DIRECT AND INTERIOR INVERSE GENERALIZED IMPEDANCE PROBLEMS FOR THE MODIFIED HELMHOLTZ EQUATION

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Dedicated to the ones I love the most, my father and my mother

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ABSTRACT

DIRECT AND INTERIOR INVERSE GENERALIZED IMPEDANCE PROBLEMS FOR THE MODIFIED HELMHOLTZ EQUATION

Our research is motivated by the classical inverse scattering problem to reconstruct impedance functions. This problem is ill-posed and nonlinear. This problem can be solved by Newton-type iterative and regularization methods. In the first part, we suggest numerical methods for resolving the generalized impedance boundary value problem for the modified Helmholtz equation. We follow some strategies to solve it. The strategies of the first method are founded on the idea that the problem can be reduced to the boundary integral equation with a hyper-singular kernel. While the strategy of the second approach makes use of the concept of numerical differentiation, the first approach treats the hyper singular integral operator by splitting off the singularity. We also show the convergence of the first method in the Sobolev sense and the solvability of the boundary integral equation. We give numerical examples which show exponential convergence for analytical data. In the second part of this work, we take into account the inverse scattering problem of reconstructing the cavity's surface impedance from sources and measurements positioned on a curve within it. For the approximate solution of an ill-posed and nonlinear problem, we propose a direct and hybrid method which is a Newton-type method based on a boundary integral equation approach for the boundary value problem for the modified Helmholtz equation. As a consequence of this, the numerical algorithm combines the benefits of direct and iterative schemes and has the same level of accuracy as a Newton-type method while not requiring an initial guess. The results are confirmed by numerical examples which show that the numerical method is feasible and effective.

ÖZET

MODİFİYE EDİLMİŞ HELMHOLTZ DENKLEMİ İÇİN DİREK VE İÇSEL TERS GENELLEŞTİRİLMİŞ EMPEDANS PROBLEMLER

Araştırmamız, empedans fonksiyonlarını yeniden yapılandırmak için klasik ters saçılma problemi tarafından motive edilmiştir. Bu problem kötü tanımlanmıştır ve lineer değildir. Bu problem Newton tipi yinelemeli ve düzenlileştirme yöntemleriyle çözülebilir. İlk bölümde, modifiye edilmiş Helmholtz denklemi için genelleştirilmiş empedans sınır değeri problemini çözmek için sayısal yöntemler öneriyoruz. Bunu çözmek için bazı stratejiler izliyoruz. İlk yöntemin stratejileri, problemin hiper-tekil bir çekirdek ile sınır integral denklemine indirgenebileceği fikri üzerine kurulmuştur. İkinci yaklaşımın stratejisi sayısal türev kavramını kullanırken, birinci yaklaşım hiper tekil integral operatörünü tekilliği bölerek ele alır. Sobolev anlamında birinci yöntemin yakınsamasını ve sınır integral denkleminin çözülebilirliğini de gösteriyoruz. Analitik veriler için üstel yakınsama gösteren sayısal örnekler veriyoruz. Bu çalışmanın ikinci bölümünde, kavitenin yüzey empedansını kaynaktan ve bunun içinde bir eğri üzerinde konumlandırılmış ölçümlerden yeniden yapılandırmanın ters saçılma problemini dikkate alıyoruz. Kötü konumlanmış ve doğrusal olmayan bir problemin yaklaşık çözümü için, modifiye edilmiş Helmholtz denklemi için sınır değer problemi için sınır integral denklemi yaklaşımına dayalı Newton tipi bir yöntem olan doğrudan ve hibrit bir yöntem öneriyoruz. Bunun bir sonucu olarak, sayısal algoritma, doğrudan ve yinelemeli şemaların faydalarını birleştirir ve ilk tahmin gerektirmeden Newton tipi bir yöntemle aynı doğruluk seviyesine sahiptir. Sonuçlar, sayısal yöntemin uygulanabilir ve etkili olduğunu gösteren sayısal örneklerle doğrulanmıştır.

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CHAPTER 1

INTRODUCTION

Many significant problems in science and engineering entail finding a solution to the equation

$$\Delta u - k^2 u = 0 \tag{1.1}$$

depending on the proper boundary conditions. This equation (1.1) is called the modified Helmholtz equation. Boundary value problems for the modified Helmholtz equation play a crucial role in various areas such as in heating and in cooling materials, in implicit marching schemes for the heat equation, Debye-Huckel theory, and the linearization of the Poisson-Boltzmann equation associated with electrostatic interactions and electric potential governed by the modified Helmholtz equation, see Cheng et al. (2006) and references therein. The Yukawa equation which is a modified version of the Helmholtz equation, appears in a number of scientific applications (see, Bin-Mohsin and Lesnic (2019); Yovanovich et al. (1988); Balakrishnan and Ramachandran (2000)). The modified Helmholtz equation called Helmholtz equation with mines occurs in many elliptic PDEs(Golberg and Chen (1998)). It is also considered as a modified Helmholtz equation when k is complex imaginary number. Possibly the most investigated problems are Laplace equation(Greenbaum et al. (1993)) $\Delta u = 0$ and the Helmholtz equation $\Delta u + k^2 u = 0$. For more knowledge about Helmholtz and Laplace equation we refer to Colton et al. (1998)'s book. There are numerous numerical techniques in the literature to solve the boundary value problems for the modified Helmholtz equation. For instance, Li (2006) introduced a technique based on plane wave functions in simply connected domain in $\mathbb{R}^2$; Bin-Mohsin and Lesnic (2012) used the fundamental solution method; Kropinski and Quaife (2011) employed fast multiple method with integral equations and dealt with singularity by hybrid Gauss-trapezoidal rule. The Chen et al. (2014) applied a singular boundary technique; Cheng et al. (2006) presented a fast multipole-accelerated integral equation; and Duruflé et al. (2006) considered the Trefftz technique. All these results take into consideration the classical boundary condition. Our research is motivated by generalized impedance boundary condition

$$\frac{\partial u}{\partial \nu} + k\left(\lambda u - \frac{d}{ds}\mu\frac{du}{ds}\right) = g \tag{1.2}$$

for the modified Helmholtz equation and also the corresponding inverse impedance problem. Direct interior problem consist of obtaining the solution to interior boundary value problem (1.1)-(1.2). This problem is well-posed problem. Here we are interested in integral equation methods based on the reduction of the problem to the boundary integral equation to solve (1.1) subject to boundary condition (1.2). Two methods are proposed to solve the problem. In the first method the hyper-singular kernel is treated by splitting off the singularity whereas in the second method the idea of numerical differentiation is used. Moreover, we first have proved the solvability of the boundary integral equation and shown the convergence of the first method in Sobolov space, which is verified by numerical examples. Cakoni and Kress (2012) and Kress (2018) examined the direct and inverse problems for the 2D Helmholtz equation, which are most pertinent to the current work. The inverse problems for an elliptic partial differential equation in a bounded domain can be roughly classified into two groups of exterior and interior problems. The exterior problems are concerned with either reconstructing the unknown boundary, Liu (2019), Bin-Mohsin and Lesnic (2012) and Marin and Karageorghis (2009), or boundary condition, source from the Cauchy data on the boundary, Yang et al. (2017), or concerned with finding the solution inside the region from the Cauchy data on the boundary or its segment, Nguyen et al. (2013). The interior inverse problem for the modified Helmholtz equation, where the unknowns are the impedance functions defining the boundary condition, is the focus of this study.

Let us introduce operator $F : D(F) \to R(F)$ described by

$$F(\lambda, \mu) = u^s|_C. \tag{1.3}$$

The inverse problem introduced above is a nonlinear equation with respect to impedance

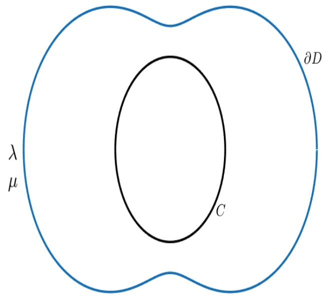

Figure 1.1.: Geometry of the domain for inverse impedance problem

functions λ and μ and it is ill-posed problem. Iterative regularization method which is Newton-type can successfully solve this issue. Our aim is to recover impedance functions in (1.2) given boundary in Figure 1.1 and measurement data u^s on C. The second chapter is concerned with a direct problem and its convergence analysis. We suggest two numerical methods for solving the boundary value problem for the modified Helmholtz equation with generalized impedance boundary condition(GIBC) and show the method is convergent in Sobolev space. The third chapter is devoted to the inverse impedance problem. We develop a numerical scheme for recovery of impedance functions and provide results which confirm uniqueness of the interior inverse impedance problem. We also give some numerical examples showing the effectiveness of the methods.

CHAPTER 2

PRELIMINARIES

This chapter includes some basic definitions and theorems which are employed in the next chapters.

2.1. PDE Part

Theorem 2.1 (Holmgren's Theorem) *Let $D \subset \mathbb{R}^2$ be bounded domain of class C^2 and $u \in C^2(D) \cap C^1(\bar{D})$ be solution to the modified Helmholtz equation in D such that*

$$u = \frac{\partial u}{\partial \nu} = 0 \quad on\ \Gamma \tag{2.1}$$

for some open subset $\Gamma \subset \partial D$. Then u vanishes identically in D.

We refer to Colton et al. (1998) for more detail.

Theorem 2.2 (Green's Theorem) *Let D be a bounded domain of class C^1 and let ν denote the unit normal vector to the boundary ∂D into the exterior of D. Then for $u \in C^1(\bar{D})$ and $\upsilon \in C^2(\bar{D})$, we have Green's first theorem*

$$\int_D (u\Delta\upsilon - grad\, u \cdot grad\, \upsilon)dx = \int_{\partial D} u\frac{\partial \upsilon}{\partial \nu} ds, \tag{2.2}$$

and for $u, \upsilon \in C^2(\bar{D})$ we have Green's second theorem

$$\int_D (u\Delta\upsilon - \upsilon\Delta u)dx = \int_{\partial D} \left(u\frac{\partial \upsilon}{\partial \nu} - \upsilon\frac{\partial u}{\partial \nu} \right) ds. \tag{2.3}$$

For more knowledge we refer to Kress (2013).

Definition 2.1 *A function $\psi \in L^2[a,b]$ is said to be posses a weak derivative $\psi' \in L^2[a,b]$ if*

$$\int_a^b \psi\varphi' dx = -\int_a^b \psi'\varphi dx \tag{2.4}$$

for all $\varphi \in C^1[a,b]$ with $\varphi(a) = \varphi(b) = 0$.

For more knowledge we refer to Kress (2013).

Theorem 2.3 (Maximum-minimum principle) *Let u satisfy differential inequality*

$$(L + h)(u) \geq 0$$

with $h \leq 0$, with L uniformly elliptic in D and with the coefficients of L and h bounded. If u attains a nonnegative maximum M at the interior point of D, then $u = M$.

More knowledge can be found in Protter and Weinberger (2012).

Theorem 2.4 *The following problem*

$$\Delta u - k^2 u = 0 \quad in\ D \tag{2.5}$$

$$u = f \quad on\ \partial D \tag{2.6}$$

has unique solution.

Theorem 2.5 *The following problem*

$$\Delta u - k^2 u = 0 \quad in\ \mathbb{R}^2 \setminus D \tag{2.7}$$

$$u = f \quad on\ \partial D \tag{2.8}$$

$$\lim_{R\to\infty} \sup_{|x|>R} u(x) = 0 \tag{2.9}$$

has unique solution.

For exterior and interior Dirichlet uniqueness theorems we refer to Quaife (2011).

2.2. Boundary Integral Equations

Definition 2.2 *Given a function $\varphi \in C(\partial D)$, the functions*

$$u(x) := \int_{\partial D} \Phi(x,y)\varphi(y)ds(y), \quad x \in \mathbb{R}^n \setminus \partial D, \tag{2.10}$$

and

$$\upsilon(x) := \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)ds(y), \quad x \in \mathbb{R}^n \setminus \partial D, \tag{2.11}$$

are called single-layer and double-layer potential with density φ where Φ represents fundamental solution of the modified Helmholtz equation.

For this we refer to Kress (2013).

Theorem 2.6 *For ∂D of class C^2, the double layer potential υ with continuous density φ can be extented from $D \to \bar{D}$ and from $\mathbb{R}^n \setminus \bar{D}$ to $\mathbb{R}^n \setminus D$ with limiting values*

$$\upsilon_{\pm}(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)ds(y) \pm \frac{1}{2}\varphi(x), \quad x \in \partial D, \tag{2.12}$$

where $\upsilon_{\pm}(x) := \lim_{h\to+0} \upsilon(x \pm h\nu(x))$ and the integral exists as an improper integral.

For more knowledge you can see Kress (2013).

Theorem 2.7 *Let ∂D be of class C^2 and $\mu \in C(\partial D)$. Then the single layer potential u with density φ is continuous throughout $\mathbb{R}^n$. On the boundary we have*

$$u(x) = \int_{\partial D} \Phi(x,y)\varphi(y)ds(y), \quad x \in \partial D \tag{2.13}$$

and we also have

$$\frac{\partial u_\pm}{\partial \nu}(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)} \varphi(y) ds(y) \mp \frac{1}{2}\varphi(x), \quad x \in \partial D. \tag{2.14}$$

For more knowledge you can see Kress (2013).

Definition 2.3 *A linear operator $S : X \to Y$ from a normed space X into a normed space Y is called compact if it maps each bounded set in X into a relatively compact set in Y.*

Definition 2.4 *X is compactly embedded in Y if the identity operator $I : X \to Y$ is compact.*

Theorem 2.8 *Let $A : X \to X$ be compact linear operator on a normed space X. Then $I - A$ is injective if and only if it is surjective. If $I - A$ is injective, then the inverse operator $(I - A)^{-1} : X \to X$ is bounded.*

For more knowledge we refer to Kress (2013). We can rewrite Theorem 2.8 in terms of the solvability of an operator equation of the second kind as follows.

Theorem 2.9 *Let $A : X \to X$ be a compact linear operator on a normed space X. If the homogeneous equation*

$$\varphi - A\varphi = 0 \tag{2.15}$$

only has the trivial solution $\varphi = 0$, then for each $f \in X$ the inhomogeneous equation

$$\varphi - A\varphi = f \tag{2.16}$$

has a unique solution $\varphi \in X$ and this solution depends continuously on f.

For more details you can see Kress (2013).

