EXACTLY SOLVABLE BURGERS TYPE EQUATIONS WITH VARIABLE COEFFICIENTS AND MOVING BOUNDARY CONDITIONS

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ABSTRACT

EXACTLY SOLVABLE BURGERS TYPE EQUATIONS WITH VARIABLE COEFFICIENTS AND MOVING BOUNDARY CONDITIONS

In this thesis, firstly, a generalized diffusion type equation is considered. A family of analytical solutions to an initial value problem on the whole line for this equation is obtained in terms of solutions to the characteristic ordinary differential equation and the standard heat model by using Wei-Norman Lie algebraic approach for finding the evolution operator of the associated diffusion type equation. Then, initial-boundary value problems on half-line and an initial-boundary value problem with moving boundary for this equation are studied. It is shown that if the boundary propagates according to an associated classical equation of motion determined by the time-dependent parameters, then the analytical solution is obtained in terms of the heat problem on the half-line. For this, a non-linear Riccati type dynamical system, that simultaneously determines the solution of the diffusion type problem and the moving boundary is solved by a linearization procedure. The mean position of the solutions, the influence of the moving boundaries and the variable parameters are examined by constructing exactly solvable models. Then, an initial value problem for a generalized Burgers type equation on whole real line is discussed. By using Cole-Hopf linearization and solution of the corresponding generalized linear diffusion type equation, a family of analytical solution is obtained in terms of solutions to the characteristic equation and the standard heat or Burgers model. Exactly solvable models are constructed and the influence of the variable coefficients are examined. Later, an initial-boundary value problem for the generalized Burgers type equation with Dirichlet boundary condition defined on the half-line is studied. Finally, an initial-boundary value problem for the generalized Burgers type equations with Dirichlet boundary condition imposed at a moving boundary is considered. The analytical solution is obtained in terms of solution to characteristic equation and the standard heat or Burgers model, if the moving boundary propagates according to an associated classical equation of motion. In order to show certain aspects of the general results, some exactly solvable models are introduced and solutions corresponding to different types of initial and homogeneous/inhomogeneous boundary conditions are discussed by examining the influence of the moving boundaries.

ÖZET

DEĞİŞKEN KATSAYILI VE HAREKET EDEN SINIR KOŞULUNA SAHİP TAM ÇÖZÜLEBİLEN BURGERS TİPİ DENKLEMLER

Bu tezde, ilk olarak genelleştirilmiş difüzyon tipi bir denklem ele alınmıştır. Tüm reel çizgi üzerinde bir başlangıç değer probleminin analitik çözümler ailesi ilgili difüzyon tipi denklemin evrim operatörünü bulmak için Wei-Norman Lie cebirsel yaklaşımı kullanılarak karakteristik adi diferensiyel denklemin ve standart ısı modelinin çözümleri cinsinden elde edilmiştir. Daha sonra bu denklem için yarım çizgideki başlangıç-sınır değer problemleri ve hareketli sınır koşullu başlangıç-sınır değer problemi çalışılmıştır. Sınır, zamana bağlı parametreler tarafından belirlenen ilişkili bir klasik hareket denklemine göre yayılırsa, o zaman analitik çözümün yarım çizgi üzerindeki klasik ısı problemi açısından elde edildiği gösterilmiştir. Bunun için difüzyon tipi problemin ve hareketli sınırın çözümünü eş zamanlı olarak belirleyen lineer olmayan Riccati tipi bir dinamik sistem bir lineerleştirme prosedürü ile çözülmüştür. Çözüm dağılımının ortalama konumu, hareketli sınırların ve değişken parametrelerin etkisi tam olarak çözülebilir modeller oluşturularak gösterilmiştir. Daha sonra, tüm reel cizgi üzerinde genelleştirilmiş bir Burgers tipi denklem için bir başlangıç değer problemi tartışılmıştır. Cole-Hopf doğrusallaştırması ve karşılık gelen genelleştirilmiş doğrusal difüzyon tipi denklemin çözümü kullanılarak, karakteristik denklem ve standart ısı veya Burgers modelinin çözümleri açısından bir analitik çözüm ailesi elde edilmiştir. Tam olarak çözülebilir modeller oluşturulur ve değişken katsayıların etkisi incelenmiştir. Daha sonra, yarım çizgi üzerinde tanımlanan Dirichlet sınır koşullu genelleştirilmiş Burgers tipi denklem için bir başlangıç-sınır değer problemi incelenmiş ve farklı başlangıç ve sınır koşullarına sahip tam olarak çözülebilir modeller sunulmuştur. Son olarak, hareketli bir sınıra dayatılan Dirichlet sınır koşuluna sahip genelleştirilmiş Burgers tipi denklemler için bir başlan-gıç sınır değer problemi ele alınmıştır. Eğer hareketli sınır ilişkili bir klasik hareket denklemine göre yayılırsa, analitik çözüm, karakteristik denklemin ve standart ısının veya Burgers modelinin çözümü cinsinden elde edilmiştir. Genel sonuçların belirli yönlerini göstermek için, tam olarak çözülebilir bazı modeller tanıtılmış ve hareketli sınırların etkisi incelenerek farklı başlangıç ve homojen/homojen olmayan sınır koşullarına karşılık gelen çözümler tartışılmıştır.

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LIST OF SYMBOLS

$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of positive real numbers
$\mathbb{R}_{\geq 0}$	The nonnegative real numbers
$r_h^\alpha(t)$	The alpha parameterization of homogeneous solutions to ordinary differential equation
$r_g^\alpha(t)$	The alpha parameterization of general solution to ordinary differential equation
$\Psi(\eta, \tau)$	Solution to heat equation
$V(\eta, \tau)$	Solution to standard Burgers equation
$\Phi(x, t)$	Solution to generalized diffusion type equation
$U(x, t)$	Solution to generalized Burgers type equation
$K(\eta, \tau)$	Kernel of heat equation
$G_D(\eta, \xi, \tau)$	Dirichlet heat kernel
$G_N(\eta, \xi, \tau)$	Neumann heat kernel
$L_g^\alpha(t)$	The Lagrange function for the alpha parameterization of general solution to ordinary differential equation
$p_g^\alpha(t)$	The generalized momentum
$H_n(\eta, \tau)$	Kampe de Feriet polynomial (or heat polynomial)
Erf(x)	Error function
Erfc(x)	Complementary error function
$M_0(t)$	Total amount of mass
$M_1(t)$	First spatial moment
$M_2(t)$	Second spatial moment
$\langle x \rangle(t)$	Mean position
$\langle x^2 \rangle(t)$	Mean square
Var(t)	Variance

LIST OF ABBREVIATIONS

ODE	Ordinary differential equation
PDE	Partial differential equation
PDE's	Partial differential equations
IC	Initial condition
IC's	Initial conditions
BC	Boundary condition
IVP	Initial value problem
IVP's	Initial value problems
IBVP	Initial-boundary value problem
IBVP's	Initial-boundary value problems
mIBVP	Initial-boundary value problem with moving boundary
KFP	Kampe de Feriet polynomials
FPE	Fokker-Planck equation
BE	Burgers equation
FBE	Forced Burgers equation
HE	Heat equation
CDE	Convection-diffusion equation
CDR	Convection-diffusion-reaction
CDRE	Convection-diffusion-reaction equation

CHAPTER 1

INTRODUCTION

Partial differential equations often express fundamental laws of nature and usually appear in the mathematical formulation of many problems in science and engineering. In particular, time-evolution problems arise in different areas such as diffusion processes, transport phenomena, fluid mechanics, wave propagation, chemical reactions and biology. The study of partial differential equations (PDE's) includes mathematical modeling, investigation of the existence and uniqueness of solutions, approximation and numerical techniques, methods for deriving analytical solutions and constructing exact closed form solutions.

As known, analytical solutions to initial-value problems (IVP's) and initial-boundary value problems (IBVP's) for *linear* PDE's are usually obtained and investigated using classical methods such as separation of variables, Fourier, Laplace, and other integral transforms, Green's functions approaches and method of images. Although commonly used, these methods have some limitations as discussed in (Fokas, 2008): (i) For higher order PDE's and non-separable boundary conditions (BC's) proper integral transforms may not exist. (ii) The integral equations arising in the Green's function approach are difficult to solve explicitly in closed form. (iii) The method of images can be applied only to problems that possess certain symmetries.

Although in the study of *linear* PDE's one may encounter certain problems, in general, their theory is well-developed. The situation with non-linear PDE's is much more complicated. There are no general techniques for solving *non-linear* PDE's and almost every non-linear equation should be investigated separately. An exception is the class of C-integrable non-linear PDE's that are directly linearizable by suitable transforms, like the Burgers equation. Another exceptional class consists of completely integrable non-linear equations that are characterized as systems possessing infinitely many conserved quantities. For such problems the Inverse Scattering Transform (Ablowitz & Segur, 1981), Hirota bilinear method (Hirota, 2004), Bäcklund transforms (Kingston & Sophocleous, 1990), were developed and applied to some well-known models such as the Korteweg-de-Vries equation (KdV), Kadomtsev-Petviashvili (KP) equation and the nonlinear Schrödinger equation. The use of Lie group symmetries is also a powerful and algorithmic method for constructing exact solutions to differential equations, but these solutions usually do not respect initial and boundary conditions.

An important point is that the methods mentioned above are mostly applied to PDE's with constant coefficients. When the problems possess space and time variable coefficients, exact and analytical solutions to the associated linear or non-linear IBVP's exist only for some special PDE's and specific BC's. In most variable parametric problems one has to use approximations and numerical techniques. However, knowing exact and explicit solutions, if exist, usually plays a significant role in understanding the underlying dynamics of the physical problems.

In this thesis, we investigate analytical and exact solutions written in terms of elementary or special functions to a large class of linear one-dimensional diffusion-type equations and generalized viscous Burgers equations with time and space variable coefficients. For these linear and non-linear PDE's first we study IVP's on the whole real line, then we discuss IBVP's on the the fixed half-line, and finally, we study IBVP's on a semi-infinite interval with a moving boundary. More precisely, the content of the thesis is as follows.

In Chapter 2, we provide some necessary concepts and brief background to the problems. We mention some well-known results and examples related to our research.

In Chapter 3, we study IVP and IBVP's for the one-dimensional linear PDE,

$$\Phi_t = \frac{1}{2\mu(t)}\Phi_{xx} - [a(t) - b(t)x]\Phi_x + \mu(t)[\frac{\omega^2(t)}{2}x^2 - f(t)x + f_0(t)]\Phi, \quad x \in \mathbb{R}, \quad t > t_0 > 0, \quad (1.1)$$

with smooth diffusion coefficient depending on time, $1/(2\mu(t))$, smooth convection coefficients which is linear in space variable x, $a(t) - b(t)x$, and reaction coefficients that depend on time and space, $\mu(t)[(\omega^2(t)/2)x^2 - f(t)x + f_0(t)]$. Together with certain initial and boundary conditions, this equation can describe convection-diffusion-reaction (CDR) processes in unsteady and inhomogeneous media. This equation is widely employed to model phenomena in many different fields in mathematics and sciences such as diffusion processes, statistical mechanics, probabilty theory, financial mathematics, population genetics, quantum chaos, modeling of biological systems, diffusion of neutrons, reaction of chemical, stochastic equation for Brownian motion and many more. It involves the change of concentration/population of one or more substances/species distributed in space under the influence of three processes: diffusion which refers to for example spreading of the contaminant from highly concentrated areas to less consantrated areas, convection which is defined as the movement of the species due to the fluid transport medium and reaction which is the process of interaction through which the species in the phenomena are generated or consumed. The solution can represent for instance concentration of a substance, heat energy or momentum. Eq.(1.1) comprises also Fokker-Planck type models in statistical mechanics and the stochastic equation for a Brownian motion. In that case solution represents the probability density that a Brownian particle is at position x, at time t, (Risken, 1984), (Englefield, 1988), (Mazo, 2002), (Paganin & Morgan, 2019).

Mathematically, this equation is special in the sense that it is the most general one-dimensional linear diffusive type PDE that can be written using generators of the Heisenberg-Weil and su(1,1) Lie algebras. The Lie algebraic or group theoretical methods for solving linear PDE's that appear both in classical and quantum problems are known for a long time. An important contribution in that direction is the work of Wei and Norman (Wei & Norman, 1963), who proposed an efficient procedure for finding the evolution operator of linear equations. In the previous works, such as in (Wolf, 1988), the Lie algebraic solution of the linear Fokker-Planck equation has been concerned, in (Dattoli, Gallardo & Torre, 1988), (Cheng & Fung, 1988), (Lo, 1991), authors have used the Wei-Norman technique together with transformation methods to solve quantum evolution problems. Recently, in (A. Büyükaşık, & Çayiç, 2019) by a straightforward application of the Wei-Norman procedure and by properly choosing the ordering of the exponential operators, the evolution operator for a quantum parametric oscillator has been found. For a constant coefficient case of the parabolic PDE (1.1), a long time ago in (Steinberg, 1977), a Lie algebraic approach was used and discussed the applications of Baker, Campbell, Haussdorf, and Zassenhaus formulas for solving the corresponding IVP. Later, for a nonautonomous diffusion-type equation of the form (1.1), authors in (Zola & et all, 2008) have used a Green's function ansatz and quite recently in (Suazo, Suslov & Guzman, 2014) by a transformation method it has been reduced to a standard form and fundamental solution has found by solving a Riccati type system. For some parabolic PDE's with variable coefficients, including Eq.(1.1), reduction transformations that lead to well-known PDE's can be found in (Polyanin, 2002), however, their formal expressions do not allow any understanding of the influence of the variable coefficients and behavior of the solutions.

In this thesis, we solve the evolution problem for the linear PDE (1.1) by the Wei-Norman Lie algebraic method. An important point in this direction is that we are able to determine the unknown coefficients of the evolution operator exactly in terms of two linearly independent homogeneous solutions and a particular solution to the linear second-order inhomogeneous characteristic equation. As a result, we obtain the evolution operator as product of simple exponential operators, and we provide analytical solution to the IVP for PDE (1.1) explicitly in terms of solutions to the characteristic equation and the standard heat model.

Then, we consider IBVP's for linear diffusion type equations on the domain $0 < x < \infty$, $0 < t < T$, and with Dirichlet, Neumann and Robin type BC's imposed at $x = 0$. As known, exact solutions of half-line IBVP's exist only for limited cases. In (Zoppou, Member & Knight, 1997), authors has presented analytical solutions to the spatially variable coefficient advection and advection-diffusion equations, written in conservative and nonconservative forms. Recently some variable coefficient diffusion type models with analytical solutions and describing concrete physical phenomena have been discussed in (Kumar, Jaiswal & Kumar, 2010), (Jaiswal, Kumar & Yadav, 2011), (Kumar & Yadav, 2013), (Kim, 2020) and references therein. Generalized parabolic type PDE's are addressed also in (Polyanin, 2002), where solutions to IBVP's are given without solving the related problems.

We note that in general PDE (1.1) does not possess space inversion symmetry and solving an IBVP on half-line is not always a straightforward task. However, for some special cases we show that the IBVP's with Dirichlet, Neumann and Robin type boundary conditions can be reduced to standard models. We provide an integral representation of the general solutions and obtain fundamental solution. Some exactly solvable models with different initial and boundary conditions are introduced and their dynamics is discussed.

In Chapter 4, motivated by the results in previous chapter, we study an IBVP for Eq.(1.1) on a semi-infinite interval $s(t) < x < \infty$ with a moving boundary $s(t)$, $0 < t < T$. As known, in many applications there are problems in which the boundaries of the domain are changing with time. In particular, diffusion and heat-conduction type problems with moving boundaries occur in different processes containing phase changes and chemical reactions, (Dankwerts, 1950). Practical applications include the progressive freezing or melting of water or liquids, solidification of melts, evaporation, laser glazing (Davis & Hill, 1982), (Gupta & Arora, 1988), crystallization, ablation problems (Mitchell, 2012), oxygen metabolism in tissues (Crank & Gupta, 1972a), (Evans & Gourlay, 1977). Such applications often comprise PDE models in which the location of the boundary is not known in advance and must be determined together with the solution. In that case, one usually speaks about moving and/or free boundary problems (Chen & Shahgholian & Vazquez, 2015), and the best known are the one-phase or two-phase Stefan-like problems, (Griffin & Coughanowr, 1965), (Crank, 1984). In general, finding solutions to such problems is a difficult task mainly because the moving boundary makes the problem nonlinear. Due to this, the study of moving boundary problems is usually based on numerical methods, approximation techniques, or pure analysis. Only a few exact analytic solutions are known for Stefan-like problems, (Bluman, 1974), (Crank, 1984), (Alexandrov & Malygin, 2006), (Johansson & Lesnic, 2011), (Salva & Tarzia, 2011), (Rodrigo & Thamwattana, 2021) .

Since finding exact solutions to such problems occurs rarely, some authors have considered the simplified problem of finding solutions for boundary conditions imposed on a priory given moving boundary (Langford, 1967), (Kartashov, Lyubov & Bartenev, 1970), (Bluman, 1974), (Tait, 1979). When such solutions are available, one has an approximation to the moving boundary problem in the sense that the solution gives a possible state of the medium, if the heat can be supplied externally in a prescribed way, (Tait, 1979). For example, in (Kartashov, Lyubov & Bartenev, 1970) authors have considered a problem in which the boundary of a crystallizing melt moves in accordance with a quadratic crystallization law $y(t) = \pm\beta t^2 + \alpha t$ and arrived at the standard diffusion equation for the distribution of concentration in the region $y(t) < x < \infty$, $t > 0$. In that case, the solution has been derived by changing to a moving coordinate system and using the Laplace transform. In (Tait, 1979), the exterior ablation problem $u_{rr} + (2\nu + 1/r)u_r - u_t = 0$ defined on the domain $R(t) < r < \infty$, $t \in (0, T)$, has been addressed using an ansatz with similarity variable $\xi = r/R(t)$, transforming conditions on the moving boundary to conditions on the fixed one. Similar standard procedures were used also to study both stationary and non-stationary Schrödinger equations with moving boundaries describing for instance the oscillations of a particle in a potential well of time-dependent width, a quantum bouncer and a cut-off oscillator (Munier & et all, 1981), (Makowski & Dembinski, 1991), (Makowski, 1992).

