $\tau(V) = \tau(R)V$ by properties of radicals. Since V is a supplement of U in M, $U + V = M$ and $U \cap V \ll V$. So $U \cap V \leq \operatorname{Rad}(V)$. Since R is a left perfect ring, it is a semilocal ring and so $\text{Rad}(V) = J(R)V$. Thus $U \cap V \leq \text{Rad}(V) = J(R)V = \tau(R)V = \tau(V)$. Hence V is a τ-supplement of U in M. □

Note that the above proof for the converse implication works for every free left R-module F, not necessarily countably generated. Moreover, since every factor module of a τ-supplemented module is τ-supplemented and every module is isomorphic to a factor module of a free module, we have:

Corollary 5.4. *If R is a ring that is a τ-reduced left R-module, then every (free) left R-module is τ-supplemented if and only if R is left perfect and $\tau(R) = J(R)$.*

It is easy to see that a radical τ on R-modules is also a radical on $R/P_\tau(R)$-modules since every $R/P_\tau(R)$-module can be considered as an R-module (with annihilator containing $P_\tau(R)$). We shall use this fact in the proof of the following theorem:

Theorem 5.5. *For a ring R with $P_\tau(R) \leq J(R)$, the following are equivalent.*

(i) *every left R-module is τ-supplemented;*
(ii) *every free left R-module is τ-supplemented;*
(iii) *the free left R-module $F = R^{(\mathbb{N})}$ is τ-supplemented;*
(iv) *the quotient ring $R/P_\tau(R)$ is left perfect and $\tau(R) = J(R)$.*

Proof. $(i) \Leftrightarrow (ii)$ follows by Corollary 4.9. $(ii) \Rightarrow (iii)$ is clear.

$(iii) \Rightarrow (iv)$: Since F is τ-supplemented, so is its factor module $\overline{F} = F/P_\tau(F) \cong (R/P_\tau(R))^{(\mathbb{N})}$. The R-module $\overline{F}$ can be considered as an $R/P_\tau(R)$-module and τ can be considered also as a radical on $R/P_\tau(R)$-modules. By Theorem 5.3, since $R/P_\tau(R)$ is τ-reduced, we obtain that the quotient ring $R/P_\tau(R)$ is left perfect and

$$\tau(R/P_\tau(R)) = J(R/P_\tau(R)).$$

Then by properties of radicals, $\tau(R/P_\tau(R)) = \tau(R)/P_\tau(R)$ and $J(R/P_\tau(R)) = J(R)/P_\tau(R)$ since $P_\tau(R) \leq J(R)$ by hypothesis. Hence $\tau(R) = J(R)$.

$(iv) \Rightarrow (ii)$: By properties of radicals, since $P_\tau(R) \leq \tau(R) = J(R)$ by hypothesis, we obtain for the left perfect quotient ring $S = R/P_\tau(R)$ that:

$$\tau(S) = \tau(R/P_\tau(R)) = \tau(R)/P_\tau(R) = J(R)/P_\tau(R) = J(R/P_\tau(R)) = J(S).$$

By Corollary 5.4, every free S-module is τ-supplemented, where we consider τ also as a radical on S-modules. Let F be a free R-module. Then $F \cong R^{(I)}$ for some index set I. By Proposition 4.8, it is enough to prove that $\overline{F} = F/P_\tau(F) \cong S^{(I)}$ is τ-supplemented. But this holds since $\overline{F}$ can be considered as a free S-module. □

6. When are all left R-modules Rad-supplemented?

Using the results of the previous sections for $\tau = \text{Rad}$, we obtain the following characterization of the rings R over which every R-module is Rad-supplemented. Of course, more work still remains to understand $P(R)$ and the condition that $R/P(R)$ is left perfect.

Theorem 6.1. *For a ring R, the following are equivalent.*

(i) *every left R-module is Rad-supplemented;*
(ii) *every reduced left R-module is Rad-supplemented;*
(iii) *every reduced left R-module is supplemented;*
(iv) *the free left R-module $R^{(\mathbb{N})}$ is Rad-supplemented;*
(v) *$R/P(R)$ is left perfect.*

Proof. $(i) \Leftrightarrow (iv) \Leftrightarrow (v)$ is obtained by Theorem 5.5 since $P(R) \leq \text{Rad}(R) = J(R)$. $(i) \Leftrightarrow (ii)$ follows by Corollary 4.9. $(iii) \Rightarrow (ii)$ holds since supplemented modules are Rad-supplemented. To prove $(ii) \Rightarrow (iii)$, take any reduced left R-module M. Then every submodule of M is also reduced and Rad-supplemented by hypothesis (ii). So M is a reduced module that is totally Rad-supplemented. By Corollary 4.12, M is totally supplemented and hence supplemented. □

The following is an example of a ring R that is not left perfect (and so not left Rad-perfect by [23, Theorem 1.5]) but where all R-modules are Rad-supplemented.

Example 6.2. Let k be a field. In the polynomial ring $k[x_1, x_2, \ldots]$ with countably many indeterminates x_n, $n \in \mathbb{Z}^+$, consider the ideal $I = (x_1^2, x_2^2 - x_1, x_3^2 - x_2, \ldots)$ generated by x_1^2 and $x_{n+1}^2 - x_n$ for each $n \in \mathbb{Z}^+$. In the quotient ring $R = k[x_1, x_2, \ldots]/I$, the maximal ideal $M = (x_1, x_2, \ldots)/I$ of R generated by all $\overline{x_n} = x_n + I$, $n \in \mathbb{Z}^+$, is the *unique* maximal ideal of R. This is because, if K is any maximal ideal of R, then $\overline{x}_1^2 = 0 \in K$ and so $\overline{x}_1 \in K$ since K is a prime ideal. Now $\overline{x}_2^2 = \overline{x}_1 \in K$ and so $\overline{x}_2 \in K$. By induction, we obtain $\overline{x}_n^2 = \overline{x}_{n-1} \in K$ and so $\overline{x}_n \in K$ for all $n \in \mathbb{Z}^+$. Therefore $K = M$, as desired. Since $\overline{x}_n = \overline{x}_{n+1}^2$ for every $n \in \mathbb{Z}^+$, we obtain $M = M^2$. So $\operatorname{Rad} M = M$ and hence $P(R) = M$. Since the ring $R/P(R) = R/M$ is a field (and so perfect), every R-module is Rad-supplemented (by Theorem 6.1). By [4, Lemma 28.3], $M = J(R)$ is not (left) T-nilpotent, and so R is not a (left) perfect ring.

In [9], it is proved that the class of rings that are Rad-supplemented lies properly between the classes of semilocal rings and semiperfect rings. Recall that a ring R is said to be a *left duo ring* if every left ideal of R is a two-sided ideal. We shall characterize the left duo rings R that are Rad-supplemented left R-modules. Firstly, we need the following lemma:

Lemma 6.3. *If R is a left duo ring and J, A, B are left ideals of R such that $A + B = R$ and $A \cap B = JA \cap JB$, then $A \cap B = J(A \cap B)$.*

Proof. Clearly $J(A \cap B) \leq A \cap B$. Conversely let $x \in A \cap B = JA \cap JB$. Since $A + B = R$, we have $a + b = 1$ for some $a \in A$ and $b \in B$. Then $x = xa + xb$ and $x = \sum_{i \in I} s_i a_i = \sum_{i \in I'} t_i b_i$ where I, I' are finite index sets, $a_i \in A$, $b_i \in B$ and $s_i, t_i \in J$. Now we have,

$$xb = \sum_{i \in I} s_i a_i b \in J(AB) \text{ and } xa = \sum_{i \in I'} t_i b_i a \in J(BA).$$

Since R is a left duo ring we have $AB \leq A \cap B$ and $BA \leq A \cap B$. So $x = xa + xb \in J(BA) + J(AB) \leq J(A \cap B)$. Thus $A \cap B \leq J(A \cap B)$. □

Theorem 6.4. *If R is a left duo ring such that $P(R) = 0$, then R is a Rad-supplemented left R-module if and only if R is semiperfect.*

Proof. If R is semiperfect, then R is a supplemented, and so a Rad-supplemented, left R-module. Conversely, suppose R is a Rad-supplemented left R-module. Then R is semilocal and R is an amply Rad-supplemented left R-module by [1, 2.2(3) and 2.6(2)]. Let A' be a left ideal of R. Since R is an amply Rad-supplemented left R-module, A' has a Rad-supplement B in R, and B has a Rad-supplement $A \leq A'$ in R. So $R = A' + B = A + B$, $A \cap B \leq A' \cap B \leq \operatorname{Rad} B$ and $A \cap B \leq \operatorname{Rad} A$. Thus $A \cap B = (\operatorname{Rad} A) \cap (\operatorname{Rad} B)$. Let $J = J(R)$. Then $A \cap B = JA \cap JB = J(A \cap B)$ by Lemma 6.3. Since R is a semilocal ring, $\operatorname{Rad}(A \cap B) = J(A \cap B)$. Then $A \cap B$ is a Rad-torsion submodule of R and so $A \cap B \leq P(R) = 0$. This gives that $R = A \oplus B$. Therefore $JB \leq J \ll R$ implies that $\operatorname{Rad}(B) = JB \ll B$ since B is a direct summand of R. Hence B is a supplement of A' in R. This shows that R is a supplemented left R-module and so R is semiperfect (see [30, 42.6]). □

Theorem 6.5. *For a left duo ring R, the following are equivalent:*

(i) *$R/P(R)$ is semiperfect;*
(ii) *the left R-module R is Rad-supplemented;*
(iii) *every finitely generated free left R-module is Rad-supplemented;*
(iv) *every finitely generated left R-module is Rad-supplemented.*

Proof. *(ii)* ⇒ *(iii)* follows by [1, 2.3(2)]. *(iii)* ⇒ *(iv)* holds since every finitely generated module is an epimorphic image of a finitely generated free module and Rad-supplemented modules are closed under epimorphic images. *(iv)* ⇒ *(ii)* is clear.

(i) ⇒ *(ii)*: Since the quotient ring $S = R/P(R)$ is semiperfect, $R/P(R)$ is a Rad-supplemented left S-module and so a Rad-supplemented left R-module. Then the left R-module R is Rad-supplemented by Proposition 4.8.

$(ii) \Rightarrow (i)$: The factor module $R/P(R)$ is also a Rad-supplemented left R-module. So the ring $S = R/P(R)$ is a Rad-supplemented left S-module with $P(S) = 0$ and so $S = R/P(R)$ is semiperfect by Theorem 6.4. □

Note that all implications except $(ii) \Rightarrow (i)$ of Theorem 6.5 hold for any ring R, while the implication $(ii) \Rightarrow (i)$ raises the question whether a Rad-supplemented ring R with $P(R) = 0$ is necessarily semiperfect.

7. Rad-supplemented Modules over Dedekind Domains

Over Dedekind domains, divisible modules coincide with injective modules as in abelian groups. Note that for a module M over a Dedekind domain R, M is divisible if and only if $\operatorname{Rad} M = M$, and this holds if and only if M is injective; see for example [2, Lemma 4.4]. This is the motivation for the definition of reduced modules in general. A module over a Dedekind domain is *reduced* if it has no nonzero divisible submodules. As in abelian groups (see for example [11, Theorem 21.3]), any module M over a Dedekind domain possesses a unique largest divisible submodule D and $M = D \oplus C$ for a reduced submodule C of M (see [16, Theorem 8]); this D is called the *divisible part* of M. Following the terminology in abelian groups, an R-module M over a Dedekind domain is said to be *bounded* if $rM = 0$ for some nonzero $r \in R$.

The structure of supplemented modules over Dedekind domains is completely determined in [32]:

Theorem 7.1. [32, Theorem 2.4. and Theorem 3.1] *Let R be a Dedekind domain with quotient field $K \neq R$. Let M be an R-module.*

- (i) *Suppose R is a local Dedekind domain, that is, a discrete valuation ring (DVR) with the unique prime element p. Then M is supplemented if and only if $M \cong R^a \oplus K^b \oplus (K/R)^c \oplus B$ for some R-module B, where a, b, c are nonnegative integers and $p^n B = 0$ for some integer $n > 0$.*
- (ii) *Suppose R is non local. Then M is supplemented if and only if M is torsion and every primary component of M is a direct sum of an artinian submodule and a bounded submodule.*

Part (i) of the above theorem for Rad-supplemented modules is obtained as follows:

Theorem 7.2. *Let R be a DVR with quotient field $K \neq R$, and p be the unique prime element. Then M is Rad-supplemented if and only if $M \cong R^a \oplus K^{(I)} \oplus (K/R)^{(J)} \oplus B$ for some R-module B, where a is a nonnegative integer, I, J are arbitrary index sets and $p^n B = 0$ for some integer $n0$.*

Proof. $(\Rightarrow)$: If M_1 is the divisible part of M, then there exists a reduced submodule M_2 of M such that $M = M_1 \oplus M_2$. Since M_2 is also Rad-supplemented, it is coatomic by Theorem 4.6. Then by [32, Lemma 2.1], $M_2 = R^a \oplus B$, for some nonnegative integer a and a bounded module B. Since M_1 is divisible, $M_1 \cong K^{(I)} \oplus (K/R)^{(J)}$ for some index sets I and J (see [16, Theorem 7]).

$(\Leftarrow)$: The module $N = K^{(I)} \oplus (K/R)^{(J)}$ is divisible, and so $\operatorname{Rad} N = N$. Then N is Rad-supplemented by Proposition 4.5. By Theorem 7.1, the module $R^a \oplus B$ is supplemented, and hence Rad-supplemented. Therefore the direct sum $R^a \oplus K^{(I)} \oplus (K/R)^{(J)} \oplus B$ is Rad-supplemented. □

Over commutative Noetherian rings we have:

Proposition 7.3. *Let R be a commutative noetherian ring and M be a reduced R-module. Then M is Rad-supplemented if and only if M is supplemented.*

Proof. Suppose M is Rad-supplemented. Then M is coatomic by Theorem 4.6, and so every submodule of M is coatomic by [33, Lemma 1.1] since R is a commutative noetherian ring. Let U be a submodule of M and V be a Rad-supplement of U in M. Then V is coatomic, and so $U \cap V \leq \operatorname{Rad} V \ll V$. Thus V is a supplement of U in M. The converse is clear. □

Since the structure of supplemented modules is known by Theorem 7.1, it is enough to characterize Rad-supplemented modules in terms of supplemented modules. Note that for an R-module M where R is a Dedekind domain, $P(M)$ equals the *divisible part* of M.

Theorem 7.4. *Let R be a Dedekind domain and M be an R-module. Then M is Rad-supplemented if and only if $M/P(M)$ is (Rad-)supplemented.*

Proof. Since R is a Dedekind domain, M has a decomposition as $M = P(M) \oplus N$ for some reduced submodule N of M. If M is Rad-supplemented, then $N \cong M/P(M)$ is also Rad-supplemented. Since N is reduced, N is supplemented by Proposition 7.3. Conversely, suppose $N \cong M/P(M)$ is Rad-supplemented. By Proposition 4.5-(ii), the submodule $P(M)$ is already Rad-supplemented. Therefore $M = P(M) \oplus N$ is Rad-supplemented as a sum of two Rad-supplemented modules. □

These characterizations can be used to give examples of Rad-supplemented modules which are not supplemented.

Example 7.5. Let R be a Dedekind domain with quotient field $K \neq R$. The R-module $M = K^{(I)}$ is Rad-supplemented for every index set I. If R is a local Dedekind domain (i.e. a DVR), then M is supplemented only when I is finite. If R is a non-local Dedekind domain, then M is not supplemented for every index set I, since M is not torsion.

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THE PROPER CLASS GENERATED BY WEAK SUPPLEMENTS

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1. INTRODUCTION

It is well-known that the class $\mathcal{S}uppl$ of short exact sequences determined by supplement submodules is proper in the sense of Buchsbaum (see [7], 20.7). In this paper we study three classes of short exact sequences: $\mathcal{S}mall$, $\mathcal{S}$ and $\mathcal{WS}$ determined by small submodules, submodules that have supplements and weak supplement submodules respectively. These classes are not proper in general, so we study the least proper classes containing them, that is the proper classes generated by these classes (see [14]). It turned out that for a hereditary ring R they generate the same proper class $\overline{\mathcal{WS}}$ and this proper class can be obtained by natural extension of $\mathcal{WS}$. We study injective, projective, coinjective and coprojective objects of $\overline{\mathcal{WS}}$. Note that injective and projective objects of this class coincide with the injective and projective objects of $\mathcal{S}mall$, $\mathcal{S}$ and $\mathcal{WS}$ (see [14]). We prove that $\overline{\mathcal{WS}}$ is coinjectively generated, so by Proposition in [3] $\operatorname{gl.dim}\overline{\mathcal{WS}} \leq 1$ over a hereditary ring. We also describe $\overline{\mathcal{WS}}$ in terms of supplement submodules: A is a $\overline{\mathcal{WS}}$-submodule of B iff there is a submodule C of B such that $A + C = B$ and $A \cap C$ is coatomic. We end the paper with some relations between $\overline{\mathcal{WS}}$ and coneat submodules.

2. PRELIMINARIES

Let $\mathcal{P}$ be a class of short exact sequences of R-modules and R-module homomorphisms. If a short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \qquad (1)$$

belongs to $\mathcal{P}$, then f is said to be a $\mathcal{P}$-*monomorphism* and g is a $\mathcal{P}$-*epimorphism* (both are said to be $\mathcal{P}$-*proper* and the short exact sequence is said to be a $\mathcal{P}$-*proper* short exact sequence.). A short exact sequence $\mathbb{E}$ is determined by each of the monomorphism f and epimorphism g uniquely up to isomorphism.

Definition 2.1. The class $\mathcal{P}$ is said to be *proper* (in the sense of Buchsbaum) if it satisfies the following conditions ([5], [10], [12]):

P-1) If a short exact sequence $\mathbb{E}$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to $\mathbb{E}$.
P-2) $\mathcal{P}$ contains all splitting short exact sequences.
P-3) The composite of two $\mathcal{P}$-monomorphisms is a $\mathcal{P}$-monomorphism if this composite is defined.

P-3') The composite of two $\mathcal{P}$-epimorphisms is a $\mathcal{P}$-epimorphism if this composite is defined.