Theorem 2.10 *Theorem 2.8 and 2.9 remain valid when $I - A$ is replaced by $S - A$, where $S : X \to Y$ is a bounded linear operator that has bounded inverse $S^{-1} : Y \to X$, i.e.,*

$S : X \to Y$ is an isomorphism, and $A : X \to Y$ is a compact linear operator from a normed space X into a normed space Y.

For more knowledge you can see Kress (2013).

2.3. Numerical Methods

Definition 2.5 (Numerical integration) *Quadrature formula computes the approximations to integrals by using numerical techniques. Consider quadrature formula of the form*

$$Q(f) := \int_a^b w(x)f(x)dx, \tag{2.17}$$

where w is weight function and continous function f over the interval $[a, b]$ and quadrature form is presented by

$$Q_n(f) := \sum_{i=1}^{n} \alpha_i^{(n)} f(x_i^{(n)}), \tag{2.18}$$

where n is distinct quadrature points and $\alpha_i^{(n)}$ quadrature weights. It is very efficient way to approximate integral with periodic functions by trapezoidal rule described by

$$\int_a^b f(x)dx \approx h\left(\frac{1}{2}f(x_0) + f(x_1) + \ldots + \frac{1}{2}f(x_n)\right). \tag{2.19}$$

To deal with singularity of the kernels we use and to have final dimensional system, we introduce quadrature operators formula

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right)(P_n\varphi)(\tau)d\tau = \sum_{j=0}^{2n-1} R_j^{(n)}(t)\varphi(t_j^{(n)}), \quad t\in[0,2\pi], \tag{2.20}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\tau-t}{2}(P_n\varphi)'(\tau)d\tau = \sum_{j=0}^{2n-1} T_j^{(n)}(t)\varphi(t_j^{(n)}), \quad t\in[0,2\pi], \tag{2.21}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\tau-t}{2}(P_n\varphi)(\tau)d\tau = \sum_{j=0}^{2n-1} I_j^{(n)}(t)\varphi(t_j^{(n)}), \quad t\in[0,2\pi], \tag{2.22}$$

where $P_n : C[0,2\pi] \to T_n$ is interpolation operator and T_n is subspace of trigonometric polynomials and with the quadrature weights

$$R_j(t) = -\frac{1}{n}\left(\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t-t_j) + \frac{1}{2n}\cos n(t-t_j)\right), \tag{2.23}$$

$$T_j(t) = -\frac{1}{n}\sum_{m=1}^{n-1} m\cos m(t-t_j) - \frac{1}{2}\cos n(t-t_j), \tag{2.24}$$

$$I_j(t) = \frac{1}{2n}\left(1-\cos n(t-t_j)\right)\cot\frac{t-t_j}{2}, \tag{2.25}$$

where $j = 0,\dots,2n-1, t_j = jh, h = \frac{\pi}{n}$.
The derivation of quadrature formulas can be seen in Kress (2013).

Theorem 2.11 (Banach-Steinhaus) *Let X and Y be Banach spaces, and let $A, A_n : X \to Y$ be bounded linear operators. Let U be a dense subspace of X. Then in order that $A_n\varphi \to A\varphi$ for all $\varphi \in X$, it is necessary and sufficient that*

- *$A_n\varphi \to A\varphi$ for all $\varphi \in U$,*
- *$\|A_n\| < C$ for all $n \in \mathbb{N}$ and for some constant C.*

Theorem 2.12 *For nonnegative integer k, let $f \in C^k_{2\pi}$ denoted by the space of k times continuously differentiable 2π periodic functions from $\mathbb{R}$ to $\mathbb{C}$ and assume that $0 \le p \le k$. Then for all $\varphi \in H^p[0,2\pi]$ the product $f\varphi$ belongs to $H^p[0,2\pi]$ and*

$$\|f\varphi\|_p \le C\left(\|f\|_\infty + \|f^k|_\infty\right)\|\varphi\|_p \tag{2.26}$$

for some constant C depending on p.

The proofs can be seen in Kress (2013).

Definition 2.6 *For a function $\varphi \in L^2[0, 2\pi]$ the series*

$$\sum_{m=-\infty}^{\infty} \hat{\varphi}_m e^{imt}, \tag{2.27}$$

where

$$\hat{\varphi}_m := \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) e^{-imt} \tag{2.28}$$

is called the Fourier series of φ, its coefficients $\hat{\varphi}_m$ are called the Fourier coefficients of φ. On $L^2[0, 2\pi]$, the mean square norm is described by the scalar product

$$(\varphi, \psi) := \int_0^{2\pi} \varphi(t)\bar{\psi}(t)dt. \tag{2.29}$$

Let $0 \leq p < \infty$. By $H^p[0, 2\pi]$ we denote the space of all functions $\varphi \in L^2[0, 2\pi]$ with the property

$$\sum_{m=-\infty}^{\infty} (1 + m^2)^p |\hat{\varphi}_m|^2 < \infty \tag{2.30}$$

for the Fourier coefficients $\hat{\varphi}_m$ of φ. The space $H^p[0, 2\pi]$ is called a Sobolev space.

Theorem 2.13 *The Sobolev space $H^p[0, 2\pi]$ is a Hilbert space with scalar product defined by*

$$(\varphi, \psi)_p := \sum_{m=-\infty}^{\infty} (1 + m^2)^p \hat{\varphi}_m \bar{\hat{\psi}}_m \tag{2.31}$$

for $\varphi, \psi \in H^p[0, 2\pi]$ with Fourier coefficients $\hat{\varphi}_m, \hat{\psi}_m$ respectively. Note that the norm of

$H^p[0, 2\pi]$ is given by

$$\|\varphi\|_p = \left(\sum_{m=-\infty}^{\infty} (1 + m^2)^p |\hat{\varphi}_m|^2 \right)^{\frac{1}{2}}. \tag{2.32}$$

The trigonometric polynomials are dense in $H^p[0, 2\pi]$.

Theorem 2.14 *if $q > p$ then $H^q[0, 2\pi]$ is dense $H^p[0, 2\pi]$ with compact embedding from $H^q[0, 2\pi]$ into $H^p[0, 2\pi]$.*

We refer to Kress (2013)'s book for more information about Sobolev space.

Definition 2.7 (Modified Bessel Functions) *Modified Bessel fucntions $K_n(x)$ and $I_n(x)$ are solution to Bessel's modified differential equation, where n and x being termed the order and argument of the function. We are interested in integer order $n = 0, 1, 2$ here. K_n is called the modified Bessel function of the second kind or alternatively the modified Hankel function and also I_n represents the modified Bessel function of the first kind. They are given by*

$$I_0(x) = 1 + \frac{x^2}{4} + \frac{x^4}{64} + \frac{x^6}{2304} + \cdots \tag{2.33}$$

$$I_1(x) = \frac{x}{2} + \frac{x^3}{16} + \frac{x^5}{384} + \frac{x^7}{18432} \cdots \tag{2.34}$$

$$I_2(x) = \frac{x^2}{8} + \frac{x^4}{96} + \frac{x^6}{3072} + \cdots \tag{2.35}$$

The limit I_n at large and small arguments are given by

$$I_n(x \to \infty) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \textit{and} \quad I_n(x \to 0) \sim \frac{1}{n!} \left(\frac{x}{2}\right)^2. \tag{2.36}$$

$h_n = I_n, K_n$ satisfy the formula for derivatives

$$\left(\frac{1}{z}\frac{d}{dx}\right)^k (x^n h_n(x)) = x^{n-k} h_{n-k}(x), \quad k = 0, 1, 2, \ldots \tag{2.37}$$

We have $I_0'(x) = I_1(x)$ and $K_0'(x) = -K_1(x)$ from derivative formula and both obeys reflection formula

$$h_{n+1}(x) = \frac{2n}{x} h_n(x) + h_{n-1}. \tag{2.38}$$

Also

$$K_0(x) = -\left(\gamma + \ln \frac{x}{2}\right) I_0(x) + 2 \sum_{j=1}^{\infty} \frac{I_{2j}(x)}{j}, \tag{2.39}$$

$$K_1(x) = \frac{1}{x} I_0(x) + \left(\gamma - 1 + \ln \frac{x}{2}\right) I_1(x) - \sum_{j=1}^{\infty} \frac{2j+1}{j^2+j} I_{2j+1}(x), \tag{2.40}$$

where γ is Euler's constant and the limit values for small and large arguments are described by

$$K_n(x \to 0) \to \ln \frac{2}{x} - \gamma \quad \text{and} \quad K_n(x \to \infty) \to \sqrt{\left(\frac{\pi}{2x}\right)} e^{-x}. \tag{2.41}$$

Many details about solution to modified Helmholtz can be found in Abramowitz and Stegun (1964) and Oldham et al. (2009).

2.4. Nonlinear and Ill-posed Problems

Definition 2.8 *Let $f : X \to Y$ be a function and X, Y be Banach spaces. f is Gateaux differentiable at x_0 if there is an operator $df(x_0, h) : X \to Y$ such that*

$$\lim_{\tau \to 0} \|f(x_0 + \tau h) - f(x_0) - \tau df(x_0, h)\| = 0, \tag{2.42}$$

where $x_0 + \tau h \in X$.

Definition 2.9 *Let X, Y be Banach spaces and and U be open set in X. The operator $f : X \to Y$ is called Frechet differentiable at $x_0 \in U \subset X$ if there is linear bounded* *operator $A : X \to Y$ such that*

$$\lim_{\|h\|\to 0} \frac{\|f(x_0+h) - f(x_0) - A(h)\|_Y}{\|h\|_X} = 0. \tag{2.43}$$

Theorem 2.15 *Let $A : X \to Y$ be bounded linear operator and let $\alpha > 0$. Then for each $f \in Y$ there exists a unique $\varphi_\alpha \in X$ such that*

$$\|A\varphi_\alpha - f\|^2 + \alpha\|\varphi_\alpha\|^2 = inf_{\varphi\in X}\left\{\|A\varphi - f\|^2 + \alpha\|\varphi\|^2\right\}. \tag{2.44}$$

The minimizer φ_α is given by the unique solution of the equation

$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f. \tag{2.45}$$

Definition 2.10 *Let $F : D \to \mathbb{R}^n$ continuously differentiable mapping with $D \subset \mathbb{R}^n$. Let F' denotes Frechet derivative and the Newton method is given by*

$$X^{k+1} = X^k - \left(F'(X^k)\right)^{-1} F(X^k), \tag{2.46}$$

where $F'(X^k)$ is Jacobian matrix.

Definition 2.11 (Morozov's Discrepancy Principle) *The stopping rule is defined with the discrepancy principle is*

$$\|Fx^\delta - y^\delta\| \leq \tau\delta, \tag{2.47}$$

where y^δ is noisy data, δ is noise level and $\tau > 1$.

Definition 2.12 (Numerical derivative) *The numerical derivative of f at x is defined by* *approximation*

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}. \tag{2.48}$$

CHAPTER 3

NUMERICAL SOLUTION OF DIRECT PROBLEM

In this chapter we suggest numerical algorithms to solve the boundary value problem for the modified Helmholtz equation with generalized impedance boundary condition. The solution of the methods are based on reduction of the problem to boundary integral equation with hyper singular-kernel. For more details about this method we refer to Kress (2014b). We describe two methods to solve the problem. In the first way the hyper-singular kernel of the integral operator is treated by splitting off the singularity approaches. As second way numerical differentiation is employed to solve it. Additionally we investigate the solvability of the boundary integral equation and convergence of the first method we proposed. The problem we consider here is stated as following.

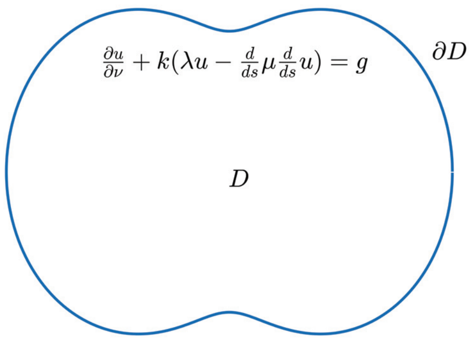

Figure 3.1.: Geometry of the Direct Problem

Let D be a simply connected and bounded domain in $\mathbb{R}^2$ with boundary ∂D of class C^3. Given $g \in H^{-\frac{1}{2}}(\partial D), \lambda > 0$ and $\mu > 0, \lambda \in C(\partial D), \mu \in C^1(\partial D)$ with $k > 0$, we consider the problem to find $u \in H^2(D)$ to the modified Helmholtz equation

$$\Delta u - k^2 u = 0 \quad \text{in } D \tag{3.1}$$

which satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} + k\left(\lambda u - \frac{d}{ds}\mu\frac{du}{ds}\right) = g \quad \text{on } \partial D \tag{3.2}$$

in the weak sense, i.e.

$$\int_{\partial D}\left(\xi\frac{\partial u}{\partial \nu} + k\lambda\xi u + k\mu\frac{d\xi}{ds}\frac{du}{ds}\right) = \int_{\partial D}\xi g ds, \quad \forall\xi \in H^{\frac{3}{2}}(\partial D), \tag{3.3}$$

where ν denotes the unit normal vector directed into the exterior of D and $\frac{d}{ds}$ is a tangential derivative. To have existence of solution for the problem we study, we should have the following theorem.

Theorem 3.1 *The boundary value problem (3.1)-(3.3) has at most one solution.*

Proof 3.1 *Let us assume that u_1 and u_2 are solutions to problem (3.1)-(3.3). Then the difference $u = u_1 - u_2$ satisfies the problem. Multiplying (3.1) by u and integrating over D, we receive*

$$\int_D u\Delta u dx - k^2\int_D u^2 dx = 0, \tag{3.4}$$

and also by the Green's first theorem (2.2) and the condition (3.3) for $\xi = u|_{\partial D}$, we have

$$-\int_D (\nabla u)^2\, dx - k^2\int_D u^2 dx - k\int_{\partial D}\lambda u^2 ds - k\int_{\partial D}\mu\left(\frac{du}{ds}\right)^2 ds = 0. \tag{3.5}$$

It follows that $u = 0$ in D since k, λ and μ are positive.

3.1. The Boundary Integral equations

In this section we describe a boundary integral equation method for solving the problem (3.1)-(3.3).