Although the subject is extensively studied for standard models, to the best of our knowledge, the variable parametric generalized diffusion type models defined on the time-dependent regions are less investigated. Since moving boundaries often and significantly affect the behavior of the systems, finding exact solutions is an essential and motivating step toward understanding their dynamics.

Therefore, in this thesis, we study an IBVP for Eq.(1.1) defined on the domain $s(t) < x < \infty$, $0 < t < T$ and with Dirichlet boundary condition (BC) imposed on the boundary $x = s(t)$. For this, passing to a moving coordinate system, the mIBVP is transformed into an IBVP on the half-line. As expected, the transformed problem is defined on a simpler domain, but the associated PDE in the new variables becomes more complicated since the boundary $s(t)$ contributes to the convection and reaction coefficients of the PDE. In that case, we prove that if the boundary function propagates according to an associated forced classical equation of motion, then the mIBVP reduces to solving a standard heat IBVP on the half-line. An important step in this direction is the solution of a nonlinear Riccati-type dynamical system, which simultaneously determines the solution of the mIBVP and the moving boundary for which this solution holds. As a result, we provide an integral representation of the solution and fundamental solution of the mIBVP explicitly in terms of solutions to characteristic equation. Then, applying it to our general results, we present exactly solvable models that can describe for example diffusion-reaction, convection-diffusion, and convection-diffusion-reaction type processes. For each model, if a family of moving boundaries propagate according to characteristic equation, then we derive families of exact solutions corresponding to different types of initial data and homogeneous boundary conditions explicitly and discuss their dynamics according to the influence of the time-variable parameters and the prescribed moving boundaries.

On the other side, it is known that the spatial moments of a solution distribution are used to characterize the geometry of the evolving distribution of, for instance, concentration such as zeroth, first and second moments. The zeroth moment gives the total amount of substance, the first moment normalized by the total amount represents the mean location of the distribution and the second central moment normalized by the total amount gives the second central moment normalized by the total amount gives the mean square position and correspondingly it is used to find the variance which is a measure of the spreading of the distribution about its mean position. Therefore, for the fundamental solution of exactly solvable models, we find the moments of the solution distribution and analyze the mean position of the distribution and motion of the moving boundary.

Next, we introduce an IBVP for the generalized diffusion type equation with Neumann and Robin boundary condition imposed at a boundary $x = s(t)$. Using the same procesure given in previous section, we obtain that the solution of mIBVP with inhomogeneous Neumann and Robin BC reduces to the standard heat IBVP with Robin type boundary condition with variable coefficient on the half line which requires solving a second kind Volterra integral equation. Similarly, we obtain the exact analytical solution and the integral representation of solution to the mIBVP's with Neumann and Robin BC's if the time-dependent boundary is prescribed in a certain way. Then, in order to show certain aspects of the general results, we present exactly solvable models for some special choices of boundary conditions.

In Chapter 5, we study an IVP and IBVP's for generalized Burgers type equations

$$U_t + \frac{\dot{\mu}(t)}{\mu(t)}U + UU_x = \frac{1}{2\mu(t)}U_{xx} - ((a(t) + b(t)x)U)_x - \omega^2(t)x + f(t), \quad x \in \mathbb{R}, \quad t > t_0 > 0, \tag{1.2}$$

for the field $U(x, t)$, with smooth coefficients of damping $\Gamma(t) = \dot{\mu}(t)/\mu(t)$, diffusion $D(t) = 1/2\mu(t)$, linear in position convection coefficient $v(x, t) = a(t) - b(t)x$, and an external forcing term $F(x, t) = -\omega^2(t)x + f(t)$. This equation can be seen as an extension of the forced Burgers equation (FBE) discussed in (Eule & Friedrich, 2006), (A. Büyükaşık & Pashaev, 2013), (A. Büyükaşık & Bozacı, 2019), and is special since it admits a direct linearization in the form of Eq.(1.1).

As known, the one-dimensional standard Burgers equation (BE), $V_t + VV_x = \nu V_{xx}$, is the simplest model combining nonlinear convection and linear dissipative effects. Due to this, it appears in many fields such as fluid dynamics (Burgers, 1948), (Burgers, 1974), gas dynamics (Hopf, 1950), (Cole, 1951), nonlinear acoustics (Lighthill, 1956), (Saichev, Gurbatov & Rudenko, 2011), (Enflo & Hedberg, 2002) and mass transport. It is commonly used to describe the nonlinear propagation of waves in dissipative media and shock formation (Bec & Khanin, 2007), distribution of matter in large-scale structures, and interface growth models (Kardar, Parisi & Yi, 1986), (Woyczynski, 1998).

From the mathematical point of view, standard BE is an integrable model that admits direct linearization and analytic solution of the IVP on the whole real line is found easily by using the Cole-Hopf transformation, (Hopf, 1950), (Cole, 1951). Then, many interesting solutions can be derived from solutions of the associated heat equation, (Saichev, Gurbatov & Rudenko, 2011), (Whitham, 1999), (Sachdev, 1987), (Benton, 1941). Later, other approaches like Hirota method (Wang & et all, 2004), Lie symmetries method (Sophocleous, 2004), (Vaganan & Kumaran, 2006), generalized Cole-Hopf transform (Vaganan & Jeyalakshmi, 2011), etc., were also used to construct exact and explicit solutions for Burgers type models.

Due to the great number of possible applications and its integrability, different generalizations of Burgers equation were introduced and discussed from different perspectives (Orlowsky & Sobczyk, 1989), (Schulze-Halberg & Jimenez, 2009), (A. Büyükaşık & Pashaev, 2013), (Zuparic & Hoek, 2019). One of them is when external forces and forcing terms with time-dependent coefficients are included in the problem, so that one can take into account inhomogeneities of media, non-uniformities of boundaries, friction forces, and for example it can model the evolution of growing interfaces (Xu & et all, 2007). Such problems were addressed also in some earlier works (Zola & et all, 2008), (Moreau & Vallee, 2006), (Suazo, Suslov & Guzman, 2014) and were treated by different approaches. Recently, the exact solvability of generalized BE including a nonlinear forcing term was addressed in (Schulze-Halberg, 2015) by introducing point transformations for linearizing the Burgers model and relating it to its Schrödinger counterpart. In (Pereira, Suazo & Trespalacios, 2018) reaction-diffusion equations and some Burgers equations were solved through Riccati equations and similarity transformations, a method of solving initial and boundary value problems for the nonlinear integrable systems was described in (Fokas & Pelloni, 2000).

Therefore, in this thesis, we solve the IVP for BE (1.2) using a generalized Cole-Hopf transformation and our results are presented in Chapter 3. We formulate the general closed-form solution of the IVP for BE (1.2) using the solutions to the characteristic equation and the standard heat and Burgers models. Using the symmetries of standard Burgers equation such as the shift of origin, change of scale, and Galilean symmetry where usually solutions obtained from each other by symmetry transformations are said to be equivalent or isomorphic, since they don't change the form of the original function (Sachdev, 1987), (Benton, 1941), we show that a family of solutions obtained by the translation and Galilean invariance of standard BE provide a more complete description of the corresponding family of generalized traveling waves. We can say that the exact and explicit results obtained in this work, provide deeper insight into the solutions of the generalized Burgers equation and lead to a better understanding of the influence of the time-variable parameters, external terms, and forcing terms on the dynamics of the nonlinear waves.

Next we note that, the Cole-Hopf transform is special also in the sense that the initial condition for Burgers equation transforms directly to the initial condition for the corresponding heat equation. Then one can easily write and analyze solutions of the Burgers initial value problem on the whole real line $-\infty < x < \infty$. However, for IBVP's posed on the infinite half-line $0 < x < \infty$ the situation is different. Depending on the type of the boundary conditions, solving the problem is not always a straightforward task. A long time ago, Rodin in his work (Rodin, 1970) discussed the IBVP for a standard Burgers equation on the half-line $0 < x < \infty$, with Dirichlet boundary condition (BC) imposed at $x = 0$, and showed that to find a solution of this IBVP, one must first solve a corresponding second-kind linear Volterra type integral equation. Since in general, its solution requires approximation techniques, Rodin discussed a technique for obtaining closed-form solutions of Burgers equation on the half-line by "sacrificing" the initial data, but retaining the exact boundary condition. By this approach, he re-obtained some well-known solutions, and then Sachdev in (Sachdev, 1987) enlarged the solution class of these problems. Calogero and De Lillo introduced a "generalized" Cole-Hopf transform for the Dirichlet Burgers problem on the half-line (Calogero & De Lillo, 1989). Then in (Biondini & De Lillo, 1991), authors obtain the explicit solution to the Burgers equation on the semiline with flux-type boundary conditions at the origin. The Neumann problem for Burgers equation is studied in (De Lilo & Sommacal, 2011) and it is reduced to a nonlinear integral equation in one independent variable, whose unique solution is proven to exist for small time. Later, Fokas and De Lillo used the unified transform method to solve the Dirichlet problem for standard BE on the half-line (Fokas & De Lillo, 2014). After the introduction of the unified transform method (Fokas method) (Fokas, 1997), there is a renewed interest in IBVP's for linear and non-linear PDE's on the infinite half-line.

We recall that the standard Burgers equation has different types of exact solutions and among them, rational-type solutions form an important class. Indeed, zeros of the polynomial solutions of the heat problem lead to pole singularities for the Burgers rational solutions. Then, the motion of these singularities corresponds formally to the motion of one-dimensional particles interacting via two-body potentials and the corresponding many-body problems are integrable, (Choodnovsky, 1977), (Calogero, 1978). For recent work on the pole dynamics, one can see (Deconinck, Kimura & Segur, 2007), and for classes of rational type solutions of the Burgers hierarchy, one can see (Kudryashov & Sinelshchikov, 2009). Analysis of dynamics of complex pole singularities and spatial analyticity properties of the solution to Burgers equation was addressed in (Senuof, 2007).

Here, we would like to notice that almost all research about IBVP's for BE, was done for the standard constant coefficient cases. In the present thesis, we consider Burgers equation both with forcing term and with time-variable coefficients. For a recent discussion on the presence of forcing terms in Burgers equation one can see (Rudenko & Hedberg, 2018). On the other side, recently there is an increasing interest in evolution equations with time variable coefficients, since they are able to reflect the varying inhomogeneities of media and non-uniformities of boundaries, as investigated in (Sophocleous, 2004), (Xu & et all, 2007).

Motivated by the above discussions, the question about exact solutions of IBVP's for forced Burgers equations with time-variable coefficients on the half-line appears naturally. Precisely, in (A. Büyükaşık & Bozacı, 2019), we studied an IBVP for Burgers equation (1.2) when $a(t) = f(t) = 0$ and $b(t) = 0$, on the half-line $0 < x < \infty$, with initial condition $U(x, t_0) = F(x)$ and Dirichlet boundary condition $U(0, t) = D(t)$, $t_0 < t < T$. We provide a solution method for the Dirichlet IBVP on the infinite half-line, introduce exactly solvable models, and construct exact solutions. For this, first, we show that the Burgers IBVP can be transformed into a linear heat problem with Robin BC at $x = 0$. Then, we obtain the analytical solution of the Burgers IBVP in terms of two independent solutions to a second-order ordinary differential equation (ODE) and a second-kind Volterra-type integral equation with a weakly singular kernel. Both ODE and the Volterra integral equation are linear, but with time-dependent coefficients, and due to this, they rarely admit exact solutions. However, exact solutions are always of considerable interest and as an application of the general results derived here, next we introduce Burgers-type model with specific damping, diffusion, and forcing coefficients and construct special classes of exact solutions. Burgers solutions satisfying smooth IC and homogeneous Dirichlet BC are derived from solutions of the associated heat problem with Neumann BC, and they are smooth on the domain $x > 0$, $t > t_0 > 0$. More interesting types of exact Burgers solutions are constructed by imposing special initial and boundary conditions. These solutions have moving singularity on the real domain, due to pole-type singularity in the initial profile. We investigate how time-dependent coefficients affect the propagation of the initial singularities, how their time-evolution is related to the given initial and boundary data, and then illustrate their dynamics explicitly.

On the other side, there are problems with moving boundaries for the standard Burgers equation. For example, in (De Lillo, 1998), the author has considered initial/boundary value problem for the Burgers equation on the semi-infinite interval $x \in [s(t), \infty)$ characterized by two sets of initial and boundary data. By introducing the generalized Hopf-Cole transformation, the problems have reduced to solving linear integral equation of Volterra second type and closed form solution has been obtained. In (Biondini & De Lillo, 2001), a method for solving Dirichlet problem for Burgers equation with moving boundary has been introduced by reducing the problem to a linear integral equation of Volterra type with mildly singular kernel. Two explicit cases such as a boundary moving with constant velocity and a rapidly oscillating boundary have been considered. Moreover, Burgers-Stefan problem in the semi-infinite domain $x \in (-\infty, s(t))$, where the moving boundary is unknown, has been studied in (Ablowitz & De Lillo, 2000). However, in these works, just the formalization of solutions have been found. Since the subject with given moving boundary is less studied for standard Burgers equation, to the best of our knowledge, the variable coefficient generalized Burgers type equation defined on time-dependent domains have been almost never investigated.

In Chapter 6, we study an initial-boundary value problem for one dimensional generalized Burgers type equation of the form (1.2) with moving boundary (mIBVP) on a time-dependent domain $s(t) < x < \infty$, $0 < t < T$ and with Dirichlet BC imposed at $x = s(t)$. Following the same procedure as in the linear problem, we transform the moving domain into the fixed one, but with the more complicated Burgers equation as expected. Then, motivated from previous works, we obtain analytical solution in terms of standard

Burgers model and correspondingly heat model provided that the boundary propagates according to an associated classical equation of motion determined by the time-dependent parameters of the Burgers equation. But, the difficulties in this mIBVP is that while the corresponding standard Burgers IBVP has Dirichlet BC on half line, the heat IBVP has variable coefficient Robin type BC which requires solving Volterra integral equation of second-kind. As known from previous discussion, solving the second-kind Volterra integral is quite intractable. However, for some specific initial and Dirichlet boundary condition imposed at $x = s(t),\ 0 < t < T$, we are able present exactly solvable models. First, we study the model for standard Burgers equation with oscillatory time-dependent forcing term. For some special choices of initial and boundary conditions, we discuss the behavior of the solution and the motion of the boundary. then, we examine unforced Burgers model with space and time-dependent convection term. For rational type singular initial data and homogeneous boundary condition, we analyze the influence of parameters which creates moving singularities in the solution.

Chapter 7 includes brief discussion and concluding remarks.

CHAPTER 2

PRELIMINARIES

This chapter briefly reviews of some properties and main results for the linear and Burgers equation.

2.1. Heat equation in one dimension

The one-dimensional standard heat equation (HE),

$$\Psi_t = \nu \Psi_{xx},$$

is the simplest linear partial differential equation with constant diffusion coefficient $\nu > 0$. Here x and t are space and time variables respectively, subscripts denotes partial derivatives with respect to x and t.

The heat equation is also known as the diffusion equation and it describes one-dimensional unsteady state thermal processes in motionless medium or solids with constant thermal diffusivity.

In this section we recall the invariance properties, the well-known solutions of the heat equation and the main results for the solution of the IVP on whole line and IBVP's with Dirichlet, Neumann and Robin BC's defined on the half-line.

Invariance properties of the heat equation

It is known that the heat equation admits some invariant transformations as follows

(a) *Translations :* If $\Psi(x,t)$ is a solution of heat equation, then so is $\Psi(x-x_0, t-t_0)$, where x_0 and t_0 are translation parameters in space and time respectively.

(b) *Scaling :* If $\Psi(x,t)$ is a solution, then $\Psi(\alpha x, \alpha^2 t)$ is also solution where α and α^2 are space and time scalied parameters for any constant α.

(c) *Differentiation :* If $\Psi(x,t)$ is a solution, then so are $\Psi_x, \Psi_t, \Psi_{xx}$ and so on.

(d) *Linear combinations :* If $\Psi_1(x,t), \Psi_2(x,t), \Psi_3(x,t), ..., \Psi_n(x,t)$ are solutions, then so is $c_1\Psi_1(x,t) + c_2\Psi_2(x,t) + c_3\Psi_3(x,t) + ... + c_n\Psi_n(x,t)$ for any constants $c_1, ..., c_n$.

These symmetries are used to generate new solutions from the given one.

2.1.1. Solutions of the heat equation

Here, we list several known solutions for the heat equation, which are used in Chapter 4, Chapter 5 and Chapter 6.