P-4) If g and f are monomorphisms, and $g \circ f$ is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism.

P-4') If g and f are epimorphisms, and $g \circ f$ is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

The set $\mathrm{Ext}_{\mathcal{P}}(C, A)$ of all short exact sequence of $\mathrm{Ext}(C, A)$ that belongs to $\mathcal{P}$ is a subgroup of the group of the extensions $\mathrm{Ext}^1_R(C, A)$.

Proposition 2.2 ([11], Proposition 1.7)**.** *An R-module N is $\mathcal{P}$-coinjective if and only if there is $\mathcal{P}$-monomorphism from N to an injective module I.*

Corollary 2.3 ([11], Proposition 1.8)**.** *If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence in a proper class $\mathcal{P}$ and B is $\mathcal{P}$-coinjective, then A is also $\mathcal{P}$-coinjective.*

Proposition 2.4 ([11], Proposition 1.12)**.** *An R-module M is $\mathcal{P}$-coprojective if and only if there is a $\mathcal{P}$-epimorphism from a projective R-module P to M.*

Let $\mathcal{M}$ and $\mathcal{J}$ be classes of modules over some ring R. The smallest proper class $\overline{k}(\mathcal{M})$ (resp. $\underline{k}(\mathcal{J})$) for which all modules in $\mathcal{M}$ (resp. $\mathcal{J}$) are coprojective (resp. coinjective) is said to be coprojectively (resp. coinjectively) generated by $\mathcal{M}$ (resp. $\mathcal{J}$).

Definition 2.5. For a proper class $\mathcal{P}$ of short exact sequences of R-modules, the global dimension of $\mathcal{P}$ is defined as

$$\mathrm{gl.\,dim}\,\mathcal{P} = \inf\{n : \mathrm{Ext}^{n+1}(C, A) = 0 \text{ for all A and C in R-modules}\}.$$

If there is no such n, then $\mathrm{gl.\,dim}\,\mathcal{P} = \infty$.

Definition 2.6. For a proper class $\mathcal{P}$ of short exact sequences of R-modules, the injective dimension of a module A with respect to $\mathcal{P}$ is defined by the formula

$$\mathrm{inj.\,dim}\,A = \inf\{n : \mathrm{Ext}^{n+1}(C, A) = 0 \text{ for all C in R-modules}\}.$$

Proposition 2.7 ([3])**.** *If R is a hereditary ring, then* $\mathrm{inj.\,dim}\,A \leq 1$ *for every proper class $\mathcal{P}$ and $\mathcal{P}$-coinjective module A.*

Proposition 2.8 ([3])**.** *If $\underline{k}(\mathcal{J})$ is closed under extensions, then* $\mathrm{gl.\,dim}\,\underline{k}(\mathcal{J}) \leq \mathrm{gl.\,dim}\,R$ *for every coinjectively generated class $\underline{k}(\mathcal{J})$.*

Corollary 2.9 ([3])**.** *If R is a hereditary ring, then* $\mathrm{inj.\,dim}\,\underline{k}(\mathcal{J}) \leq 1$ *for every coinjectively generated class $\underline{k}(\mathcal{J})$.*

For more information about coprojectively and coinjectively generated proper classes see [1], [2] and [3]. The following propositions give the relation between projective (resp. injective) modules with respect to a class $\mathcal{E}$ of short exact sequences and with respect to the proper class $< \mathcal{E} >$ generated by $\mathcal{E}$.

Proposition 2.10 ([14], Propositions 2.3 and 2.4)**.**

(a) $\pi(\mathcal{E}) = \pi(< \mathcal{E} >)$

(b) $\iota(\mathcal{E}) = \iota(< \mathcal{E} >)$.

3. THE LEAST PROPER CLASS CONTAINING $\mathcal{WS}$

Let $\mathcal{S}$ be the class of all short exact sequences (called by Zöschinger κ-elements in [15])

$$E : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{2}$$

such that $\operatorname{Im} f$ has a supplement in B, i.e. a minimal element in the set $\{V \subset B \mid V + \operatorname{Im} f = B\}$. We denote by $\mathcal{WS}$ the class of short exact sequences (2), where $\operatorname{Im} f$ has (is) a weak supplement in B, i.e. there is a submodule K of B such that $\operatorname{Im} f + K = B$ and $\operatorname{Im} f \cap K \ll B$ and by $\mathcal{S}mall$ the class of short exact sequences (2) where $\operatorname{Im} f \ll B$.

If X is a $\mathcal{S}mall$-submodule of an R-module Y, then Y is a supplement of X in Y, so X is $\mathcal{S}$-submodule of Y. If U is a $\mathcal{S}$-submodule of an R-module Z, then a supplement V of U in Z is also a weak supplement, therefore U is a $\mathcal{WS}$-submodule of Z. These arguments give us the relation $\mathcal{S}mall \subseteq \mathcal{S} \subseteq \mathcal{WS}$ for any ring R. Neither of classes $\mathcal{S}mall$, $\mathcal{S}$ and $\mathcal{WS}$ need be a proper class in general as shows the following example.

Example 3.1. Let $R = \mathbb{Z}$ and consider the composition $\beta \circ \alpha$ of the monomorphisms $\alpha : 2\mathbb{Z} \longrightarrow \mathbb{Z}$ and $\beta : \mathbb{Z} \longrightarrow \mathbb{Q}$ where α and β are the corresponding inclusions. Then the short exact sequence $0 \longrightarrow 2\mathbb{Z} \xrightarrow{\beta\circ\alpha} \mathbb{Q} \longrightarrow \mathbb{Q}/2\mathbb{Z} \longrightarrow 0$ is in $\mathcal{S}mall$, but the short exact sequence $0 \longrightarrow 2\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ is not in $\mathcal{WS}$ as $2\mathbb{Z}$ has not a weak supplement in $\mathbb{Z}$.

This example shows that $\operatorname{Ext}_{\mathcal{WS}}(\cdot, \cdot)$ is not a subfunctor of $\operatorname{Ext}(\cdot, \cdot)$ since the elements from $\mathcal{WS}$ are not preserved with respect to the first variable. We extend the class $\mathcal{WS}$ to the class $\overline{\mathcal{WS}}$, which consists of all images of $\mathcal{WS}$-elements of $\operatorname{Ext}(C', A)$ under $\operatorname{Ext}(f, 1_A) : \operatorname{Ext}(C', A) \longrightarrow \operatorname{Ext}(C, A)$ for all homomorphism $f : C \longrightarrow C'$. We will prove that $\overline{\mathcal{WS}}$ is the least proper class containing $\mathcal{WS}$. To prove that $\overline{\mathcal{WS}}$ is a proper class we will use the Theorem 1.1 in [9] that states that a class $\mathcal{P}$ of short exact sequences is proper if $\operatorname{Ext}_{\mathcal{P}}(C, A)$ is a subfunctor of $\operatorname{Ext}_R(C, A)$, $\operatorname{Ext}_{\mathcal{P}}(C, A)$ is a subgroup of $\operatorname{Ext}_R(C, A)$ for every R-modules A, C and the composition of two $\mathcal{P}$-monomorphisms ($\mathcal{P}$-epimorphisms) is a $\mathcal{P}$-monomorphism ($\mathcal{P}$-epimorphism).

Definition 3.2. A short exact sequence $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is said to be extended weak supplement if there is a short exact sequence $E' : 0 \xrightarrow{f} A \longrightarrow B' \longrightarrow C' \longrightarrow 0$ such that $\operatorname{Im} f$ has (is) a weak supplement and there is a homomorphism $g : C \longrightarrow C'$ such that $E = g^*(E')$, that is there is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 : E \\
 & & \| & & \downarrow & & \downarrow g & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B' & \longrightarrow & C' & \longrightarrow & 0 : E'
\end{array}$$

The class of all extended weak supplement short exact sequences will be denoted by $\overline{\mathcal{WS}}$. So $\mathrm{Ext}_{\overline{\mathcal{WS}}}(C, A) = \{ E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \mid E = g^*(E')$ for some $E' : 0 \longrightarrow A \longrightarrow B \longrightarrow C' \longrightarrow 0 \in \mathcal{WS}$ and $g : C \to C' \}$.

Lemma 3.3. *If $f : A \longrightarrow A'$, then $f_* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C, A')$ preserves $\mathcal{WS}$-elements.*

Proof. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence in the class $\mathcal{WS}$ and $f : A \longrightarrow A'$ be an arbitrary homomorphism. We have the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & C & \longrightarrow & 0 : E \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \| & & \\ 0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \longrightarrow & C & \longrightarrow & 0 : E_1 \end{array}$$

where $E_1 = f_*(E)$.

If V is a weak supplement of $\mathrm{Im}\,\alpha$ in B, then $\mathrm{Im}\,\alpha + V = B$ and $\mathrm{Im}\,\alpha \cap V \ll B$. Then $f'(V) + \mathrm{Im}\,\alpha' = B'$ by push out diagram and $f'(V) \cap \mathrm{Im}\,\alpha' = f'(\mathrm{Im}\,\alpha \cap V) \ll f'(B) \subseteq B'$. So $E_1 \in \mathcal{WS}$. □

Lemma 3.4. *If $f : A \longrightarrow A'$, then $f_* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C, A')$ preserves $\overline{\mathcal{WS}}$-elements.*

Proof. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence in the class $\overline{\mathcal{WS}}$ and $f : A \longrightarrow A'$ be an arbitrary homomorphism. Then $E = g^*(E')$ for some $E' : 0 \longrightarrow A \longrightarrow B_1 \longrightarrow C_1 \longrightarrow 0 \in \mathcal{WS}$ and a homomorphism $g : C \longrightarrow C_1$. Therefore $E_1 = f_*(E) = f_* \circ g^*(E') = g^* \circ f_*(E') = g^*(E'_1)$ where $E'_1 = f_*(E') \in \mathcal{WS}$ by Lemma 3.3, and so $g^*(E'_1) = E_1 \in \overline{\mathcal{WS}}$. □

Lemma 3.5. *For every homomorphism $g : C' \longrightarrow C$, the homomorphism $g^* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C', A)$ preserves $\overline{\mathcal{WS}}$-elements.*

Proof. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence in the class $\overline{\mathcal{WS}}$ and $g : C' \longrightarrow C$ be an arbitrary homomorphism.Then $E = f^*(E_1)$ for some $E_1 \in \mathcal{WS}$ and homomorphism $f : C \longrightarrow C_1$. Therefore $E' = g^* \circ f^*(E_1) = (f \circ g)^*(E_1)$. Since $E_1 \in \mathcal{WS}$, $E' \in \overline{\mathcal{WS}}$. □

Corollary 3.6. *Every multiple of a $\overline{\mathcal{WS}}$-element of $\mathrm{Ext}(C, A)$ is again a $\overline{\mathcal{WS}}$-element.*

Proposition 3.7. *If $E_1, E_2 \in \mathrm{Ext}_{\mathcal{WS}}(C, A)$, then $E_1 \oplus E_2 \in \mathrm{Ext}_{\mathcal{WS}}(C \oplus C, A \oplus A)$.*

Proof. Let $E_1, E_2 \in \mathrm{Ext}_{\mathcal{WS}}(C, A)$, then there exist a submodule V_i in B_i such that $V_i + A = B_i$ and $V_i \cap A \ll B_i$, $i = 1, 2$. Then

$$E_1 \oplus E_2 : 0 \longrightarrow A \oplus A \longrightarrow B_1 \oplus B_2 \longrightarrow C \oplus C \longrightarrow 0 \in \mathcal{WS}$$

since $(A \oplus A) + (V_1 \oplus V_2) = B_1 \oplus B_2$ and $(A \oplus A) \cap (V_1 \oplus V_2) = (V_1 \cap A) \oplus (V_2 \cap A) \ll B_1 \oplus B_2$. □

Corollary 3.8. *The $\overline{\mathcal{WS}}$-elements of $\mathrm{Ext}(C, A)$ form a subgroup.*

Proof. Let E_1, $E_2 \in \operatorname{Ext}_{\overline{\mathcal{WS}}}(C, A)$. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
E_1 \oplus E_2 : 0 & \longrightarrow & A \oplus A & \longrightarrow & B_1 \oplus B_2 & \longrightarrow & C \oplus C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
E_1' \oplus E_2' : 0 & \longrightarrow & A \oplus A & \longrightarrow & B_1' \oplus B_2' & \longrightarrow & C_1' \oplus C_2' & \longrightarrow & 0
\end{array}$$

where E_1 and E_2 are the image of short exact sequences E_1' and E_2' from $\mathcal{WS}$ respectively. $E_1' \oplus E_2'$ is $\mathcal{WS}$-element by Proposition 3.7 and so $E_1 \oplus E_2$ is $\overline{\mathcal{WS}}$-element. Since $E_1 + E_2 = \nabla_A(E_1 \oplus E_2)\Delta_C$ where $\Delta_C : c \mapsto (c, c)$ is the the diagonal map and $\nabla_A : (a_1, a_2) \mapsto a_1 + a_2$ is the codiagonal map, $E_1 + E_2$ is in $\overline{\mathcal{WS}}$ by Lemma 3.4 and Lemma 3.5. ☐

Now by Theorem 1.1 in [9] to prove that $\overline{\mathcal{WS}}$ class is a proper class it remains only to show that the composition of two $\overline{\mathcal{WS}}$-monomorphisms (or epimorphisms) is a $\overline{\mathcal{WS}}$-monomorphism (or epimorphism). Firstly we prove some useful results.

Lemma 3.9. *Let $A \subseteq B \subseteq C$ be R-modules. If A is direct summand in B and B has a weak supplement in C, then the short exact sequence $0 \longrightarrow A \longrightarrow C \longrightarrow C/A \longrightarrow 0$ is in $\overline{\mathcal{WS}}$.*

Proof. Let $B = A \oplus B'$. We have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
E_1 : 0 & \longrightarrow & A & \longrightarrow & A \oplus B' & \longrightarrow & B' & \longrightarrow & 0 \\
 & & \| & & \downarrow f & & \downarrow & & \\
E_2 : 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & B_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C' & = & C' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

E_3

By the codiagonal map $\nabla_C : (c_1, c_2) \mapsto c_1 + c_2$ and the monomorphism $f_A \oplus f_{B'} : (a, b') \mapsto (f(a), f(b'))$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
E_1' : 0 & \longrightarrow & A \oplus B' & \xrightarrow{f_A \oplus f_{B'}} & C \oplus C & \longrightarrow & B_1 \oplus D & \longrightarrow & 0 \\
 & & \| & & \downarrow \nabla_C & & \downarrow & & \\
E_3 : 0 & \longrightarrow & A \oplus B' & \xrightarrow{f} & C & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

Since E_3 is in $\mathcal{WS}$, E_1' is in $\overline{\mathcal{WS}}$. By the monomorphisms $f_A \oplus 1_{B'} : (a, b') \mapsto (f(a), b')$ and $1_C \oplus f_{B'} : (c, b') \mapsto (c, f(b'))$, we have the following commutative diagram with exact rows:

$$
\begin{array}{lccccccccc}
E_2' : 0 & \longrightarrow & A \oplus B' & \xrightarrow{f_A \oplus 1_{B'}} & C \oplus B' & \longrightarrow & B_1 & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle 1_C \oplus f_{B'}} & & \downarrow & & \\
E_1' : 0 & \longrightarrow & A \oplus B' & \xrightarrow{f_A \oplus f_{B'}} & C \oplus C & \longrightarrow & B_1 \oplus D & \longrightarrow & 0
\end{array}
$$

E_2' is in $\overline{\mathcal{WS}}$, by Lemma 3.5. Finally, the following diagram is commutative with exact rows and by Lemma 3.4, E_2 is in $\overline{\mathcal{WS}}$.

$$
\begin{array}{lccccccccc}
E_2' : 0 & \longrightarrow & A \oplus B' & \xrightarrow{f_A \oplus 1_{B'}} & C \oplus B' & \longrightarrow & B_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_A \oplus 0_{B'}} & & \downarrow & & \Vert & & \\
E_2 : 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & B_1 & \longrightarrow & 0
\end{array}
$$

□

Lemma 3.10. *The composition of an $\mathcal{S}$mall-epimorphism and a $\mathcal{WS}$-epimorphism is a $\mathcal{WS}$-epimorphism.*

Proof. Let $f : B \to B'$ be a small epimorphism and $h : B' \to C$ be a $\mathcal{WS}$-epimorphism; i.e. we have a commutative exact diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{h \circ f} & C & \longrightarrow & 0 : E \\
 & & \downarrow & & \downarrow{\scriptstyle f} & & \Vert & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \xrightarrow{h} & C & \longrightarrow & 0 : E_1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & & \\
 & & & & E_2 & & & &
\end{array}
$$

with $E_2 \in \mathcal{S}mall$ and $E_1 \in \mathcal{WS}$. Then without of loss generality we can assume that $K \ll B$ and A/K has a weak supplement in B/K. So there is a submodule D/K of B/K such that $D/K + A/K = B/K$ and $(D \cap A)/K \ll B/K$. Therefore we have $A + D = B$ and $A \cap D \ll B$, i.e. A has a weak supplement in B. □

Lemma 3.11. *Let R be hereditary ring. For a $\overline{\mathcal{WS}}$ class of short exact sequences of R modules, the composition of an $\mathcal{S}mall$-epimorphism and a $\overline{\mathcal{WS}}$-epimorphism is a $\overline{\mathcal{WS}}$-epimorphism.*

Proof. Let $f : B \to B'$ be a small epimorphism and $h : B' \to C$ be a $\overline{\mathcal{WS}}$-epimorphism; i.e. we have a commutative exact diagram:

$$\begin{array}{ccccccccl}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{h \circ f} & C & \longrightarrow & 0 : E \\
 & & \downarrow{\scriptstyle g} & & {\scriptstyle f}\downarrow & & \| & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \xrightarrow{h} & C & \longrightarrow & 0 : E_1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & & \\
 & & & & E_2 & & & &
\end{array}$$

with $E_2 \in \mathit{Small}$ and $E_1 \in \overline{\mathcal{WS}}$. Then there is a commutative diagram with exact rows and with $E_3 \in \mathcal{WS}$:

$$\begin{array}{ccccccccl}
0 & \longrightarrow & A' & \longrightarrow & B' & \xrightarrow{h} & C & \longrightarrow & 0 : E_1 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 : E_3
\end{array}$$