We seek a solution of (3.1)-(3.3) in the form of a single layer potential

$$u(x) = \int_{\partial D} \Phi(x, y)\varphi(y)ds(y), \quad x \in D \tag{3.6}$$

where $\varphi \in H^{\frac{1}{2}}(\partial D)$ and $\Phi(x, y) = \frac{1}{2\pi} K_0(k|x - y|)$ is a fundamental solution of modified Helmholtz equation in $\mathbb{R}^2$ with

$$K_0(x) = -\left(\ln \frac{x}{2} + \gamma\right) I_0(x) + 2 \sum_{k=1}^{\infty} \frac{I_{2k}(x)}{k} \tag{3.7}$$

where $\gamma = 0.5772156\ldots$ and K_0, I_0 are modified Bessel functions of second and the first kind of order zero respectively. For this we refer to Oldham et al. (2009). The modified Bessel function of the first kind with order zero is

$$I_0(x) = 1 + \frac{x^2}{4} + \frac{x^4}{64} + \ldots \tag{3.8}$$

We note that I_0 is analytic function whereas K_0 has singularity at $x = 0$. Let us now derive the boundary integral equation. By substituting (3.6) into boundary condition (3.2) on ∂D with aid of limiting values for single layer potential (2.14), we end up with

$$K'\psi + \frac{1}{2}\psi + k\left(\lambda - \frac{d}{ds}\mu\frac{d}{ds}\right) S\psi = g \quad \text{on } \partial D \tag{3.9}$$

where $S : H^{\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$ and $K' : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ are bounded integral operators introduced by

$$(S\psi)(x) = \int_D \Phi(x, y)\psi(y)ds(y) \text{ and } \quad (K'\psi)(x) = \int_{\partial D} \frac{\partial\Phi(x, y)}{\partial\nu(x)}\psi(y)ds(y), \ x \in \partial D. \tag{3.10}$$

More details about bounded operators can be seen in McLean (2000); Kress (2013). We provide the following theorem to show injectivity of the integral equation (3.9).

Theorem 3.2 *For each $g \in H^{-\frac{1}{2}}(\partial D)$, the boundary integral equation (3.9) has a unique solution $\psi \in H^{\frac{1}{2}}(\partial D)$ under the condition $\lambda > 0, \mu > 0$ and $k > 0$.*

Proof 3.2 *Equation (3.9) can be reformulated in the equivalent form*

$$(A_1 + A_2)\psi = -\frac{1}{\mu}g, \tag{3.11}$$

where

$$A_1\psi = k\left(\frac{d^2}{ds^2}S\psi + \int_{\partial D} S\psi ds\right), \tag{3.12}$$

$$A_2\psi = \frac{k}{\mu}\frac{d\mu}{ds}\frac{d}{ds}S\psi - k\frac{\lambda}{\mu}S\psi - \frac{1}{\mu}\left(K'\psi + \frac{1}{2}\psi\right) - k\int_{\partial D} S\psi ds. \tag{3.13}$$

The bounded invertibility of $A_1 : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is shown by Cakoni and Kress (2012); Kress (2018) and the operator $A_2 : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is compact since (3.10) is bounded and the map $H^{\frac{1}{2}}(\partial D) \hookrightarrow H^{-\frac{1}{2}}(\partial D)$ is compactly embedding(Kress (2013)). In order to show (3.11) is one-to-one, assume that $\psi \in H^{\frac{1}{2}}(\partial D)$ such that

$$(A_1 + A_2)\psi = 0, \tag{3.14}$$

and define the function

$$u(x) := \int_{\partial D} \Phi(x, y)\psi(y)ds(y), \quad x \in \mathbb{R}^2 \setminus \partial D. \tag{3.15}$$

Since u satisfies the modified Helmholtz equation (3.1) and the homogeneous boundary condition (3.3) and by uniqueness Theorem 3.1, we have $u = 0$ in D. Moreover, u solves equation (3.1) in the exterior of D. That follows from $u(x) = O\left(\frac{1}{|x|}\right)$ as $|x| \to \infty$. The uniqueness of exterior Dirichlet problem(Quaife (2011)) yields $u = 0$ in $\mathbb{R}^2 \setminus \bar{D}$. From the limiting case for the normal derivative of the single layer potential (2.14), we obtain $\psi =$

0. *That implies our assertion. By the Riesz-Fredholm theory(Kress (2013)) the boundary integral equation (3.11) has a unique solution.*

3.2. Parametrization of the Integral Equations

We consider boundary ∂D is analytic with 2π periodic representation of the form

$$\partial D = \{z(t) = (z_1(t), z_2(t)) : t \in [0, 2\pi)\}, \tag{3.16}$$

where $z_i : \mathbb{R} \to \mathbb{R}^2, i = 1, 2$ is 2π periodic and analytic function with $|z'(t)| > 0$ for all t. The parametrized single layer operator is introduced by

$$(\tilde{S}\tilde{\psi})(t) = \frac{1}{2\pi}\int_0^{2\pi} K_0(k|z(t) - z(\tau)|)\tilde{\psi}(\tau)d\tau, \quad t \in [0, 2\pi] \tag{3.17}$$

where $|z'(\tau)|d\tau$ is equivalent to arc length ds and also $\tilde{\psi}(\tau) = \psi(z(\tau))|z'(\tau)|$. The kernel of the operator S has a weakly singular since series (3.7) includes singularity at $x = 0$. That creates issues with numerical calculation. Kress and Sloan (1993) suggested a method to deal with this singularity. We adjust this method to the integral operators we presented here. We begin with splitting off singularity of the kernel. By adding and subtracting

$$\frac{1}{2}\ln\left(4\sin^2\frac{t-\tau}{2}\right) \tag{3.18}$$

to the kernel of operator $\tilde{S}$ which can be decomposed into

$$F_1(t, \tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + F_2(t, \tau), \tag{3.19}$$

where

$$F_1(t,\tau) = -\frac{1}{4\pi} I_0(k|z(t) - z(\tau)|), \tag{3.20}$$

$$F_2(t,\tau) = \frac{1}{2\pi} K_0(k|z(t) - z(\tau)|) - F_1(t,\tau) \ln\left(4 \sin^2 \frac{t-\tau}{2}\right) \tag{3.21}$$

with smooth diagonal terms

$$F_1(t,t) = -\frac{1}{4\pi}, \quad F_2(t,t) = -\frac{1}{2\pi}\left(\gamma + \ln \frac{k}{2}|z'(t)|\right). \tag{3.22}$$

The parametrized operator $\tilde{K}'$ is described by

$$(\tilde{K}'\tilde{\psi})(t) = \int_0^{2\pi} M(t,\tau)\tilde{\psi}(\tau) d\tau \tag{3.23}$$

with continuous kernel

$$M(t,\tau) = \frac{1}{|z'(t)|} \begin{cases} [z'(t)]^{\perp} \cdot \frac{z''(t)}{4\pi}, & t = \tau \\ \frac{1}{2\pi} k [z'(t)]^{\perp} \cdot (z(t) - z(\tau)) \frac{K_1(k|z(t)-z(\tau)|)}{|z(t)-z(\tau)|}, & t \neq \tau \end{cases}$$

and where $[z'(t)]^{\perp} = (z_2'(t), -z_1'(t))$ also using parametrization $\frac{d}{ds}\mu\frac{d}{ds}S \circ z = \frac{1}{|z'|}\frac{d}{dt}\frac{\tilde{\mu}}{|z'|}\frac{d}{dt}\tilde{S}$ we rewrite the boundary integral equation (3.9) in the form

$$\frac{1}{b}\tilde{K}\tilde{\psi} + \frac{\tilde{\psi}}{2b} + \frac{k}{b}\tilde{\lambda}|z'|\tilde{S}\tilde{\psi} + \frac{a}{b}\frac{d\tilde{S}\tilde{\psi}}{dt} + \frac{d^2\tilde{S}\psi}{dt^2} = h, \tag{3.24}$$

where

$$a(t) = \frac{k\tilde{\mu}(t)z'(t) \cdot z''(t)}{|z'(t)|^3} - \frac{k}{|z'(t)|}\frac{\tilde{\mu}(t)}{dt}, \quad b(t) = -\frac{k\tilde{\mu}(t)}{|z'(t)|}, \quad h(t) = \frac{|z'(t)|g(z(t))}{b(t)}.$$

The boundary integral (3.24) includes continuous, weakly singular and hyper singular kernels of operator. The technique in Kress (2013)[Chapter 13] is used to treat the hyper singular kernel. The tangential derivative of single layer operator is

$$\frac{d(\tilde{S}\varphi)}{ds} = \frac{k}{2\pi|z'(t)|}\int_0^{2\pi} K'_0(k|z(t)-z(\tau|)\frac{z'(t)\cdot(z(t)-z(\tau)}{|z(t)-z(\tau)|}\varphi(z(\tau))|z'(\tau)|d\tau,\ t\in[0,2\pi] \tag{3.25}$$

with the help of expansion (2.38) and (2.39), (3.25) can be rewritten as

$$\begin{aligned}\frac{d\tilde{S}\psi(z(t))}{ds} &= -\frac{1}{2\pi}\frac{1}{|z'(t)|}\int_0^{2\pi}\frac{z'(t)\cdot(z(t)-z(\tau))}{|z(t)-z(\tau)|^2}I_0(k|z(t)-z(\tau)|)\psi(z(\tau))|z'(\tau)|d\tau\\ &-\frac{k}{2\pi}\frac{1}{|z'(t)|}\int_0^{2\pi}\frac{z'(t)\cdot(z(t)-z(\tau))}{|z(t)-z(\tau)|}I'_0(k|z(t)-z(\tau)|)\left(\ln\left(k\frac{|z(t)-z(\tau)|}{2}\right)+\alpha\right)\psi(z(\tau))|z'(\tau)|d\tau\\ &+\frac{1}{2\pi}\frac{1}{|z'(t)|}\int_0^{2\pi}\underbrace{\frac{d}{dt}\left(2\sum_{n=1}^{\infty}\frac{I_{2n}(k|z(t)-z(\tau|)}{n}\right)}_{Continous}\psi(z(\tau))|z'(\tau)|d\tau.\end{aligned}$$

We are not interested in continuous kernel which goes to 0 as $\tau\to t$. We are going to treat the singularity in the rest of kernels which are not continuous. By using Taylor expansion

$$z(t)-z(\tau) = (t-\tau)z'(\tau)+(t-\tau)^2\int_0^1(1-\lambda)z''(\tau+\lambda(t-\tau))d\lambda \tag{3.26}$$

and with the aid of expansion (3.7) and (3.8), the hyper singular kernel of the operator can be translated into

$$\frac{d\tilde{S}\tilde{\psi}}{dt}(t) = \frac{1}{4\pi}\int_0^{2\pi}\cot\frac{\tau-t}{2}\tilde{\psi}(\tau)d\tau+\int_0^{2\pi}L(t,\tau)\tilde{\psi}(\tau)d\tau \tag{3.27}$$

and by splitting off singular of kernel L, one obtain

$$L(t,\tau) = L_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)+L_2(t,\tau), \tag{3.28}$$

where the continuous terms are

$$L_1(t,\tau) = \frac{k}{4\pi}\frac{z'(t)\cdot(z(t)-z(\tau))I_0'(k|z(t)-z(\tau)|)}{|z(t)-z(\tau)|}, \tag{3.29}$$

$$L_2(t,\tau) = \frac{1}{2\pi}\frac{dK_0(k|z(t)-z(\tau)|)}{dt} - L_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) - \frac{1}{4\pi}\cot\frac{\tau-t}{2} \tag{3.30}$$

which have limit values as $\tau \to t$

$$L_1(t,t) = 0, \tag{3.31}$$

$$L_2(t,t) = -\frac{1}{4\pi}\frac{z'(t)\cdot z''(t)}{|z'(t)|^2}. \tag{3.32}$$

Moreover, $\frac{d^2\tilde{S}\tilde{\psi}}{dt^2}$ presented by

$$\frac{d^2\tilde{S}\tilde{\psi}}{dt^2}(t) = \frac{1}{2\pi}\frac{d^2}{dt^2}\int_0^{2\pi} K_0(k|z(t)-z(\tau)|)\tilde{\psi}(\tau)d\tau \tag{3.33}$$

can be rewritten, with the aid of partial integration, Taylor expansion (3.26) and series expansion (3.7) and (3.8), as follows.

$$\frac{d^2\tilde{S}\tilde{\psi}}{dt^2}(t) = \frac{1}{4\pi}\int_0^{2\pi}\cot\frac{\tau-t}{2}\tilde{\psi}'(\tau)d\tau + \int_0^{2\pi} N(t,\tau)\tilde{\psi}(\tau)d\tau \tag{3.34}$$

with

$$N(t,\tau) = N_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + N_2(t,\tau), \tag{3.35}$$

where

$$N_1(t,\tau) = \frac{1}{4\pi} k I_1(k|z(t)-z(\tau)|) \frac{z'(t)\cdot(z(t)-z(\tau))^2}{|z(t)-z(\tau)|^3} \tag{3.36}$$

$$- k^2 I_0(k|z(t)-z(\tau)|) \frac{z'(t)\cdot(z(t)-z(\tau))^2}{4\pi|z(t)-z(\tau)|^2} \tag{3.37}$$

$$+ k I_1(k|z(t)-z(\tau)|) \left(\frac{z''(t)\cdot(z(\tau)-z(t)) - |z'(t)|^2}{4\pi|z(t)-z(\tau)|} + \frac{(z'(t)\cdot(z(t)-z(\tau)))^2}{4\pi|z(t)-z(\tau)|^3} \right) \tag{3.38}$$

and

$$N_2(t,\tau) = \frac{1}{2\pi} \frac{d^2 K_0(k|t-\tau|)}{dt^2} + \frac{1}{8\pi} \frac{1}{\sin^2 \frac{t-\tau}{2}} - N_1(t,\tau) \ln\left(4\sin^2 \frac{t-\tau}{2}\right) \tag{3.39}$$

with diagonal terms

$$N_1(t,t) = -k^2 \frac{|z'(t)|^2}{8\pi}, \tag{3.40}$$

$$N_2(t,t) = \frac{1}{2\pi} \left(-k^2 \frac{|z'(t)|^2}{4} - \gamma \frac{k^2|z'(t)|^2}{2} - k^2 \frac{|z'(t)|^2}{2} \ln\left(\frac{k}{2}|z'(t)|\right) \right) \tag{3.41}$$

$$+ \frac{6(z'(t)\cdot z''(t))^2 - |z'(t)|^4 - 4|z'(t)|^2 z'(t)\cdot z'''(t) - 3|z'(t)|^2|z''(t)|^2}{12|z'(t)|^4}. \tag{3.42}$$

We note that for the static case continuous representation of regular part of a mixed second order tangential derivative is found by Erhard (2005). After some calculations of the limit values of kernels, we are able to solve boundary integral equation (3.9) numerically. Now the parametrized eqution (3.24) can be rewritten, where all singularities appear explicitly

$$\begin{aligned} &\frac{1}{4\pi}\int_0^{2\pi} \cot\left(\frac{\tau-t}{2}\right)\tilde{\psi}'(\tau)d\tau + \frac{1}{4\pi}\frac{a(t)}{b(t)}\int_0^{2\pi} \cot\left(\frac{\tau-t}{2}\right)\tilde{\psi}(\tau)d\tau \\ &+ \frac{1}{b(t)}\int_0^{2\pi} \left(H_1(t,\tau)\ln\left(4\sin^2\left(\frac{t-\tau}{2}\right)\right) + H_2(t,\tau) + |z'(t)|M(t,\tau)\right)\tilde{\psi}(\tau)d\tau \\ &+ \frac{\tilde{\psi}(t)}{2b(t)} = h(t), \quad 0 \le t \le 2\pi. \end{aligned} \tag{3.43}$$

Here

$$H_i(t,\tau) = k\tilde{\lambda}(t)|z'(t)|F_i(t,\tau) + a(t)L_i(t,\tau) + b(t)N_i(t,\tau) \tag{3.44}$$

are analytic functions for $i = 1, 2$.