1) Similarity solutions

If $\Psi(x,t)$ is a solution of the heat equation, then we look for a one-parameter transformation of variables x, t and Ψ under which the heat equation becomes invariant. Particularly, we denote

$$\Psi(x,t) = \left(\frac{1}{\sqrt{2t}}\right)^{-c} \tilde{\Psi}\left(\frac{x}{\sqrt{2t}}, \frac{1}{2}\right), \qquad c \in \mathbb{R}, \tag{2.1}$$

where the right hand side depends on single variable. For the new similarity variable $z = x/\sqrt{2t}$, which is a dimensionless parameter, let us define $f(z) = \tilde{\Psi}(\frac{x}{\sqrt{2t}}, \frac{1}{2})$, where $f(z)$ is an unknown function of z to be determined. Then substituting $\Psi(x,t) = \left(1/\sqrt{2t}\right)^{-c} f(z)$ into heat equation, we obtain second order ODE $f'' + 2zf' - 2cf = 0$ for $\nu = 1/2$. For the special case $c = n$ for $n \in \mathbb{N}$, this equation becomes $f'' + 2zf' - 2nf = 0,\ -\infty < z < \infty$, which has solutions as follows

$$\tilde{h}_n^-(z) = \int_0^\infty e^{-(z-y)^2} y^n dy, \quad -\infty < z < \infty, \tag{2.2}$$

$$\tilde{h}_n^+(z) = \int_0^\infty e^{-(z+y)^2} y^n dy, \quad -\infty < z < \infty, \tag{2.3}$$

$$H_n^k(z) = \int_{-\infty}^\infty e^{-(z-y)^2} y^n dy. \quad -\infty < z < \infty. \tag{2.4}$$

Using similarity variable $z = x/\sqrt{2t}$ and by changing variable $\sqrt{2t}y \rightarrow \xi$, we have

$$\tilde{h}_n^-\left(\frac{x}{\sqrt{2t}}\right) = \int_0^\infty \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2t}}\left(\frac{\xi}{\sqrt{2t}}\right)^n d\xi, \tag{2.5}$$

$$\tilde{h}_n^+\left(\frac{x}{\sqrt{2t}}\right) = \int_0^\infty \frac{e^{-\frac{(x+\xi)^2}{2t}}}{\sqrt{2t}}\left(\frac{\xi}{\sqrt{2t}}\right)^n d\xi, \tag{2.6}$$

$$H_n^k\left(\frac{x}{\sqrt{2t}}\right) = \int_{-\infty}^\infty \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2t}}\left(\frac{\xi}{\sqrt{2t}}\right)^n d\xi, \tag{2.7}$$

and define, (Widder, 1975)

$$h_n^-(x,t) = \frac{1}{\sqrt{\pi}}(\sqrt{2t})^n \tilde{h}_n^-\left(\frac{x}{\sqrt{2t}}\right) = \int_0^\infty \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}}\xi^n d\xi, \tag{2.8}$$

$$h_n^+(x,t) = \frac{1}{\sqrt{\pi}}(\sqrt{2t})^n \tilde{h}_n^+\left(\frac{x}{\sqrt{2t}}\right) = \int_0^\infty \frac{e^{-\frac{(x+\xi)^2}{2t}}}{\sqrt{2\pi t}}\xi^n d\xi, \tag{2.9}$$

$$H_n^k(x,t/2) = \frac{1}{\sqrt{\pi}}(\sqrt{2t})^n H_n^k\left(\frac{x}{\sqrt{2t}}\right) = \int_{-\infty}^\infty \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}}\xi^n d\xi, \tag{2.10}$$

where function (2.10) is Kampé de Fériet polynomials (KFP) or heat polynomials, defined by

$$H_n^k(x,t/2) = n!\sum_{m=0}^{[n/2]} \frac{(t/2)^m}{m!(n-2m)!}x^{n-2m}. \tag{2.11}$$

The first few KFP in explicit form are $H_0(x,t) = 1$, $H_1(x,t) = x$, $H_2(x,t) = x^2 + t$, $H_3(x,t) = x^3 + 3xt$, $H_4(x,t) = x^4 + 6x^2t + 3t^2$. Clearly, these functions are similarity solutions which satisfy (2.1). In function (2.9), replacing $\xi \to -\xi$, we get

$$h_n^+(x,t) = (-1)^n \int_{-\infty}^0 \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}}\xi^n d\xi.$$

Then it's easily seen that for even n, i.e $n = 2p$ for $p = 0, 1, 2, \cdots$, we have even KFP in terms of h_n^+ and h_n^-,

$$H_{2p}^k(x,t) = h_{2p}^-(x,t) + h_{2p}^+(x,t), \quad p = 0, 1, 2, \ldots,$$

and for odd n, say $n = 2p + 1$, we have odd KFP

$$H^k_{2p+1}(x,t) \;=\; h^-_{2p+1}(x,t) - h^+_{2p+1}(x,t), \quad p = 0, 1, 2, \ldots.$$

For fixed $t > 0$ and at $x = 0$, we have

$$\begin{aligned}
h^-_n(0,t) &= h^+_n(0,t) = \int_0^\infty \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} \xi^n d\xi = \frac{2^{\frac{n-1}{2}} t^{\frac{n+1}{2}} \Gamma[\frac{n+1}{2}]}{\sqrt{2\pi t}}, \\
H^k_{2p}(0,t) &= h^-_{2p}(0,t) + h^+_{2p}(0,t) = 2\int_0^\infty \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} \xi^{2p} d\xi, \\
H^k_{2p+1}(0,t) &= h^-_{2p+1}(0,t) - h^+_{2p+1}(0,t) = 0.
\end{aligned}$$

For fixed $x \in (-\infty, \infty)$ and as $t \to 0$, $h^-_n(x,0) = x^n$, $h^+_n(x,0) = 0$, $H^k_n(x,0) = x^n$.

Also, we can write Kampe de Feriet polynomials in terms of Hermite polynomials

$$H_m(x,t) = \frac{t^{m/2}}{(2i)^m} h_m\Big(\frac{ix}{\sqrt{2t}}\Big), \qquad m = 1, 2, \ldots, \tag{2.12}$$

where $h_m(\xi)$ are Hermite polynomials defined by

$$\exp\Big[2\xi s - s^2\Big] = \sum_{m=0}^{\infty} (s^m/m!) h_m(\xi).$$

Thus, the points where KFP vanish can be found in terms of the zeros of the Hermite polynomials. Let $\xi^{(l)}_m \in \mathbb{R}$, $l = 1, 2, ..., m$, denote the zeros of Hermite polynomial $h_m(\xi)$, so that for each fixed m, one has $h_m(\xi^{(l)}_m) = 0$, for all $l = 1, 2, ..., m$. From relation (2.12) it follows that

$$H_m(x,t) = 0 \iff x = -i\xi^{(l)}_m \sqrt{2t}, \qquad l = 1, 2, \ldots, m. \tag{2.13}$$

For $\xi^{(l)}_m = 0$, we have zeros in real domain.

2) Exponential solutions

The standard heat equation has family of exponential type solutions in the form

$$\Psi(x,t) = e^{\alpha x + \alpha^2 t}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.14}$$

for real or complex constant α. The following functions

$$e^{-\alpha^2 t}\cos(\alpha x), \quad e^{-\alpha^2 t}\sin(\alpha x), \quad e^{\alpha^2 t}\cosh(\alpha x), \quad e^{\alpha^2 t}\sinh(\alpha x), \tag{2.15}$$

are also solutions to the heat equation.

In particular, the standard heat equation has solutions

$$\Psi_i(x,t) = \exp[p_i(x,t)], \qquad p_i(x,t) = -2\alpha_2^{(i)}(x - (\alpha_1^{(i)} + \alpha_2^{(i)} t)), \qquad i = 1,2,\ldots,k, \tag{2.16}$$

for arbitrary real constants $\alpha_1^{(i)}$ and $\alpha_2^{(i)}$, $i = 1,2,\ldots,k$. The linear superposition of solutions (2.16) given as

$$\Psi_{\alpha(k)}(x,t) = \exp[p_1(x,t)] + \exp[p_2(x,t)] + ... + \exp[p_k(x,t)], \tag{2.17}$$

for the index $\alpha(k) = (\alpha^{(1)}, \alpha^{(2)}, ..., \alpha^{(k)})$, where $\alpha^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)}) \in \mathbb{R}^2$ for each $i = 1,2,...,k$, is also a solution for the heat equation.

3) Error and complementary error functions

Heat equation has also solutions

$$1 + \mathrm{Erf}\left(\frac{x}{\sqrt{4\nu t}}\right), \qquad \mathrm{Erfc}\left(\frac{x}{\sqrt{4\nu t}}\right), \tag{2.18}$$

where Erf(x) is the error function and Erfc(x) is the complementary error function defined respectively by

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du, \qquad \mathrm{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du, \tag{2.19}$$

with $\mathrm{Erf}(0) = 0,\ \mathrm{Erf}(\infty) = 1,\ \mathrm{Erfc}(0) = 1,\ \mathrm{Erfc}(\infty) = 0.$

Using invariance properties one can enlarge the class of solutions to the heat equation.

2.1.2. Initial and boundary value problems for the heat equation

In this section, we recall solutions to the IVP on whole real line and IBVP's defined on the half-line with Dirichlet, Neumann and Robin type boundary conditions for the heat equation, which are used during this thesis.

1) An Initial Value Problem

An initial value problem on whole real line $-\infty < x < \infty$ for the heat equation is defined

$$\begin{cases} \Psi_t = \frac{1}{2}\Psi_{xx}, & -\infty < x < \infty, \quad t > 0, \\ \Psi(x,0) = f(x), & -\infty < x < \infty, \end{cases} \tag{2.20}$$

where $f(x)$ is an arbitrary given smooth and bounded initial data.

Definition. The function $G(x, \xi, t)$ is a fundamental solution for IVP (2.20) if it satisfies the IVP

$$G_t = \frac{1}{2} G_{xx}, \qquad -\infty < x < \infty, \quad t > 0, \tag{2.21a}$$

$$G(x, \xi, 0) = \delta(x - \xi), \quad -\infty < x < \infty, \tag{2.21b}$$

where $\delta(x - \xi)$ is the Dirac-delta distribution centered at $x = \xi$. Applying the Fourier transform, the fundamental solution is obtained as

$$G(x, \xi, t) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(x-\xi)^2}{2t}\right], \tag{2.22}$$

which is also known as the source solution, Green's function or the heat kernel.

It is not difficult to verify the properties:

a. $G(x, \xi, t)$ satisfies the heat equation

b. For $x, \xi \in \mathbb{R}$

$$\lim_{t\to 0} G(x, \xi, t) = \delta(x - \xi).$$

During this thesis we denote the heat kernel as $G(x, \xi, t) \equiv K(x, \xi, t)$. Therefore, if the fundamental solution is known, then solution of IVP (2.20) for arbitrary initial data $f(x)$ is formally found as

$$\Psi(x, t) = \int_{-\infty}^{\infty} K(x, \xi, t) f(\xi) d\xi, \tag{2.23}$$

where $K(x, \xi, t)$ is given in (2.22).

Properties of heat kernel

1. $K(x, t) > 0, \quad t > 0.$

2. $\lim_{t\to 0} K(x, t) = 0, \quad x \neq 0.$

3. $\lim_{t\to 0^+} K(0, t) = \infty.$

4. $\int_{-\infty}^{\infty} K(x, t) dx = 1.$

2) Initial-Boundary Value Problems on the Half-line

(a) Dirichlet IBVP

The IBVP defined on the half-line with initial condition at time $t = 0$ and the Dirichlet boundary condition at $x = 0$ is defined by

$$\begin{cases} \Psi_t = \frac{1}{2}\Psi_{xx}, & 0 < x < \infty, \quad t > 0, \\ \Psi(x,0) = f(x), & 0 < x < \infty, \\ \Psi(0,t) = g(t), & t > 0, \end{cases} \tag{2.24}$$

where $f(x)$ and $g(t)$ are given sufficiently smooth functions of x and t respectively. Here one assumes $\Psi(x,t) \to 0$ and $\Psi_x(x,t) \to 0$ as $x \to \infty$. Then, the IBVP (2.24) has solution representation in integral form, (Widder, 1975), (Cannon, 1984), given as

$$\Psi(x,t) = \int_0^\infty \left(\frac{e^{-\frac{(x-\xi)^2}{2t}} - e^{-\frac{(x+\xi)^2}{2t}}}{\sqrt{2\pi t}} \right) f(\xi) d\xi + \int_0^t \left(\frac{x}{t-t'} \right) \frac{e^{-\frac{x^2}{2(t-t')}}}{\sqrt{2\pi(t-t')}} g(t') dt'. \tag{2.25}$$

<u>*Fundamental Solution*</u> : In particular, when we take the initial condition $f(x) = \delta(x - x_0)$, $x_0 > 0$, and homogeneous boundary data $g(t) = 0$,

$$G_t = \frac{1}{2} G_{xx}, \qquad 0 < x < \infty, \quad t > 0, \tag{2.26a}$$

$$G(x, x_0, 0) = \delta(x - x_0), \quad x,\ x_0 \in (0, \infty), \tag{2.26b}$$

$$G(0, x_0, t) = 0, \quad t > 0, \tag{2.26c}$$

then we obtain the fundamental solution (or Green's function) of the IBVP (2.24) as

$$G_D(x, x_0; t) = \frac{1}{\sqrt{2\pi t}} \left(\exp\left[-\frac{(x-x_0)^2}{2t} \right] - \exp\left[-\frac{(x+x_0)^2}{2t} \right] \right), \tag{2.27}$$

where $G_D(x, x_0, t) = K(x - x_0, t) - K(x + x_0, t)$ denotes the Dirichlet heat kernel.

One can easily verify the following properties:

a. $G_D(x, x_0, t)$ satisfies the heat equation

b. For $x > 0$, $x_0 > 0$ we have $\lim_{t\to 0} G_D(x, x_0, t) = \delta(x - x_0)$.

c. $G_D(x, x_0, t)$ satisfies the Dirichlet BC, i.e.

$$G_D(0, x_0, t) = K(-x_0, t) - K(+x_0, t) = 0.$$

Then, the solution (2.25) can be written in closed form

$$\Psi(x,t) = \int_0^\infty G_D(x,\xi,t) f(\xi) d\xi - \int_0^t \partial_x K(x, t-t') g(t') dt', \qquad (2.28)$$

where $\partial_x K(x,t)$ is the partial derivative of heat kernel with respect to x.

(b) Neumann IBVP

The IBVP with Neumann boundary condition

$$\begin{cases} \Psi_t = \frac{1}{2}\Psi_{xx}, & 0 < x < \infty, \quad t > 0, \\ \Psi(x,0) = f(x), & 0 < x < \infty, \\ \Psi_x(0,t) = h(t), & t > 0, \end{cases} \qquad (2.29)$$

where $f(x)$ and $h(t)$ are given functions, has solution, (Widder, 1975), (Cannon, 1984),

$$\Psi(x,t) = \int_0^\infty \left(\frac{e^{-\frac{(x-\xi)^2}{2t}} + e^{-\frac{(x+\xi)^2}{2t}}}{\sqrt{2\pi t}} \right) f(\xi) d\xi - \int_0^t \frac{e^{-\frac{x^2}{2(t-t')}}}{\sqrt{2\pi(t-t')}} h(t') dt'. \qquad (2.30)$$

<u>*Fundamental Solution*</u> : When $f(x) = \delta(x - x_0)$, $0 < x < \infty$, $x_0 > 0$, and $h(t) = 0$, the fundamental solution to the IBVP (2.29) is found as

$$G_N(x, x_0; t) = \frac{1}{\sqrt{2\pi t}} \left(\exp\left[-\frac{(x-x_0)^2}{2t} \right] + \exp\left[-\frac{(x+x_0)^2}{2t} \right] \right), \qquad (2.31)$$

where $G_N(x,\xi,t) = K(x+\xi,t) + K(x-\xi,t)$ denotes Neumann heat kernel. Similarly, it can be satisfied the properties of the Neumann heat kernel. Therefore, the solution (2.25) can be also written in closed form

$$\Psi(x,t) = \int_0^\infty G_N(x,\xi,t)f(\xi)d\xi - \int_0^t K(x,t-t')h(t')dt'. \tag{2.32}$$

(c) Robin type IBVP

The solution to the IBVP with homogeneous Robin BC on the half-line

$$\begin{cases} \Psi_t = \frac{1}{2}\Psi_{xx}, \quad 0 < x < \infty, \quad t > 0, \\ \Psi(x,0) = f(x), \quad 0 < x < \infty, \\ \Psi_x(0,t) + \beta(t)\Psi(0,t) = 0, \quad t > 0, \end{cases} \tag{2.33}$$

where $f(x)$ and $\beta(t)$ are given functions with $\beta(t) \neq 0$, can be found as follows, (Cannon, 1984):

Assume temporary we know $\Psi_x(0,t) = h(t)$ in order to have IBVP with Neumann boundary condition (2.29). Then using solution (2.30), we obtain $\Psi(0,t)$ as follows

$$\Psi(0,t) = 2\int_0^\infty \Big(\frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}}\Big)f(\xi)d\xi - \int_0^t \frac{h(t')}{\sqrt{2\pi(t-t')}}dt', \tag{2.34}$$

and substituting $\Psi(0,t)$ and $\Psi_x(0,t) = h(t)$ into Robin BC in (2.33), we get

$$h(t) + \beta(t)\Big(2\int_0^\infty \Big(\frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}}\Big)f(\xi)d\xi - \int_0^t \frac{h(t')}{\sqrt{2\pi(t-t')}}dt'\Big) = 0, \tag{2.35}$$

which is a second-kind Volterra type integral equation for the unknown function $h(t)$. It is also called as second-kind Volterra type integral equation with singular kernel $K(t,t') = 1/\sqrt{t-t'}$. If we can solve it explicitly, we will fix $\Psi_x(0,t) = h(t)$, so that the solution to the heat IBVP (2.33) can be found. Notice that, while the first integral in (2.35) is known for the given smooth initial function $f(x)$. the second integral in (2.35) is unknown. So, in what follows we need to recall the Volterra integral equation.