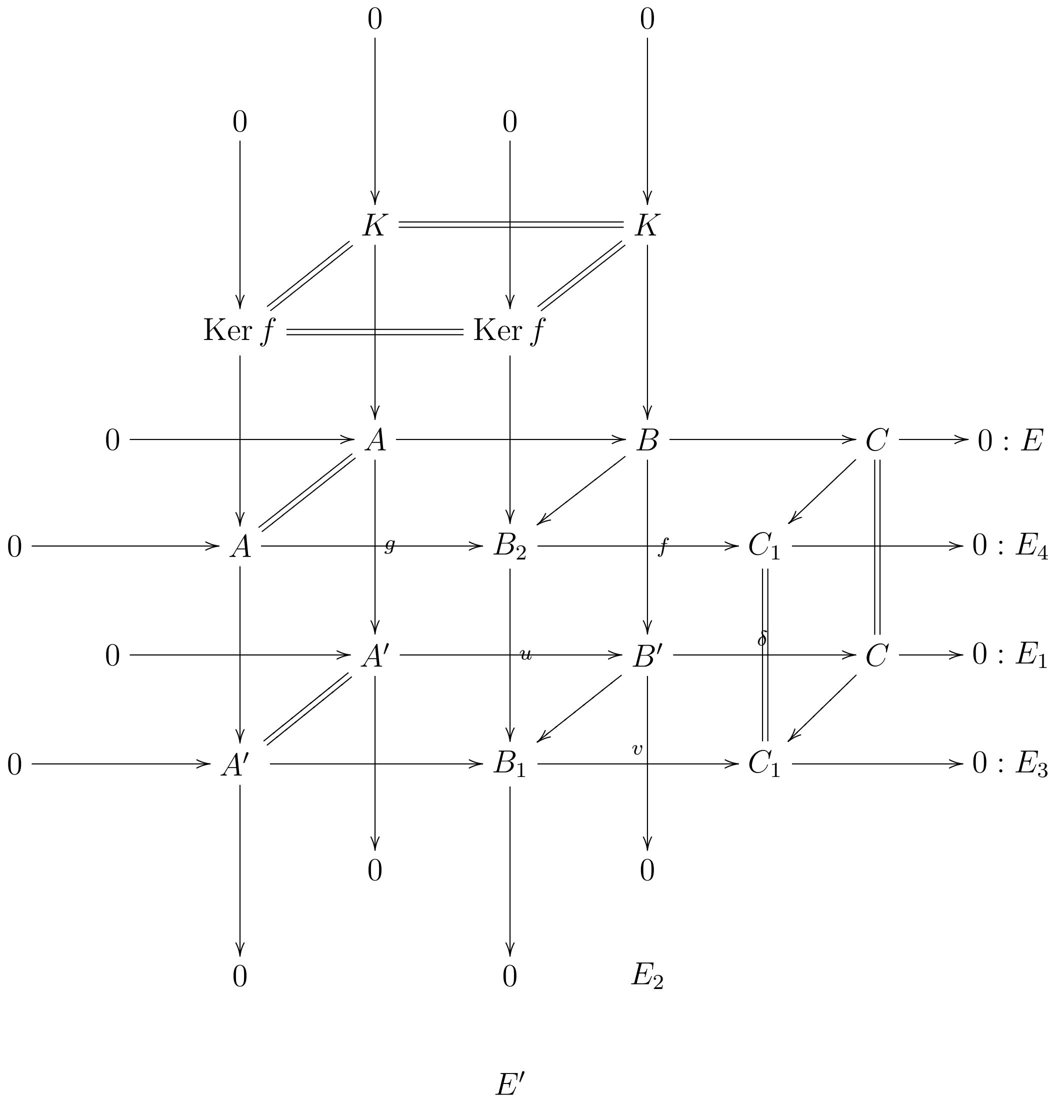
Since R is hereditary the homomorphism

$$\mathrm{Ext}^1(1_{C_1}, g) : \mathrm{Ext}^1(C_1, A) \to \mathrm{Ext}^1(C_1, A')$$

is an epimorphism therefore

$$E_3 = \mathrm{Ext}^1(1_{C_1}, g)(E_4)$$

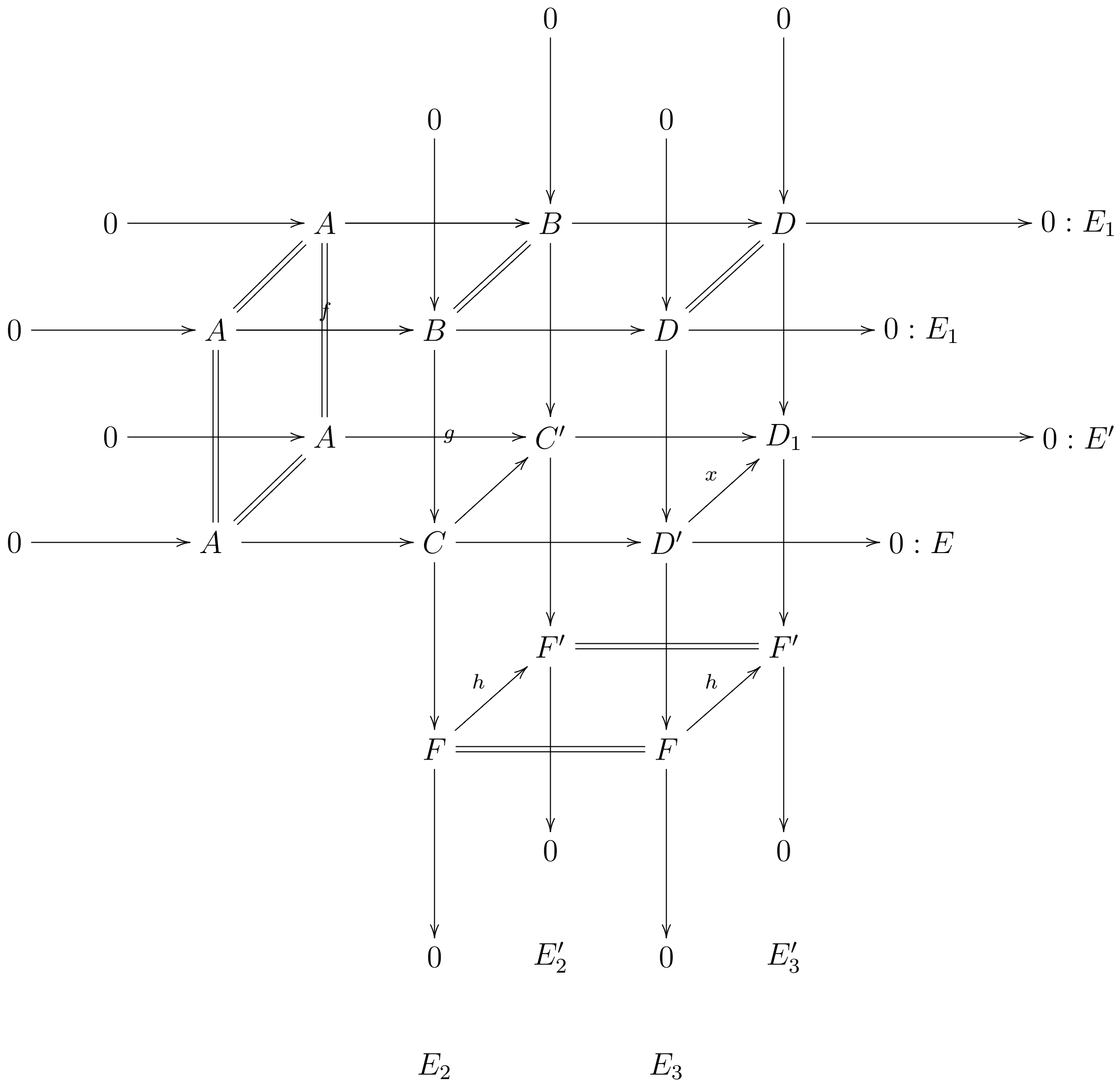
for some $E_4 : 0 \longrightarrow A \longrightarrow B_2 \longrightarrow C_1 \longrightarrow 0$. Then we have the following commutative exact diagram:



Since $K = \operatorname{Ker} f \ll B$, u is $\mathcal{S}mall$ epimorphism. Therefore $v \circ u$ is a $\mathcal{WS}$-epimorphism by Lemma 3.10, i.e. $E_4 \in \mathcal{WS}$. Then $E \in \overline{\mathcal{WS}}$. □

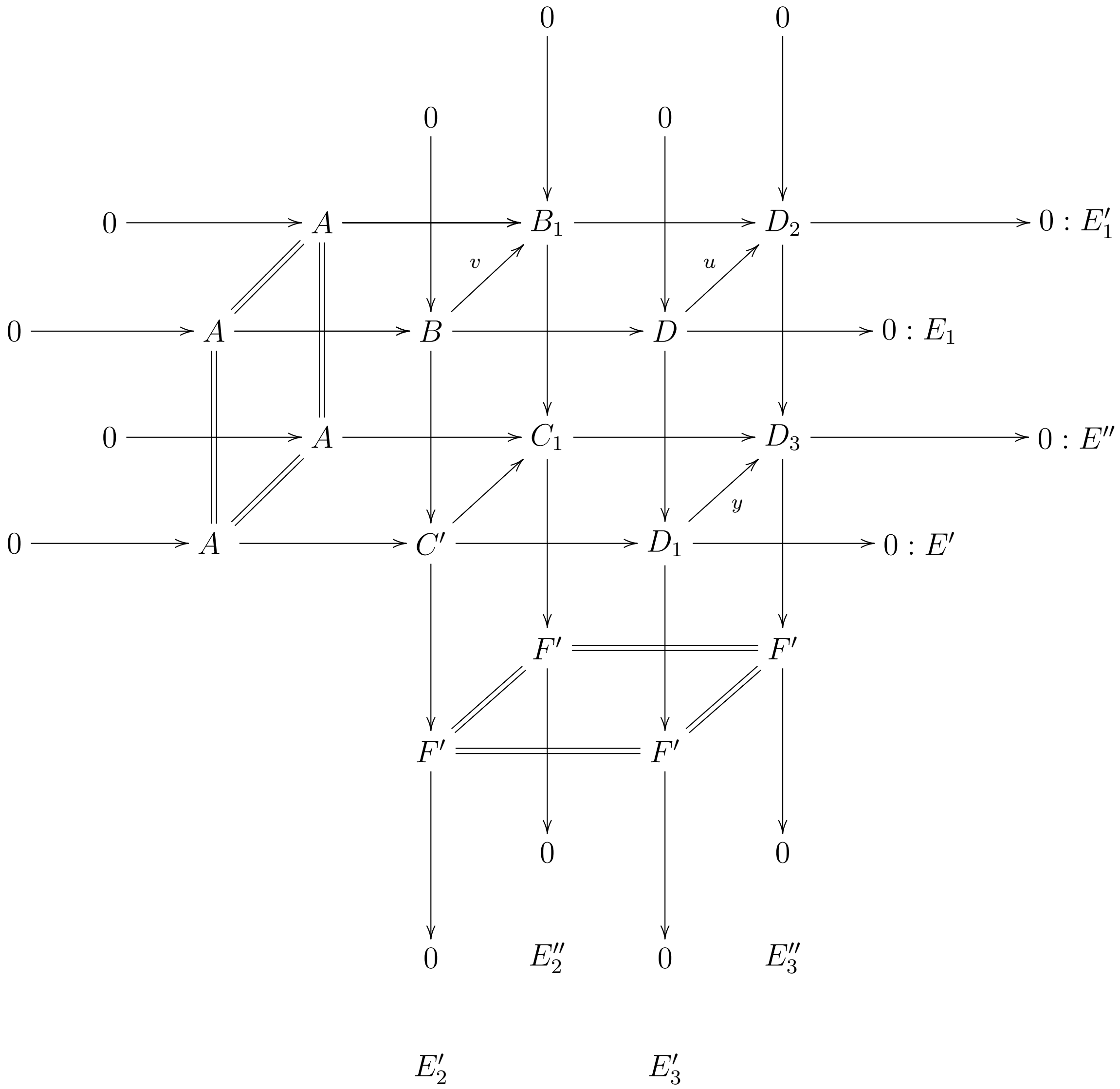
Theorem 3.12. *If R is a hereditary ring, $\overline{\mathcal{WS}}$ is a proper class.*

Proof. By Lemma 3.4, Lemma 3.5, Corollary 3.8, $\operatorname{Ext}_{\overline{\mathcal{WS}}}(C, A)$ is an E-functor in the sense Buttler and Horrocks (1961). By Theorem 1.1 in [9], it is sufficient to show that the composition of two $\overline{\mathcal{WS}}$ monomorphism is a $\overline{\mathcal{WS}}$ monomorphism. Let $f : A \to B$ and $g : B \to C$ be $\overline{\mathcal{WS}}$-monomorphisms. Then for the short exact sequence $E_2 : 0 \longrightarrow B \xrightarrow{g} C \longrightarrow F \longrightarrow 0 \in \overline{\mathcal{WS}}$ we have $E_2 = h^*(E_2')$ for some $E_2' : 0 \longrightarrow B \longrightarrow C' \longrightarrow F' \longrightarrow 0 \in \mathcal{WS}$ and homomorphism $h : F \to F'$. Therefore we have a commutative diagram with exact rows and columns:



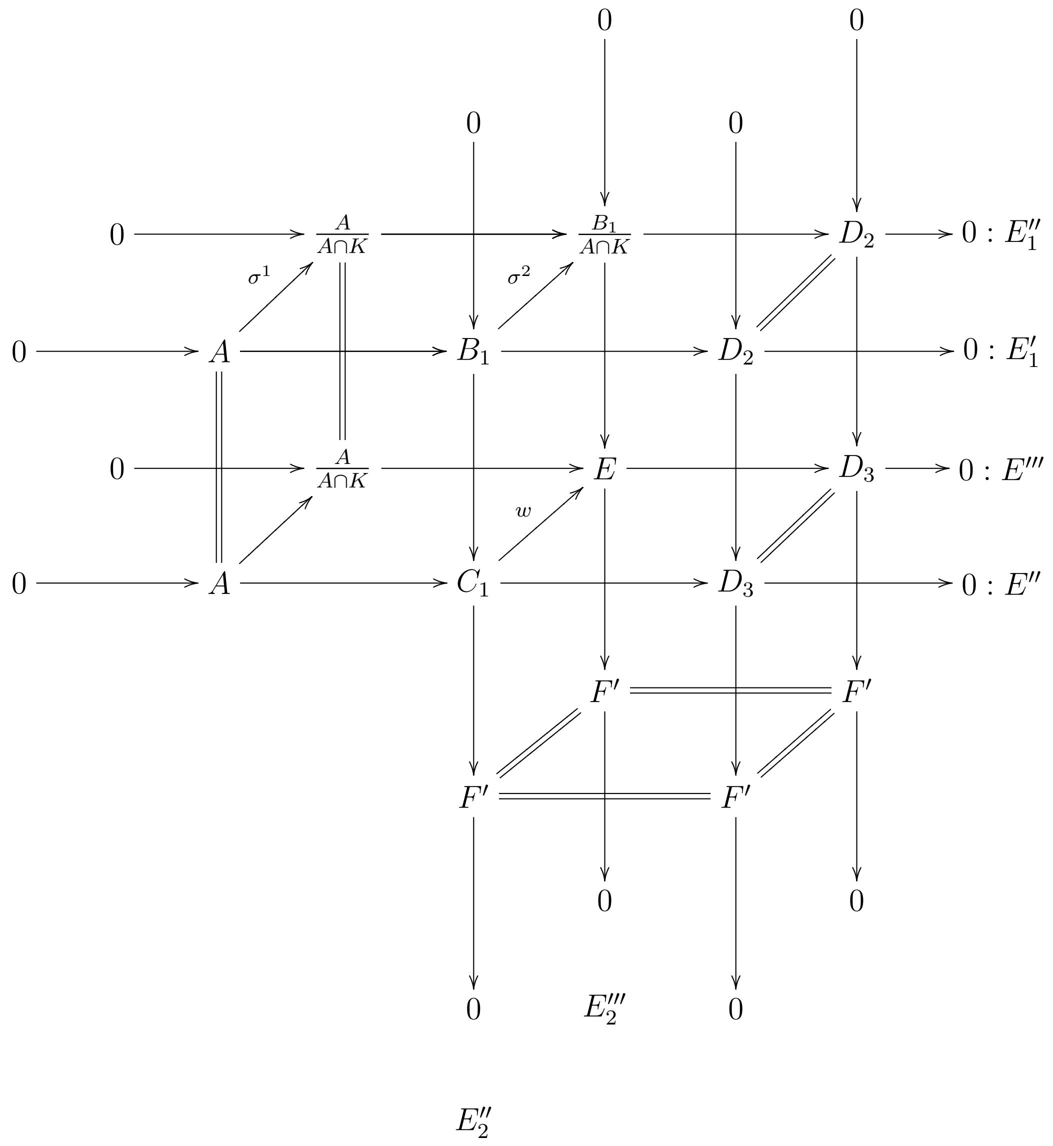
where E_2 and E_3 are images of E_2' and E_3' respectively under the first variable. Now for the short exact sequence $E_1 : 0 \longrightarrow A \xrightarrow{f} B \longrightarrow D \longrightarrow 0 \in \overline{\mathcal{WS}}$ we have $E_1 = u^*(E_1')$ for some $E_1' : 0 \longrightarrow A \longrightarrow B_1 \longrightarrow D_2 \longrightarrow 0 \in \mathcal{WS}$ and homomorphism $u : D \to D_2$.