Theorem 3.3 *For any $h \in H^{-\frac{1}{2}}[0, 2\pi]$ and $\tilde{\lambda} \in C[0, 2\pi]$, $\tilde{\mu} \in C^1[0, 2\pi]$, $\tilde{\lambda} > 0$, $\tilde{\mu} > 0$, the integral equation (3.24) has unique solution $\tilde{\psi} \in H^{\frac{3}{2}}[0, 2\pi]$ which depends continuously on the data.*

Proof 3.3 *To investigate solvability of the parametrized integral equation (3.24), we describe the following operators*

$$\begin{aligned}
(T\tilde{\psi})(t) &= \frac{1}{4\pi}\int_0^{4\pi} \cot\frac{\tau - t}{2}\tilde{\psi}'(\tau)d\tau + \frac{1}{2\pi}\int_0^{2\pi} \tilde{\psi}(\tau)d\tau, \qquad (3.45)\\
(B_1\tilde{\psi})(t) &= \frac{1}{b(t)}\int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right)H_1(t,\tau)\tilde{\psi}(\tau)d\tau,\\
(B_2\tilde{\psi})(t) &= \int_0^{2\pi} H_2(t,\tau)\tilde{\psi}(\tau)d\tau + \frac{|z'(t)|}{b(t)}\int_0^{2\pi} M(t,\tau)\tilde{\psi}(\tau)d\tau + \frac{\tilde{\psi}(t)}{2b(t)} - \frac{1}{2\pi}\int_0^{2\pi} \tilde{\psi}(\tau)d\tau,\\
(B_3\tilde{\psi})(t) &= \frac{1}{4\pi}\int_0^{2\pi} \cot\frac{\tau - t}{2}\tilde{\psi}(\tau)d\tau
\end{aligned}$$

and now set $B = B_1 + B_2 + \frac{a}{b}B_3$.
The operator $T : H^p[0, 2\pi] \to H^{p-1}[0, 2\pi]$ is bounded and its inverse is bounded for all $p \geq 0$. From Kress (1995) and Kress (2013) considering trigonometric monomials $u_m(t) = e^{imt}$ are eigenfunction of operator T, we have

$$Tu_m = \beta_m u_m \tag{3.46}$$

for $m \in \mathbb{Z}$ with $\beta_m = -\frac{|m|}{2}, m \neq 0$ and $\beta_0 = 1$. Consequently, it shows that $T : H^p[0, 2\pi] \to H^{p-1}[0, 2\pi]$ is bounded and its existence of inverse operator $T^{-1} : H^{p-1}[0, 2\pi] \to H^p[0, 2\pi]$

given by

$$T^{-1}u_m = \frac{1}{\beta_m}u_m. \tag{3.47}$$

The operator $B : H^p[0, 2\pi] \to H^{p-1}[0, 2\pi]$ is compact since $B_3 : H^p[0, 2\pi] \to H^p[0, 2\pi]$ is bounded(Kress (2013)). We can infer that from Theorem 3.2 ,$T + B$ is injective. For this reason, $T + B$ has bounded inverse by Theorem 2.10.

3.3. Numerical Approaches

In this section we briefly describe numerical approaches to solve the parametrized integral equation (3.24). Namely, we approximate the operators we considered in (3.43) with quadrature rule based on trigonometric interpolation. After separating the kernels of integrals in (3.43) into an analytical and singular part, we substitute quadrature operators for the singular part. For more information on quadrature operators, we refer to Kress (2013). Recall the trigonometric interpolation operator $P_n : H^p[0, 2\pi] \to \mathbb{T}_n$, $n \in \mathbb{N}$ with $2n$ equidistant interpolation points

$$t_i^{(n)} = \frac{i\pi}{n}, \quad i = 0, \ldots, 2n - 1. \tag{3.48}$$

There is an important error estimate

$$\|P_n\tilde{\psi} - \tilde{\psi}\|_q \leq \frac{C}{n^{p-q}}\|\tilde{\psi}\|, \quad 0 \leq q \leq p, p > \frac{1}{2} \tag{3.49}$$

which holds for all $\tilde{\psi} \in H^p$,see details in Kress (2013). If the function is 2π periodic and analytic, the interpolation error (3.49) decays exponentially. Now we introduce the following quadrature rules based on trigonometric interpolation.

$$\int_0^{2\pi} h(\tau)d\tau \approx \frac{\pi}{n}\sum_{i=0}^{2n-1} h(t_i^{(n)}), \tag{3.50}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\tau-t}{2}\tilde{\psi}'(\tau)d\tau \approx \sum_{i=0}^{2n-1} T_{1,i}(t)\tilde{\psi}(t_i^{(n)}), \tag{3.51}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\tau-t}{2}\tilde{\psi}(\tau)d\tau \approx \sum_{i=0}^{2n-1} T_{2,i}(t)\tilde{\psi}(t_i^{(n)}), \tag{3.52}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\left(4\sin^2\frac{\tau-t}{2}\right)\tilde{\psi}(\tau)d\tau \approx \sum_{i=0}^{2n-1} R_i(t_i^{(n)}) \tag{3.53}$$

with quadrature weights

$$T_{1,i}^{(n)}(t) = -\frac{1}{n}\sum_{m=1}^{n-1} m\cos m(t-t_i^{(n)}) - \frac{1}{2}\cos n(t-t_i^{(n)}), \tag{3.54}$$

$$T_{2,i}^{(n)}(t) = \frac{1}{2n}\left(1-\cos n(t-t_i^{(n)})\right)\cot\frac{t-t_i^{(n)}}{2}, \tag{3.55}$$

$$R_i^{(n)}(t) = -\frac{1}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t-t_i^{(n)}) - \frac{1}{2n^2}\cos n(t-t_i^{(n)}). \tag{3.56}$$

The derivation of equations (3.54), (3.55) and (3.56) can be found in Kress (2013). The integral equation (3.43) is reduced to full discrete system

$$(T + P_nB_n)\tilde{\psi}_n(t_i) = (P_nh_n)(t_i), \quad i = 1,\dots,2n \tag{3.57}$$

where $B_n = B_{1,n} + B_{2,n} + B_{3,n}$,

$$(B_{1,n}\tilde{\psi})(t) = \frac{1}{b(t)}\int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right)\left(P_n(H_1(t,\cdot)\tilde{\psi})\right)(\tau)d\tau, \tag{3.58}$$

$$(B_{2,n}\tilde{\psi})(t) = \frac{1}{b(t)}\int_0^{2\pi}\left(P_n(H_2(t,\cdot)\tilde{\psi})\right)(\tau)d\tau + \frac{1}{b(t)}\int_0^{2\pi}\left(P_n(M(t,\cdot)\tilde{\psi})\right)d\tau + \frac{1}{2b(t)}\tilde{\psi}(t), \tag{3.59}$$

$$(B_{3,n}\tilde{\psi})(t) = \frac{1}{4\pi}\int_0^{2\pi} \cot\frac{\tau-t}{2}(P_n\tilde{\psi})(\tau)d\tau. \tag{3.60}$$

We select t_i for $i = 1, \dots, 2n$ as collocation points since we wish to compare two techniques. This choice gives guarantee to approximate the derivative of 2π periodic function by derivative the unique interpolatory trigonometric polynomial of degree n without $\sin(nt)$.

Theorem 3.4 *Under the assumption that λ, μ and ∂D are analytic the fully discrete collocation method (3.57) converges in $H^p[0, 2\pi]$ for each $p > 3/2$.*

Proof 3.4 *From Kress (2013)[Theorem 12.18], the convergence of weakly singular kernels of operators for $B_{1,n}$ to the kernel B_1 given by*

$$\|B_{1,n}\tilde{\psi} - B\tilde{\psi}\|_{q+1} \leq \frac{c}{n^{p-q}}\|\tilde{\psi}\|_p, p > \frac{1}{2} \tag{3.61}$$

for all $\tilde{\psi} \in H^p[0, 2\pi]$ and some constant depending on p and q. Also same convergence estimate works for analytic kernel of $B_{2,n}$. If we integrates $B_{3,n}$ for trigonometric polynomials of degree less then or equal to n exactly Kress (2013)[page 164] and Kirsch and Ritter (1999), then we have $B_n\tilde{\psi} \to B\tilde{\psi}$ as $n \to \infty$ for all $\tilde{\psi} \in \mathbb{T}_n \subset H^p[0, 2\pi]$. Banach-Steinhaus Theorem 2.11 states that $B_n\tilde{\psi} \to B\tilde{\psi}$ for all $\tilde{\psi} \in H^p[0, 2\pi]$ because $\mathbb{T}_n$ is dense in $H^p[0, 2\pi]$. From Kress (2013)[Theorem, 11.8] interpolation polynomials $P_n : H^p[0, 2\pi] \to H^p[0, 2\pi]$ are bounded for $p > \frac{1}{2}$. By employing (3.61) and boundedness of the interpolation operator, it is obvious that

$$\|P_n(B_{1,n} - B_1)\tilde{\psi}\|_{p-1} \leq \frac{c}{n}\|\tilde{\psi}\|_p, \quad p > \frac{3}{2}. \tag{3.62}$$

By the same approach, the estimate (3.62) can be done for the operator B_2 with analytic kernels. The boundedness of P_n for $p > \frac{3}{2}$ and Theorem 2.12 yields

$$\|P_n\left(\frac{a}{b}(B_{3,n} - B_3)\,\tilde{\psi}\right)\|_{p-1} \leq c\|(B_{3,n} - B_3)\tilde{\psi}\|_{p-1}, \tag{3.63}$$

where a/b is analytic. We have convergence for all trigonometric polynomials since $B_{3,n}$ is derived by precisely integrating a trigonometric polynomial. With the aid of the Banach-Steinhaus theorem, we have $P_nB_n\tilde{\psi} \to P_nB\tilde{\psi}$ as $n \to \infty$ for all $\tilde{\psi} \in H^p[0, 2\pi]$. Then Kress

(2013)[Corollary 13.13] gives what is desired.

We have provided some information regarding the first solution method to the problem until now. As a second strategy, we use the concept of numerical differentiation in place of splitting off singularities in the kernels of integral operators resulting from tangential derivatives of single layer operators suggested Kress (2014a). Introducing the derivative $D_n := P_n'$ of trigonometric interpolation operator, integro-differential opperator is approximated as follows

$$\left(\frac{1}{|z'|}\frac{d}{dt}\frac{\tilde{\mu}}{|z'|}\frac{d}{dt}\tilde{S}\tilde{\psi}_n\right)(t_i)\Big|_{i=1,\dots,2n} \approx \frac{1}{|z'(t_i)|}D_n diag\left(\frac{\tilde{\mu}(t_i)}{|z'(t_i)|}\right)D_n(\tilde{S}\tilde{\psi})(t_i)\Big|_{i=1,\dots,2n} \tag{3.64}$$

for $\tilde{\psi} \in \mathbb{T}_n$,

$$(D_n g)(t_i) = \sum_{k=0}^{2n-1} d_{k-i}^{(n)}(t_k)g(t_k), \quad i = 0, 1, \dots, 2n-1, \tag{3.65}$$

and

$$d_i^{(n)} = \begin{cases} \frac{(-1)^i}{2}\cot\frac{i\pi}{2n}, & i = \mp 1, \dots, \mp(2n-1) \\ 0, & i = 0. \end{cases} \tag{3.66}$$

3.4. Numerical Examples

In this section we provide some illustrations to test both methods. Let us assume that boundary ∂D in Figure 3.2 is parameterized by

$$z(t) = \left(2\cos t - 2\cos^2 t + 1, 5\sin t - \cos t \sin t\right), \quad 0 \le t \le 2\pi \tag{3.67}$$

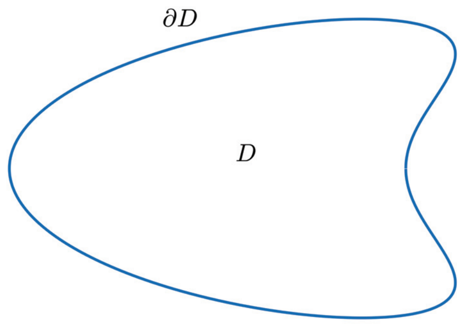

Figure 3.2.: Domain D

and coefficient k is chosen as $\frac{1}{2}$. The impedance functions we considered are

$$\tilde{\lambda}(t) = -\sin(|z(t)|) + 4.5 \quad \text{and} \quad \tilde{\mu}(t) = -2\cos(|z(t)|) + 4.5. \tag{3.68}$$

In all examples $u^{\dagger}$ denotes the exact solution and it is considered a point source with location $x_1 = (2, 0.4)$

$$u^{\dagger}(x) = \Phi(x, x_1), \quad x \in D, x_1 \in \mathbb{R}^2 \setminus \bar{D}. \tag{3.69}$$

Additionally, the approximate solutions u_1 and u_2 to the first and second methods, respectively, are taken into consideration. The parametrized measurement curve given by

$$\Pi_m = \left(\rho(t) = (5\cos t, 5\sin t), t \in \left[\frac{2\pi}{3}, \frac{7\pi}{6}\right]\right) \tag{3.70}$$

to analyze numerical convergence of the proposed methods.