2.1.3. Volterra integral equation

There exists two types of linear Volterra integral equations :

First-kind Volterra integral equation :
The linear Volterra integral equations of the first-kind have the form

$$u(x) = \int_a^x K(x,t)y(t)dt, \tag{2.36}$$

where $y(x)$ is the unknown function for $a \leq x \leq b$, $K(x,t)$ is the kernel of the integral equation, and $u(x)$ is a given function. The functions $y(x)$ and $u(x)$ are usually assumed to be continuous or square integrable on $[a,b]$. The kernel $K(x,t)$ is usually assumed either to be continuous on the square $S = a \leq x \leq b,\ a \leq t \leq b$ or to satisfy the condition

$$\int_a^b \int_a^b K^2(x,t)dxdt = B^2 < \infty, \tag{2.37}$$

where B is a constant.

Second-kind Volterra integral equation :
The second-kind Volterra integral equations have the form

$$u(x) = y(x) - \int_a^x K(x,t)y(t)dt, \tag{2.38}$$

where $y(x)$ is the unknown function, $K(x,t)$ is the kernel of the integral equation, and $u(x)$ is a given function. The function classes to which $y(x)$, $u(x)$ and $K(x,t)$ belong are mentioned above.

Depending on the kernel, one can have special forms of the Volterra integral equations as follows:

The integral equation

$$f(t) = \int_0^t \frac{1}{\sqrt{t-\tau}} u(\tau) d\tau, \tag{2.39}$$

where $f(t)$ is given and $u(t)$ is unknown as a special form of first-kind Volterra integral equation with weakly singular kernel $K(t,\tau) = 1/\sqrt{t-\tau}$, where $K(t,\tau) \to \infty$ as $\tau \to t$. This equation can be solved by applying the Laplace transform and then by inverse Laplace transform so that we have

$$u(t) = \frac{1}{\pi} \frac{d}{dt} \int_0^t \frac{f(\tau)}{\sqrt{t-\tau}} d\tau. \tag{2.40}$$

Clearly, the formula (2.40) will be used for solving special form of first-kind Volterra integral equation (2.39). It's known that for some special functions $f(t)$, the solution (2.40) can be obtained explicitly as follows:

Case (i) : For $f(t) = t^{n+1/2}$, n is a positive integer, we have solutions for $u(t)$ as follows

$$\begin{aligned} &n = 1, \quad f(t) = \tfrac{4}{3} t^{3/2} \ \Rightarrow \ u(t) = t, \\ &n = 2, \quad f(t) = \tfrac{16}{15} t^{5/2} \ \Rightarrow \ u(t) = t^2, \\ &n = 3, \quad f(t) = \tfrac{32}{35} t^{7/2} \ \Rightarrow \ u(t) = t^3, \\ &\vdots \end{aligned} \tag{2.41}$$

In general, $n = 1, 2, 3, \ldots,$

$$f(t) = \frac{2^{n+1}\Gamma(n+1)}{1.3.5\cdots(2n+1)} t^{n+1/2} \ \Rightarrow \ u(t) = t^n. \tag{2.42}$$

Case (ii) : For $f(t) = t^n$, n is a positive integer, we have solutions for $u(t)$ as follows

$$
\begin{aligned}
&n = 1, \quad f(t) = \tfrac{1}{2}\pi t \;\;\Rightarrow\;\; u(t) = t^{1/2},\\
&n = 2, \quad f(t) = \tfrac{3}{8}\pi t^2 \;\;\Rightarrow\;\; u(t) = t^{3/2},\\
&n = 3, \quad f(t) = \tfrac{5}{16}\pi t^3 \;\;\Rightarrow\;\; u(t) = t^{5/2},\\
&\vdots
\end{aligned}
\tag{2.43}
$$

and in general, we have

$$
f(t) = \frac{\Gamma(n+1/2)}{\Gamma(n+1)}\sqrt{\pi}t^n \;\;\Rightarrow\;\; u(t) = t^{n-1/2}, \qquad n = 1, 2, 3, ... \tag{2.44}
$$

On the other side, a special form of Volterra integral equations of the second-kind with kernel $K(t, t') = 1/\sqrt{t - t'}$ are given by

$$
Q(t) = F(t) + \int_0^t \frac{\beta}{\sqrt{t - t'}} Q(t')dt', \qquad t \in [0, T], \tag{2.45}
$$

where $F(t)$ is known and $Q(t)$ is unknown with constant β. To solve this type of integral equation, one can use again Laplace transform or method of successive approximations.

<u>*Special case*</u> : If $\beta(t) = -k,\; k \in \mathbb{R}$, then the Robin IBVP has solution, (Carslaw & Jaeger, 1959),

$$
\Psi(x, t) = \int_0^\infty G(x, \xi, t) f(\xi) d\xi - \nu \int_0^t G(x, 0, t - t') h(t') dt', \tag{2.46}
$$

where

$$
G(x, \xi, t) = \frac{1}{\sqrt{4\pi\nu t}}\left(\exp\left[-\frac{(x-\xi)^2}{2t}\right] + \exp\left[-\frac{(x+\xi)^2}{2t}\right] - 2k\int_0^\infty \exp\left[-\frac{(x+\xi+y)^2}{4\nu t} - ky\right] dy\right).
$$

The improper integral may be calculated by the formula, (Carslaw & Jaeger, 1959),

$$\int_0^\infty \exp\left[-\frac{(x+\xi+y)^2}{4\nu t} - ky\right] dy = \sqrt{\pi\nu t}\exp\left[\nu k^2 t + k(x+\xi)\right] \mathrm{Erfc}\left[\frac{x+\xi}{\sqrt{4\nu t}} + k\sqrt{\nu t}\right].$$

In particular, (Carslaw & Jaeger, 1959), when initial condition is taken as $f(x) = c_0,\ c_0 \in \mathbb{R}$ and $\beta(t) = h,\ h \in \mathbb{R}$, the exact solution to the Robin IBVP (2.33) is given as,

$$\Psi(x,t) = c_0\left(\mathrm{Erf}\left[\frac{x}{\sqrt{2t}}\right] + e^{-hx}e^{h^2t/2}\mathrm{Erfc}\left[\frac{x-ht}{\sqrt{2t}}\right]\right). \tag{2.47}$$

2.2. Linear transport phenomena

The linear transport equation, or advection equation, with constant speed $c \in \mathbb{R}$ is $u_t + cu_x = 0$, where u is a function of two variables (x,t), $x \in \mathbb{R},\ t > 0$ and subscripts denote partial derivatives.

This equation is seen in fields of chemical processes, physics, biology and engineering. It describes, for instance, the propagation of a wave without changing of shape with speed c or models the concentration of substance flowing in a fluid at a constant rate c, or is used as a transport of a scalar field such as material properties or temperature.

For any initial function $f(x) \in C^1(\mathbb{R})$, the corresponding IVP

$$\begin{cases} u_t + cu_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases} \tag{2.48}$$

has implicit solution $u(x,t) = f(x-ct)$, which is called traveling wave solution. Here the sign of c characterizes the direction of propagation of wave, i.e. if $c > 0$, then the wave propagates to the positive direction of x-axis, say moving to the right through x-direction, if $c < 0$, then it moves to the negative x-direction, i.e. the propagation to the left in space.

2.2.1. Convection-diffusion-reaction equation

The modification of linear diffusion equation with additional terms corresponding to convection or reaction or both, with variable coefficients or constant coefficients, is known as linear parabolic type diffusion equations of the form

$$\Phi_t = d(x,t)\Phi_{xx} + c(x,t)\Phi_x + r(x,t)\Phi + R(x,t), \tag{2.49}$$

with coefficients of diffusion $d(x,t) > 0$, convection $c(x,t)$ and reaction $r(x,t)$ and with source term $R(x,t)$.

The convection-diffusion-reaction equation is widely employed to model phenomena in many different fields in mathematics and sciences such as diffusion processes, statistical mechanics, probability theory, financial mathematics, population genetics, quantum chaos, modeling of biological systems, diffusion of neutrons, reaction of chemical, stochastic equation for Brownian motion and many more.

For instance, it involves the change of concentration of one or more substances distributed in space under the influence of three processes:
1) *diffusion*, which refers to spreading of the contaminant from highly concentrated areas to less consantrated areas,
2) *convection* which is defined as the movement of the concentration due to the fluid transport medium,
3) *reaction* which is the process of interaction through which the concentration in the phenomena are generated or consumed.

In view of its broad applicability, it is thus desirable to obtain analytic solutions of this equation for as many systems as possible. However, just as any equation in sciences, solving the diffusion-convection-reaction equation exactly is in general intractable, except in a few simplified cases for instant by taking constant coefficients.

Constant coefficients case : One can see the transformation method for the contant coefficients of the equation (2.49) as follows :

When $d(x,t) = \nu > 0,\ c(x,t) = c,\ r(x,t) = r$ for $c, r \in \mathbb{R}$ and $R(x,t) = 0$ we have

$$\Phi_t = \nu\Phi_{xx} + c\Phi_x + r\Phi. \tag{2.50}$$

The substitution $\Phi(x,t) = e^{\beta t+\mu x}\Psi(x,t)$, where $\beta = r - c^2/(4\nu)$ and $\mu = -c/(2\nu)$, transforms the equation (2.50) to the heat equation $\Psi_t = \nu\Psi_{xx}$. Or the transformation $\Phi(x,t) = e^{ct}\Psi(y,t),\ y = x + ct$ leads to the heat equation $\Psi_t = \nu\Psi_{yy}$.

The exact solutions to the IVP and IBVP's with Dirichlet, Neumann and Robin type BC's for the equation (2.50) are given in Appendix B.

Convection-diffusion equation

When $r(x,t) = R(x,t) = 0$, we have convection-diffusion equation, where the solution $\Phi(x,t)$ describes the heat transfer in a moving medium and the velocity of motion is an arbitrary function of time and space.

The simplified linear model of nonlinear Navier-Stokes equation for fluid flow, drift-diffusion equations of semi-conductor device modeling and Black-Sholes equation from financial modeling have this form.

The convection-diffusion equation comprises also some other well-known models such as Kolmogorov equation and Fokker-Planck equation. The well-known one-dimensional Fokker Planck (FP) equation has the form

$$\Phi_t = -\left(A(x,t)\Phi\right)_x + \frac{1}{2}\left(B(x,t)\Phi\right)_{xx}, \tag{2.51}$$

where $\Phi(x,t)$ mostly represents the probability density; A and B are differentiable functions. This is the basic equation in the theory of continuous Markovian processes. The Ornstein-Uhlenbeck process, Rayleigh-type process and Klein-Kramers or Kramers equation describing the Brownian motion in a potential are of special interests of FP equation:

(*a*) *Ornstein-Uhlenbeck process*

$$\Phi_t = (kx\Phi)_x + \frac{1}{2}D\Phi_{xx}, \quad k \in \mathbb{R},\ D > 0.$$

(*b*) *Rayleigh-type process*

$$\Phi_t = \left(\left(\gamma x - \frac{\nu}{x}\right)\Phi\right)_x + \frac{1}{2}\nu\Phi_{xx}, \quad \nu > 0, \gamma \in \mathbb{R}.$$

The exact solutions to the IVP and IBVP's for convection-diffusion equation are given in Appendix B.

2.3. Nonlinear transport phenomena

The nonlinear transport equation is $u_t + c(u)u_x = 0$, where $x \in \mathbb{R}$, $t > 0$, and $c(u)u_x$ represents the nonlinear convection term with non-constant velocity $c(u)$. For the arbitrary initial function $f(x)$, the corresponding IVP

$$\begin{cases} u_t + c(u)u_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases} \tag{2.52}$$

has implicit solution

$$u(x,t) = f(x - c(u)t), \tag{2.53}$$

which can be found by method of characteristics. Here the dependence of c on u produces a gradual nonlinear distortion of the wave profile as it propagates in the medium, i.e shape of the wave changes. In the case $c(u) > 0$, higher values of u will propagate faster than lower values, see Fig.2.2b when $c(u) = u$. This leads to a wave steepening, since upstream values will advances faster than downstream values and there occurs discontinuities which is called a shock since their process of development is identical to that of shock waves in gas dynamics.

2.3.1. Inviscid Burgers equation

When we take the velocity $c(u) = u$, we get the well-known inviscid Burgers equation which is a quasilinear hyperbolic partial differential equation written as

$$u_t + uu_x = 0, \qquad x \in \mathbb{R}, \quad t > 0. \tag{2.54}$$

The corresponding IVP for this equation is

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}. \end{cases} \tag{2.55}$$

The solution to the IVP (2.55) can be found by using solution of linear transport problem (2.48) for the case $c(u) = u$. Here the speed of translation of the wave depends on u, so different parts of the wave will move with different speeds, causing it to distort as it propagates.

From method of characteristics, u is constant along the characteristics curves. So,

$$u(x, t) = f(x - ut) = f(\xi), \tag{2.56}$$

where ξ is constant and $x - ut = \xi$. Therefore $x = f(\xi)t + \xi$ are characteristics lines which has slope $f(\xi)$ for the initial data f. Let us write solution as

$$F(x, t, u) = u - f(x - ut). \tag{2.57}$$

We recall Implicit Function Theorem stated as : For continuously differentiable function $F : \mathbb{R}^3 \to \mathbb{R}$ defining a surface $F(x, t, u) = 0$, if

$$\frac{\partial F}{\partial u}(x_0, t_0, u_0) \neq 0, \tag{2.58}$$

where (x_0, t_0, u_0) is a point on F, then there exists a unique differentiable function $u = u(x,t)$ in the neighbourhood (x_0, t_0) satisfying $F(x, t, u(x,t)) = 0$, and $u_x = -F_u/F_x$, we can write $u(x,t) = f(x - u(x,t)t)$. So we have

$$u_x = \frac{f'(\xi)}{1 + f'(\xi)t}, \tag{2.59}$$

where $\xi = x - ut$. If $1 + f'(\xi)t \to 0$, then $u_x(x,t)$ approaches infinity and the solution $u(x,t)$ develops discontinuity at $1 + f'(\xi)t = 0$. Since $t > 0$, the equality $1 + f'(\xi)t = 0$ holds when $f'(\xi) < 0$ and therefore we can say that the solution $u(x,t)$ has discontinuity at $t = -1/f'(\xi)$.

Briefly, the time at which the solution first develops a discontinuity is called a critical or breaking time and is given by

$$t_b = min\{-\frac{1}{f'(\xi)} \mid f'(\xi) < 0\}. \tag{2.60}$$

In what follows, one can see the behavior of solution to the IVP 2.55 corresponding to different initial datas.

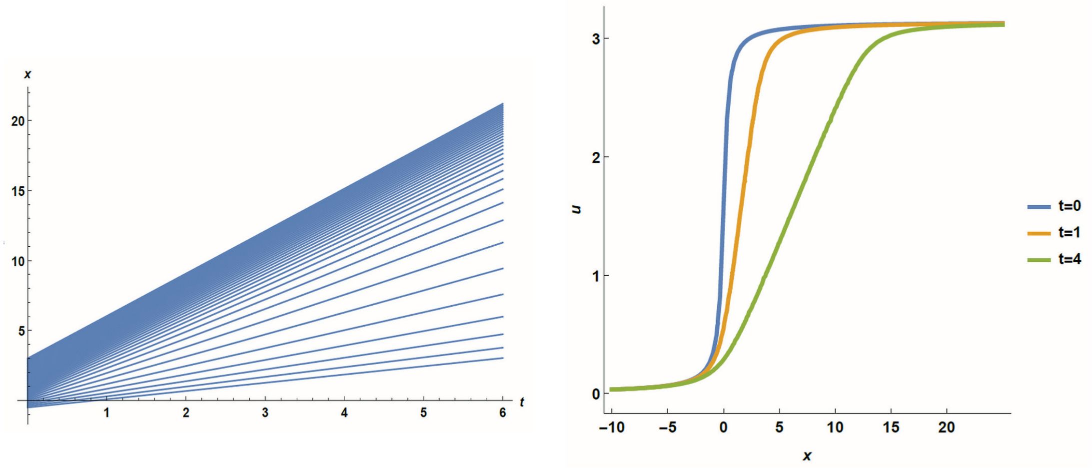
Example 2.1 ***(Rarefaction wave :)*** *Consider the following IVP*

$$\begin{cases} u_t + uu_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = \arctan(3x) + \pi/2, \quad x \in \mathbb{R}, \end{cases} \tag{2.61}$$

where the initial data is taken as $f(x) = \arctan(3x) + \pi/2$. Therefore solution becomes

$$u(x,t) = \arctan(3(x - ut)) + \frac{\pi}{2}, \tag{2.62}$$

and since $f'(\xi) = 3/(1 + 9\xi^2) > 0$, there is no breaking time, i.e the shock behavior doesn't occur. The characteristic velocities on the left are smaller than those on the right and the characteristics will diverge, see Fig.2.1a. This proper solution is a *rarefaction wave*, which is a nonlinear wave that smoothly connects the left and the right state. We illustrate the behavior of rarefaction wave (2.62) at different times in Fig.2.1b.



(a) (b)

Figure 2.1 (a) The characteristics lines for $\xi = -0.5, \xi = 3, \xi = 0.1$. (b) The behavior of solution (2.62) at $t = 0,\ t = 1,\ t = 4$.