Therefore we have a commutative diagram with exact rows and columns:



where $E_2''=v_*(E_2')$, $E_3''=u_*(E_3')$. Without lost of generality we can assume that $A\le B_1\le C_1$. Since $E_1'\in\mathcal{WS}$, there is a submodule K of B_1 such that $A+K=B_1$ and $A\cap K\ll B_1$. Then $A/(A\cap K)\oplus K/(A\cap K)=B_1/A\cap K$, that is, $A/(A\cap K)$ is direct summand in

$B_1/(A \cap K)$. Then we have the following diagram with exact rows and columns:



where $\sigma^1 : A \to A/(A \cap K)$ and $\sigma^2 : B_1 \to B_1/(A \cap K)$ are canonical epimorphisms, $E_1'' = \sigma_*^1(E_1')$, $E_2''' = \sigma_*^2(E_2'')$. Since $E_2' \in \mathcal{WS}$, E_2'' and E_2''' are in $\mathcal{WS}$. By Lemma 3.9, $E''' \in \mathcal{WS}$. By 3×3 Lemma $\operatorname{Ker} w = \operatorname{Ker} \sigma^2 = A \cap K \ll C_1$. Therefore by Lemma 3.11 $E'' \in \overline{\mathcal{WS}}$. Now $E = (y \circ x)^*(E''') \in \mathcal{WS}$ by Lemma 3.5. □

Corollary 3.13. *If R is hereditary, then $\langle \mathcal{S}mall \rangle = \langle \mathcal{S} \rangle = \langle \mathcal{WS} \rangle = \overline{\mathcal{WS}}$.*

Proof. The equivalence $\langle \mathcal{S}mall \rangle = \langle \mathcal{S} \rangle = \langle \mathcal{WS} \rangle$ had been proved in [8]. Since $\langle \mathcal{WS} \rangle$ is the least proper class containing $\mathcal{WS}$ and $\mathcal{WS}$ is contained in the proper class $\overline{\mathcal{WS}}$, $\langle \mathcal{WS} \rangle \subseteq \overline{\mathcal{WS}}$. Conversely, let $\mathbb{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \overline{\mathcal{WS}}$. Then there exists a short exact sequence $\mathbb{E}'$ in $\mathcal{WS}$ such that the following diagram is commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 : \mathbb{E} \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 : \mathbb{E}' \end{array}$$

Then $\mathbb{E}' \in \langle \mathcal{WS} \rangle$ and since $\langle \mathcal{WS} \rangle$ is proper class, $\mathbb{E} \in \langle \mathcal{WS} \rangle$ and we have that $\overline{\mathcal{WS}} \subseteq \langle \mathcal{WS} \rangle$. This completes the proof. □

4. HOMOLOGICAL OBJECTS OF $\overline{\mathcal{WS}}$

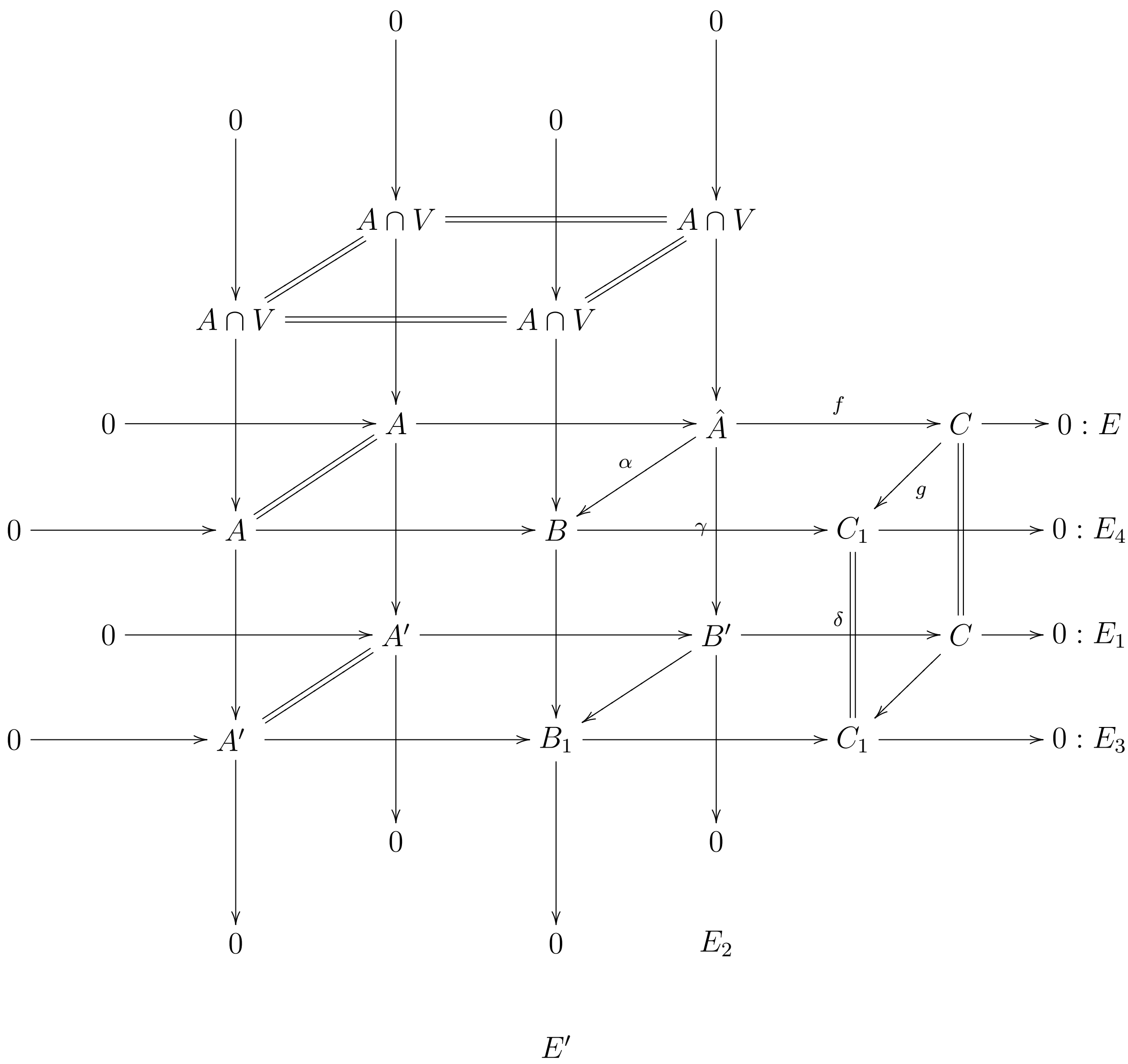
In this section, R denotes a Dedekind domain which is not a field and K denotes its field of fractions, we will denote the set of maximal ideals of R by Ω.

4.1. **Coinjective Modules With Respect to $\overline{\mathcal{WS}}$.**

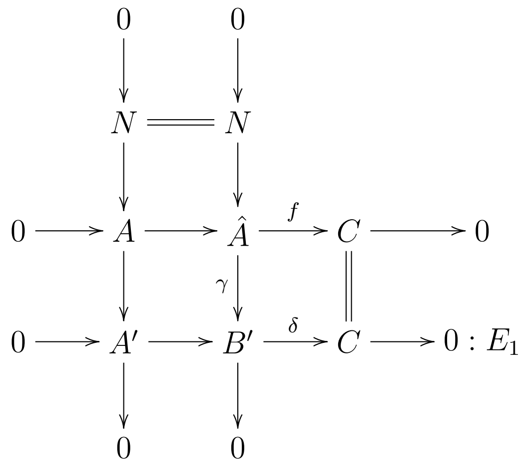
Lemma 4.1. *Let R be a Dedekind ring. For an R-module A the following are equivalent:*

(*i*) *A is $\overline{\mathcal{WS}}$-coinjective.*
(*ii*) *There is a submodule N of A such that N is small in the injective hull $\hat{A}$ of A and A/N is injective.*
(*iii*) *A has a weak supplement in its injective hull $\hat{A}$.*

Proof. ($i \Rightarrow ii$) Let E be $\overline{\mathcal{WS}}$-element. By definition of $\overline{\mathcal{WS}}$, E is an image of a $\mathcal{WS}$-element, say E_4, such that $g^*(E_4) = E$. Then, there exist a submodule V of B such that $A + V = B$ and $A \cap V \ll B$. Since epimorphic image of a injective module is injective, $A/A \cap V$ which is direct summand of a epimorphic image of $\hat{A}$ is injective. And since A is essential in its injective hull $\hat{A}$, α is a monomorphism. So $\hat{A}$ is an injective submodule of B' and, $\hat{A}$ is a direct summand of B', and so $A \cap V \ll \hat{A}$. Then we obtain the following commutative diagram where $E', E_2 \in Small$ and $E_1, E_3 \in \mathcal{S}plit$.



$(ii \Rightarrow iii)$ By the hypothesis, we obtain the following diagram where $E \in Small$ and $E_1 \in \mathcal{S}plit$.



E

Then γ is a $Small$-epimorphism and δ is a $\mathcal{S}plit$-epimorphism. So $f = \delta \circ \gamma$ is $\mathcal{WS}$-epimorphism by Lemma 3.10.

$(iii \Rightarrow i)$ By Proposition 2.2, since every $\mathcal{WS}$-element is an $\overline{\mathcal{WS}}$-element. □

Definition 4.2. A module M is said to be coatomic if $\mathrm{Rad}(M/U) \neq M/U$ for every proper submodule U of M or equivalently every proper submodule of M is contained in a maximal submodule of M.

Lemma 4.3. [18]*, Lemma 2.1] For an R-module M the following are equivalent:*

(i) *M has a weak supplement in its injective hull $\hat{M}$.*
(ii) *There is an injective module I containing M such that M has a supplement in I.*
(iii) *There is an extension N of M, such that M is a direct summand in N and N has a supplement in its injective hull $\hat{N}$.*
(iv) *M has a dense coatomic submodule.*

Proposition 4.4. [17]*,Proof of Lemma 3.3] Let A, B be R-modules and $A \subseteq B$. Then $A \ll B$ if and only if A is coatomic and $A \subseteq \mathrm{Rad}\, B$.*

Proposition 4.5. *If there is a Small-monomorphism from a module A to any module A', then A is a $\overline{\mathcal{WS}}$-coinjective module.*

Proof. Without of loss generality we can assume that $A \ll A'$. Then A is small in injective hull A'. Thus A is $\overline{\mathcal{WS}}$-coinjective by Proposition 2.2. □

Corollary 4.6. *Every coatomic module is a $\overline{\mathcal{WS}}$-coinjective.*

Proof. Every coatomic submodule is small in its injective hull by Proposition 4.4. Then by Proposition 4.5, every coatomic module is a $\overline{\mathcal{WS}}$-coinjective. □

The converse of Corollary 4.6 is not true in general. For example the $\mathbb{Z}$-module $\mathbb{Q}$ is a weakly supplemented module and every submodule of $\mathbb{Q}$ is $\overline{\mathcal{WS}}$-coinjective. If we assume that every proper submodule of $\mathbb{Q}$ is coatomic, then we come to the conclusion that $\mathbb{Q}$ is hollow. But $\mathbb{Q}$ is not hollow and so $\mathbb{Q}$ has a $\overline{\mathcal{WS}}$-coinjective proper submodule which is not coataomic. And also the group of p-adic numbers, J_p, is $\overline{\mathcal{WS}}$-coinjective but not coatomic.

Proposition 4.7. *Let R be a domain. Then every bounded R-module is $\overline{\mathcal{WS}}$-coinjective.*

Proof. Let B be a bounded R-module and I be an injective hull of B. We will show that $B \ll I$. Suppose $B + X = I$ for some $X \subset I$. Since B is bounded, there exists $0 \neq r \in R$ such that $rB = 0$. Then $I = rI = rB + rX = rX$, since I is divisible. Therefore $X = I$ and so $B \ll I$. I is $\langle \mathcal{S}mall \rangle$-coinjective, since it is injective. Then B is $\langle \mathcal{S}mall \rangle$-coinjective by Corollary 2.3. ☐

Lemma 4.8. [8]*, Lemma 4.5] Let S be a DVR, A be a reduced torsion S-module and B be a bounded submodule of A. If A/B is divisible, then A is also bounded.*

Lemma 4.9. *Let M is torsion and reduced module over a Discrete Valuation Ring . Then M is $\overline{\mathcal{WS}}$-coinjective iff M is coatomic.*

Proof. ($\Rightarrow$)Since M is $\overline{\mathcal{WS}}$-coinjective, M has a dense coatomic submodule N by Lemma 4.3. Since M is torsion , N is torsion. Since N is coatomic, $N = B + R^n$ with $p^m B = 0$ for some $n \in N$ [15]. Since N is torsion $R^n = 0$ and N is bounded. By Lemma 4.8, M is bounded and so it is coatomic.
($\Leftarrow$)Since any coatomic module is small in its injective hull, it is $\langle \mathcal{S}mall \rangle$-coinjective and also it is $\overline{\mathcal{WS}}$-coinjective. ☐

Definition 4.10. A module M is called radical-supplemented, if $\mathrm{Rad}(M)$ has a supplement in M.

Zschinger proved that If M has a weak supplement in its injective hull, then $T(M)$ is radical-supplemented and there exists $n \geq 0$ with $\mathbf{p} - \mathrm{Rank}(M/T(M)) \leq n$ for all maximal ideals $\mathbf{p}$ in [18]. From this we obtain the following Corollary by Proposition 2.2.

Corollary 4.11. *If M is a $\overline{\mathcal{WS}}$-coinjective, then $T(M)$ is radical-supplemented and there exists $n \geq 0$ with $\mathbf{p} - \mathrm{Rank}(M/T(M)) \leq n$ for all maximal ideals $\mathbf{p}$.*

Zöschinger proved that the class of R-modules, which have a weak supplement in their injective hull is closed under factor modules and group extensions. This class contains all torsion-free modules with finite rank in [18]. From this we obtain the following Corollary by Proposition 2.2.

Corollary 4.12. *The class of R-modules, which $\overline{\mathcal{WS}}$-coinjective is closed under factor modules and group extensions. This class contains torsion-free modules with finite rank.*

Corollary 4.13. *Every finitely generated module is $\overline{\mathcal{WS}}$-coinjective.*

Proof. Every finitely generated module is small in its injective hull. ☐

Theorem 4.14. *Let $\mathcal{J}$ be a class of modules which $\overline{\mathcal{WS}}$-coinjective. Then, $\underline{k}(\mathcal{J}) = \overline{\mathcal{WS}}$.*

Proof. ($\supseteq$) Let E_1 be a $\overline{\mathcal{WS}}$-element. Then, there is a $\mathcal{WS}$-element E_2 such that the following diagram is commutative.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 : E_1 \\
 & & \| & & \downarrow & & \downarrow g & & \\
0 & \longrightarrow & A & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 : E_2
\end{array}$$

There exist a submodule V of B_1 such that $A + V = B_1$ and $A \cap V \ll B_1$. So, we obtain the following diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A \cap V & = & A \cap V & & & & \\
 & & \downarrow f & & \downarrow & & & & \\
0 & \longrightarrow & A & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 : E_2 \\
 & & \downarrow f' & & \downarrow & & \| & & \\
0 & \longrightarrow & A/A \cap V & \longrightarrow & B_1/A \cap V & \longrightarrow & C_1 & \longrightarrow & 0 \in \mathcal{S}plit \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

If we apply the functor $\mathrm{Hom}(C_1, \ )$, we obtain the following

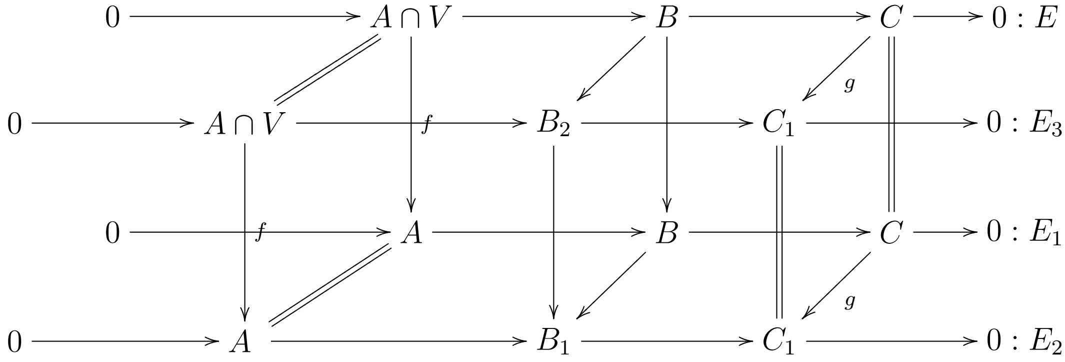
$$0 \longrightarrow \mathrm{Hom}(C_1, A \cap V) \longrightarrow \mathrm{Hom}(C_1, A) \longrightarrow \mathrm{Hom}(C_1, A/A \cap V) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}(C_1, A \cap V) \xrightarrow{f_*} \mathrm{Ext}(C_1, A) \xrightarrow{f'_*} \mathrm{Ext}(C_1, A/A \cap V) = 0$$

Then, f_* is epimorphism and so there exist $E_3 \in \mathrm{Ext}(C_1, A \cap V)$ such that $f_*(E_3) = E_2$. Since the following square is commutative:

$$\begin{array}{ccc}
\mathrm{Ext}(C_1, A \cap V) & \xrightarrow{g^*} & \mathrm{Ext}(C, A \cap V) \\
\downarrow f_* & & \downarrow f_* \\
\mathrm{Ext}(C_1, A) & \xrightarrow{g^*} & \mathrm{Ext}(C, A)
\end{array}$$

$g^* \circ f_*(E_3) = E_1 = f_* \circ g^*(E_3)$. Hence, we obtain the following diagram.



Since $A \cap V \ll B_1$, $A \cap V$ is $\overline{\mathcal{WS}}$-coinjective by Proposition 4.5. Then $E \in \underline{k}(\mathcal{J})$ and since $\underline{k}(\mathcal{J})$ is subfunctor, $E_1 \in \underline{k}(\mathcal{J})$.

($\subseteq$) $\underline{k}(\mathcal{J}) \subseteq \overline{\mathcal{WS}}$ is trivial. □

By the Propositions 2.7 and 2.8, we obtain that the following Corollaries:

Corollary 4.15. *The* global dimension*of* $\overline{\mathcal{WS}}$ *is* $\text{gl.dim}\,\overline{\mathcal{WS}} \leq 1$.

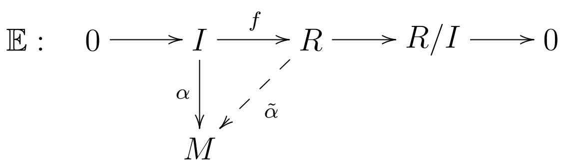
Corollary 4.16. $\text{inj.dim}\, A \leq 1$ *for every* $\overline{\mathcal{WS}}$*-coinjective module* A.