Example 3.1 *The boundary value problem (3.1)-(3.3) solved by two numerical approaches* *with known exact solution in order to compare methods. The problem reads*

$$\Delta u - k^2 u = 0 \quad \textit{in } D \tag{3.71}$$

$$\frac{\partial u}{\partial \nu} + k\left(\lambda u - \frac{d}{ds}\mu\frac{du}{ds}\right) = g \quad \textit{on } \partial D \tag{3.72}$$

$$g(x) = \frac{\partial \Phi(x, x_1)}{\partial \nu} + k\left(\lambda(x)\Phi(x, x_1) - \frac{d}{ds}\mu(x)\frac{d\Phi(x, x_1)}{ds}\right), \quad x \in \partial D. \tag{3.73}$$

By solving the problem with two methods we obtain the following Table 3.1. We present the maximum absolute errors at points $y \in \Pi_m$ in the Table 3.1.

Table 3.1.: Error analysis for Example 3.1

n	$\|u_1 - u^\dagger\|_{\Pi_{m,\infty}}$	$\|u_2 - u^\dagger\|_{\Pi_{m,\infty}}$
8	*1.24e-03*	*3.54e-03*
16	*1.37e-05*	*4.72e-05*
32	*9.90e-09*	*4.23e-07*
64	*1.00e-15*	*1.52e-12*

$\|u_1 - u^\dagger\|_{\Pi_{m,\infty}}$ *indicates the maximum error for the first method and* $\|u_2 - u^\dagger\|_{\Pi_{m,\infty}}$ *denotes the maximum error for the second approach based on numerical differentiation.For the case of analytic boundary and data, the theoretical investigation predicts that the numerical error of the first scheme decreases exponentially and it is confirmed by Table 3.1. The convergence is slower for the second method because the approximation of the hyper singular kernel is inaccurate, as can be shown in Table 3.1. Indeed, from (3.46),we have*

$$\frac{1}{4\pi}\frac{d^2}{dt^2}\int_0^{2\pi} \ln\left(\sin^2\frac{t-\tau}{2}\right)\cos n\tau d\tau = -\frac{n}{2}\cos nt, \quad n \in \mathbb{N} \tag{3.74}$$

whereas for its approximation from (3.66), we obtain

$$D_n D_n \frac{1}{2\pi}\int_0^{2\pi} \ln\left(\sin^2\frac{t-\tau}{2}\right)\cos n\tau d\tau = 0, \quad n \in \mathbb{N}. \tag{3.75}$$

By including additional weights in the calculation of $D_n D_n \tilde{S}$ or by selecting an odd num-

ber of interpolation and collocation points, the drawback of the trigonometric differention can be overcome. For more knowledge about this, we refer to Kress (2014a).The boundary value problem is solved numerically for the next experiment in the scenario where the exact solution is unknown. We choose boundary data g to be given by

$$g(x) = \Phi(x, x_1), \quad x \in \partial D, x_1 = (3.2) \in \mathbb{R}^2 \setminus \bar{D}. \tag{3.76}$$

The other parameters remain the same. Table 3.2 exhibits the value of the solution of the

Table 3.2.: Numerical solution

n	u_1
8	0.012063279277905
16	0.012284634729342
32	0.012285858740215
64	0.012285858683054
128	0.012285858683054

boundary value problem at the point $y = (0, 0.5) \in D$ *via the first method. We observe that the number of correct digits of the exact solution doubles when the number of grid points is increased twofold.*

In the last experiment the disk of radius 2 centered at the origin as a domain is considered and the constant impedance functions $\tilde{\mu} = 1, \tilde{\lambda} = 1$ *are chosen. Moreover the boundary data g is given by*

$$g(t) = \frac{3}{\pi} \arcsin(\sin t) + 0.04 \cos(16t) + 0.02 \cos(8t) - 0.02 \cos(32t). \tag{3.77}$$

We contrast the differences in error between the Dirichlet traces of the results produced by the two methods, which are shown in a Table 3.3 by columns $\|u_1 - u_2\|_{\Pi_{\partial D,\infty}}$*. The solution found by the first technique with* $n = 256$ *can also be taken into consideration since the first approach converges, which is guaranteed by the theorem and supported by the previous two examples.*

Table 3.3.: Error analysis

n	$\|u_1 - u_2\|_{\Pi_{\partial D},\infty}$	$\|u_1 - u^\dagger\|_{\Pi_m,\infty}$	$\|u_2 - u^\dagger\|_{\Pi_m,\infty}$
8	*2.69e-03*	*1.03-01*	*1.00e-01*
16	*3.60e-03*	*3.35e-02*	*3.71e-02*
32	*1.06e-03*	*1.60e-03*	*2.65e-03*
64	*7.00e-10*	*3.93-04*	*3.93e-04*
128	*2.31e-11*	*7.99e-05*	*7.99e-05*

The results of first part published by Ivanyshyn Yaman and Özdemir (2021).

CHAPTER 4

INVERSE IMPEDANCE PROBLEM FOR THE MODIFIED HELMHOLTZ EQUATION IN TWO DIMENSIONS

In this chapter we consider classic inverse problem to reconstruct impedance functions from measurements of the sources. Let $D \subset \mathbb{R}^2$ be bounded simply connected domain with C^3 boundary ∂D and with positive impedance functions $\lambda \in C(\partial D), \mu \in C^1(\partial D)$. For a given function $h \in C(\partial D)$ and constant $k > 0$ let $u^s \in H^2(D)$ satisfy the modified Helmholtz equation

$$\Delta u^s - k^2 u^s = 0 \quad \text{in } D \tag{4.1}$$

and the boundary condition

$$\frac{\partial u^s}{\partial \nu} + k\left(\lambda u^s - \frac{d}{ds}\mu\frac{du^s}{ds}\right) = -\frac{\partial \Phi(\cdot, x^*)}{\partial \nu} - k\left(\lambda \Phi(\cdot, x^*) - \frac{d}{ds}\mu\frac{d\Phi(\cdot, x^*)}{ds}\right) \quad \text{on } \partial D \tag{4.2}$$

i.e, total field $u = u^s + \Phi(\cdot, x^*)$ satisfies

$$\frac{\partial u}{\partial \nu} + k\left(\lambda u - \frac{d}{ds}\mu\frac{du}{ds}\right) = 0 \quad \text{on } \partial D \tag{4.3}$$

where ν denotes the unit normal vector directed into exterior of D and $\frac{d}{ds}$ is tangential derivative. The well posedness of the interior direct problem is investigated in Chapter 3 and it is well known it has a unique solution $u^s \in H^2(D)$. Let x^* be on closed curve $C \subset D$ and measure u^s for $x^* \in C$.

The inverse problem we are interested is to recover λ and μ simultaneously from knowledge of the sources and measured data $u^s|_C = f$ placed on the curve C in the Figure 4.1. To do that we need to ask what is the minimum amount of sources is necessary to uniquely reconstruct impedance functions simultaneously. The following counter ex-

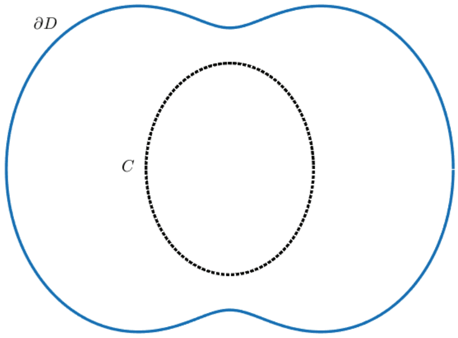

Figure 4.1.: Domain D with measurement C

ample, inspired by counterexample in Kress (2018) depicts nonuniqueness issues. The modified Helmholtz equation is considered in the disk of radius R centered at the origin and two sources chosen to be modified Bessel function I_n, K_n of the first kind and the second kind, respectively, with order $n \in \mathbb{N}$. Total fields in this case have following formulation

$$u_1(r, \theta) = (I_n(kr) + b_n K_n(kr)) \cos\theta, \tag{4.4}$$

$$u_2(r, \theta) = (I_n(kr) + b_n K_n(kr)) \sin\theta. \tag{4.5}$$

The following boundary condition represents (4.3) in polar coordinates of the form

$$\frac{\partial u}{\partial r} - \frac{\mu}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \lambda u = 0 \quad \text{on } r = R \tag{4.6}$$

under condition λ and μ being positive constant and employing (4.4), (4.5) and (4.6), we have the coefficients

$$b_n = -\frac{R^2 I_n'(kR) + (R^2\lambda + n^2\mu) I_n(kR)}{R^2 K_n'(kR) + (R^2\lambda + n^2\mu) K_n(kR)}. \tag{4.7}$$

The well posedness of the direct problem guarantees that the denominator does not vanish.

Apparently, there are many combinations λ and μ that yield the same value b_n. Thus, two sources are not sufficient to reconstruct impedance functions. Now we can give uniqueness theorem.

Theorem 4.1 *The values $u_1^s|_C, u_2^s|_C, u_3^s|_C$ corresponding to the three different sources x_1^*, x_2^*, x_3^* on the measurement C uniquely determine the impedance functions λ and μ for known ∂D.*

Proof 4.1 *We first show that the corresponding solution $u_1|_{\partial D}, u_2|_{\partial D}, u_1|_{\partial D}$ are linearly independent. Recall that $u_i = u_i^s + \Phi(x_i^*, \cdot), i = 1, 2, 3$ solves*

$$\Delta u_i^s - k^2 u_i^s = 0 \quad in\ D \tag{4.8}$$
$$\frac{\partial u_i}{\partial \nu} + k\left(\lambda u_i - \frac{d}{ds}\mu\frac{d}{ds}u_i\right) = 0 \quad on\ \partial D. \tag{4.9}$$

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ with assumption $\alpha_1\alpha_2\alpha_3 \neq 0$. Define $u := \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$. Assume that $u \equiv 0$. The function u solves the modified Helmholtz equation in domain

$$D \setminus \overline{\left(B(x_1^*, \epsilon) \cup B(x_2^*, \epsilon) \cup B(x_3^*, \epsilon)\right)} \tag{4.10}$$

where $B(x_i^*, \epsilon) = \{x, |x - x_i| < \epsilon\}$ for $i = 1, 2, 3$. Also u satisfies the homogeneous generalized impedance boundary condition and therefore, $\frac{\partial u}{\partial \nu}|_{\partial D} = 0$. From Holmgren's theorem 2.1 and analyticity, we have

$$u = 0 \quad in\ D \setminus (B(x_1^*, \epsilon) \cup B(x_2^*, \epsilon) \cup B(x_3^*, \epsilon)) \tag{4.11}$$

which can be rewritten as

$$\alpha_1 u_1^s + \alpha_2 u_2^s + \alpha_3 u_3^s = -\left(\alpha_1\Phi(x_1^*, \cdot) + \alpha_2\Phi(x_2^*, \cdot) + \alpha_3\Phi(x_3^*, \cdot)\right) on\ \partial B(x_1^*, \epsilon), \tag{4.12}$$
$$\alpha_1 u_1^s + \alpha_2 u_2^s + \alpha_3 u_3^s = -\left(\alpha_1\Phi(x_1^*, \cdot) + \alpha_2\Phi(x_2^*, \cdot) + \alpha_3\Phi(x_3^*, \cdot)\right) on\ \partial B(x_2^*, \epsilon), \tag{4.13}$$
$$\alpha_1 u_1^s + \alpha_2 u_2^s + \alpha_3 u_3^s = -\left(\alpha_1\Phi(x_1^*, \cdot) + \alpha_2\Phi(x_2^*, \cdot) + \alpha_3\Phi(x_3^*, \cdot)\right) on\ \partial B(x_3^*, \epsilon). \tag{4.14}$$

The absolute value of the right-hand sides (4.12)-(4.14) increase as $\epsilon \to 0$, that is, the solutions u_1^s, u_2^s, u_3^s to the modified Helmholtz equation take their maximum or minimum values inside D. Hence we arrive at a contradiction with the maximum principle and for that reason the functions $\Phi(x_1^*, \cdot), \Phi(x_2^*, \cdot)$ and $\Phi(x_3^*, \cdot)$ are linearly independent and the right hand sides cannot vanish. The proof finishes with the result of Cakoni and Kress (2012)[Theorem 3.1].

Therefore, we can reconstruct uniquely λ and μ with at least three Cauchy pairs.

4.1. Synthetic Data

Synthetic data are obtained by solving the integral equation for Green's approach whereas the inverse solver is based on the single layer approach. Direct and inverse solver should be different due to obtaining unrealistically good reconstructions otherwise we commit inverse crime. To avoid committing such a crime, we choose Green's representation formula

$$u^s(x) = \int_{\partial D} \left(\frac{\partial u^s}{\partial \nu} \Phi(x, y) - u^s(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right) ds, x \in D \tag{4.15}$$

which is solution to (4.1)-(4.2). By employing jump relations for the single-layer 2.14 and double-layer 2.12 potentials and the boundary condition (4.2), we obtain

$$\left(K + \frac{I}{2} + kS \left(\lambda - \frac{d}{ds} \mu \frac{d}{ds} \right) \right) u^s|_{\partial D} = \int_{\partial D} \Phi(\cdot, y) h(y) ds(y) \quad \text{on } \partial D \tag{4.16}$$

where $S : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D), K : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ and $A : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(C)$ are bounded integral operators described by

$$(S\varphi)(x) = \int_{\partial D} \Phi(x, y) \varphi(y) ds(y), \quad x \in \partial D, \tag{4.17}$$

$$(K\varphi)(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu} \varphi(y) ds(y), \quad x \in \partial D \tag{4.18}$$

and

$$(A\varphi)(x) = \int_{\partial D} \Phi(x,y)\varphi(y) ds(y), \quad x \in C \tag{4.19}$$

where $\Phi(x,y) = \frac{1}{2\pi} K_0(k|x-y|)$ with the modified Bessel function of the second kind K_0.

For each $h \in H^{-\frac{1}{2}}(\partial D)$, the boundary integral equation (4.16) has unique solution $u^s|_{\partial D} \in H^{-\frac{1}{2}}(\partial D)$ under condition $\lambda > 0, \mu > 0, k > 0$.

We have adjoint boundary integral equation (4.16) in Chapter 3 which is

$$\left(K' + \frac{I}{2} + k \left(\lambda - \frac{d}{ds} \mu \frac{d}{ds} \right) S \right) \psi = g. \tag{4.20}$$

From Ivanyshyn Yaman and Özdemir (2021)[Theorem 2.1], the boundary integral equation (4.20) has Fredholm index zero. More specifically, the solution exists and it is unique. The proof can be concluded from Fredholm alternative Kress (2013)[Corollary 4.18].