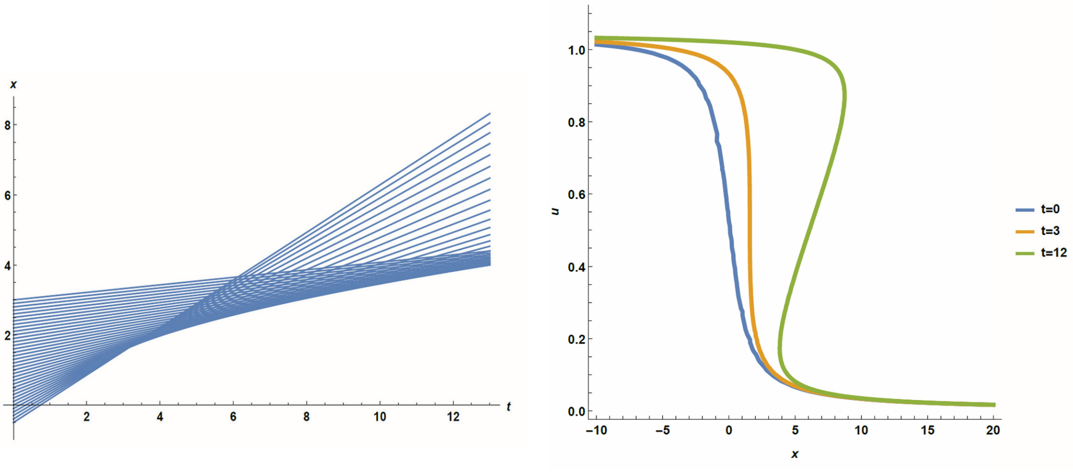
Example 2.2 ***(Shock-traveling wave):*** *The IVP*

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = \pi/6 - \arctan(x)/3, & x \in \mathbb{R}. \end{cases} \tag{2.63}$$

has solution

$$u(x,t) = \frac{\pi}{6} - \frac{\arctan(x - ut)}{3}. \tag{2.64}$$

When $f'(\xi) = -1/3(1+\xi^2) < 0$, there occurs shock behavior after breaking time at $t_b = min\{3(1+\xi^2)\} = 3$, and behavior of solution at times $t = 0,\ t = 3$ and $t = 12$ is shown in Fig.2.2b.



(a) (b)

Figure 2.2 (a) The characteristics lines for $\xi = -0.5,\ \xi = 3,\ \xi = 0.1$. (b) The behavior of solution (2.64) at times $t = 0,\ t = 3$ and $t = 12$.

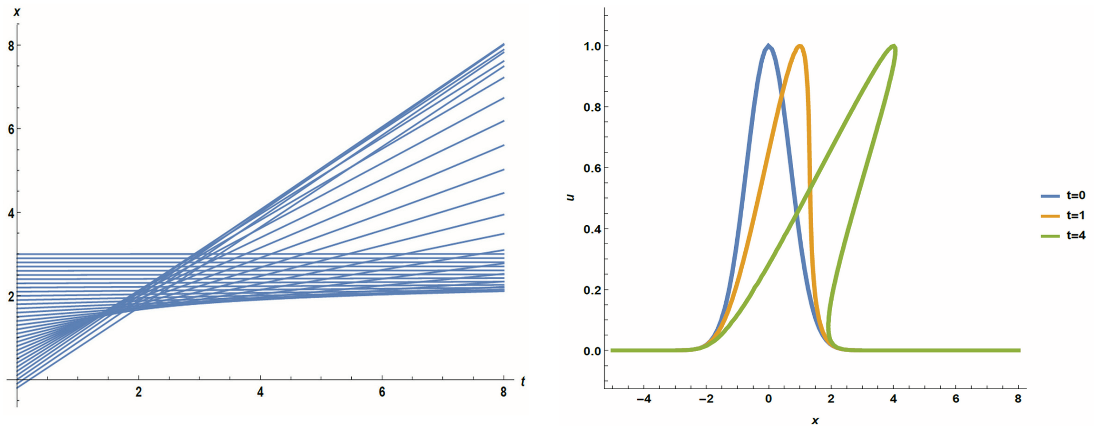
Example 2.3 *The IVP with Gaussian initial data*

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = e^{-x^2}, & x \in \mathbb{R}. \end{cases} \tag{2.65}$$

has solution

$$u(x,t) = e^{-(x-ut)^2}. \tag{2.66}$$

Since $f'(\xi) = -2\xi e^{-\xi^2}$ for $\xi > 0$, there exists breaking time at $t_b = min\{e^{\xi^2}/(2\xi)\}$, which occurs at $\xi = 1/\sqrt{2}$. The characteristics lines and the behavior of solution at times $t = 0, t = 1$ and $t = 4$ is shown in Fig.2.3a and Fig.2.3b respectively.



(a) (b)

Figure 2.3 (a) The characteristics lines for $\xi = -0.5$, $\xi = 3$, $\xi = 0.1$. (b) The behavior of solution (2.66) at times $t = 0, t = 1$ and $t = 4$.

2.3.2. Viscous Burgers equation

The viscous Burgers equation which we called the standard Burgers equation, is a well-known nonlinear partial differential equation

$$u_t + uu_x = \nu u_{xx}, \tag{2.67}$$

where $u(x, t)$ is a field, $t > 0$ is the time, x is the space variable, $\nu > 0$ is viscosity coefficient.

The equation describes time evolution of balance between nonlinear convection and diffusion (or dissipation). The convection term is responsible for the steepening and shock-formation, while the diffusive term has the smoothing effect on the shock discontinues. No matter how small, the diffusion term always prevents the formation of shocks. Hence the viscous BE with smooth initial data always leads to smooth solutions for all $t > 0$. This is in contrast with the inviscid BE ($\nu = 0$), where a smooth initial data can lead to the jump discontinuity and shocks.

Usually the viscous BE describes the transport of quantity u with velocity u. For example, in nonlinear convection-diffusion processes $u(x,t)$ may describe solute concentration. In fluid dynamics one can speak about momentum transport and $u(x,t)$ can represent the velocity field or local velocity of fluid particles. We mostly prefer the language of wave dynamics, and speak about nonlinear diffusive waves, where $u(x,t)$ describes both the wave profile and its velocity.

Here, we present some properties and well-known exact solutions for the standard Burgers equation such as single / multi -shock traveling waves, triangular-shaped wave, N-shaped wave and rational type solutions.

2.3.2.1. Symmetries of Burgers equation

(a) (Translation) If $u(x,t)$ is a solution of (2.67), then so is $u(x-x_0, t-t_0)$, where $x \to x - x_0$ and $t \to t - t_0$ are translations in space and time respectively.

(b) (Scaling) If $u(x,t)$ is a solution, then $\alpha u(\alpha x, \alpha^2 t)$ is solution, where αx, $\alpha^2 t$ and αu are scalied variables for arbitrary number α.

(c) (Galilean invariance) If $u(x,t)$ is a solution, then so is $\alpha + u(x - \alpha t, t)$.

2.3.2.2. An initial value problem on whole real line

The initial value problem for standard Burgers equation on whole real line is defined by (Bateman, 1915)

$$\begin{cases} u_t + uu_x = \nu u_{xx}, & -\infty < x < \infty, \quad 0 < t < T, \\ u(x,0) = u_0(x), & -\infty < x < \infty. \end{cases} \tag{2.68}$$

By Cole-Hopf transformation (Cole, 1951), (Hopf, 1950)

$$u(x,t) = -2\nu \frac{\Psi_x(x,t)}{\Psi(x,t)} = -2\nu(\ln \Psi)_x, \tag{2.69}$$

the Burgers equation in (2.68) is reduced to heat equation $\Psi_t = \nu \Psi_{xx}$ and the IC in (2.68) directly transforms to IC for heat equation, $\Psi(x,0) = \exp\left[-\nu \int^{x} u(x',0)dx'\right]$.

By using the integral representation of solution to IVP for heat equation (2.23), the IVP (2.68) has solution in integral representation form

$$u(x,t) = -2\nu \frac{\int_{-\infty}^{\infty} K_x(x-\xi,t) e^{-\frac{1}{2\nu}\int^{\xi} u_0(x')dx'} d\xi}{\int_{-\infty}^{\infty} K(x-\xi,t) e^{-\frac{1}{2\nu}\int^{\xi} u_0(x')dx'} d\xi},$$

where $K(x,t)$ is the heat kernel and $K_x(x,t)$ is the partial derivative of heat kernel wrt x.

Therefore, many solutions can be derived from solutions of the associated HE.

2.3.2.3. Some exact solutions

The standard Burgers equation has many physically interesting exact solutions in explicit form, such as traveling shock and multi-shock waves, diffusive waves (triangular-shaped and N-shaped waves) and rational type solutions, see (Whitham, 1999) and (A. Büyükaşık & Pashaev, 2013). Here, we recall these exact solutions used in Chapter 5.

1) Shock traveling wave solution : The standard Burgers equation corresponding to α-parametrized initial condition $u_\alpha(x,0) = \alpha_2 - A\tanh\left[A(x-\alpha_1)\right]$, for arbitrary constants $A > 0$, α_1, α_2, has shock traveling wave solution

$$u_\alpha(x,t) = \alpha_2 - A\tanh\left[A(x-(\alpha_1+\alpha_2 t))\right]. \tag{2.70}$$

The solution (2.70) is a wave of constant amplitude, moving with constant speed and without changing shape. Parameter A controls the amplitude and steepness of the shock profile, while its "center" propagates according to $x = \alpha_1 + \alpha_2 t$, with initial position α_1 and velocity α_2. Note that position of the center is described by a function of the form $x_\alpha(t) = \alpha_1 r_1(t) + \alpha_2 r_2(t)$, where $r_1(t) = 1$ and $r_2(t) = t$ are two independent solutions of $\ddot{r}(t) = 0$, satisfying the initial conditions $r_1(0) = 1$, $\dot{r}_1(0) = 0$; $r_2(0) = 0$, $\dot{r}_2(0) = 1$, (Atılgan Büyükaşık & Bozacı, 2021). Moreover, (2.70) is a wavefront type solution satisfying boundary conditions $u_\alpha(-\infty,t) = \alpha_2 + A \equiv c_2$, $u_\alpha(+\infty,t) = \alpha_2 - A \equiv c_1$, $t > 0$. Then, $\alpha_2 = (c_1+c_2)/2$ and $A = (c_2-c_1)/2$, show how the velocity α_2 and the amplitude

A are related by the maximum and minimum values of the field $u_\alpha(x,t)$. The behavior of the solution can be seen in Fig. 2.4a.

2) Multi-shock traveling wave solution : Using solution (2.16) for the standard heat equation, then standard BE (2.67) has solution of the form

$$u_{\alpha(k)}(x,t) = 2\frac{\alpha_2^{(1)}\exp[p_1(x,t)] + \alpha_2^{(2)}\exp[p_2(x,t)] + ... + \alpha_2^{(k)}\exp[p_k(x,t)]}{\exp[p_1(x,t)] + \exp[p_2(x,t)] + ... + \exp[p_k(x,t)]}, \qquad (2.71)$$

which depends on $2k-$ free parameters, and we use index notation $\alpha(k) = (\alpha^{(1)}, \alpha^{(2)}, ..., \alpha^{(k)})$, where $\alpha^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)}) \in \mathbb{R}^2$ for each $i = 1, 2, ..., k$. For $k = 2$ and certain choice of constants, one obtains one-shock wave. When $k > 2$ one expects formation of multi-shock wave solutions. In that case, it is known that, for certain values of the free parameters inelastic interactions can occur in (2.71), such that when a number of shocks are produced, shocks with higher speed can overtake the shocks with smaller speed, merge at certain time and then continue to propagate as a single shock. Such inelastic interactions in collisions or fusion processes of multiple-front waves are often addressed in literature and one can see (Whitham, 1999), (Wang & et all, 2004), (Xu & et all, 2007), (Veksler & Zarmi, 2005). In Fig.2.4b, we plot two-shock wave solution for certain parameters.

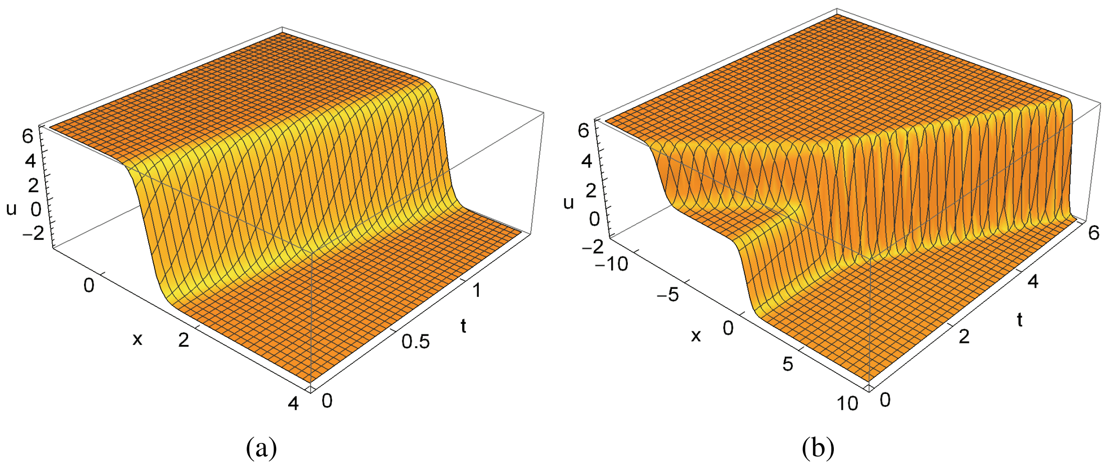


Figure 2.4 (a) Solution (2.70) with $A = 5$, $\alpha_1 = 1$, $\alpha_2 = 1$. (b) Solution (2.71) with $\alpha_1^1 = -3$, $\alpha_2^1 = -1$, $\alpha_1^2 = 3$, $\alpha_2^2 = 1.2$, $\alpha_1^3 = -3$ and $\alpha_2^3 = 3$.

3) Triangular-shaped wave solution : The Burgers equation has triangular wave solution, (Saichev, Gurbatov & Rudenko, 2011), (Whitham, 1999), (Sachdev, 1987), which is a similarity solution as follows

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} \left(\frac{(e^R - 1)\exp\left[-\frac{x^2}{2t}\right]}{1 + \frac{1}{2}(e^R - 1)\,\mathrm{Erfc}\left[\frac{x}{\sqrt{2t}}\right]} \right),$$

corresponding to the IC $\lim_{t\to 0} u(x,t) = (e^R - 1)\delta(x)$, where R is real constant, called Reynolds number, and $\mathrm{Erfc}[a] = (2/\sqrt{\pi}) \int_a^\infty \exp\left(-s^2\right) ds$ is the complimentary error function.

Using scaling invariance of Burgers equation, we can generate a family of nonlinear diffusive traveling waves for the standard BE

$$u_\alpha(x,t) = \alpha_2 + \frac{1}{\sqrt{2\pi t}} \left(\frac{(e^R - 1)\exp\left[-\frac{(x-(\alpha_1+\alpha_2 t))^2}{2t}\right]}{1 + \frac{1}{2}(e^R - 1)\,\mathrm{Erfc}\left[\frac{x-(\alpha_1+\alpha_2 t)}{\sqrt{2t}}\right]} \right), \quad (\alpha_1, \alpha_2) \in \mathbb{R}^2. \tag{2.72}$$

As time approaches zero, this solution weakly converges to shifted Dirac-delta function, that is $\lim_{t\to 0} u_\alpha(x,t) = \alpha_2 + (e^R - 1)\delta(x - \alpha_1)$. Also, we note that solutions (2.72) are wave packets localized in space with $u_\alpha(\pm\infty, t) = \alpha_2$, $t > 0$, propagating with constant speed, decreasing amplitude and spreading in time. The behavior of solution is seen in Fig.2.5a.

4) N-shaped wave solution : For the solution to the heat equation of the form

$$\Psi(x,t) = 1 + \sqrt{\frac{c}{t}} \exp\left[-\frac{x^2}{2t}\right], \tag{2.73}$$

which has delta function behavior as $t \to 0$, the corresponding N-shaped wave solution for Burgers equation is obtained by using Cole-Hopf transform as given in the form,

$$u(x,t) = \left(\frac{x}{t}\right) \left(\frac{\sqrt{\frac{c}{t}} \exp\left[-\frac{x^2}{2t}\right]}{1 + \sqrt{\frac{c}{t}} \exp\left[-\frac{x^2}{2t}\right]} \right), \tag{2.74}$$

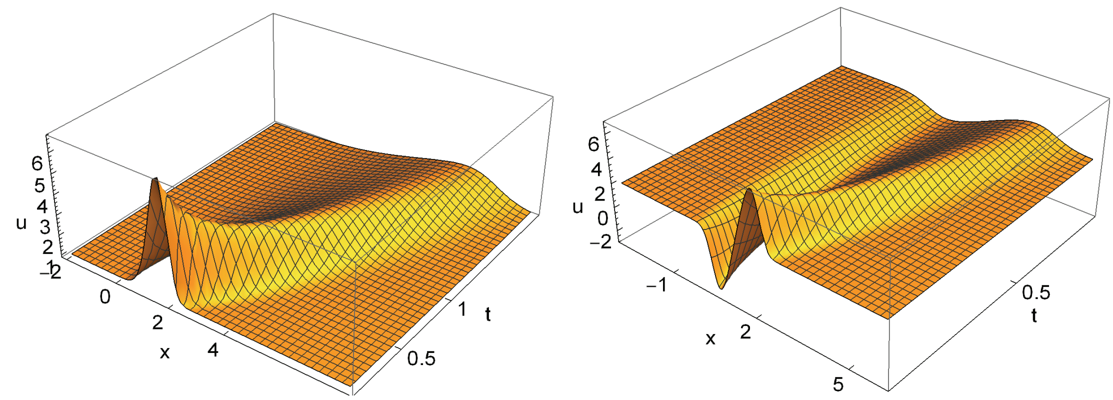
where $c > 0$. Since $\Psi(x,t) \to \delta(x)$ as $t \to 0$, it is a little hard to interpret the solution

(2.74) as an initial value problem on $u(x,t)$.