4.2. Injective Modules with Respect to $\overline{\mathcal{WS}}$.

Corollary 4.17. *Over a Dedekind domain* R, $\overline{\mathcal{WS}}$*-injective modules are only the injective* R*-modules.*

Proof. Let M be a $\overline{\mathcal{WS}}$-injective module and I be any ideal of Dedekind domain R. Since R is Dedekind domain, R is noetherian ring and so I is finitely generated. $\mathbb{E} : 0 \longrightarrow I \xrightarrow{f} R \longrightarrow R/I \longrightarrow 0$ in $\overline{\mathcal{WS}}$ by Corollary 4.13. Since M is $\overline{\mathcal{WS}}$-injective module; for every homomorphism $\alpha : I \longrightarrow M$, there exists a homomorphism $\tilde{\alpha} : R \longrightarrow M$ such that $\tilde{\alpha} \circ f = \alpha$. We have the following commutative diagram,

(3)



Since for any left ideal I of R-homomorphism: $I \to M$ can be extended to an R-homomorphism: $R \to M$, then M is injective R-module by Baer's criterion ([13], Theorem 3.3.5). □

We obtain the following Corollary by using Proposition 2.10 from Corollary 4.17.

Corollary 4.18. $\mathcal{WS}$*-injective modules are only the injective* R*-modules.*

4.3. **Projective and Coprojective Submodules with Respect to $\overline{\mathcal{WS}}$.** For $\overline{\mathcal{WS}}$-projective modules, we obtain the following criteria:

Lemma 4.19. *If C is any module such that $\operatorname{Ext}_R(C, A') = 0$ for every coatomic module A', then C is an $\overline{\mathcal{WS}}$-projective module.*

Proof. An R -module C is $\mathcal{P}$-*projective* if and only if $\operatorname{Ext}_{\mathcal{P}}(C, A) = 0$ for all R-modules A. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence in the class $\overline{\mathcal{WS}}$. In Proof of Theorem 4.14, it was shown that every elements of $\overline{\mathcal{WS}}$ is an image of a short exact sequence with starting a coatomic module such as

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & B_1 & \longrightarrow & C & \longrightarrow & 0 : E_1 \\
 & & \downarrow f & & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 : E
\end{array}$$

where f is a monomorphism from a coatomic module A' to A.

Since A' is coatomic module, E_1 is in $\mathcal{S}plit$ with respect to our assumption. Then $E = f_*(E_1) = 0$. This completes the proof. □

Corollary 4.20. *Every finitely presented module is $\overline{\mathcal{WS}}$-coprojective.*

Proof. Let a finitely presented module F. There is a epimorphism from a projective module P to F, $f : P \to F$. Since F is finitely presented, P and $\operatorname{Ker} f$ is finitely generated. Thus $\operatorname{Ker} f$ is $\overline{\mathcal{WS}}$-coinjective by Corollary 4.13. Then F is $\overline{\mathcal{WS}}$-coprojective by Proposition 2.4. □

4.4. **Coinjective Submodules with Respect to $\overline{\mathcal{WS}}$ over DVR.** In the following part R is always a discrete valuation ring with quotient field $K \neq R$ and the maximal ideal (p).

Corollary 4.21. *If $M/\operatorname{Rad}(M)$ is simple, M is $\overline{\mathcal{WS}}$-coinjective.*

Proof. Zschinger proved that if $M/\operatorname{Rad}(M)$ is simple, then M has a supplement in every extension N with N/M is torsion in [17]. Since every module is essential in its injective hull, M is essential in $E(M)$ and also $E(M)/M$ is torsion. So M has a supplement in its injective hull. Then M is $\overline{\mathcal{WS}}$-coinjective by Proposition 2.2. □

Theorem 4.22 ([17], Theorem 3.1)**.** *For an R-module M the following are equivalent:*

(*a*) *M is radical-supplemented.*
(*b*) $\operatorname{Rad}^n(M) = \operatorname{Rad}^{n+1}(M)$ *is finitely generated for some $n \geq 0$.*
(*c*) *The basic-submodule of M is coatomic.*
(*d*) *$M = T(M) \oplus X$ where the reduced part of $T(M)$ is bounded and $X/\operatorname{Rad}(X)$ is finitely generated.*

Lemma 4.23 ([17], Lemma 3.2)**.** (*a*) *The class of radical-supplemented R-modules is closed under factor modules, pure submodules and extensions.*
(*b*) *If M is radical-supplemented and M/U is reduced, then U is also radical-supplemented.*
(*c*) *Every submodule of M is radical-supplemented if and only if $T(M)$ is supplemented and $M/T(M)$ has finite rank.*

By Lemma 4.1, Theorem 4.22 and Lemma 4.23, we obtain the following Corollary.

Corollary 4.24. *For an R-module M the following are equivalent:*

(a) *M is $\overline{\mathcal{WS}}$-coinjective.*
(b) *M is radical-supplemented.*
(c) *$M = T(M) \oplus X$ where the reduced part of $T(M)$ is bounded and $X/\operatorname{Rad}(X)$ is finitely generated.*
(d) *The class of $\overline{\mathcal{WS}}$-coinjective R-modules is closed under factor modules, pure submodules and extensions.*
(e) *Every submodule of M is $\overline{\mathcal{WS}}$-coinjective if and only if $T(M)$ is supplemented and $M/T(M)$ has finite rank.*

5. COATOMIC SUPPLEMENT SUBMODULES

Throughout this chapter all rings are hereditary rings, unless otherwise stated. In this chapter, we define the notion "coatomic supplement" and give some results about the relation between coatomic supplement and supplement submodules.

Let U be a submodule of an R-module M. If there exists a submodule V of M such that $M = U + V$ and $U \cap V$ is coatomic then U is called a *coatomic supplement* of V in M. We study the class Σ of σ-exact sequences where an element $\mathbb{E} : 0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ of $\operatorname{Ext}_R(C, A)$ is called σ-exact if $\operatorname{Im}\alpha$ has a coatomic supplement in B.

Lemma 5.1. *If $f : A \longrightarrow A'$, then $f_* : \operatorname{Ext}(C,\, A) \longrightarrow \operatorname{Ext}(C,\, A')$ preserves σ -element.*

Proof. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence in $\operatorname{Ext}(C, A)$ and $f : A \longrightarrow A'$ be an arbitrary homomorphism. The following diagram is commutative with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & C & \longrightarrow & 0 : E \\ & & \downarrow f & & \downarrow f' & & \| & & \\ 0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \longrightarrow & C & \longrightarrow & 0 : E_1 \end{array}$$

where $f_*(E) = E_1$. If V is a coatomic supplement of $\operatorname{Im}\alpha$ in B, then $\operatorname{Im}\alpha + V = B$ and $V \cap \operatorname{Im}\alpha$ is coatomic. Then $f'(V) + \operatorname{Im}\alpha' = B'$ by pushout diagram and $f'(V) \cap \operatorname{Im}\alpha' = f'(V \cap \operatorname{Im}\alpha)$ is coatomic, since $V \cap \operatorname{Im}\alpha$ is coatomic and homomorphic image of a coatomic module is coatomic. □

Lemma 5.2. *If $g : C' \longrightarrow C$, then $g^* : \operatorname{Ext}(C,\, A) \longrightarrow \operatorname{Ext}(C',\, A)$ preserves σ-elements.*

Proof. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short exact sequence in $\operatorname{Ext}(C, A)$ and $g : C' \longrightarrow C$ be an arbitrary homomorphism. The following diagram is commutative with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \longrightarrow & 0 : E_1 \\ & & \| & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 : E \end{array}$$

where $g^*(E) = E_1$.

Let V be a coatomic supplement of $\operatorname{Ker}\beta$ in B, i.e. $\operatorname{Ker}\beta + V = B$ and $V \cap \operatorname{Ker}\beta$ is coatomic. Then $g'^{-1}(V) + \operatorname{Ker}\beta' = B'$ by pullback diagram. Since g' induces an isomorphism between $D' = g'^{-1}(V) \cap \operatorname{Ker}\beta'$ and $D = V \cap \operatorname{Ker}\beta$ and epimorphic image of coatomic module coatomic, D' is coatomic. □

Corollary 5.3. *Every multiple of a σ-element of $\operatorname{Ext}(C, A)$ is again a σ-element.*

Theorem 5.4. *The class Σ of σ-elements coincide with the class $\overline{\mathcal{WS}}$ of $\overline{\mathcal{WS}}$-elements.*

Proof. Assume that A has a coatomic supplement in B, then there exists a submodule V of B such that $B = A + V$ and $A \cap V$ is coatomic. So, the following diagram is commutative with exact columns and rows:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & A \cap V & = & A \cap V & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 : E \\ & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \| & & \\ 0 & \longrightarrow & A/A \cap V & \longrightarrow & B/A \cap V & \xrightarrow{\alpha} & C & \longrightarrow & 0 : E_1 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Clearly α is $\mathcal{S}plit$-epimorphism and since coatomic module is $\overline{\mathcal{WS}}$-coinjective, γ is $\overline{\mathcal{WS}}$-epimorphism. Then, the composition $\alpha \circ \gamma$ is an $\overline{\mathcal{WS}}$-epimorphism. So, E is a $\overline{\mathcal{WS}}$-element. To prove the converse, let $E \in \overline{\mathcal{WS}}$, then there is E_1 in the class $\mathcal{WS}$ such that the following diagram is commutative with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & C & \longrightarrow & 0 : E \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha'} & B' & \longrightarrow & C' & \longrightarrow & 0 : E_1 \end{array}$$

If V is weak supplement of $\operatorname{Im}\alpha'$ in B', then $\operatorname{Im}\alpha' + V = B'$ and $\operatorname{Im}\alpha' \cap V \ll B'$ and so $\operatorname{Im}\alpha' \cap V$ is coatomic by Proposition 4.4. By Lemma 5.2, E is σ-element. □

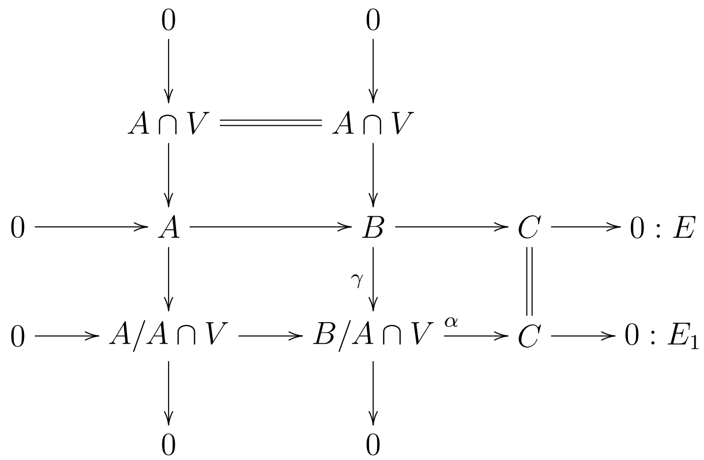
Let R be a discrete valuation ring with quotient field $K \neq R$ and the maximal ideal (p). If A is a coatomic submodule of B, then it does not need to be small in B, but, since $B/\operatorname{Rad}(B)$ semisimple, from

$$X/\operatorname{Rad}(B) \oplus (A + \operatorname{Rad}(B))/\operatorname{Rad}(B) = B/\operatorname{Rad}(B)$$

nevertheless follows that $X + A = B$ with $X \cap A$ small in B. So, every coatomic submodule has a weak supplement in every extension.

Lemma 5.5. *$\mathcal{WS}$ form a proper class over the Discrete Valuation Ring .*

Proof. Assume that A has a coatomic supplement in B, then there exists a submodule V of B such that $B = A + V$ and $A \cap V$ is coatomic. So, the following diagram is commutative with exact column and rows:



Since $A \cap V$ is coatomic, γ is $\mathcal{WS}$-epimorphism. Then, the composition $\alpha \circ \gamma$ is $\mathcal{WS}$-epimorphism. So, E is $\mathcal{WS}$-element. □

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CONEAT SUBMODULES OVER DEDEKIND DOMAINS

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Abstract. We deal with two proper classes defined by means of complements (closed submodules) and supplements in modules and their relations with the neat and coneat short exact sequences of modules. For a Dedekind domain W, if $\operatorname{Rad} W = 0$, then the proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ is *strictly* between $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$ and $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$. When $\operatorname{Rad} W \neq 0$, still $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \neq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$, but $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$. If W is a Dedekind domain such that $\operatorname{Rad} W = 0$ and W is not a field, then the functors $\operatorname{Ext}_{\mathcal{S}uppl_{W\text{-}\mathcal{M}od}}$ and $\operatorname{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}}$ are not factorizable as $W\text{-}\mathcal{M}od \times W\text{-}\mathcal{M}od \xrightarrow{Ext_W} W\text{-}\mathcal{M}od \xrightarrow{H} W\text{-}\mathcal{M}od$ for any functor $H : W\text{-}\mathcal{M}od \longrightarrow W\text{-}\mathcal{M}od$. Neat submodules of a torsion module over a Dedekind domain coincide with its coneat submodules. $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojectives are only torsion-free W-modules.

1. Introduction

Throughout this article, by a ring we shall mean an associative ring with unity; R will denote such a general ring, unless otherwise stated. We shall consider unital left R-modules; R-module will mean left R-module. $R\text{-}\mathcal{M}od$ denotes the category of all left R-modules. $\mathbb{Z}$ denotes the ring of integers. Group will mean abelian group only. By W, we denote a commutative Dedekind domain. All definitions not given here can be found in [7], [28] and [3].

A submodule A of a module B is said to *have a complement in* B if there exits a submodule K of B maximal with respect to $K \cap A = 0$. A submodule A of a module B is said to be a *complement in* B if A is a complement of some submodule of B. It is said that A is *closed in* B if A has *no* proper essential extension in B and it is known that closed submodules and complements in a module coincide (see [7, §1]).

Dually, a submodule A of a module B is said to *have a supplement in* B if there exits a submodule K of B minimal with respect to $K + A = M$; equivalently $K + A = M$ and $K \cap A$ is *small*(=*superfluous*) in K (which is denoted by $K \cap A \ll K$, meaning that for *no* proper submodule X of K, $K \cap A + X = K$). A submodule A of a module B is said to be a *supplement in* B if A is a supplement of some submodule of B. Unlike complements a submodule of a module may *not* have any supplements. If every submodule of a module has a supplement, then it is said to be a *supplemented module*. For the definitions and related properties see [28, §41] and [5].

We deal with complements (closed submodules) and supplements in unital R-modules for an associative ring R with unity using relative homological algebra via the known two dual proper classes of short exact sequences of R-modules and R-module homomorphisms, $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$, and related other proper classes like $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$. $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$ [$\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$] consists of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of R-modules and R-module homomorphisms such that $\operatorname{Im}(f)$ is a complement [resp. supplement] in B. $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ [$\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$] consists of all short exact sequences of R-modules and

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R-module homomorphisms with respect to which every simple module is projective [resp. every module with zero radical is injective]. In the case of modules over Dedekind domains, we shall investigate the relation of these proper classes; the inclusion relations among them and when they are equal. We shall extend some of the results for abelian groups in [2] to modules over Dedekind domains. See [2, §3] for some of the properties of these proper classes that we shall use.

[13, Corollary 1 and 6] gives the following interesting result (the equality from [12, Theorem 5] as a Dedekind domain is a C-ring): For a Dedekind domain W,

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od},$$

where the inclusion is strict if W is *not* a field. So if A is a *supplement* in a W-module B where W is a Dedekind domain, then A is a *complement*. We shall prove that for a Dedekind domain W that is not a field,

(i) If $\operatorname{Rad} W = 0$, then

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

(ii) If $\operatorname{Rad} W \neq 0$, then

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

The proper class $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ is both projectively generated, injectively generated and flatly generated by simple modules (see [2, Theorem 3.7] and Theorem 3.9). One of the main steps in the proof is this fact follows from [24, Lemmas 4.4, 5.2 and Theorem 5.1]. Another consequence of [24, Theorem 5.1] is that for a Dedekind domain W, and W-modules A, C,

$$\operatorname{Ext}_{\mathcal{C}ompl_{W\text{-}\mathcal{M}od}}(C, A) = \operatorname{Ext}_{\mathcal{N}eat_{W\text{-}\mathcal{M}od}}(C, A) = \operatorname{Rad}(\operatorname{Ext}_W(C, A)).$$

(see [2, Theorem 3.8]). But for supplements and coneat submodules, we shall show that this is not possible if W is a Dedekind domain such that $\operatorname{Rad} W = 0$ and W is not a field: the functors $\operatorname{Ext}_{\mathcal{S}uppl_{W\text{-}\mathcal{M}od}}$ and $\operatorname{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}}$ are *not* factorizable as

$$W\text{-}\mathcal{M}od \times W\text{-}\mathcal{M}od \xrightarrow{Ext_W} W\text{-}\mathcal{M}od \xrightarrow{H} W\text{-}\mathcal{M}od$$

for any functor $H : W\text{-}\mathcal{M}od \longrightarrow W\text{-}\mathcal{M}od$. This extends the result for abelian groups given in [2, Theorem 6.3]. To every proper class $\mathcal{A}$, we have a relative $\operatorname{Ext}_{\mathcal{A}}$ functor and for the proper class $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$, this functor behaves badly in this factorizabililty sense unlike for $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$.

For a Dedekind domain W, a partial converse of the inclusion $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ is the following: A *finitely generated torsion* submodule of a W-module is a complement if and only if it is a supplement (see [2, Theorem 4.1]). We shall show that for a *torsion* W-module B, neat submodules and coneat submodules of B coincide.

We also note the coinjectives and coprojectives of these proper classes for a Dedekind domain W:

(1) $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coinjectives (and so $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$-coinjectives and $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-coinjectives) are only injective W-modules.
(2) $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojectives are only torsion-free W-modules.
(3) If $\operatorname{Rad} W = 0$, then $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$-coprojectives and $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-coprojectives are only projective W-modules.