Parametrization of the boundary. We introduce the parameterized boundaries and the involved integral operators. We assume that

$$\partial D := \{z(t) = (z_1(t), z_2(t)) : 0 \le t \le 2\pi\} \tag{4.21}$$

and

$$C := \{\zeta(t) = (\zeta_1(t), \zeta_2(t)) : 0 \le t \le 2\pi\} \tag{4.22}$$

with 2π periodic functions $z, \zeta : \mathbb{R} \to \mathbb{R}^2$ such that z, ζ are injective on $[0, 2\pi)$ and satisfy $|z'(t)| > 0$ and $|\zeta'(t)| > 0$ for all t. The parameterized integral operators are denoted $\tilde{S}, \tilde{K}$

and represented by

$$(\tilde{K}\tilde{\varphi})(t) = \int_0^{2\pi} \frac{\partial K_0(k|z(t) - z(\tau)|)}{\partial \nu(z(\tau))} \tilde{\varphi}(\tau)|z'(\tau)|d\tau \tag{4.23}$$

and

$$(\tilde{S}\tilde{\varphi})(t) = \int_0^{2\pi} K_0(k|z(t) - z(\tau)|)\tilde{\varphi}(\tau)|z'(\tau)|d\tau. \tag{4.24}$$

Approximation The parameterized boundary integral equation (4.16) is solved by method which has been initially developed by Kress and Sloan (1993) and after the kernel of integral operators $\tilde{K}, \tilde{S}$ are divided, they are evaluated by quadrature operators based on trigonometric interpolation formula, Kress (2018) and Ivanyshyn Yaman and Özdemir (2021). Moreover, the numerical computation of the tangential derivatives is accomplished by trigonometric interpolation polynomial P_n, i.e the approximation can be calculated in the formula

$$\frac{1}{|z'|}\frac{d}{dt}\frac{\mu \circ z}{|z'|}\frac{d}{dt}\tilde{S}\tilde{\varphi} \approx \frac{1}{|z'|}P_n'\frac{\mu \circ z}{|z'|}P_n'\tilde{S}\tilde{\varphi}. \tag{4.25}$$

For a detailed study of the use of numerical derivative of integral equations with trigonometric interpolation, we refer to Kress (2013).

4.2. Solution Methods for Inverse Impedance Problem

In this section we describe two inversion methods. One of the method called direct method which is the extension of the technique suggested Cakoni and Kress (2012). The uniqueness Theorem 4.1 indicates the requirement of three sources which create three fields u_1, u_2 and u_3. Multiplying the boundary condition (4.3) for u_1 by u_2 and the boundary condition (4.3) for u_2 by u_1 and taking difference, we obtain

$$k\frac{d}{ds}\mu\left(u_1\frac{du_2}{ds}-u_2\frac{du_1}{ds}\right)=u_1\frac{\partial u_2}{\partial\nu}-u_2\frac{\partial u_1}{\partial\nu}\quad\text{on }\partial D. \tag{4.26}$$

Then μ can be obtained by integrating (4.26) from a fixed point x_0 to any $x\in\partial D$ as following.

$$k\mu(x)\left(u_1(x)\frac{du_2(x)}{ds}-u_2(x)\frac{du_1(x)}{ds}\right)-k\mu(x_0)\left(u_1(x_0)\frac{du_2(x_0)}{ds}-u_2(x_0)\frac{du_1(x_0)}{ds}\right) \tag{4.27}$$
$$=\int_{x_0}^{x}\left(u_1(y)\frac{\partial u_2(y)}{\partial\nu}-u_2(y)\frac{\partial u_1(y)}{\partial\nu}\right)ds\quad\text{on }\partial D. \tag{4.28}$$

Following similar steps for pairs u_2,u_3 and u_3,u_1 as above, we have

$$k\mu(x)\left(u_2(x)\frac{du_3(x)}{ds}-u_3(x)\frac{du_2(x)}{ds}\right)-k\mu(x_0)\left(u_2(x_0)\frac{du_3(x_0)}{ds}-u_3(x_0)\frac{du_2(x_0)}{ds}\right) \tag{4.29}$$
$$=\int_{x_0}^{x}\left(u_2(y)\frac{\partial u_3(y)}{\partial\nu}-u_3(y)\frac{\partial u_2(y)}{\partial\nu}\right)ds\quad\text{on }\partial D \tag{4.30}$$

and

$$k\mu(x)\left(u_3(x)\frac{du_1(x)}{ds}-u_1(x)\frac{du_3(x)}{ds}\right)-k\mu(x_0)\left(u_3(x_0)\frac{du_1(x_0)}{ds}-u_1(x_0)\frac{du_3(x_0)}{ds}\right) \tag{4.31}$$
$$=\int_{x_0}^{x}\left(u_3(y)\frac{\partial u_1(y)}{\partial\nu}-u_1(y)\frac{\partial u_3(y)}{\partial\nu}\right)ds\quad\text{on }\partial D. \tag{4.32}$$

To implement the direct inversion method, we approximate μ by trigonometric polynomials of degree m as follows.

$$\mu(t)\approx\sum_{k=0}^{m}a_k\cos kt+\sum_{k=1}^{m}b_k\sin kt. \tag{4.33}$$

Choosing equidistant mesh

$$h = \frac{\pi}{n}, \quad n \in \mathbb{N} \tag{4.34}$$

and collocation points for (4.27), (4.29) and (4.31) as $t_j = hj, j = 1, 2, \ldots 2n$ and computing the integration in (4.27), (4.29) and (4.31) by trapezoidal rule with equidistant points, we get a linear system

$$\begin{bmatrix} c_1(t_1) - c_1(t_0) & (c_1(t_1) - c_1(t_0))(\cos(t_0) - 1) & (c_1(t_1) - c_2(t_0))\sin(t_1) & (c_1(t_1) - c_1(t_0))\cos(2t_1) & (c_1(t_1) - c_1(t_0))\sin(2t_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1(t_{2n-1}) - c_1(t_0) & (c_1(t_{2n-1}) - c_1(t_0))(\cos(t_{2n-1}) - 1) & (c_1(t_{2n-1}) - c_1(t_0))\sin(t_{2n-1}) & (c_1(t_{2n-1}) - c_1(t_0))\cos(2t_{2n-1}) & (c_1(t_{2n-1}) - c_1(t_0))\sin(2t_{2n-1}) \\ c_2(t_1) - c_2(t_0) & (c_2(t_1) - c_2(t_0))(\cos(t_0) - 1) & (c_2(t_1) - c_2(t_0))\sin(t_1) & (c_2(t_1) - c_2(t_0))\cos(2t_1) & (c_2(t_1) - c_2(t_0))\sin(2t_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_2(t_{2n-1}) - c_2(t_0) & (c_2(t_{2n-1}) - c_2(t_0))(\cos(t_{2n-1}) - 1) & (c_2(t_{2n-1}) - c_2(t_0))\sin(t_{2n-1}) & (c_2(t_{2n-1}) - c_2(t_0))\cos(2t_{2n-1}) & (c_2(t_{2n-1}) - c_2(t_0))\sin(2t_{2n-1}) \\ c_3(t_1) - c_3(t_0) & (c_3(t_1) - c_3(t_0))(\cos(t_0) - 1) & (c_3(t_1) - c_2(t_0))\sin(t_1) & (c_3(t_1) - c_3(t_0))\cos(2t_1) & (c_3(t_1) - c_3(t_0))\sin(2t_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_3(t_{2n-1}) - c_3(t_0) & (c_3(t_{2n-1}) - c_3(t_0))(\cos(t_0) - 1) & (c_3(t_{2n-1}) - c_2(t_0))\sin(t_{2n-1}) & (c_3(t_{2n-1}) - c_3(t_0))\cos(2t_{2n-1}) & (c_3(t_{2n-1}) - c_3(t_0))\sin(2t_{2n-1}) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \\ d_{2n-1} \\ \vdots \\ d_{4n} \\ \vdots \\ d_{6n-3} \end{bmatrix}$$

where c_1, c_2, c_3 and $d = d_1, \cdots, d_{6n-3}$ for $t_j, j = 1, \cdots, 2n$ represent left hand side of the equations (4.27), (4.29), (4.31) and their right hand side respectively after plugging approximation of μ above. Then this overdetermined system is solved by least square sense. Similarly we approximate λ by trigonometric polynomial of degree m and having boundary condition (4.3) for all three solutions u_1, u_2 and u_3 at collocation points, we obtain a linear system for $2m + 1$ Fourier coefficients of λ via least square method. To recover λ and μ we need to determine the image of inverse operator

$$F_{Cauchy} : u^s|_C \to \left(u|_{\partial D}, \frac{\partial u}{\partial \nu}|_{\partial D} \right). \tag{4.35}$$

This can be performed by representing the solution (4.1)-(4.2) as single layer potential and solving the ill-posed set of Fredholm equations of the first kind

$$(A\varphi_i)(x) = u_i^s(x), \quad x \in C, i = 1, 2, 3 \tag{4.36}$$

for the case of three sources. The operator A has analytic kernel and it is compact therefore (4.36) is severely ill-posed.

Implementation of Regularization method

Hadamard (2003) gave definition of the ill-posedness for boundary and initial value problems for partial differential equations. These properties:

- Existence of the solution.
- Uniqueness of the solution.
- Continuous dependence of the solution on the data.

A problem is called well-posedness if these three conditions above are met. Otherwise it is called ill-posedness. Equation (4.36) is ill-posed since A^{-1} is unbounded inverse. In order to treat this problem, there are many type of regularization methods such as Landweber, truncated SVD and the Tikhonov method etc. One of the most successful regularization methods is probably the Tikhonov method. In our work we consider Tikhonov regularization method described as follows from Kress (2013)'s book.

Theorem 4.2 *Let $A : X \to Y$ is be bounded injective linear operator and let $\alpha > 0$. Then for each $u^s \in Y$ there exists a unique $\varphi_\alpha \in X$ such that*

$$||A\varphi_\alpha - u^s||^2 + \alpha||\varphi_\alpha||^2 = \inf_{\varphi \in X} \left\{ ||A\varphi - u^s||^2 + \alpha||\varphi||^2 \right\}$$

equivalent unique solution of th equation given by

$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*u^s. \tag{4.37}$$

In our case we will apply Tikhonov H^s regularization method described by

$$(\alpha D_s + A^*A)\varphi = A^*u^s \tag{4.38}$$

equivalent to solution to minimizer of the equation

$$\|A\varphi_\alpha - u^s\|_{L^2}^2 + \alpha\|\varphi_\alpha\|_{H^s}^2 = \inf_{\varphi\in H^s}\left\{\|A\varphi - u^s\|_{L^2}^2 + \alpha\|\varphi\|_{H^s}^2\right\}, \tag{4.39}$$

where α is regularization parameter and D_s is diagonal matrix introduced in thesis of Hohage (1999). We choose $s = 2$ in all experiments. In order to apply Tikhonov regularization technique to solve (4.36), we have the following lemma.

Lemma 4.1 *The integral operator $A : L^2(\partial D) \to L^2(C)$ is injective and has dense range.*

Proof 4.2 *Define the function*

$$\upsilon(x) := \int_{\partial D} \Phi(x, y)\varphi(y)ds(y), \quad x \in D \tag{4.40}$$

which satisfies equation $\Delta\upsilon - k\upsilon^2 = 0$ in the region bounded by curve C and assume $\upsilon|_C = 0$. Then the analyticity gives rise to $\upsilon = 0$ in D. Hence, $\upsilon^-|_{\partial D} = 0$ and $\frac{\partial\upsilon}{\partial\nu}|_{\partial D} = 0$ in D. Also single layer potential υ fulfills the exterior Dirichlet boundary value problem for modified Helmholtz equation with behavior for $\upsilon(x) = O\left(\frac{1}{|x|}\right)$ as $|x| \to \infty$. For extended investigation, we refer to Quaife (2011). It follows that uniqueness of the exterior Dirichlet boundary value problem yields $\upsilon = 0$ in $\mathbb{R}^2 \setminus D$. By the aid of limiting case of single layer potential (2.14), we have $\varphi = 0$ on ∂D, i.e. A is injective. We continue similarly to prove denseness. It is well known that A is dense in $L^2(C)$ if and only if its annihilator is trivial in $L^2(C)$. The adjoint operator $A^* : L^2(C) \to L^2(\partial D)$ is defined by

$$(A^*\sigma)(x) = \overline{\int_C \Phi(x, y)\overline{\sigma(y)}ds(y)}, \quad x \in \partial D \tag{4.41}$$

and consider the function

$$\rho(x) = \int_C \Phi(x, y)\overline{\sigma(y)}ds(y), \quad x \in \mathbb{R}^2 \setminus C. \tag{4.42}$$

Let $A^*\sigma = 0$, then the function vanishes on ∂D and satisfies the modified Helmholtz *equation in $\mathbb{R}^2 \setminus C$. The uniqueness of exterior Dirichlet problem leads to $\rho = 0$ and $\frac{\partial \rho}{\partial \nu} = 0$. The function ρ has analytic kernel in the region by the contours C and ∂D and plus the Cauchy pair $\left(\rho|_{\partial D}, \frac{\partial \rho}{\partial \nu}|_{\partial D}\right)$ vanish on ∂D. By Holmgren's theorem 2.1, $\rho = 0$ in the same region. Finally, considering $\frac{\partial \rho}{\partial \nu}$ we receive the homogeneous Fredholm integral equation of second kind. Then using (Kress, 2013, Theorem 6.21), ρ has to be trivial and therefore the operator A^* is one-to-one.*

This theorem depicts the ill-posedness of the inverse problem which is consequence of the compactness property of the mapping $\partial D \rightarrow C$.