Then, after translation and Galilean transform, the family of solutions becomes of the form, (Atılgan Büyükaşık & Bozacı, 2021)

$$u_\alpha(x,t) = \alpha_2 + \left(\frac{x-(\alpha_1+\alpha_2 t)}{t}\right)\left(\frac{\sqrt{\frac{c}{t}}\exp\left[-\frac{(x-(\alpha_1+\alpha_2 t))^2}{2t}\right]}{1+\sqrt{\frac{c}{t}}\exp\left[-\frac{(x-(\alpha_1+\alpha_2 t))^2}{2t}\right]}\right), \quad (\alpha_1,\alpha_2)\in\mathbb{R}^2. \quad (2.75)$$

These solutions are diffusive waves traveling with constant speed, and localized in space with $u_\alpha(\pm\infty,t)=\alpha_2$, and shown in Fig.2.5b.



(a) (b)

Figure 2.5 (a) Solution (2.72) with $A = 5,\ \alpha_1 = 1,\ \alpha_2 = 1$. (b) Solution (2.75) with $\alpha_1 = 1, \alpha_2 = 2,\ c = 50$.

<u>5) Rational type wave solution</u> : The rational type solution is obtained by Kampe de Feriet polynomial solution for the heat equation, (A. Büyükaşık & Pashaev, 2013),

$$u_n(x,t) = -\frac{n\,H_{n-1}(x,t)}{H_n(x,t)}, \quad n = 1,2,3,...,$$

corresponding to pole type singularity in initial data $u(x,0) = -n/x$.

The $\alpha-$parameterized family of rational type solutions is, (Atılgan Büyükaşık &

Bozacı, 2021)

$$ {}_{\alpha}u_m(x,t) = \alpha_2 - m\frac{H_{m-1}(x-(\alpha_1+\alpha_2 t),t)}{H_m(x-(\alpha_1+\alpha_2 t),t)}. \tag{2.76} $$

Using the relation (2.12), we have zeros of KFP as

$$ H_m(x-(\alpha_1+\alpha_2 t),t) = 0 \Longleftrightarrow x = (\alpha_1+\alpha_2 t) - i\xi_m^{(l)}\sqrt{2t}, \qquad l = 1,2,...,m, \tag{2.77} $$

showing that in general, singularities of (2.76) appear in the complex domain. Only for odd m, say $m = 2p+1$, $p = 0,1,2,...$, Hermite polynomial $h_{2p+1}(\xi)$ has root at $\xi = 0$, that's $\xi_{2p+1}^{(0)} = 0$. Therefore, Burgers solution ${}_{\alpha}u_{2p+1}(x,t)$ has moving singularity whose position is described by $x = \alpha_1 + \alpha_2 t$.

For $m = 3$, we obtain solution explicitly

$$ {}_{\alpha}u_3(x,t) = \alpha_2 - 3\frac{(x-(\alpha_1+\alpha_2 t))^2 + t}{(x-(\alpha_1+\alpha_2 t))^3 + 3\,(x-(\alpha_1+\alpha_2 t))\,t}, \tag{2.78} $$

which has moving singularity on the line $x = 1 + t$, initially located at $x = 1$.

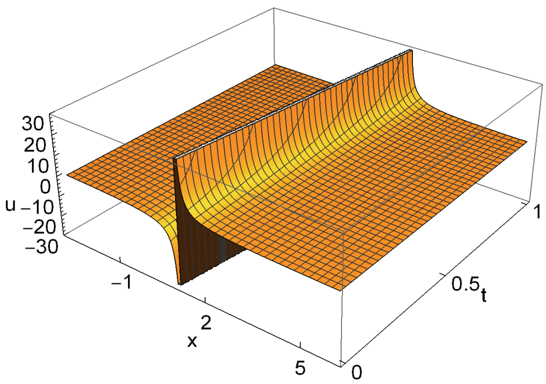


Figure 2.6 Solution (2.78) with $\alpha_1 = \alpha_2 = 1$.

Also notice that, using superposition of heat polynomials $\sum_{m=0}^{N} a_m H_m(x,t)$, where a_m's are real constants, one can write more general rational solutions of the standard BE

$$ {}_{\alpha}u_N(x,t) = \alpha_2 - \frac{\sum_{m=1}^{N} m\, a_{m-1} H_{m-1}\left(x - (\alpha_1 + \alpha_2 t), t\right)}{\sum_{m=0}^{N} a_m H_m\left(x - (\alpha_1 + \alpha_2 t), t\right)}. \tag{2.79} $$

2.4. Lie Group and Lie Algebra

Definition 2.1 *(Group)*

A group is a pair $G = \{S, \circ\}$ where S is a set and $\circ$ is an operation mapping pairs of elements in S to elements in S i.e. $\circ : S \times S \to S$ and satisfying the following conditions:

1. *Existence of an identity: $\exists\, e \in S$ such that $e \circ a = a \circ e$, $\forall a \in S$.*
2. *Existence of inverses : $\forall a \in S, \exists\, a^{-1} \in S$ such that $a \circ a^{-1} = a^{-1} \circ a = e$.*
3. *Associativity : $\forall a, b, c \in S$, $a \circ (b \circ c) = (a \circ b) \circ c = a \circ b \circ c$.*

Definition 2.2 *(Algebra)*

An algebra is a vector space $\mathcal{V}$ over a field $\mathcal{F}$ on which a multiplication $(\times)$ between vectors has been defined (yielding a vector in $\mathcal{V}$) such that for all $u, v, w \in \mathcal{V}$ and $\alpha \in \mathcal{F}$:

1. $(\alpha u) \times v = \alpha(u \times v) = u \times (\alpha v)$.
2. $(u + v) \times w = (u \times w) + (v \times w)$ *and* $w \times (u + v) = (w \times u) + (w \times v)$.

Definition 2.3 *(Lie group)*

A Lie group is a finite dimensional smooth manifold G together with a group structure on G, such that the multiplication $G \times G \to G$ and the attaching of an inverse $g \to g^{-1} : G \to G$ are smooth maps.

Definition 2.4 *(Lie algebra)*

An algebra $\mathscr{L}$ is called a Lie algebra endowed with a binary operation, the Lie bracket $[\ .\ ] : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$, that is anti-symmetry and bilinear respectively

1. $[X,Y] = -[Y,X]$,

2. $[aX + bY, Z] = a[X,Z] + b[Y,Z], \qquad [Z, aX + bY] = a[Z,X] + b[Z,Y]$,

and satisfies Jacobi identity

3. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$,

for arbitrary scalars a, b and $X, Y, Z \in \mathscr{L}$, (Wybourne, 1974).

<u>$su(1,1)$ **Lie algebra**</u> :

$su(1,1)$ Lie algebra has generators, (Wybourne, 1974)

$$\hat{K}_- = -\frac{i}{2}\frac{\partial^2}{\partial x^2}, \qquad \hat{K}_+ = \frac{i}{2}x^2, \qquad \hat{K}_0 = \frac{1}{2}(x\frac{\partial}{\partial x} + \frac{1}{2}),$$

satisfying the commutation relations

$$[\hat{K}_-, \hat{K}_+] = 2\hat{K}_0, \qquad [\hat{K}_+, \hat{K}_0] = -\hat{K}_+, \qquad [\hat{K}_-, \hat{K}_0] = \hat{K}_-,$$

which can be satisfied for an arbitrary function $\phi(x)$, as follows

$$\begin{aligned} [\hat{K}_-, \hat{K}_+]\phi(x) &= \left[-\frac{i}{2}\frac{\partial^2}{\partial x^2}, \frac{i}{2}x^2\right]\phi(x) \\ &= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2}(x^2\phi(x)) - x^2\frac{\partial^2}{\partial x^2}\phi(x)\right) = \left(\frac{1}{2} + x\frac{\partial}{\partial x}\right)\phi(x) \\ &= 2\hat{K}_0\phi(x). \end{aligned} \tag{2.80}$$

$$\begin{aligned} [\hat{K}_+, \hat{K}_0]\phi(x) &= \left[\frac{i}{2}x^2, \frac{1}{2}(x\frac{\partial}{\partial x} + \frac{1}{2})\right]\phi(x) \\ &= -\hat{K}_+\phi(x). \end{aligned} \tag{2.81}$$

By the same way, other commutation relation can be calculated. These commutation relations are used in Chapter 3.

Heisenberg-Weyl Lie algebra :

Heisenberg-Weyl Lie algebra has generators

$$\hat{E}_1 = ix, \qquad \hat{E}_2 = \frac{\partial}{\partial x}, \qquad \hat{E}_3 = i\hat{I},$$

satisfying the commutation relations.

$$[\hat{E}_1, \hat{E}_2] = -\hat{E}_3, \qquad [\hat{E}_1, \hat{E}_3] = 0, \qquad [\hat{E}_2, \hat{E}_3] = 0,$$

which can be found as follows

$$\begin{aligned} [\hat{E}_1, \hat{E}_2]\phi(x) &= \left[ix, \frac{\partial}{\partial x}\right]\phi(x) \\ &= i\left(x\frac{\partial}{\partial x}\phi(x) - \frac{\partial}{\partial x}(x\phi(x))\right) = -i\phi(x) \\ &= -\hat{E}_3\phi(x). \end{aligned} \tag{2.82}$$

Similarly the other commutation relations can be shown.

The generators $\hat{K}_-, \hat{K}_+\hat{K}_0$, and $\hat{E}_1, \hat{E}_2, \hat{E}_3$, have the following mutual commutation relations :

$$\begin{aligned} &[\hat{E}_1, \hat{K}_-] = -\hat{E}_2, \qquad [\hat{E}_1, \hat{K}_+] = 0, \qquad [\hat{E}_1, \hat{K}_0] = -\frac{1}{2}\hat{E}_1, \\ &[\hat{E}_2, \hat{K}_-] = 0, \qquad [\hat{E}_2, \hat{K}_+] = \hat{E}_1, \qquad [\hat{E}_2, \hat{K}_0] = \frac{1}{2}\hat{E}_2, \\ &[\hat{E}_3, \hat{K}_-] = 0, \qquad [\hat{E}_3, \hat{K}_+] = 0, \qquad [\hat{E}_3, \hat{K}_0] = 0. \end{aligned}$$

2.4.1. Wei-Norman Lie Algebraic Approach

Wei and Norman proposed an efficient procedure for finding evolution operator of linear equations, (Wei & Norman, 1963). Supppose $\hat{U}(t)$ and $\hat{H}(t)$ are explicitly time-dependent operators and

$$\begin{cases} \frac{d}{dt}\hat{U}(t) = \hat{H}(t)\hat{U}(t), \\ \hat{U}(t_0) = \hat{I}, \end{cases} \tag{2.83}$$

where H and U are given and unknown linear operators respectively and $\hat{I}$ is identity operator.

It is known that if the linear operator $\hat{H}(t)$ can be expressed in the form

$$\hat{H}(t) = \sum_{i=1}^{n} a_i(t)H_i \ ,$$

where H_i's, $i = 1, ..., n$ are time-independent operators of finite dimension n, then there exists a neighborhood of $t = t_0$, in which the solution of the equation (2.83) may be expressed in the form

$$\hat{U}(t) = \prod_{i=1}^{n} \text{Exp}\,[g_i(t)H_i],$$

where H_i, $i = 1, ..., n$ is a basis for $\mathscr{L}$, and the $g_i(t)$ are scalar functions of time. Moreover, the $g_i(t)$, $i = 1, ..., n$ depend only on the Lie algebra $\mathscr{L}$ and the $a_i(t)$, satisfing a nonlinear system of first-order differential equations.

Exponential operators

The actions of exponential operators on a given function $\phi(x)$ can be seen as in the followings:

Shifting operator :

$$\exp\left[\lambda\frac{\partial}{\partial x}\right]\phi(x) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\frac{\partial^n}{\partial x^n}\phi(x) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\phi^{(n)}(x) = \phi(x+\lambda), \tag{2.84}$$

for arbitrary constant parameter λ.

Dilatation operator :

$$\begin{aligned}\exp\left[\lambda x\frac{\partial}{\partial x}\right]\phi(x) &= \sum_{n=0}^{\infty}\frac{\lambda^n x^n}{n!}\frac{\partial^n}{\partial x^n}\phi(x) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}x^n\phi^{(n)}(x) \\ &= \phi(e^{\lambda}x), \end{aligned} \tag{2.85}$$

for arbitrary constant parameter λ.

Proposition 2.1 *For a given function $\phi_0(x) \in C^{\infty}$, we have*

$$\exp\left[\frac{z}{2}\frac{\partial^2}{\partial x^2}\right]\phi_0(x) = \phi(x,z),$$

where $\phi(x,z)$ satisfies the IVP for the standard heat equation

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\phi(x,z) = \frac{\partial}{\partial z}\phi(x,z), \tag{2.86a}$$

$$\phi(x,z)|_{z=0} = \phi(x,0) \equiv \phi_0(x). \tag{2.86b}$$

Proof Suppose $\phi(x,z)$ satisfies (2.86a). Then we have

$$\exp\left[\frac{\lambda}{2}\frac{\partial^2}{\partial x^2}\right]\phi(x,z) = \exp\left[\lambda\frac{\partial}{\partial z}\right]\phi(x,z).$$

It follows that

$$\begin{aligned}&\exp\left[\frac{\lambda}{2}\frac{\partial^2}{\partial x^2}\right]\phi_0(x) = \exp\left[\frac{\lambda}{2}\frac{\partial^2}{\partial x^2}\right]\phi(x,z)|_{z=0} = \exp\left[\lambda\frac{\partial}{\partial z}\right]\phi(x,z)|_{z=0} \\ &= \phi(x,z+\lambda)|_{z=0} = \phi(x,\lambda).\end{aligned}$$

□

CHAPTER 3

GENERALIZED DIFFUSION TYPE EQUATIONS WITH VARIABLE COEFFICIENTS

In this chapter, we study IVP and IBVP's for one dimensional generalized diffusion type equation of the form

$$\Phi_t = \frac{1}{2\mu(t)}\Phi_{xx} - [a(t) - b(t)x]\Phi_x + \mu(t)[\frac{\omega^2(t)}{2}x^2 - f(t)x + f_0(t)]\Phi, \quad x \in \mathbb{R}, \ t > t_0 > 0, \tag{3.1}$$

with diffusion coefficient depending on time, and convection and reaction coefficients that depend on time and space. First, we consider IVP defined on whole real line $-\infty < x < \infty$, and obtain analytical solution in terms of solutions to the characteristic ordinary differential equation and standard heat model. Second, we study IBVP's defined on the half-line $0 < x < \infty, \ t > 0$, with Dirichlet, Neumann and Robin boundary conditions. Then using our general results, we introduce exactly solvable models and investigate influence of the variable parameters.

3.1. Initial Value Problem on the Whole Real Line

In this section, we consider an IVP defined on $-\infty < x < \infty, \ t > t_0 > 0$ for the generalized diffusion type equation

$$\begin{cases} \Phi_t = \frac{1}{2\mu(t)}\Phi_{xx} - [a(t) - b(t)x]\Phi_x + \mu(t)[\frac{\omega^2(t)}{2}x^2 - f(t)x + f_0(t)]\Phi, & x \in \mathbb{R}, \ t > t_0 > 0, \\ \Phi(x, t_0) = \Phi_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{3.2}$$

where coefficients $\mu(t) > 0, \ \omega^2(t) > 0, \ a(t), \ b(t), \ f(t)$ and $f_0(t)$ are given real-valued smooth functions depending on time and initial data $\Phi_0(x)$ at time $t = t_0$ is given smooth and bounded function of x.