Neat subgroups of abelian groups (introduced by [15, pp. 43-44]) have been generalized to modules in [27, 9.6] (and [26, §3]); this is the above definition that we have taken. Dually, coneat submodules have been introduced in [22] and [2]; as defined above, a monomorphism $f : K \to L$ is called *coneat* if each module M with $\operatorname{Rad} M = 0$ is *injective* with respect to it, that is, the Hom sequence

$$\operatorname{Hom}(L, M) \to \operatorname{Hom}(K, M) \to 0$$

is exact. See [22, Proposition 3.4.2] or [5, 10.14] or [1, 1.14] for the following characterization of coneat submodules: For a submodule A of a module B, A is coneat in B if and only if there exists a submodule $K \leq B$ such that $A + K = B$ and $A \cap K \leq \operatorname{Rad} A$ (or $A \cap K = \operatorname{Rad} A$).

This is like the usual supplement condition except that, instead of $U \cap V \ll V$ ($U \cap V$ *small* in V), the condition $U \cap V \leq \mathrm{Rad}(V)$ is required. For submodules U and V of a module M, the submodule V is said to be a Rad-*supplement* of U in M or U is said to *have a* Rad-*supplement* V in M if $U + V = M$ and $U \cap V \leq \mathrm{Rad}(V)$. So a submodule V of a module M is a *coneat* submodule of M if and only if V is a *Rad-supplement* of a submodule U of M in M. In [5, §10 and 20.7–8] and [1], the properties of τ-supplements are also investigated where τ is a radical for *R-$\mathcal{M}$od*.

Proper classes of monomorphisms and short exact sequences were introduced in [4] to do *relative* homological algebra. In [26, Remark after Proposition 6], it is pointed out that supplement submodules induce a proper class of short exact sequences (the term 'low' is used for supplements dualizing the term 'high' used in abelian groups). See also [5, 20.7] for a proof of that. [13] uses the terminology 'cohigh' for supplements and gives more general definitions for proper classes of supplements related to another given proper class (motivated by the considerations as pure-high extensions and neat-high extensions in [14]). For the definition and properties of *proper classes*, see [25], [21, Ch. 12, §4], [5, §10], [1], [27] and [23]. The terminology and notation for proper classes are given in the next section.

2. Terminology and notation for proper classes

Let $\mathcal{A}$ be a class of short exact sequences of R-modules and R-module homomorphisms. If a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1}$$

belongs to $\mathcal{A}$, then f is said to be an $\mathcal{A}$-*monomorphism* and g is said to be an $\mathcal{A}$-*epimorphism* (both are said to be $\mathcal{A}$-*proper* and the short exact sequence is said to be an $\mathcal{A}$-*proper* short exact sequence.). The class $\mathcal{A}$ is said to be *proper* if it satisfies the following conditions (see Ch. 12, §4 in [21] or [27] or [25]):

(1) If a short exact sequence $\mathbb{E}$ is in $\mathcal{A}$, then $\mathcal{A}$ contains every short exact sequence isomorphic to $\mathbb{E}$.
(2) $\mathcal{A}$ contains all splitting short exact sequences.
(3) The composite of two $\mathcal{A}$-monomorphisms is an $\mathcal{A}$-monomorphism if this composite is defined. The composite of two $\mathcal{A}$-epimorphisms is an $\mathcal{A}$-epimorphism if this composite is defined.
(4) If g and f are monomorphisms, and $g \circ f$ is an $\mathcal{A}$-monomorphism, then f is an $\mathcal{A}$-monomorphism. If g and f are epimorphisms, and $g \circ f$ is an $\mathcal{A}$-epimorphism, then g is an $\mathcal{A}$-epimorphism.

For a proper class $\mathcal{A}$ of R-modules, call a submodule A of a module B an $\mathcal{A}$-submodule of B, if the inclusion monomorphism $i_A : A \to B$, $i_A(a) = a$, $a \in A$, is an $\mathcal{A}$-monomorphism. We denote this by $A \leq_{\mathcal{A}} B$.

An important example for proper classes in abelian groups is $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$: The proper class of all short exact sequences (1) of abelian groups and abelian group homomorphisms such that $\mathrm{Im}(f)$ is a pure subgroup of B, where a subgroup A of a group B is *pure* in B if $A \cap nB = nA$ for all integers n (see §26-30 in [9] for purity in abelian groups). The proper class $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$ forms one of the origins of *relative* homological algebra; it is the reason why proper classes are also called *purities* (as in [23], [11], [12], [13]).

Denote by $\mathcal{A}$ a proper class of R-modules. An R-module M is said to be $\mathcal{A}$-*projective* [$\mathcal{A}$-*injective*] if it is projective [resp. injective] with respect to all short exact sequences in $\mathcal{A}$, that is, $\mathrm{Hom}(M, \mathbb{E})$ [resp. $\mathrm{Hom}(\mathbb{E}, M)$] is exact for every $\mathbb{E}$ in $\mathcal{A}$. Denote all $\mathcal{A}$-projective [$\mathcal{A}$-injective] modules by $\pi(\mathcal{A})$ [resp. $\iota(\mathcal{A})$]. For a given class $\mathcal{M}$ of modules, denote by $\pi^{-1}(\mathcal{M})$ [$\iota^{-1}(\mathcal{M})$], the largest proper class $\mathcal{A}$ for which each $M \in \mathcal{M}$ is $\mathcal{A}$-projective [resp. $\mathcal{A}$-injective]; it is called the proper class *projectively generated* [resp. *injectively generated*] by $\mathcal{M}$. A *right* R-module M is said to be $\mathcal{A}$-*flat* if M is flat with respect to every short exact sequence $\mathbb{E} \in \mathcal{A}$, that is, $M \otimes \mathbb{E}$ is exact for every $\mathbb{E}$ in $\mathcal{A}$. Denote all $\mathcal{A}$-flat *right* R-modules by $\tau(\mathcal{A})$. For a given class

$\mathcal{M}$ of *right* R-modules, denote by $\tau^{-1}(\mathcal{M})$ the class of all short exact sequences $\mathbb{E}$ of R-modules and R-module homomorphisms such that $M \otimes \mathbb{E}$ is exact for all $M \in \mathcal{M}$. $\tau^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{A}$ of (left) R-modules for which each $M \in \mathcal{M}$ is $\mathcal{A}$-flat. It is called the proper class *flatly generated* by the class $\mathcal{M}$ of *right* R-modules. When the ring R is commutative, there is no need to mention the sides of the modules since a right R-module may also be considered as a left R-module and vice versa. An R-module C is said to be $\mathcal{A}$-*coprojective* if *every* short exact sequence of R-modules and R-module homomorphisms *ending* with C is in the proper class $\mathcal{A}$. An R-module A is said to be $\mathcal{A}$-*coinjective* if *every* short exact sequence of R-modules and R-module homomorphisms *starting* with A is in the proper class $\mathcal{A}$. See [25, §1-3,8] for these concepts in relative homological algebra in categories of modules.

For a proper class $\mathcal{A}$ and R-modules A, C, denote by $\mathrm{Ext}^1_{\mathcal{A}}(C, A)$ or just by $\mathrm{Ext}_{\mathcal{A}}(C, A)$, the equivalence classes of all short exact sequences in $\mathcal{A}$ which start with A and end with C, i.e. a short exact sequence in $\mathcal{A}$ of the form (1). This turns out to be a subgroup of $\mathrm{Ext}_R(C, A)$ and a bifunctor $\mathrm{Ext}^1_{\mathcal{A}} : R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ is obtained which is a subfunctor of Ext^1_R (see Ch. 12, §4-5 in [21]).

Using the functor $\mathrm{Ext}_{\mathcal{A}}$, the $\mathcal{A}$-projectives, $\mathcal{A}$-injectives, $\mathcal{A}$-coprojectives, $\mathcal{A}$-coinjectives are simply described as extreme ends for the subgroup $\mathrm{Ext}_{\mathcal{A}}(C, A) \leq \mathrm{Ext}_R(C, A)$ being 0 or the whole of $\mathrm{Ext}_R(C, A)$:

(1) An R -module C is $\mathcal{A}$-*projective* if and only if
$$\mathrm{Ext}_{\mathcal{A}}(C, A) = 0 \text{ for all } R\text{-modules } A.$$
(2) An R -module C is $\mathcal{A}$-*coprojective* if and only if
$$\mathrm{Ext}_{\mathcal{A}}(C, A) = \mathrm{Ext}_R(C, A) \text{ for all } R\text{-modules } A.$$
(3) An R -module A is $\mathcal{A}$-*injective* if and only if
$$\mathrm{Ext}_{\mathcal{A}}(C, A) = 0 \text{ for all } R\text{-modules } C.$$
(4) An R -module A is $\mathcal{A}$-*coinjective* if and only if
$$\mathrm{Ext}_{\mathcal{A}}(C, A) = \mathrm{Ext}_R(C, A) \text{ for all } R\text{-modules } C.$$

Note also the following property that we shall use for the coprojective modules with respect to an injectively generated proper class:

Proposition 2.1. *(*[25, Proposition 9.4]*) If $\mathcal{A}$ is an injectively generated proper class of R-modules, then for an R-module C, the condition $\mathrm{Ext}^1_R(C, J) = 0$ for all $\mathcal{A}$-injective J is equivalent to C being $\mathcal{A}$-coprojective.*

More directly:

Proposition 2.2. *If $\mathcal{A} = \iota^{-1}(\mathcal{M})$ for a class $\mathcal{M}$ of R-modules, then for an R-module C, the condition $\mathrm{Ext}^1_R(C, M) = 0$ for all $M \in \mathcal{M}$ is equivalent to C being $\mathcal{A}$-coprojective.*

Proof. Suppose C is a $\mathcal{A}$-coprojective module. Let $M \in \mathcal{M}$. Take an element $[\mathbb{E}] \in \mathrm{Ext}^1_R(C, M)$:
$$\mathbb{E} : \qquad 0 \longrightarrow M \longrightarrow B \longrightarrow C \longrightarrow 0$$
Since C is $\mathcal{A}$-coprojective, $\mathbb{E} \in \mathcal{A}$. Then $\mathbb{E}$ splits because M, being an element of $\mathcal{M}$, is $\mathcal{A}$-injective as $\mathcal{A} = \iota^{-1}(\mathcal{M})$. Hence $[\mathbb{E}] = 0$ as required. Thus $\mathrm{Ext}^1_R(C, M) = 0$.

Conversely, suppose for an R-module C, $\mathrm{Ext}^1_R(C, M) = 0$ for all $M \in \mathcal{M}$. Take any short exact sequence $\mathbb{E}$ of R-modules ending with C:
$$\mathbb{E} : \qquad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$
Applying $\mathrm{Hom}(-, M)$, we obtain the following exact sequence by the long exact sequence connecting Hom and Ext:
$$0 \longrightarrow \mathrm{Hom}(C, M) \longrightarrow \mathrm{Hom}(B, M) \longrightarrow \mathrm{Hom}(A, M) \longrightarrow \mathrm{Ext}^1_R(C, M) = 0$$
So $\mathrm{Hom}(\mathbb{E}, M)$ is exact for every $M \in \mathcal{M}$. This means $\mathbb{E} \in \iota^{-1}(\mathcal{M}) = \mathcal{A}$. □

For a proper class $\mathcal{A}$ of R-modules, let us say that $\mathrm{Ext}_{\mathcal{A}}$ is *factorizable* as
$$R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \xrightarrow{Ext_R} \mathcal{A}b \longrightarrow \mathcal{A}b\ ,$$

if it is a composition $H \circ \mathrm{Ext}_R$ for some functor $H : \mathcal{A}b \longrightarrow \mathcal{A}b$: the diagram

$$\begin{array}{ccc} R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od & \xrightarrow{Ext_{\mathcal{A}}} & \mathcal{A}b \\ & {\scriptstyle Ext_R}\searrow \quad \nearrow{\scriptstyle H} & \\ & \mathcal{A}b & \end{array}$$

is commutative, that is, for all R-modules A, C,

$$\mathrm{Ext}_{\mathcal{A}}(C, A) = H(\mathrm{Ext}_R(C, A)).$$

When the ring R is *commutative*, since the functor Ext_R can be considered to have range $R\text{-}\mathcal{M}od$, we say that $\mathrm{Ext}_{\mathcal{A}}$ is *factorizable* as

$$R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \xrightarrow{Ext_R} R\text{-}\mathcal{M}od \longrightarrow R\text{-}\mathcal{M}od\ ,$$

if it is a composition $H \circ \mathrm{Ext}_R$ for some functor $H : R\text{-}\mathcal{M}od \longrightarrow R\text{-}\mathcal{M}od$: the diagram

$$\begin{array}{ccc} R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od & \xrightarrow{Ext_{\mathcal{A}}} & R\text{-}\mathcal{M}od \\ & {\scriptstyle Ext_R}\searrow \quad \nearrow{\scriptstyle H} & \\ & R\text{-}\mathcal{M}od & \end{array}$$

is commutative, that is, for all R-modules A, C,

$$\mathrm{Ext}_{\mathcal{A}}(C, A) = H(\mathrm{Ext}_R(C, A)).$$

3. The proper classes $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$, $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$, $\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ for a ring R

We have,

$$\begin{aligned} \mathcal{N}eat_{R\text{-}\mathcal{M}od} &= \pi^{-1}(\{\text{all semisimple } R\text{-modules}\}) \\ &= \pi^{-1}(\{M |\, \mathrm{Soc}\, M = M,\ M \text{ an } R\text{-module}\}), \end{aligned}$$

where $\mathrm{Soc}\, M$ is the socle of M, that is the sum of all simple submodules of M. Dualizing this, we have defined the proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$ as said in the introduction by

$$\begin{aligned} \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} &= \iota^{-1}(\{\text{all } R\text{-modules with zero radical}\}) \\ &= \iota^{-1}(\{M |\, \mathrm{Rad}\, M = 0,\ M \text{ an } R\text{-module}\}). \end{aligned}$$

If A is a $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-submodule of an R-module B, denote this by $A \leq_{c\mathcal{N}} B$ and say that A is a *coneat submodule* of B, or that the submodule A of the module B is *coneat in* B.

Every module M with $\mathrm{Rad}\, M = 0$ is $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-injective that is M is injective with respect to every short exact sequence in $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$. Thus supplement submodules are always coneat submodules by the definition of coneat submodules. For any ring R (see [2, Proposition 3.5]),

$$\mathcal{S}uppl_{R\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} \subseteq \iota^{-1}(\{\text{ all (semi-)simple } R\text{-modules}\}).$$

Proposition 3.1. *Given an R-module A, denote by $\mathrm{E}(A)$ the injective envelope of A. Then the monomorphism*

$$\begin{aligned} f : A &\longrightarrow \mathrm{E}(A) \oplus (A/\,\mathrm{Rad}\, A) \\ x &\longmapsto (x, x + \mathrm{Rad}\, A) \end{aligned}$$

is a $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-monomorphism and $\mathrm{E}(A) \oplus (A/\,\mathrm{Rad}\, A)$ is $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-injective.

Proof. From the module $B := \mathrm{E}(A) \oplus (A/\,\mathrm{Rad}\, A)$, we clearly have a projection $B \longrightarrow A/\,\mathrm{Rad}\, A$ and any map $A \longrightarrow M$, with $\mathrm{Rad}\, M = 0$, factors through $A \longrightarrow A/\,\mathrm{Rad}\, A$. □

Corollary 3.2. *An R-module M is $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-injective if and only if it is a direct summand of a module of the form $E \oplus A$, where E is an injective R-module and A is an R-module with* $\mathrm{Rad}\, A = 0$.

Proof. (⇐) is clear since a module with zero radical is $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-injective, and injective modules are of course $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-injective.
(⇒): By Proposition 3.1, we can embed any R-module M as a $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-submodule into a $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-injective module of the form $E \oplus A$, where E is an injective R-module and A is an R-module with $\operatorname{Rad} A = 0$:

$$M \leq_{c\mathcal{N}} E \oplus A \quad \text{and} \quad E \oplus A \text{ is } \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}\text{-injective.}$$

If M is a $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$-injective R-module, then M is a direct summand of $E \oplus A$. □

Proposition 3.3. [25, Lemma 6.1] *Let A be a submodule of an R-module B and $i_A : A \hookrightarrow B$ be the inclusion map. For a* right *ideal I of R, $A \cap IB = IA$ if and only if*

$$R/I \otimes A \xrightarrow{1_{R/I} \otimes i_A} R/I \otimes B$$

is monic.

A ring R is said to be a *left quasi-duo ring* if each *maximal* left ideal is a two-sided ideal.

Lemma 3.4. [13, Lemma 3] *Let R be a left quasi-duo ring. Then for each module M,*

$$\operatorname{Rad} M = \bigcap_{P \underset{max.}{\leq} {}_R R} PM,$$

where the intersection is over all maximal left *ideals of R.*

Proposition 3.5. *Let R be a left quasi-duo ring. Then,*

$$\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} \subseteq \tau^{-1}(\{R/P | P \text{ maximal left ideal of } R\})$$

Proof. The proof is the proof in [13, Proposition 1] where it has been shown that

$$\mathcal{S}uppl_{R\text{-}\mathcal{M}od} \subseteq \tau^{-1}(\{R/P | P \text{ maximal left ideal of } R\}).$$

Take a short exact sequence $\mathbb{E} \in \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$:

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

Without loss of generality, suppose that A is a submodule of B and f is the inclusion homomorphism. So A is a coneat submodule of the module B. By Proposition 3.3, to end the proof it suffices to show that $A \cap PB = PA$ for each maximal left ideal P of R.

Since A is a coneat submodule of B, there exists a submodule K of B such that $A + K = B$ and $A \cap K \leq \operatorname{Rad} A$. Then,

$$\begin{aligned} A \cap PB &= A \cap P(A+K) \leq A \cap (PA + PK) = PA + A \cap PK \\ &\leq PA + A \cap K \leq PA + \operatorname{Rad} A = PA, \end{aligned}$$

where the last equality follows from Lemma 3.4, since each maximal left ideal is assumed to be a two-sided ideal. So $A \cap PB \leq PA$, and since the converse is clear, we obtain $A \cap PB = PA$ as required. □

Proposition 3.6. [13, Proposition 4] *Let R be a ring that can be embedded in an R-module S such that* $\operatorname{Rad} S = R$. *Then:*

(i) *For each module M, there exists a module H such that* $\operatorname{Rad} H = M$.
(ii) *If, in addition, the R-module S/R is* semisimple, *then an* essential *extension H of the module M such that H/M is a semisimple module can be taken such that* $\operatorname{Rad} H = M$.