As a second method we introduce a hybrid method which is combination of the direct method and the regularized Newton method. The reconstructions are obtained by direct method will be used as an initial guess for the a regulerized Newton-type method. To begin the second part of the hybrid method, we introduce nonlinear operator equation

$$F(\lambda_m, \mu_m) = u^s|_C, \quad u^s|_C = [u_1^s, u_2^s, u_3^s]^T \tag{4.43}$$

the inverse problem consists of solving nonlinear operator equation (4.43) for the unknown λ and μ. λ_m, μ_m are the Fourier coefficient vector corresponding to the trigonometric approximation of (4.33). To reconstruct λ and μ with a given ∂D more accurately, we will apply the regularized Newton method to equation (4.43). The linearized form of equation (4.43) is given by

$$F(\lambda_m, \mu_m) + F'_{\lambda_m}(\lambda_m, \mu_m)\varsigma_m + F'_{\mu_m}(\lambda_m, \mu_m)\eta_m = u^s|_C, \tag{4.44}$$

where F'_{λ_m} and F'_{μ_m} are Frechet derivative. The determination of the Frechet derivatives F'_{λ_m} and F'_{μ_m} in the hybrid method has been obtained by numerical differentiating with step size $h = 10^{-5}$. The analysis of the Frechet derivative for generalized impedance boundary condition with Helmholtz equation is studied in Bourgeois et al. (2012). The ill-posedness of the linearized equation (4.44) is reflected through the ill-posedness of the data equation (4.43). It is solved by regularized least square for update $\lambda_m + \varsigma_m$ and $\mu_m + \eta_m$. For iteration as a stopping rule Morozov's discrepancy principle is employed until the the following condition

$$\|F(\lambda_m, \mu_m) - f^\delta\| < \tau\delta, \quad \text{for some fixed constant } \tau > 1 \tag{4.45}$$

is satisfied and δ is noise level. The second method motivates the following algorithm.

Algorithm for hybrid method

input : λ_0, μ_0 and ∂D; % λ_0, μ_0 obtained by direct method

while $\|F(\lambda_m, \mu_m) - f^\delta\| < \tau\delta$

J_λ, J_μ; % compute Jacobien

$F(\lambda_m, \mu_m) + F'_{\lambda_m}(\lambda_m, \mu_m)\varsigma_m + F'_{\mu_m}(\lambda_m, \mu_m)\eta_m = f$; % solve for ς_m, η_m

$\lambda_m \leftarrow \varsigma_m + \lambda_m$; % update λ_m

$\mu_m \leftarrow \eta_m + \mu_m$; % update η_m

end

Output : λ, μ.

4.3. Numerical Results

In this section we provide some numerical results to exhibit the effectiveness and accuracy of the reconstruction methods illustrated in previous section. We compare the performance of the methods described for the inverse impedance problem. For the implementation the parameterized impedance functions are given by

$$\lambda(t) = \frac{1}{1 - 0.1\sin(2t)} \quad \text{and} \quad \mu(t) = \frac{1}{1 + 0.3\cos(t)} - 0.15, \quad t \in [0, 2\pi]. \tag{4.46}$$

In all experiments the regularization parameter is chosen by trial and error and the wave number is taken $k = \frac{1}{2}$. The number of quadrature and measurement points for producing the Cauchy data are chosen as $2n = 80$ and the impedance functions are approximated by trigonometric polynomial of degree $m = 2$. Also the density function φ in the data equation (4.36) is approximated by trigonometric polynomial with degree $m = 12$. To generate the perturbed data, some normally distributed noise was added to synthetic data with respect to L^2 norm. Equally distant sources points are considered as

$$x^+ = 0.9\,(\cos 0, \sin 0)\,, x^\times = 0.9\left(\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}\right), x^* = 0.9\left(\cos\frac{4\pi}{3}, \sin\frac{4\pi}{3}\right). \tag{4.47}$$

Example 4.1 *The first example studied is here a peanut parameterized by*

$$\partial D = \left\{0.8\sqrt{\cos^2(t) + 0.25\sin^2(t)}(\cos(t), \sin(t)),\ 0 \le t \le 2\pi\right\}. \tag{4.48}$$

The numerical computations of the functions (4.46) in the following table are evaluated by hybrid method. Table 4.1 consist of numerical error with L^2 norm where the choice of the involved regularization parameters is $\alpha_\lambda = \alpha_\mu = 10^{-8} \times 0.8^j$ where j is iteration number. Approximation of λ and μ in the Table 4.1 correspond to λ_{app}, μ_{app} respectively. The

Table 4.1.: Errors for the exact data

Iteration	$\|\lambda - \lambda_{app}\|_{L^2}/\|\lambda\|_{L^2}$	$\|\mu - \mu_{app}\|_{L^2}/\|\mu\|_{L^2}$	$\|u^s - F(\lambda,\mu)\|_{L^2(C)}/\|u^s\|_{L^2(C)}$
0	*0.0185*	*0.0321*	*0.0134*
1	*0.0036*	*0.0067*	*4.8371e-04*
2	*0.0035*	*0.0067*	*2.8069e-04*

numerical relative error decreases as seen in the Table 4.1. The third column is calculated for the source point x^+. That table shows reconstructions of λ and μ are reasonable according to numerical error in the Table 4.1.

Example 4.2 *In this example the source points (4.47) are considered. The left and middle plots of Figure 4.2 demonstrates the solution to (4.1)-(4.2) for the source point x^+.*

Three sources and two different forms of the domain, convex and concave, are represented by variation of the appropriate fields on the measuring circle C in the right-hand side plot of Figure 4.2. In specifically, the dashed lines correspond to the elliptical domain, the solid lines represent fields for the peanut, and the line color represents the source taken into consideration. One can see that a convex domain's field amplitude is

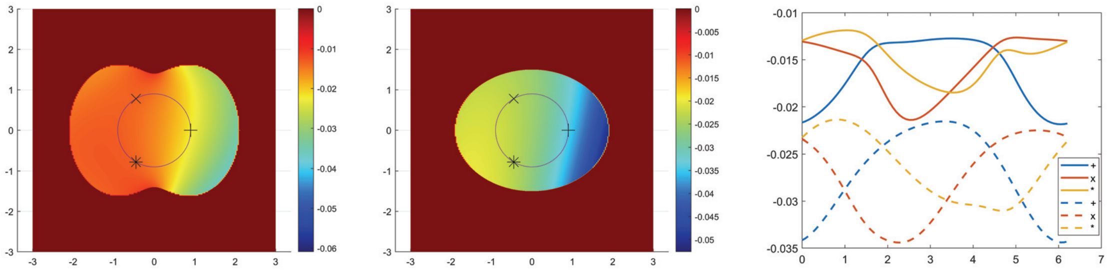

Figure 4.2.: Fields in the domain and on the measurement circle

higher than that of a convex domain with a comparable size. The fields generated by two symmetric points sources x^{x} *and* x^{*} *differ more for the convex domain, which might be a reason why the impedance reconstructions for the convex domain are more accurate.*

Example 4.3 *The second example considered here is reconstruction from the exact data for the parameterized ellipse*

$$z(t) = (1.9\cos(t), 1.5\sin(t)). \tag{4.49}$$

The reconstructions for the direct method are done with regularization parameter $\alpha_\varphi = 10^{-10}$. *For the hybrid method regularization parameter* $\alpha_\mu = \alpha_\lambda = 10^{-8}\left(\frac{4}{5}\right)^j$ *are considered with iteration number* $j = 2$. *There is no much difference between reconstructions of the methods since the domain is very smooth for ellipse. Hybrid method is slightly better than another method.*

Also we have exact reconstructions for peanut (4.48) with both methods. The hybrid method introduced in section (4.2) gives more accurate reconstructions than the another method considered here. The regularized parameters for hybrid taken as $\alpha_\varphi = 10^{-10}$ *and* $\alpha_\lambda = \alpha_\mu = 10^{-8}\left(\frac{4}{5}\right)^j$ *with iteration number* $j = 2$. *The direct method includes parameter* $\alpha_\varphi = 10^{-10}$.

To understand the shift of the function obtained by using the direct reconstruction algorithm we analyze the Cauchy data and the tangential derivative for the total field evaluated on ∂D *in Table 4.2. We chose* λ, μ, *defined in (4.46) and a point source located at* $x^{*} = (0.9, 0)$ *in* D. *The approximate field* $\hat{u}|_{\partial D}$ *is obtained by solving problem (4.36) and*