Proposition 3.1 *If $r_1(t)$, $r_2(t)$ are two independent homogeneous solutions and $r_p(t)$ is a particular solution of the following characteristic ODE*

$$\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \left[\omega^2(t) + \left(\dot{b}(t) + \frac{\dot{\mu}(t)}{\mu(t)}b(t) - b^2(t)\right)\right] r = \dot{a}(t) - a(t)\left[b(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right] + f(t), \quad (3.3)$$

satisfying initial conditions $r_1(t_0) = r_0 \neq 0$, $\dot{r}_1(t_0) = -b(t_0)r_0$, $r_2(t_0) = 0$, $\dot{r}_2(t_0) = r_0/\mu(t_0)$ and $r_p(t_0) = 0$, $\dot{r}_p(t_0) = a(t_0)$ respectively, then the IVP (3.2) has solution of the form

$$\begin{aligned}
\Phi(x,t) &= \sqrt{\frac{r_1(t_0)}{r_1(t)}} \times \exp\left[-p_p(t)r_p(t)\right] \times \exp\left[\int_{t_0}^{t}\left(L_p(t') - \frac{b(t')}{2}\right)dt'\right] \\
&\times \exp\left[\int_{t_0}^{t}\left(a(t')p_p(t') - \mu(t')\left(f(t')r_p(t') - f_0(t')\right)\right)dt'\right] \\
&\times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)\left(x - r_p(t)\right)^2\right] \times \exp\left[-p_p(t)\left(x - r_p(t)\right)\right] \\
&\times \Psi(\eta_p(x,t), \tau(t)), \qquad (3.4)
\end{aligned}$$

where $L_p(t)$ is a Lagrangian type function given by

$$L_p(t) = \frac{\mu(t)}{2}\left(\left(\dot{r}_p(t) + b(t)r_p(t) - a(t)\right)^2 - \omega^2(t)r_p^2(t) + 2f(t)r_p(t)\right), \quad (3.5)$$

and generalized momentum function

$$p_p(t) = \mu(t)\left(\dot{r}_p(t) + b(t)r_p(t) - a(t)\right), \quad (3.6)$$

one has also coordinate transformation $(x,t) \to (\eta, \tau)$

$$\eta_p(x,t) = \frac{r_1(t_0)}{r_1(t)}(x - r_p(t)), \qquad \tau(t) = \frac{r_2(t)}{r_1(t)}, \qquad t > t_0 > 0, \quad (3.7)$$

and $\Psi(\eta, \tau)$ *is solution of the IVP on whole real line for the standard heat equation*

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \quad -\infty < \eta < \infty, \quad \tau > 0, \\ \Psi(\eta, 0) = \Phi(\eta, t_0), \quad -\infty < \eta < \infty. \end{cases} \tag{3.8}$$

Proof To apply Wei-Norman algebraic approach, (Wei & Norman, 1963), first we write IVP (3.2) as

$$\begin{cases} \frac{\partial}{\partial t}\Phi(x, t) = \hat{T}(t)\Phi(x, t), \quad -\infty < x < \infty, \quad t > t_0, \\ \Phi(x, t_0) = \Phi_0(x), \quad -\infty < x < \infty, \end{cases} \tag{3.9}$$

where we have

$$\hat{T}(t) = \frac{1}{2\mu(t)}\frac{\partial^2}{\partial x^2} - a(t)\frac{\partial}{\partial x} + b(t)\left(x\frac{\partial}{\partial x} + \frac{1}{2}\hat{I}\right) + \mu(t)\left(\frac{\omega^2(t)}{2}x^2 - f(t)x + d(t)\right)\hat{I},$$

which is a linear second-order differential operator with variable coefficients. Here we used the brief notation $d(t) \equiv f_0(t) - b(t)/(2\mu(t))$. The operator $\hat{T}$ can be expressed as a finite linear combination of a closed Lie algebra generators, that is

$$\hat{T}(t) = i\frac{1}{\mu(t)}\hat{K}_- + 2b(t)\hat{K}_0 - a(t)\hat{E}_2 - i\mu(t)\left(\omega^2(t)\hat{K}_+ - f(t)\hat{E}_1 + d(t)\hat{E}_3\right), \tag{3.10}$$

where the operators

$$\hat{E}_1 = ix, \qquad \hat{E}_2 = \frac{\partial}{\partial x}, \qquad \hat{E}_3 = i\hat{I},$$

are generators of the Heisenberg-Weyl Lie algebra and operators

$$\hat{K}_- = -\frac{i}{2}\frac{\partial^2}{\partial x^2}, \qquad \hat{K}_+ = \frac{i}{2}x^2, \qquad \hat{K}_0 = \frac{1}{2}(x\frac{\partial}{\partial x} + \frac{1}{2}),$$

are generators of $su(1,1)$ Lie algebra. Here, we notice that, PDE in (3.2) is special in the sense that it is the most general equation with variable coefficients that can be written as a linear combination of the generators of $su(1,1)$ and Heisenberg-Weyl Lie algebras.

Then, the evolution operator $\hat{W}(t,t_0)$ for IVP (3.2) can be found by solving the operator problem

$$\begin{cases} \frac{d}{dt}\hat{W}(t,t_0) = \hat{T}(t)\hat{W}(t,t_0), \quad t \geq t_0, \\ \hat{W}(t_0,t_0) = \hat{I}. \end{cases} \tag{3.11}$$

We assume that the evolution operator can be written as product of exponential operators

$$\hat{W}(t,t_0) = e^{i\gamma(t)\hat{E}_3} \times e^{ip(t)\hat{E}_1} \times e^{-\beta(t)\hat{E}_2} \times e^{i\rho(t)\hat{K}_+} \times e^{2g(t)\hat{K}_0} \times e^{i\tau(t)\hat{K}_-}, \tag{3.12}$$

where $\rho(t), g(t), \tau(t), p(t), \beta(t), \gamma(t)$ are unknown real-valued functions of time t to be determined. Performing time-differentiation of the assumed evolution operator like for the product of ordinary functions, but preserving the ordering of the operators such as

$$\begin{aligned} \frac{d}{dt}\left(e^{f_1(t)\hat{A}} \times e^{f_2(t)\hat{B}}\right) &= \left(\frac{d}{dt}e^{f_1(t)\hat{A}}\right) \times e^{f_2(t)\hat{B}} + e^{f_1(t)\hat{A}} \times \left(\frac{d}{dt}e^{f_2(t)\hat{B}}\right) \\ &= \dot{f}_1(t)\hat{A}e^{f_1(t)\hat{A}}e^{f_2(t)\hat{B}} + \dot{f}_2(t)e^{f_1(t)\hat{A}}\hat{B}e^{f_2(t)\hat{B}}, \end{aligned} \tag{3.13}$$

we get

$$\begin{aligned} \frac{d}{dt}\hat{W} &= i\dot{\gamma}(t) \times \hat{E}_3 \times e^{i\gamma(t)\hat{E}_3} \times e^{ip(t)\hat{E}_1} \times e^{-\beta(t)\hat{E}_2} \times e^{i\rho(t)\hat{K}_+} \times e^{2g(t)\hat{K}_0} \times e^{i\tau(t)\hat{K}_-} \\ &+ i\dot{p}(t) \times e^{i\gamma(t)\hat{E}_3} \times \hat{E}_1 \times e^{ip(t)\hat{E}_1} \times e^{-\beta(t)\hat{E}_2} \times e^{i\rho(t)\hat{K}_+} \times e^{2g(t)\hat{K}_0} \times e^{i\tau(t)\hat{K}_-} \\ &- \dot{\beta}(t) \times e^{i\gamma(t)\hat{E}_3} \times e^{ip(t)\hat{E}_1} \times \hat{E}_2 \times e^{-\beta(t)\hat{E}_2} \times e^{i\rho(t)\hat{K}_+} \times e^{2g(t)\hat{K}_0} \times e^{i\tau(t)\hat{K}_-} \\ &+ i\dot{\rho}(t) \times e^{i\gamma(t)\hat{E}_3} \times e^{ip(t)\hat{E}_1} \times e^{-\beta(t)\hat{E}_2} \times \hat{K}_+ \times e^{i\rho(t)\hat{K}_+} \times e^{2g(t)\hat{K}_0} \times e^{i\tau(t)\hat{K}_-} \\ &+ 2\dot{g}(t) \times e^{i\gamma(t)\hat{E}_3} \times e^{ip(t)\hat{E}_1} \times e^{-\beta(t)\hat{E}_2} \times e^{i\rho(t)\hat{K}_+} \times \hat{K}_0 \times e^{2g(t)\hat{K}_0} \times e^{i\tau(t)\hat{K}_-} \\ &+ i\dot{\tau}(t) \times e^{i\gamma(t)\hat{E}_3} \times e^{ip(t)\hat{E}_1} \times e^{-\beta(t)\hat{E}_2} \times e^{i\rho(t)\hat{K}_+} \times e^{2g(t)\hat{K}_0} \times \hat{K}_- \times e^{i\tau(t)\hat{K}_-}. \end{aligned} \tag{3.14}$$

Then, using Baker-Campbell-Hausdorff relation given by

$$e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}} = \hat{B} + \xi[\hat{A},\hat{B}] + \frac{\xi^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{\xi^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + ... \tag{3.15}$$

where $\hat{A}, \hat{B}$ are two non-commuting operators and ξ is a parameter, we obtain equivalent form of (3.14) as follows

$$\begin{aligned}
\frac{d}{dt}\hat{W} = &\Big(\left[i\dot{\tau}(t)e^{-2g(t)}\right]\hat{K}_- + \left[2\dot{g}(t) + 2\dot{\tau}(t)\rho(t)e^{-2g(t)}\right]\hat{K}_0 \\
&+ \left[-\dot{\beta}(t) - \dot{g}(t)\beta(t) + \dot{\tau}(t)p(t)e^{-2g(t)} - \dot{\tau}(t)\rho(t)\beta(t)e^{-2g(t)}\right]\hat{E}_2 \\
&+ \left[i\dot{\rho}(t) - 2i\rho(t)\dot{g}(t) - i\dot{\tau}(t)\rho^2(t)e^{-2g(t)}\right]\hat{K}_+ \\
&+ \left[i\dot{p}(t) - i\dot{\rho}(t)\beta(t) - i\dot{g}(t)p(t) + 2i\rho(t)\dot{g}(t)\beta(t) - i\dot{\tau}(t)\rho(t)p(t)e^{-2g(t)} + i\dot{\tau}(t)\rho^2(t)\beta(t)e^{-2g(t)}\right]\hat{E}_1 \\
&+ [i\dot{\gamma}(t) + i\dot{\beta}(t)p(t) + i\dot{\rho}(t)\frac{\beta^2(t)}{2} + i\dot{g}(t)\beta(t)p(t) - i\rho(t)\dot{g}(t)\beta^2(t) - \frac{i}{2}\dot{\tau}(t)p^2(t)e^{-2g(t)} \\
&+ i\rho(t)\beta(t)p(t)\dot{\tau}(t)e^{-2g(t)} - \frac{i}{2}\rho^2(t)\beta^2(t)\dot{\tau}(t)e^{-2g(t)}]\hat{E}_3\Big)\hat{W}.
\end{aligned} \tag{3.16}$$

When we compare the right sides of equations (3.11) and (3.16), we obtain that $\hat{W}(t,t_0)$ is the required evolution operator, if the unknown functions satisfy the nonlinear system of six first-order differential equations

$$\begin{aligned}
\dot{\rho}(t) + \frac{\rho^2(t)}{\mu(t)} - 2b(t)\rho(t) + \mu(t)\omega^2(t) = 0, &\qquad \rho(t_0) = 0, \\
\dot{g}(t) + \frac{\rho(t)}{\mu(t)} - b(t) = 0, &\qquad g(t_0) = 0, \\
\dot{\tau}(t) - \frac{e^{2g(t)}}{\mu(t)} = 0, &\qquad \tau(t_0) = 0,
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
\dot{\beta}(t) + b(t)\beta(t) = a(t) + \frac{p(t)}{\mu(t)}, &\qquad \beta(t_0) = 0, \\
\dot{p}(t) - b(t)p(t) = \mu(t)f(t) - \mu(t)\omega^2(t)\beta(t), &\qquad p(t_0) = 0, \\
\dot{\gamma}(t) = -\frac{p^2(t)}{2\mu(t)} - a(t)p(t) + \frac{\mu(t)\omega^2(t)}{2}\beta^2(t) - \mu(t)d(t), &\qquad \gamma(t_0) = 0.
\end{aligned} \tag{3.18}$$

We note that, (3.17) and (3.18) are two independent systems, one for ρ, g, τ and second for p, β, γ. System (3.17) can be solved by realizing that the first line is an initial value problem for a nonlinear Riccati equation, and using substitution $\rho(t) = \mu(t)\left[\dot{r}(t)/r(t) + b(t)\right]$, it transforms to the linear second-order homogeneous differential equation

$$\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \left[\omega^2(t) + \left(\dot{b}(t) + \frac{\dot{\mu}(t)}{\mu(t)}b(t) - b^2(t)\right)\right] r = 0, \tag{3.19}$$

with initial conditions $r(t_0) = r_0 \neq 0,\ \ \dot{r}(t_0) = -b(t_0)r_0$, whose solution we denote by $r_1(t)$. Assuming that all coefficients in Eq.(3.19) are continuous on time interval containing $t_0 > 0$, by $r_2(t)$ we denote a second solution of Eq.(3.19) satisfying the initial conditions $r_2(t_0) = 0,\ \ \dot{r}_2(t_0) = r_0/\mu(t_0)$, and using Abel's formula we can write

$$r_2(t) = r_0^2 r_1(t) \int_{t_0}^{t} \frac{1}{\mu(t')r_1^2(t')} dt.$$

Also, we assume $r_1(t) > 0$ for $r_0 > 0$ and $r_2(t) > 0$ throughout this thesis. Then, we have $g(t) = \ln(r_1(t_0)/r_1(t))$ for $r_1(t) > 0$. Therefore, we get the solution to the third equation in system (3.17) as

$$\tau(t) = r_0^2 \int_{t_0}^{t} \frac{dt'}{\mu(t')r_1^2(t')}.$$

As a result, we obtain the solution of system (3.17) in terms of two independent solutions $r_1(t)$ and $r_2(t)$ of the homogeneous equation (3.19) as follows

$$\begin{aligned} \rho(t) &= \mu(t)\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right), \\ g(t) &= \ln\left(\frac{r_1(t_0)}{r_1(t)}\right), \\ \tau(t) &= \frac{r_2(t)}{r_1(t)}. \end{aligned} \tag{3.20}$$

On the other hand, taking derivative of the first equation in system (3.18) we obtain

$$\ddot{\beta}(t) + \frac{\dot{\mu}(t)}{\mu(t)}\dot{\beta}(t) + \left[\omega^2(t) + \left(\dot{b}(t) + \frac{\dot{\mu}(t)}{\mu(t)}b(t) - b^2(t)\right)\right]\beta(t) = \dot{a}(t) - a(t)\left[b(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right] + f(t). \tag{3.21}$$

Thus, $\beta(t)$ is a particular solution of the inhomogeneous Eq.(3.19), satisfying initial conditions

$$\beta(t_0) = 0, \quad \dot{\beta}(t_0) = a(t_0),$$

and we denote this solution by $\beta(t) \equiv r_p(t)$. It follows that the solution of the system (3.18) is

$$\begin{aligned}
\beta(t) &= r_p(t), \qquad (3.22)\\
p(t) &= \mu(t)\left(\dot{r}_p(t) + b(t)r_p(t) - a(t)\right) \equiv p_p(t),\\
\gamma(t) &= -\int_{t_0}^{t}\left(\frac{(p_p(t'))^2}{2\mu(t')} + a(t')p_p(t') - \frac{\mu(t')\omega^2(t')}{2}r_p(t') + \mu(t')f_0(t') - \frac{b(t')}{2}\right)dt'.
\end{aligned}$$

Now, after finding all unknown functions in (3.12), the exact form of the evolution operator in terms of $r_1(t), r_2(t), r_p(t)$ and $p_p(t)$ is obtained as follows

$$\begin{aligned}
\hat{W}(t,t_0) &= \exp\left[-\gamma(t)\hat{I}\right] \times \exp\left[-p_p(t)x\right]\\
&\times \exp\left[-r_p(t)\frac{\partial}{\partial x}\right] \times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)x^2\right] \qquad (3.23)\\
&\times \exp\left[\ln\left(\frac{r_0}{r_1(t)}\right)(x\frac{\partial}{\partial x} + \frac{1}{2})\right] \times \exp\left[\frac{r_2(t)}{2r_1(t)}\frac{\partial^2}{\partial x^2}\right].
\end{aligned}$$

Using the expressions

$$\exp\left(\frac{z}{2}\frac{\partial^2}{\partial x^2}\right)\Psi_0(x) = \Psi(x,z),$$

where $\Psi(x,z)$ satisfies the IVP

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\Psi(x,z) = \frac{\partial}{\partial z}\Psi(x,z), \qquad \Psi(x,0) = \Psi_0(x) \equiv \Phi(x,t_0),$$

and expressions for the shift and dilatation operators respectively,

$$\exp\left(\lambda\frac{\partial}{\partial x}\right)\Psi(x,z) = \Psi(x+\lambda,z), \qquad \exp\left(\lambda x\frac{\partial}{\partial x}\right)\Psi(x,z) = \Psi(e^{\lambda}x,z),$$

the evolution operator (3.23) is applied to the initial function $\Phi(x, t_0)$, that's $\Phi(x, t) = \hat{W}(t, t_0)\Phi(x, t_0)$ and we obtain solution (3.4) of the generalized diffusion IVP (3.2). □

3.1.1. On the characteristic equation

Notice that for the Lagrangian type function of the form (3.5), it is not difficult to show that using the Euler-Lagrange equation

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{r}}\Big) - \frac{\partial L}{\partial r} = 0, \tag{3.24}$$

one recovers the Newtonian equation of motion given by (3.3). Also, the generalized (conjugate) momentum is defined as

$$p(t) \equiv \frac{\partial L}{\partial \dot{r}} = \mu(t)\Big(\dot{r}(t) + b(t)r(t) - a(t)\Big). \tag{3.25}$$

In the absence of convection ($a(t) = 0$, $b(t) = 0$) and for constant $\mu(t) = m$, one gets the standard momentum $p(t) = m\,\dot{r}(t)$. In particular, if $\dot{r}(t) = a(t) - b(t)r(t)$, then the generalized momentum is zero.

3.1.2. Integral representation and fundamental solution

Using the integral representation (2.23) for the solution to the IVP for standard heat equation, we can write the solution to IVP (3.2) in integral form

$$\begin{aligned}
\Phi(x, t) &= \sqrt{\frac{r_1(t_0)}{r_1(t)}} \times \exp\left[-p_p(t)r_p(t)\right] \times \exp\left[\int_{t_0}^{t}\Big(L_p(t') - \frac{b(t')}{2}\Big)dt'\right] \\
&\times \exp\left[\int_{t_0}^{t}\Big(a(t')p_p(t') - \mu(t')\big(f(t')r_p(t') - f_0(t')\big)\Big)dt'\right] \\
&\times \exp\left[-\frac{\mu(t)}{2}\Big(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\Big)\big(x - r_p(t)\big)^2\right] \times \exp\left[-p_p(t)\big(x - r_p(t)\big)\right] \\
&\times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\tau(t)}} \exp\left[-\frac{(\eta_p(x, t) - \xi)^2}{2\tau(t)}\right]\Phi_0(\xi)d\xi,
\end{aligned} \tag{3.26}$$

where $\eta_p(x,t)$ and $\tau(t)$ are given in (3.7). And for the bounded initial data $\Phi_0(x)$, the integral in solution (3.26) converges.