A module M is said to be a *small module* if it is a small submodule of a module containing it, equivalently if it is a small submodule of its injective envelope. See [19] for *small modules.*

A ring R is said to be a *left small ring* if R, considered as a (left) R-module, is a small R-module, equivalently R is small in its injective envelope $\mathrm{E}(R)$. It is noted in [20, Proposition 3.3] that a ring R is left small, if and only if, $\operatorname{Rad} E = E$ for every injective R-module E, if and only if, $\operatorname{Rad} \mathrm{E}(R) = \mathrm{E}(R)$.

Proposition 3.7. [13, Corollary 5] *If R is a ring that can be embedded in an R-module S such that* $\operatorname{Rad} S = R$ *and S/R is a semisimple R-module (and R is essential in S), then R is a left small ring, so* $\operatorname{Rad} E = E$ *for every injective R-module E and in particular no injective R-module is finitely generated.*

Proposition 3.8. *A left quasi-duo domain which is not a division ring is a left small ring.*

Proof. Let R be left quasi-duo domain which is not a division ring and E be an injective R-module. Since E is injective, it is also a divisible R-module (by for example [6, Proposition 4.7.8]). Since R is not a division ring, any maximal left ideal P of R is nonzero and so $PE = E$ as E is divisible. By Lemma 3.4,

$$\operatorname{Rad} E = \bigcap_{P \underset{max.}{\leq} {}_R R} PE = \bigcap_{P \underset{max.}{\leq} {}_R R} E = E.$$

□

A ring R is said to be *semilocal* if $R/\operatorname{Rad} R$ is a semisimple ring, that is a left (and right) semisimple R-module. See for example [18, §20]. Such rings are also called as rings semisimple modulo its radical as in [3, in §15, pp. 170-172].

Theorem 3.9. *If R is a semilocal ring, then*

$$\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} = \iota^{-1}(\{\textit{all (semi-)simple } R\textit{-modules}\}).$$

Proof. For any ring R, the left side is contained in the right side by [2, Proposition 3.5]. We prove equality for a semilocal ring R. By [3, Corollary 15.18], for every R-module A, $A/\operatorname{Rad} A$ is semisimple. So every R-module M with $\operatorname{Rad} M = 0$ is semisimple. Conversely, every semisimple R-module has zero radical (for any ring R). Hence,

$$\{M \mid \operatorname{Rad} M = 0,\ M \text{ an } R\text{-module}\} = \{ \text{ all semisimple } R\text{-modules } \}.$$

So,

$$\begin{aligned} \mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} &= \iota^{-1}(\{M \mid \operatorname{Rad} M = 0,\ M \text{ an } R\text{-module}\}) \\ &= \iota^{-1}(\{ \text{ all semisimple } R\text{-modules } \}). \end{aligned}$$

The reason for the equality

$$\iota^{-1}(\{\text{all semisimple } R\text{-modules}\}) = \iota^{-1}(\{\text{all simple } R\text{-modules}\})$$

comes from the characterization of semilocal rings in [3, Proposition 15.17]: every product of simple left R-modules is semisimple. Denote $\iota^{-1}(\{\text{all semisimple } R\text{-modules}\})$ shortly by $\mathcal{A}$ and $\iota^{-1}(\{\text{all simple } R\text{-modules}\})$ shortly by $\mathcal{A}'$. Clearly $\mathcal{A} \subseteq \mathcal{A}'$. Conversely, it suffices to show that every semisimple R-module M is injective with respect to the proper class $\mathcal{A}'$. Since M is a semisimple R-module, $M = \bigoplus_{\lambda \in \Lambda} S_\lambda$ for some index set Λ and simple submodules S_λ of M. Then $M \leq N := \prod_{\lambda \in \Lambda} S_\lambda$. The right side N is also a semisimple R-module (by [3, Proposition 15.17]). So its submodule M is a direct summand of N. But N, being a product of simple modules which are injective with respect to the proper class $\mathcal{A}'$, is injective with respect to proper class $\mathcal{A}'$. Then so is its direct summand M as required. □

In [8], $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$-coinjective modules have been called *absolutely complement* modules and $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$-coprojective modules have been called *absolutely co-complement* modules. Similarly, $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-coinjective modules have been called *absolutely supplement* modules and $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-coprojective modules have been called *absolutely co-supplement* modules in [8]. For some properties of these modules, see [8, Chapters 3-4].

Remember the construction of an injective envelope of a module. It is seen from this construction that a module is injective if and only if it has no proper essential extension, that is, it is a closed submodule of every module containing it (see for example [21, Proposition III.11.2]). Since closed submodules and complement submodules of a module coincide, that means the following:

Theorem 3.10. *(by* [8, Proposition 4.1.4]*)* $\mathcal{C}ompl_{R\text{-}\mathcal{M}od}$*-coinjective modules are only injective modules.*

Dually, $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-coprojectives are only projectives if the ring R has zero Jacobson radical:

Theorem 3.11. *If* $\operatorname{Rad} R = 0$*, then* $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$*-coprojective modules are only projective modules.*

Proof. Suppose M is a $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-coprojective module. There exists an epimorphism $g : F \longrightarrow M$ from a free module F. So, for $H := \operatorname{Ker} g$ and f the inclusion homomorphism, we obtain the following short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow H \xrightarrow{f} F \xrightarrow{g} M \longrightarrow 0$$

Since M is $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-coprojective, $\mathbb{E}$ is in $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$. So, H is a supplement in F. Clearly, $\operatorname{Rad} H \leq \operatorname{Rad} F$. Since $\operatorname{Rad} F = JF$ for $J := \operatorname{Rad} R$, the Jacobson radical of R (by for example [18, Proposition 24.6-(3)]), we obtain that $\operatorname{Rad} F = 0$ as $J = 0$ by our assumption. Hence $\operatorname{Rad} H = 0$. Then the short exact sequence $\mathbb{E} \in \mathcal{S}uppl_{R\text{-}\mathcal{M}od}$ splits since modules with zero radical are $\mathcal{S}uppl_{R\text{-}\mathcal{M}od}$-injective by [2, Proposition 3.5]. Then, $F \cong H \oplus M$, and so M is also a projective module. □

This proof, in fact, gives the following:

Theorem 3.12. *If* $\operatorname{Rad} R = 0$*, then* $\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od}$*-coprojective modules are only projective modules.*

4. The proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ for a Dedekind domain W

Throughout this section, let W be a Dedekind domain and suppose it is *not* a field to exclude the trivial cases. Note firstly the following properties of Dedekind domains that we shall use.

Proposition 4.1. *(by* [6, Proposition 10.6.9]*) Any torsion W-module M over a Dedekind domain W is a direct sum of its primary parts in a unique way:*

$$M = \bigoplus_{0 \neq P \underset{max.}{\leq} W} M_P,$$

where for each nonzero prime ideal P of W (so P is a maximal ideal of W),

$$M_P = \{x \in M | P^n x = 0 \text{ for some } n \in \mathbb{Z}^+\}$$

*is the P-*primary part *of the W-module M.*

For a nonzero prime ideal P of a Dedekind domain W, we say that a W-module M is P-*primary* if $M = M_P$.

Proposition 4.2. *Let W be a Dedekind domain, P be a nonzero prime ideal of W and M be a P-primary W-module. Then* $\operatorname{Rad} M = PM$*.*

Proposition 4.3. *Let W be a Dedekind domain which is not a field. For an injective W-module E,* $\operatorname{Rad} E = E$*.*

Proposition 4.4. *Any nonzero torsion module over a Dedekind domain has a simple submodule.*

Theorem 4.5. *(by* [17, Theorem 3] *and* [16, Theorem 2-(b)]*, or by* [10, Theorem VI.1.14]*)* Projective *modules over Dedekind domains which are* not *finitely generated are* free*.*

Proposition 4.6. *For a Dedekind domain W which is not a field, the following are equivalent:*

(i) $\operatorname{Rad} W \neq 0$*,*
(ii) *W is semilocal,*
(iii) *W has only finitely many maximal ideals,*
(iv) *W is a PID (principal ideal domain) with only finitely many maximal ideals.*

Proposition 4.7. [10, Exercise I.5.5-(c)] *For a commutative domain R, an ideal J of R and any R-module M,*

$$\mathrm{Ext}_R(J^{-1}/R, M) \cong M/JM,$$

if J is an invertible ideal.

Corollary 4.8. *For a Dedekind domain W, a* nonzero *ideal J of W and any W-module M,*

$$\mathrm{Ext}_W(W/J, M) \cong M/JM$$

Proof. Since W is a Dedekind domain, the nonzero ideal J of W is invertible. So, the result follows from Proposition 4.7 since $J^{-1}/W \cong W/J$ by [24, Lemma 4.4]. □

We will show that if $\mathrm{Rad}\, W = 0$, then the proper class $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ is *strictly* between $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$ and $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$. When $\mathrm{Rad}\, W \neq 0$, still $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \neq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$, but $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$. To prove that $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$, we will follow mainly the proofs in [13, Theorems 6-7, Propositions 4-5] for the Dedekind domain W, which simplifies some steps and for which some missing steps in [13, proofs of Theorem 6 and Proposition 5] can be done.

For a Dedekind domain W, the proper class $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ is injectively generated by simple W-modules:

Proposition 4.9. *For a Dedekind domain W,*

$$\mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \iota^{-1}(\{W/P | P \textit{ maximal ideal of } W\}).$$

Proof. Denote $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ shortly by $\mathcal{C}$:

$$\mathcal{C} = \iota^{-1}(\{M | M \text{ is a homogenous semisimple } W\text{-module}\}).$$

Let $\mathcal{C}'$ be the proper class

$$\mathcal{C}' = \iota^{-1}(\{W/P | P \text{ maximal ideal of } W\}).$$

Clearly $\mathcal{C} \subseteq \mathcal{C}'$. Conversely, it suffices to show that every homogenous semisimple W-module M is injective with respect to the proper class $\mathcal{C}'$. Since M is a homogenous semisimple W-module, $M = \bigoplus_{\lambda\in\Lambda} S_\lambda$ for some index set Λ and simple submodules S_λ of M such that for some maximal left ideal P of R, $S_\lambda \cong R/P$ for every $\lambda \in \Lambda$. Then $M \leq N := \prod_{\lambda\in\Lambda} S_\lambda$. Since $PN = 0$, N may be considered as a vector space over the field W/P. If α is the dimension of the W/P-vector space N, then N is isomorphic to a direct sum of α copies of W/P. So N is a homogenous semisimple W-module. Since N is semisimple, its submodule M is a direct summand of N. But $N = \prod_{\lambda\in\Lambda} S_\lambda$, being a product of simple modules which are injective with respect to the proper class $\mathcal{C}'$, is injective with respect to proper class $\mathcal{C}'$. Then so is its direct summand M as required. □

Proposition 4.10. *For a Dedekind domain W,*

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subseteq \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

Proof. By [2, Proposition 3.5], $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$. By Proposition 3.5,

$$\mathcal{C}o\text{-}\mathcal{N}eat_{R\text{-}\mathcal{M}od} \subseteq \tau^{-1}(\{W/P | P \text{ maximal ideal of } W\}).$$

By [2, Theorem 3.7], the right side equals $\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$. □

After two lemmas, we give an example of a $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphism which is *not* a $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$-monomorphism.

Lemma 4.11. *(by* [13, Theorem 7, Proposition 4, Corollary 5]*) Let W be a Dedekind domain which is not a field and Q the field of fractions of W. Let $S \leq Q$ be the submodule of the W-module Q such that $S/W = \mathrm{Soc}(Q/W)$. Then:*

(i) $\mathrm{Rad}\, S = W$ *and S/W is a semisimple W-module,*

(ii) *For a free W-module* $F := \bigoplus_{\lambda\in\Lambda} W$ *for some index set* Λ*, take the* W*-module* $A := \bigoplus_{\lambda\in\Lambda} S$*. Then* $\operatorname{Rad} A = F$ *and* $A/\operatorname{Rad} A$ *is a semisimple* W*-module.*

Proof. (i) Since $S/W = \operatorname{Soc}(Q/W)$, it is clearly semisimple. So $\operatorname{Rad}(S/W) = 0$. Hence $\operatorname{Rad} S \leq W$.
Let P be a maximal ideal of W. Since W is not a field, $P \neq 0$. So P is an invertible ideal, that is, for the submodule $P^{-1} \leq Q$, $PP^{-1} = W$. Hence P^{-1}/W is a homogenous semisimple W-module with each simple submodule isomorphic to W/P. So, $P^{-1}/W \leq \operatorname{Soc}(Q/W) = S/W$, which implies that $P^{-1} \leq S$. So

$$W = PP^{-1} \leq PS.$$

Then, by Lemma 3.4,

$$\operatorname{Rad} S = \bigcap_{P \underset{max.}{\leq} {}_R R} PS \geq W.$$

Thus, $\operatorname{Rad} S = W$.
(ii) $\operatorname{Rad} A = \bigoplus_{\lambda\in\Lambda} \operatorname{Rad} S = \bigoplus_{\lambda\in\Lambda} W = F$ and $A/\operatorname{Rad} A = \bigoplus_{\lambda\in\Lambda}(S/W)$ is semisimple. □

Lemma 4.12. *(by* [19, Lemma 6]*) Let* W *be a Dedekind domain which is not a field and* Q *the field of fractions of* W*. There exists an epimorphism* $g : F \longrightarrow Q$ *from a free* W*-module* $F := \bigoplus_{\lambda\in\Lambda} W$ *for some index set* Λ*. The free* W*-module* $F := \bigoplus_{\lambda\in\Lambda} W$ *is* not *a small* W*-module, and so the index set* Λ *is necessarily* infinite*.*

Proof. Let $H := \operatorname{Ker} g$. Then $F/H \cong Q$. By [19, Lemma 6], F is *not* a small W-module since $F/H \cong Q$ is a nonzero injective module. In fact, this is simply because if F is a small module, then F is small in its injective envelope $\mathrm{E}(F)$ by [19, Theorem 1]. So, the quotient module F/H is small in $\mathrm{E}(F)/H$. But since $F/H \cong Q$ is injective, F/H is a direct summand of $\mathrm{E}(F)/H$ which contradicts with F/H being small in $\mathrm{E}(F)/H$.
Since Q is injective, $\operatorname{Rad} Q = Q$ by Proposition 4.3. So the finitely generated submodule W of $\operatorname{Rad} Q = Q$ is small in Q. If the index set Λ were finite, then $W \ll Q$ would imply $F = \bigoplus_{\lambda\in\Lambda} W \ll \bigoplus_{\lambda\in\Lambda} Q$ so that F would be a small module. □

Example 4.13. (by [13, Proposition 5]) Let W be a Dedekind domain which is not a field and Q the field of fractions of W. Consider the W-modules

$$F := \bigoplus_{\lambda\in\Lambda} W = \operatorname{Rad} A \leq A := \bigoplus_{\lambda\in\Lambda} S \leq \bigoplus_{\lambda\in\Lambda} Q = \mathrm{E}(A),$$

where,

(i) $S \leq Q$ is the W-module given as in Lemma 4.11 such that $S/W = \operatorname{Soc}(Q/W)$,
(ii) the free W-module $F := \bigoplus_{\lambda\in\Lambda} W$ is as in Lemma 4.12 for some *infinite* index set Λ such that there exists an epimorphism $g : F \longrightarrow Q$,
(iii) $\mathrm{E}(A)$ denotes the injective envelope of A.

Then the monomorphism

$$\begin{aligned} f : A &\longrightarrow \mathrm{E}(A) \oplus (A/\operatorname{Rad} A) \\ x &\longmapsto (x, x + \operatorname{Rad} A) \end{aligned}$$

is a $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphism but *not* a $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$-monomorphism. So $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \neq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$.

Proof. By Lemma 4.11, $\operatorname{Rad} A = F$. By Proposition 3.1, f is a $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphism and $\mathrm{E}(A) \oplus (A/\operatorname{Rad} A)$ is $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-injective.

Suppose for the contrary that f is a $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$-monomorphism.

Let $M := f(A)$ and $N := \mathrm{E}(A) \oplus (A/\operatorname{Rad} A)$. Then M is a supplement in N. That means there exits a submodule $K \leq N$ such that

$$M + K = N \quad \text{and} \quad M \cap K \ll M.$$

Let $C := M \cap K$. Since $C \ll M$, $C \leq \operatorname{Rad} M = \operatorname{Rad} f(A) \cong \operatorname{Rad} A = F$, so C is also a projective W-module. Suppose C is not finitely generated. Then by Theorem 4.5, C is free. So, rank of C is at most $|\Lambda|$, the rank of F. But, rank of C cannot be $|\Lambda|$ because then $C \cong F$ would be a small module, contradicting that F is not a small module by Lemma 4.12. Since rank of C is strictly less than Λ, C has a basis whose cardinality is strictly less than Λ. Thus C has a *generating set* whose whose cardinality is strictly less than Λ, if C is not finitely generated. But that is also true if C is finitely generated since Λ is an infinite set. So, in any case, C has a generating set $Y = \{y_\gamma | \gamma \in \Gamma\}$ for some index set Γ such that $|\Gamma| < |\Lambda|$.

As $C \ll M$,

$$C \leq \operatorname{Rad} M \leq \operatorname{Rad} N = \operatorname{Rad}(\mathrm{E}(A) \oplus (A/\operatorname{Rad} A)) = \operatorname{Rad} \mathrm{E}(A) \leq \mathrm{E}(A).$$

So,

$$\begin{aligned}(\mathrm{E}(A)/C) \oplus (A/\operatorname{Rad} A) &\cong (\mathrm{E}(A) \oplus (A/\operatorname{Rad} A))/C = N/C = (M+K)/C \\ &\cong (M/C) \oplus (K/C)\end{aligned}$$

Since the left side is $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-injective, so is the direct summand M/C of the right side. Hence by Corollary 3.2, M/C is a direct summand of a module of the form $E_1 \oplus A_1$, where E_1 is an injective W-module and A_1 is a W-module such that $\operatorname{Rad} A_1 = 0$. So there exists a submodule X of $E_1 \oplus A_1$ such that $(M/C) \oplus X = E_1 \oplus A_1$. Then, since radical of an injective W-module is equal to itself (by Proposition 4.3), we obtain that

$$((\operatorname{Rad} M)/C) \oplus \operatorname{Rad} X = (\operatorname{Rad}(M/C)) \oplus \operatorname{Rad} X = \operatorname{Rad} E_1 \oplus \operatorname{Rad} A_1 = E_1 \oplus 0 = E_1.$$

So $\operatorname{Rad} M/C$ is an injective module as it is a direct summand of an injective module.