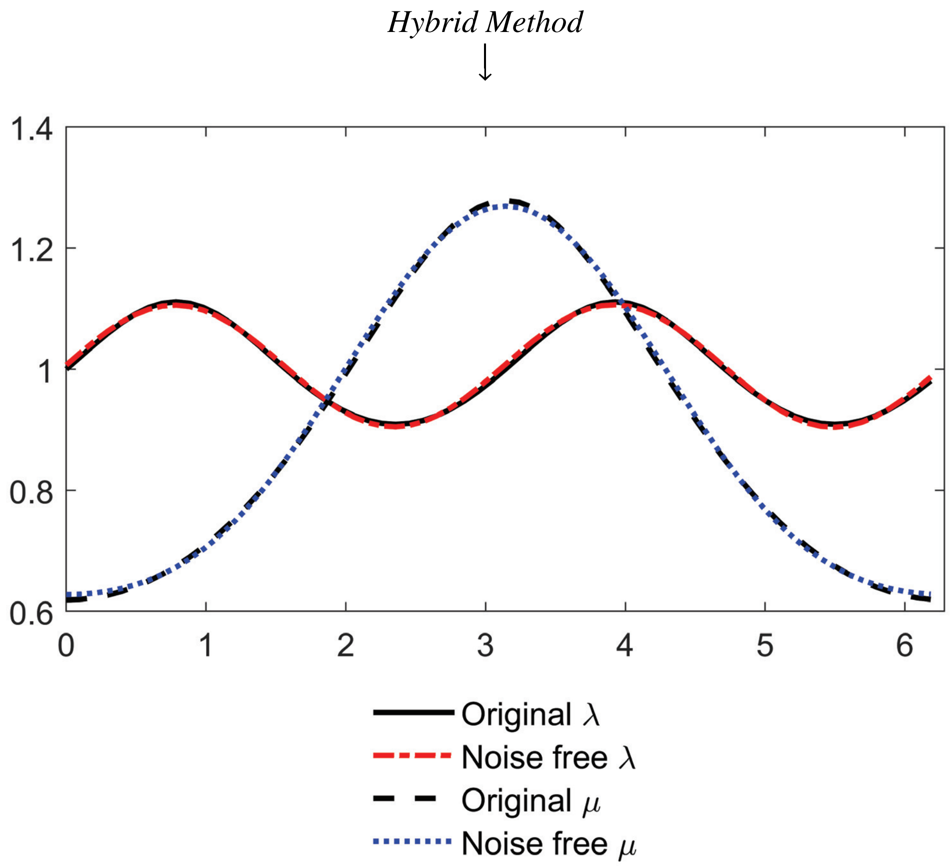

Figure 4.3.: Reconstruction from the exact data of λ and μ for ellipse by hybrid

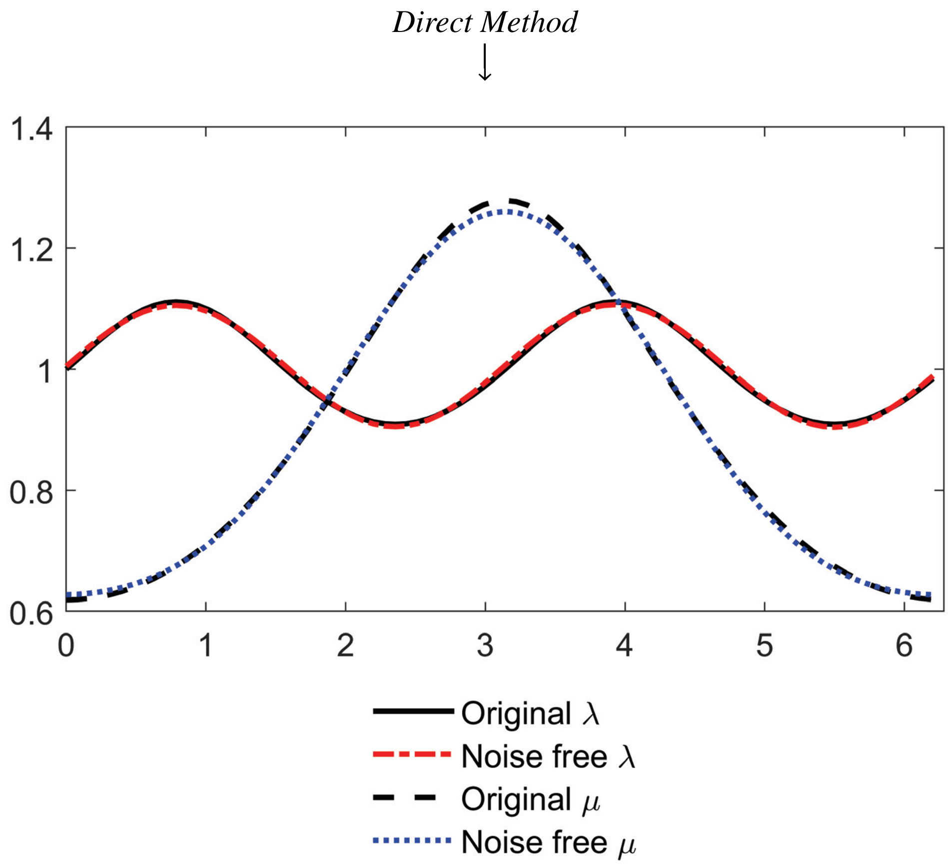

Figure 4.4.: Reconstruction from the exact data of λ and μ for ellipse by direct

Hybrid Method

↓

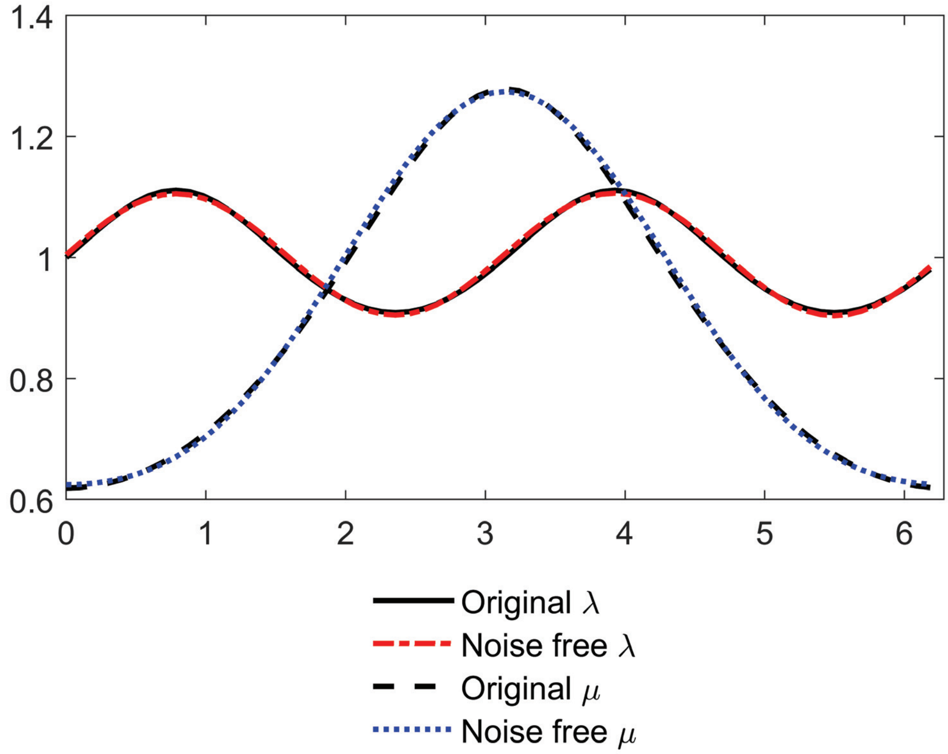

Figure 4.5.: Reconstruction from the exact data of λ and μ for peanut by hybrid

Direct Method

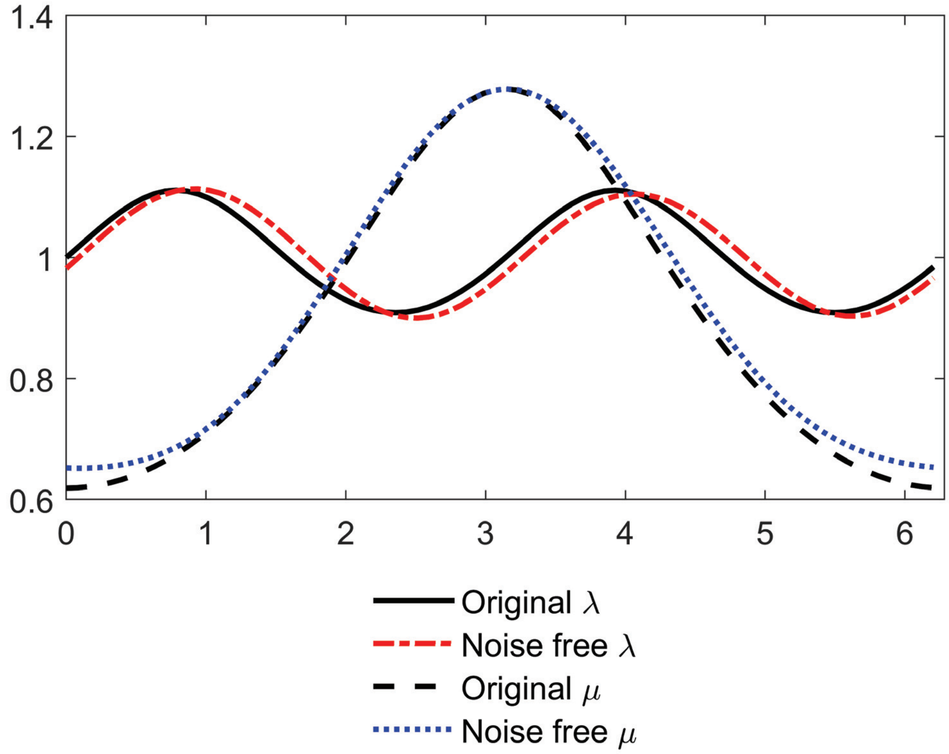

Figure 4.6.: Reconstruction from the exact data of λ and μ for peanut by direct

exact $u|_{\partial D}$ is computed by (4.16) in Table 4.2. We notice that the accuracy of u and the normal derivative of u is better than the tangential derivative since their computation involves numerical differentiation. In the presence of normally distributed noise with respect to L^2

Table 4.2.: Numerical error for peanut-shaped domain

$2n$	$\|\hat{u}-u\|_{L^2(\partial D)}/\|u_{\partial D}\|_{L^2(\partial D)}$	$\|\frac{\partial \hat{u}}{\partial \nu}-\frac{\partial u}{\partial \nu}\|_{L^2(\partial D)}/\|\frac{\partial u}{\partial \nu}\|_{L^2(\partial D)}$	$\|\frac{d\hat{u}}{ds}-\frac{du}{ds}\|_{L^2(\partial D)}/\|\frac{du}{ds}\|_{L^2(\partial D)}$
16	*0.0328*	*0.2306*	*0.0934*
32	*0.0070*	*0.1282*	*0.0720*
64	*0.0031*	*0.0493*	*0.0527*

norm, we consider the regularized parameters for the hybrid taken as $\alpha_\varphi = 10^{-6}$ and $\alpha_\lambda = \alpha_\mu = 10^{-4}\left(\frac{4}{5}\right)^j$ with iteration j number changing between 1 and 15. The parameters $\alpha_\varphi = 10^{-6}$ is chosen for direct method. Also we have employed discrepancy principle (2.47) with $\tau = 0.99$ to stop the hybrid scheme. Here τ is chosen slightly smaller than 1 since the amount of input data is much higher than the number of unknowns. In general Figure 4.7 and 4.8 show that the reconstructions deteriorate when noise level increases slightly. In other words, we observe that the interior inverse impedance problem is very sensitive to noise. Moreover, sources close to each other makes reconstruction worse and there is a analogous effect while the measurement circle and the boundary of the domain are near to each other. Due to these reasons equally spaced source distribution and the measurement circle, which is not in proximity to the boundary, were preferred. In the Figure 4.7 we choose average reconstructions out of ten.

These results are submitted as " An interior inverse generalized impedance problem for the modified Helmholtz equation in two dimensions". by Ivanyshyn Yaman and Özdemir.

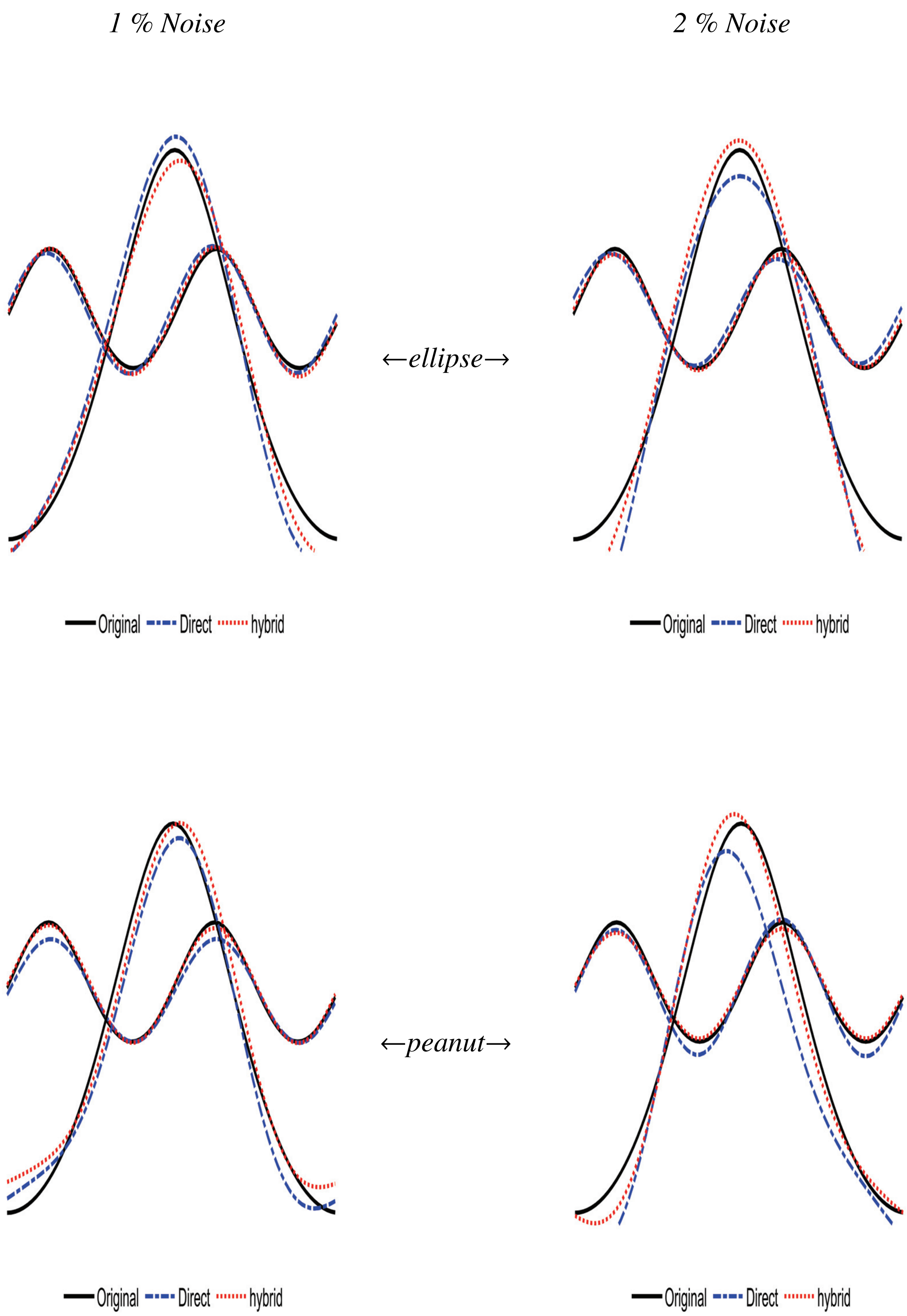

Figure 4.7.: Average reconstructions of λ and μ with noisy data out of ten

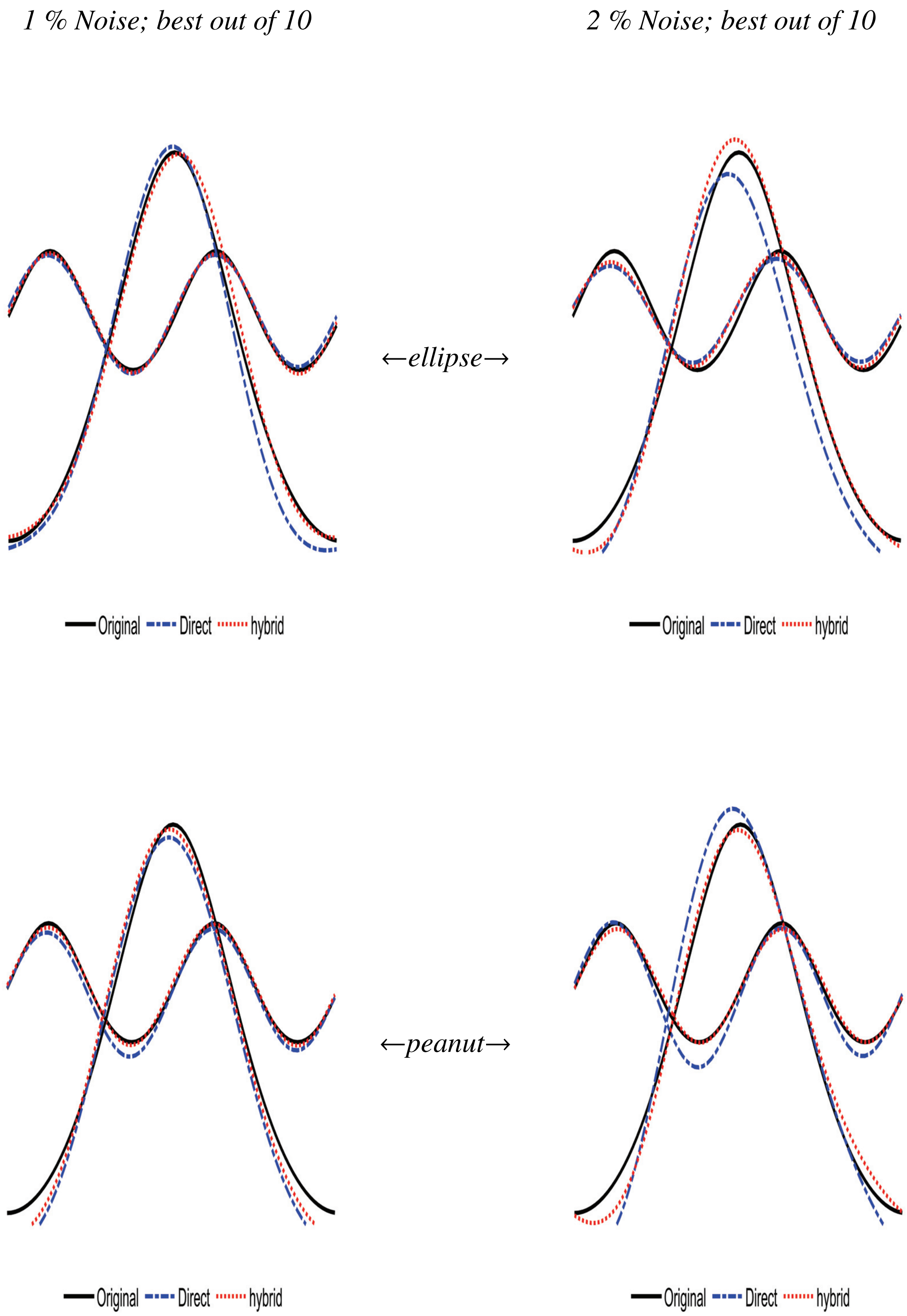

Figure 4.8.: Best reconstructions of λ and μ with noisy data out of ten

CHAPTER 5

CONCLUSION

In this thesis, the numerical solution methods for the direct and inverse problems are proposed. We demonstrate the existence and uniqueness of the solution to the GIBC problem for the two-dimensional modified Helmholtz equation and the solvability of the resulting boundary integral equation are established. The numerical solution approach is suggested using boundary integral equations. The technique relies on detaching singularities from integral operator kernels. Numerical examples are used to demonstrate and validate the convergence in the Sobolev space. If all the input data are analytic, the numerical solution converges super-algebraically. The methodology based on trigonometric differentiation and the numerical method is contrasted. The potential drawbacks of the second strategy are discussed and demonstrated with examples. Moreover, we indicate that the impedance functions are determined uniquely by the knowledge of three sources and measured data situated on an inner curve. We provide the direct method and a more complex hybrid strategy for the problem's numerical solution. The hybrid method, a Newton-type approach, uses the direct method's solution as an initial guess and hence yields more accurate reconstructions. The proposed hybrid method for the interior inverse problem provides accurate reconstructions in the case of the data with the low noise level without a priori information about the impedance functions.

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- Olha Ivanyshyn Yaman, Gazi Özdemir, "An interior inverse generalized impedance problem for the modified Helmholtz equation in two dimensions", Manuscript submitted.