But $\operatorname{Rad} M \cong F$ is a free W-module of rank $|\Lambda|$ and C has a generating set $Y = \{y_\gamma | \gamma \in \Gamma\}$ with $|\Gamma| < |\Lambda|$. Let $\{x_\lambda | \lambda \in \Lambda\}$ be a basis for the free W-module $\operatorname{Rad} M$. Express each y_γ in terms of the basis elements x_λ, $\lambda \in \Lambda$, for $\operatorname{Rad} M$. Let F_1 be the submodule of the free W-module $\operatorname{Rad} M$ spanned by the basis elements x_λ which occur with a nonzero coefficient in the expansion of at least one y_γ, $\gamma \in \Gamma$. Then F_1 has rank $\leq |\Gamma|$. Let F_2 be the submodule of the free W-module $\operatorname{Rad} M$ spanned by the remaining x_λ's. Then $\operatorname{Rad} M = F_1 \oplus F_2$ and $F_2 \neq 0$ as we have strict inequality for the cardinalities: $|\Gamma| < |\Lambda|$. Since $C \leq F_1$,

$$\operatorname{Rad} M/C \cong (F_1/C) \oplus F_2.$$

This implies that F_2 is also an injective W-module since $\operatorname{Rad} M/C$ is so. But a nonzero free W-module is *not* injective, because radical of an injective W-module is equal to itself (by Proposition 4.3) but a nonzero free module has proper radical (more generally any nonzero projective module has proper radical, see for example [3, Proposition 17.14]). This contradiction ends the proof. □

For a Dedekind domain W which is not a field, $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ only when $\operatorname{Rad} W \neq 0$:

Lemma 4.14. *Let W be a Dedekind domain such that* $\operatorname{Rad} W \neq 0$. *Then*

$$\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

Proof. The second equality holds for any Dedekind domain W by [2, Theorem 3.7]. Suppose $\operatorname{Rad} W \neq 0$. Then by Proposition 4.6, W is a semilocal ring. So by Theorem 3.9,

$$\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \iota^{-1}(\{\text{all simple } W\text{-modules}\}).$$

By Proposition 4.9,

$$\iota^{-1}(\{\text{all simple } W\text{-modules}\}) = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

□

Lemma 4.15. *Let W be a Dedekind domain which is not a field such that* $\operatorname{Rad} W = 0$*. For any maximal ideal P in W, there exits a short exact sequence $\mathbb{E} \in\in \operatorname{Ext}_W(W/P^2, W)$ which is in $\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ but* not *in $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$, and hence* not *in $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$.*

Proof. By Corollary 4.8, for the ideal $J = P^2$ we obtain

$$\operatorname{Ext}_W(W/P^2, W) = \operatorname{Ext}_W(W/J, W) \cong \operatorname{Ext}_W(J^{-1}/W, W) \cong W/JW = W/P^2$$

Denote $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$, $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ by $\mathcal{C}$, $\mathcal{S}$ and $c\mathcal{N}$ respectively. By [2, Theorem 3.8],

$$\operatorname{Ext}_{\mathcal{C}}(W/P^2, W) = \operatorname{Rad}(\operatorname{Ext}_W(W/P^2, W)) \cong \operatorname{Rad}(W/P^2) = P/P^2 \neq 0.$$

But $\operatorname{Ext}_{\mathcal{S}}(W/P^2, W) \leq \operatorname{Ext}_{c\mathcal{N}}(W/P^2, W) = 0$ since $\operatorname{Rad} W = 0$ by our assumption (the $\leq$ follows since $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ by Proposition 4.10). Take a *nonzero* element $[\mathbb{E}] \in \operatorname{Ext}_{\mathcal{C}}(W/P^2, W)$. Then $\mathbb{E}$ is in $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ but *not* in $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$. □

Theorem 4.16. *Let W be a Dedekind domain which is not a field.*

(i) *If* $\operatorname{Rad} W = 0$*, then*

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

(ii) *If* $\operatorname{Rad} W \neq 0$*, then*

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subsetneqq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

Proof. By Proposition 4.10,

$$\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subseteq \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}.$$

By Example 4.13, $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \neq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$.

(i) If $\operatorname{Rad} W = 0$, then $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \neq \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ by Lemma 4.15.

(ii) If $\operatorname{Rad} W \neq 0$, then by Lemma 4.14, $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$.

□

Theorem 4.17. *Let W be a Dedekind domain. Take a W-module B and a submodule $A \leq B$. Suppose A is a* finitely generated torsion *W-module. Then A is neat in B if and only if A is coneat in B.*

Proof. By Theorem 4.16, we already have $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od}$. So ($\Leftarrow$) holds for any W-module A. Conversely, if A is a *finitely generated torsion* W-module and A is neat in B (so complement in B), then by [2, Theorem 4.1], A is a supplement in B, hence A is coneat in B since $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ by Proposition 4.10. □

For a Dedekind domain W, the functor $\operatorname{Ext}_{\mathcal{C}ompl_{W\text{-}\mathcal{M}od}}$ is factorizable as

$$W\text{-}\mathcal{M}od \times W\text{-}\mathcal{M}od \xrightarrow{Ext_W} W\text{-}\mathcal{M}od \xrightarrow{\operatorname{Rad}} W\text{-}\mathcal{M}od$$

by [2, Theorem 3.8], but:

Theorem 4.18. *Let W be a Dedekind domain which is not a field such that* $\operatorname{Rad} W = 0$*. Then the functors* $\operatorname{Ext}_{\mathcal{S}uppl_{W\text{-}\mathcal{M}od}}$ *and* $\operatorname{Ext}_{\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}}$ *are* not *factorizable as*

$$W\text{-}\mathcal{M}od \times W\text{-}\mathcal{M}od \xrightarrow{Ext_W} W\text{-}\mathcal{M}od \xrightarrow{H} W\text{-}\mathcal{M}od$$

for any functor $H : W\text{-}\mathcal{M}od \longrightarrow W\text{-}\mathcal{M}od$.

Proof. Denote $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$, $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ by $\mathcal{C}$, $\mathcal{S}$ and $c\mathcal{N}$ respectively.

Suppose for the contrary that $\text{Ext}_{\mathcal{S}}$ is factorizable as

$$W\text{-}\mathcal{M}od \times W\text{-}\mathcal{M}od \xrightarrow{Ext_W} W\text{-}\mathcal{M}od \xrightarrow{H} W\text{-}\mathcal{M}od$$

for some functor $H : W\text{-}\mathcal{M}od \longrightarrow W\text{-}\mathcal{M}od$. So for all W-modules A and C, $\text{Ext}_{\mathcal{S}}(C, A) = H(\text{Ext}_W(C, A))$. Let P be a maximal ideal of W. In the proof of Lemma 4.15, we have found that

$$\text{Ext}_W(W/P^2, W) \cong W/P^2 \quad \text{and} \quad \text{Ext}_{\mathcal{S}}(W/P^2, W) = 0.$$

This implies that $H(W/P^2) \cong H(\text{Ext}_W(W/P^2, W)) = \text{Ext}_{\mathcal{S}}(W/P^2, W) = 0$, hence $H(W/P^2) = 0$. But also $\text{Ext}_W(W/P^2, W/P^2) \cong W/P^2$ by Corollary 4.8. By [2, Theorem 4.1], since W/P^2 is a finitely generated torsion W-module, we obtain

$$\begin{aligned} \text{Ext}_{\mathcal{S}}(W/P^2, W/P^2) &= \text{Ext}_{\mathcal{C}}(W/P^2, W/P^2) \\ &= \text{Rad}(\text{Ext}_W(W/P^2, W/P^2)) \cong P(W/P^2) = P/P^2 \neq 0. \end{aligned}$$

So in this case $H(W/P^2) \cong H(\text{Ext}_W(W/P^2, W/P^2)) = \text{Ext}_{\mathcal{S}}(W/P^2, W/P^2) \cong P/P^2 \neq 0$. This contradiction shows that $\text{Ext}_{\mathcal{S}uppl_{W\text{-}\mathcal{M}od}}$ is *not* factorizable.

Similarly, $\text{Ext}_{c\mathcal{N}}$ is *not* factorizable. In the above proof, just replace $\mathcal{S}$ by $c\mathcal{N}$. Note that $\text{Ext}_{c\mathcal{N}}(W/P^2, W) = 0$ since $\text{Rad}\, W = 0$, and

$$\text{Ext}_{c\mathcal{N}}(W/P^2, W/P^2) = \text{Ext}_{\mathcal{C}}(W/P^2, W/P^2)$$

by Theorem 4.17 as W/P^2 is a finitely generated torsion W-module and $\mathcal{N}eat_{W\text{-}\mathcal{M}od} = \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ by [2, Theorem 3.7]. □

The neat submodules of a torsion module over a Dedekind domain coincides with its coneat submodules:

Theorem 4.19. *Let W be a Dedekind domain. Let B be a* torsion *W-module, and A any submodule of B. Then A is neat in B if and only if A is coneat in B.*

Proof. ($\Leftarrow$) always holds (for any module B) by Proposition 4.10. Conversely, suppose A is neat in B. To exclude the trivial cases suppose that W is not a field, so its maximal ideals are nonzero. To show that A is coneat in B, we must show that for every W-module M with $\text{Rad}\, M = 0$, any homomorphism $f : A \longrightarrow M$ can be extended to B. Since B is a torsion W-module, so is its submodule A, hence $f(A)$ is also a torsion W-module. So, without loss of generality, we may suppose that M is also a torsion W-module. Decompose A, B and M into their P-primary parts by Proposition 4.1: $A = \bigoplus_P A_P$, $B = \bigoplus_P B_P$ and $M = \bigoplus_P M_P$, where the index P runs through all *nonzero* prime ideals of W, hence P runs through all maximal ideals of W. For each maximal ideal P of W, let $f_P : A_P \longrightarrow M_P$ be the restriction of f to A_P, with range restricted to M_P also (note that $f(A_P) \leq M_P$). Since $0 = \text{Rad}\, M = \bigoplus_P \text{Rad}\, M_P = \bigoplus_P PM_P$ by Proposition 4.2, we have $PM_P = 0$ for each maximal ideal P. So, each M_P is a $\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-injective module by [2, Theorem 3.7]. Suppose each A_P is neat in B_P. Then there exists $\tilde{f}_P : B_P \longrightarrow M_P$ extending $f_P : A_P \longrightarrow M_P$. Define $\tilde{f} : B \longrightarrow M$, by $\tilde{f}(\sum_P b_P) = \sum_P \tilde{f}_P(b_P)$ for each $\sum_P b_P \in \bigoplus_P B_P = B$ where $b_P \in B_P$ for every maximal ideal P. Then $\tilde{f} : B \longrightarrow M$ is the required homomorphism extending $f : A \longrightarrow M$:

$$\begin{array}{ccc} A = \bigoplus_P A_P & \leq_c & \bigoplus_P B_P = B \\ {\scriptstyle f=\bigoplus_P f_P}\downarrow & \swarrow {\scriptstyle \tilde{f}=\bigoplus_P \tilde{f}_P} & \\ M = \bigoplus_P M_P & & \end{array}$$

Thus, it only remains to show that each A_P is neat in B_P which follows since $\mathcal{N}eat_{W\text{-}\mathcal{M}od}$ is a proper class: A_P is neat in A as it is a direct summand of A, and A is neat in B. So, A_P is neat in B as the composition of two $\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphisms is a $\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphism by proper class axioms. Since $A_P \leq B_P \leq B$, we have that the composition $A_P \hookrightarrow B_P \hookrightarrow B$ of inclusion monomorphisms is a $\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphism, so the first inclusion monomorphism $A_P \hookrightarrow B_P$ must also be a $\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-monomorphism by proper class axioms. $\square$

5. Coinjectives and coprojectives with respect to $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$, $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$ and $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$

By Theorem 3.10, we already know that $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coinjectives are only injective W-modules. Since $\mathcal{S}uppl_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od} \subseteq \mathcal{C}ompl_{W\text{-}\mathcal{M}od}$ for a Dedekind domain W, we have that $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$-coinjectives and $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-coinjectives are also only injective W-modules. By Theorems 3.11 and 3.12, if $\operatorname{Rad} W = 0$, then $\mathcal{S}uppl_{W\text{-}\mathcal{M}od}$-coprojectives and $\mathcal{C}o\text{-}\mathcal{N}eat_{W\text{-}\mathcal{M}od}$-coprojectives are only projective W-modules.

Theorem 5.1. *For a Dedekind domain W, $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojectives are only torsion-free W-modules.*

Proof. Firstly, each torsion-free W-module C is $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojective because every short exact sequence

$$\mathbb{E}: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

of W-modules is in $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$: By [2, Theorem 3.7],

$$\mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \mathcal{N}eat_{W\text{-}\mathcal{M}od} = \pi^{-1}(\{W/P | P \text{ maximal ideal of } W\})$$

So, it suffices to show that every simple module W/P, where P is a maximal ideal of W, is projective with respect to $\mathbb{E}$. But that is clear since the image of a homomorphism $\alpha : W/P \longrightarrow C$ is torsion as W/P is torsion, so there is no homomorphism $W/P \longrightarrow C$ except the zero homomorphism which of course extends to $W/P \longrightarrow B$ as the zero homomorphism.
Conversely suppose C is a $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojective W-module. Since the proper class

$$\mathcal{C}ompl_{W\text{-}\mathcal{M}od} = \iota^{-1}(\{W/P | P \text{ maximal ideal of } W\})$$

is injectively generated by all simple W-modules (by Proposition 4.9), we know that a W-module C is $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojective if and only if $\operatorname{Ext}^1_W(C,S) = 0$ for all simple W-modules S by Proposition 2.2. Suppose for the contrary that C is not torsion-free. Hence there exists $0 \neq c \in C$ such that $Ic = 0$ for some nonzero ideal I of W. Consider the submodule Wc of C. Since Wc is a torsion module, it has a simple submodule S by Proposition 4.4. Say $S \cong W/P$ for some maximal ideal P of W. Consider the short exact sequence

$$0 \longrightarrow S \xrightarrow{f} C \xrightarrow{g} C/S \longrightarrow 0$$

where f is the inclusion homomorphism and g is the natural epimorphism. By the long exact sequence connecting Hom and Ext, we have the following exact sequence:

$$\ldots \longrightarrow \operatorname{Ext}^1_W(C,S) = 0 \longrightarrow \operatorname{Ext}^1_W(S,S) = 0 \longrightarrow \operatorname{Ext}^2_W(C/S,S) = 0 \longrightarrow \ldots$$

Here $\operatorname{Ext}^1_W(C,S) = 0$ because C is $\mathcal{C}ompl_{W\text{-}\mathcal{M}od}$-coprojective and $\operatorname{Ext}^2_W(C/S,S) = 0$ since $\operatorname{Ext}^2_W = 0$ as W is a hereditary ring. Thus the above exact sequence implies that $\operatorname{Ext}^1_W(S,S) = 0$ which is the required contradiction since $\operatorname{Ext}^1_W(S,S) \cong \operatorname{Ext}^1_W(W/P, W/P) \cong W/P \neq 0$ by Corollary 4.8. $\square$

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TÜBİTAK
PROJE ÖZET BİLGİ FORMU

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Proje Başlığı: Tümleyen ve Bütünleyen Modüllerin Homolojik Özellikleri
Proje Yürütücüsü ve Araştırmacılar: Doç.Dr. Dilek YILMAZ Prof.Dr. Refail ALİZADE Yrd. Doç. Dr. Engin BÜYÜKAŞIK Yrd. Doç. Dr. Engin MERMUT
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Destekleyen Kuruluş(ların) Adı ve Adresi:
Projenin Başlangıç ve Bitiş Tarihleri: 1 Mart 2008- 1 Mart 2010
Öz (en çok 70 kelime) Sırasıyla zayıf tümleyen altmodül, küçük altmodül ve tümleyeni bulunan altmodüllerle tanımlanan *Wsupp*, *Small* ve *S* kısa tam dizi sınıfları ele alınmıştır. Bu sınıfların hiçbiri öz sınıf oluşturmuyor. Projede bu sınıfların ürettikleri öz sınıfların aynı olduğu ve kalıtsal halka üzerinde bu öz sınıfın *Wsupp* sınıfının bir doğal genelleşmesi olduğu kanıtlanmıştır. Ayrıca bu öz sınıfın eş atomik modüller cinsinden başka bir betimlenmesi de verilmiştir. Bu öz sınıfın eşinjektif modülleri için bir kriter geliştirilmiş ve bu kriter yardımıyla bazı durumlarda eşinjektif modülleri betimlenmiştir. Kalıtsal halka üzerinde söz konusu öz sınıfın eşinjektif üretilen olduğu ve global boyutunun 1'den fazla olmadığı kanıtlanmıştır.
Anahtar Kelimeler: Proper Class, Supplement Submodule.
Fikri Ürün Bildirim Formu Sunuldu mu? Evet ☐ Gerekli Değil ☒ Fikri Ürün Bildirim Formu'nun tesliminden sonra 3 ay içerisinde patent başvurusu yapılmalıdır.
Projeden Yapılan Yayınlar: **1.** E. Büyükaşık & D. Pusat-Yılmaz, Modules Whose Maximal Submodules has Supplements, Hacettepe Journal of Science and Engineering (SCI) yayına kabul edildi. (Ek-1). **2.** E. Büyükaşık&E. Mermut & S. Özdemir, Rad-Supplemented Modules, submitted to Houston Journal of Mathematics (SCI). (Ek-2) **3.** R. Alizade & Y. Demirci & Y. Durgun & D.Pusat-Yılmaz, The Proper Class Generated by Weak Supplements, bir dergiye gönderilecek. (Ek-3) . **4.** R. Alizade & E. Mermut, Coneat Submodules over Dedekind Domains, bir dergiye gönderilecek. (Ek-4).