Fundamental solution : For the initial condition $\Phi_0(x) = \delta(x-x_0)$, where $\delta(x-x_0)$ is a shifted Dirac delta distribution, we obtain corresponding fundamental solution as follows

$$
\begin{aligned}
K(x,t;x_0,t_0) &= \frac{\sqrt{r_0}}{\sqrt{2\pi r_2(t)}} \times \exp\left[-p_p(t)r_p(t)\right] \times \exp\left[\int_{t_0}^{t}\left(L_p(t') - \frac{b(t')}{2}\right)dt'\right] \\
&\times \ \exp\left[\int_{t_0}^{t}\left(a(t')p_p(t') - \mu(t')\left(f(t')r_p(t') - f_0(t')\right)\right)dt'\right] \\
&\times \ \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)\left(x - r_p(t)\right)^2\right] \times \exp\left[-p_p(t)\left(x - r_p(t)\right)\right] \\
&\times \ \exp\left[-\frac{\left(r_0(x - r_p(t)) - x_0 r_1(t)\right)^2}{2r_1(t)r_2(t)}\right].
\end{aligned} \tag{3.27}
$$

Therefore, solution of IVP (3.2) can be formulated also as

$$
\Phi(x,t) = K(x,t;x_0,t_0) * \Phi_0(x) = \int_{-\infty}^{\infty} K(x,t;\xi,t_0)\ \Phi_0(\xi)d\xi,
$$

where $K(x,t;x_0,t_0) * \Phi_0(x)$ is the convolution of the fundamental solution and initial function.

Here, we see that $b(t)$ influences the amplitude and spreading of the solution (3.27), while $f_0(t)$ affects just the amplitude. In the absence of the reaction term with quadratic in x coefficient ($\omega(t) = 0$), for given $b(t)$ we have the relation $\dot{r}_1(t)/r_1(t) = -b(t)$, so that the Gaussian term in above solution vanishes as expected. Indeed, in that case, take any real-valued smooth function $b(t)$ and suppose it satisfies $\dot{b}(t) + \dot{\mu}(t)/\mu(t)b(t) - b^2(t) = \Lambda^2(t)$, so that the characteristic equation (3.47) becomes $\ddot{r} + \dot{\mu}(t)/\mu(t)\dot{r} + \Lambda^2(t)r = 0$. On the other side, Riccati equation for $b(t)$ can be linearized by letting $b(t) = -\dot{y}(t)/y(t)$, which gives $\ddot{y} + \dot{\mu}(t)/\mu(t)\dot{y} + \Lambda^2(t)y = 0$. Since ODEs are the same, it follows that we are allowed to replace the original $b(t)$ by $b(t) = -\dot{r}_1(t)/r_1(t)$.

Also we notice that if $a(t) = f(t) = 0$ which implies $r_p(t) = 0$ and correspondingly $p_p(t) = L_p(t) = 0$, then in that case the fundamental solution becomes

$$
\begin{aligned}
K(x,t;x_0,t_0) &= \frac{\sqrt{r_0}}{\sqrt{2\pi r_2(t)}} \times \exp\left[-\int_{t_0}^{t}\left(\frac{b(t')}{2} - \mu(t')f_0(t')\right)dt'\right] \\
&\times \ \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)x^2\right] \times \ \exp\left[-\frac{(r_0 x - x_0 r_1(t))^2}{2r_1(t)r_2(t)}\right]. \qquad (3.28)
\end{aligned}
$$

3.1.3. Spatial moments, mean and variance of the fundamental solution

The first three spatial moments of a solution distribution $\Phi(x,t)$ are defined as

$$
M_0(t) = \int_{D_x} \Phi(x,t)dx, \quad M_1(t) = \int_{D_x} x\,\Phi(x,t)dx, \quad M_2(t) = \int_{D_x} x^2\,\Phi(x,t)dx, \quad (3.29)
$$

where D_x is the spatial domain of the problem. These moments are usually used to describe the shape and behavior of the distribution profile. If $\Phi(x,t)$ is the mass density function of the solute (or concentration of a substance), then zeroth moment $M_0(t)$ gives the total mass of the solute (or amount of the substance) contained in D_x at time t. The first moment $M_1(t)$ normalized by the total mass gives the mean location of the distribution in D_x, or say the "center of mass", denoted by $\langle x\rangle(t)$, where the weighted relative positions sum to zero. The second central moment about the mean, $M_2(t)$, normalized by the total mass gives the mean square position denoted by $\langle x^2\rangle(t)$. And the variance which is a measure of the spreading of the distribution about its mean position, in other words the deviation from the center of the mass, is defined as

$$
\mathrm{Var}(t) = \int_{-\infty}^{\infty} (x - \langle x\rangle)^2\, K(x,t)dx = \langle x^2\rangle(t) - \langle x\rangle^2(t), \quad (3.30)
$$

or in terms of the moments it can be written as

$$
\mathrm{Var}(t) = \frac{M_2(t)}{M_0(t)} - \left(\frac{M_1(t)}{M_0(t)}\right)^2. \quad (3.31)
$$

In what follows, we provide explicit results for the fundamental solution of the

IVP (3.2) for the convection-diffusion-reaction equation when $\omega(t) = 0$, which implies also $b(t) = -\dot{r}_1(t)/r_1(t)$. Without loss of generality we take $r_0 = 1$ and for simplicity, we take $t_0 = 0$, so that initial data is $\Phi(x, 0) = \delta(x - x_0)$ and we denote the fundamental solution as $K(x, t; x_0, t_0) \equiv K(x, t; x_0)$. In that case, the total initial mass or concentration amount is equal to

$$\int_{-\infty}^{\infty} \Phi(x, 0)dx = \int_{-\infty}^{\infty} \delta(x - x_0)dx = 1.$$

Therefore, the corresponding fundamental solution can be written in the form

$$K(x, x_0; t) = \frac{A_0(t)\sqrt{r_1(t)}}{\sqrt{2\pi r_2(t)}} \times \exp\Big(-p_p(t)(x - r_p(t))\Big) \times \exp\Big(-\frac{(x - r_p(t) - x_0 r_1(t))^2}{2r_1(t)r_2(t)}\Big),$$

where $A_0(t)$ denotes the product of the exponential terms in $K(x, x_0; t)$ that depend only on time. Then, we find the moments as follows.

a) Zeroth Moment

First we compute the zeroth moment

$$\begin{aligned}
M_0(t) &= \int_{-\infty}^{\infty} K(x, x_0; t)dx \\
&= \frac{A_0(t)\sqrt{r_1(t)}}{\sqrt{2\pi r_2(t)}} \int_{-\infty}^{\infty} \exp\Big(-p_p(t)(x - r_p(t))\Big) \times \exp\Big(-\frac{(x - r_p(t) - x_0 r_1(t))^2}{2r_1(t)r_2(t)}\Big)dx, \\
&= \frac{A_0(t)\sqrt{r_1(t)}}{\sqrt{2\pi r_2(t)}} \exp\Big(\frac{1}{2}p_p(t)r_1(t)\Big(p_p(t)r_2(t) - 2x_0\Big)\Big) \\
&\times \int_{-\infty}^{\infty} \exp\Big(-\frac{(x - r_p(t) - x_0 r_1(t) + r_1(t)r_2(t)p_p(t))^2}{2r_1(t)r_2(t)}\Big)dx,
\end{aligned}$$

which finally becomes

$$M_0(t) = A_0(t)r_1(t) \times \exp\Big(\frac{1}{2}p_p^2(t)r_1(t)r_2(t) - x_0 p_p(t)r_1(t)\Big). \tag{3.32}$$

In particular, if $f(t) = 0$ which implies $p_p(t) = 0$ and $L_p(t) = 0$ since $\dot{r}_p(t) = a(t) - b(t)r_p(t)$, we have total mass

$$M_0(t) = r_1(t)\exp\Big(\int_{t_0}^{t}\mu(t')f_0(t')dt'\Big), \tag{3.33}$$

depending on $r_1(t) > 0$, $\mu(t) > 0$ and the first-order reaction rate $f_0(t)$. For instance, assume $r_1(t) = 1$, then the followings happen :

- when $f_0(t) = 0$, then total mass is conserved with $M_0(t) = 1$ for all $t > 0$,
- when $f_0(t) < 0$, there is "loss of mass" due to reaction,
- when $f_0(t) > 0$, there is "gain of mass" due to reaction.

b) First Moment and Mean Position

Next, we compute the first spatial moment

$$\begin{aligned}
M_1(t) &= \int_{-\infty}^{\infty} x\, K(x, x_0; t)dx \\
&= \frac{A_0(t)\sqrt{r_1(t)}}{\sqrt{2\pi r_2(t)}}\exp\Big(\frac{1}{2}p_p^2(t)r_1(t)r_2(t) - x_0 p_p(t)r_1(t)\Big) \\
&\times \int_{-\infty}^{\infty} x\ \exp\Big[-\frac{(x - r_p(t) - x_0 r_1(t) + r_1(t)r_2(t)p_p(t))^2}{2r_1(t)r_2(t)}\Big]dx, \\
&= \frac{A_0(t)r_1(t)}{\sqrt{\pi}}\exp\Big(\frac{1}{2}p_p^2(t)r_1(t)r_2(t) - x_0 p_p(t)r_1(t)\Big) \\
&\times \Big(\int_{-\infty}^{\infty}\sqrt{2r_1(t)r_2}y\, e^{-y^2}dy + \Big(r_p(t) + x_0 r_1(t) - r_1(t)r_2(t)p_p(t)\Big)\int_{-\infty}^{\infty} e^{-y^2}dy\Big).
\end{aligned}$$

Therefore, we get the first moment in terms of zeroth moment and the solutions to the characteristic ODE as follows

$$M_1(t) = M_0(t)\Big(r_p(t) + x_0 r_1(t) - r_1(t)r_2(t)p_p(t)\Big).$$

• ***Mean position* :** Then, normalizing the first moment by zeroth moment gives the mean position

$$\langle x\rangle_f(t) \equiv \frac{M_1(t)}{M_0(t)} = r_p(t) + x_0 r_1(t) - \sigma(t)p_p(t), \qquad \sigma(t) = r_1(t)r_2(t), \tag{3.34}$$

and if $f(t) = 0$, then mean position is $\langle x\rangle_0(t) = r_p(t) + x_0 r_1(t)$, that is the deformation due to reaction $f(t)$ in (3.34) has disappeared as expected.

c) Second Moment and Variance

Second spatial moment is found as

$$
\begin{aligned}
M_2(t) &= \int_{-\infty}^{\infty} x^2\, K(x, x_0; t) dx \\
&= \frac{A_0(t)\sqrt{r_1(t)}}{\sqrt{2\pi r_2(t)}} \exp\Big(\frac{1}{2} p_p(t) r_1(t) \big(p_p(t) r_2(t) - 2x_0\big)\Big) \\
&\times \int_{-\infty}^{\infty} x^2 \exp\Big(- \frac{(x - r_p(t) - x_0 r_1(t) + r_1(t) r_2(t) p_p(t))^2}{2 r_1(t) r_2(t)} \Big) dx, \\
&= \frac{A_0(t) r_1(t)}{\sqrt{\pi}} \exp\Big(\frac{1}{2} p_p(t) r_1(t) \big(p_p(t) r_2(t) - 2x_0\big)\Big) \\
&\times \int_{-\infty}^{\infty} \Big(\sqrt{2 r_1(t) r_2} y + r_p(t) + x_0 r_1(t) - r_1(t) r_2(t) p_p(t)\Big)^2 e^{-y^2} dy,
\end{aligned}
$$

which becomes

$$
M_2(t) = M_0(t) \times \Big(r_1(t) r_2(t) + \big(r_p(t) + x_0 r_1(t) - r_1(t) r_2(t) p_p(t)\big)^2 \Big). \tag{3.35}
$$

Then we obtain corresponding mean-square position

$$
\langle x^2\rangle_f \equiv \frac{M_2(t)}{M_0(t)} = \sigma(t) + (r_p(t) + x_0 r_1(t) - \sigma(t) p_p(t))^2. \tag{3.36}
$$

• **Variance :** Using the formula (3.31), we get the variance about the mean as follows

$$
Var(t) = \sigma(t). \tag{3.37}
$$

As expected, the variance $\sigma(t) = r_1(t) r_2(t)$ depends only on the homogenous solutions of the characteristic equation determined by the coefficients $\mu(t)$ and $b(t)$, while the mean position $\langle x\rangle_f(t)$ depends also on the external forcing parameters $a(t)$ and $f(t)$, due to convection and reaction term with linear in position coefficient in the diffusion model. The reaction rate $f_0(t)$ has influence on the total mass, but it does not affect the mean and variance. The computations of the mean and variance for the general case when $\omega(t) \neq 0$ and of solution distributions corresponding to different initial data can be computed in a similar way, when necessary.

In what follows we study the exactly solvable model to investigate the influence of parameters and time evolution of the center of the distribution.

Example 3.1 *Consider the IVP defined by convection-diffusion equation and Dirac-delta initial data as follows*

$$\begin{cases} \Phi_t = \frac{e^{-\gamma t}}{2}\Phi_{xx} - ((a_0 \sin(\omega t) - \beta x)\,\Phi)_x\,, & x \in \mathbb{R}, \quad t > 0, \\ \Phi(x,0) = \delta(x - x_0), \quad x \in \mathbb{R}, \end{cases} \tag{3.38}$$

where we have exponentially decaying diffusion coefficient, $\mu(t) = e^{\gamma t}$, $\gamma > 0$ *and sinusoidal convection term with frequency* $\omega > 0$, *amplitude* $a_0 \geq 0$, *and* $\beta \in \mathbb{R}$.

Then the corresponding ODE is

$$\ddot{r} + \gamma\dot{r} + (\gamma\beta - \beta^2)r = F_0\cos(\omega t + \theta), \qquad t > 0, \tag{3.39}$$

where we denote $F_0 = a_0\sqrt{\omega^2 + (\beta - \gamma)^2}$ and $\theta = \arctan((\beta - \gamma)/\omega)$. We note that, the sinusoidal velocity in diffusion equation has generated external periodic force in (3.39) as expected, with amplitude F_0 and phase shifting θ, both depending on frequency $\omega > 0$ and $\beta \in \mathbb{R}$. For the discriminant $\Delta = (\gamma - 2\beta)^2$ and $r_0 = 1$, we have homogeneous and particular solutions respectively

$$\begin{aligned} r_1(t) &= e^{-\beta t}, \\ r_2(t) &= \frac{1}{\gamma - 2\beta}\left(e^{-\beta t} - e^{-(\gamma-\beta)t}\right), \\ r_p(t) &= \frac{a_0\omega(\omega^2 + (\beta-\gamma)^2)}{\Omega}e^{-\beta t} - \frac{(\omega^2 - \gamma\beta + \beta^2)F_0}{\Omega}\cos(\omega t + \theta) + \frac{\gamma\omega F_0}{\Omega}\sin(\omega t + \theta), \end{aligned} \tag{3.40}$$

where $\gamma \neq 2\beta$ and $\Omega = (\omega^2 - \gamma\beta + \beta^2)^2 + (\gamma\omega)^2$. Since the parameter β has influence on all solutions to the characteristic equation given in (3.40), the behavior of these solutions change according to the the followings :

(i) When $\beta < 0$ and $\gamma > 0$, all solutions tends to infinity as $t \to \infty$.

(ii) If $\beta = 0$ and $\gamma > 0$, then $r_1(t)$ becomes constant function, $r_2(t) \to 1/\gamma$ and $r_p(t)$ just oscillates as time increases.

(iii) In the condition $0 < \gamma < \beta$, while $r_1(t)$ goes to zero, $r_2(t)$ tends to infinity as $t \to \infty$, and the particular solution keeps oscillating in time.

On the other hand notice that we have $b(t) = -\dot{r}_1(t)/r_1(t)$ and $r_p(t)$ satisfies $\dot{r}_p(t) = a(t) - b(t)r_p(t)$. Therefore we have $p_p(t) = L_p(t) = 0$ and obtain the fundamental solution to the problem (3.38) as follows

$$K(x, x_0; t) = \sqrt{\frac{\gamma - 2\beta}{2\pi(e^{-2\beta t} - e^{-\gamma t})}} \times \exp\left[-\frac{(x - r_p(t) - x_0 e^{-\beta t})^2}{2(e^{-2\beta t} - e^{-\gamma t})/(\gamma - 2\beta)}\right]. \tag{3.41}$$

Then, we get total amount of mass as

$$M_0(t) = \int_{-\infty}^{\infty} K(x, x_0; t)dx = 1, \tag{3.42}$$

which shows that the total mass is conserved. The first moment is

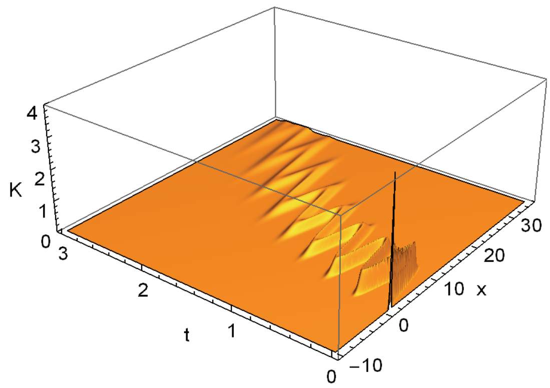
$$M_1(t) = \int_{-\infty}^{\infty} x\, K(x, x_0; t)dx = r_p(t) + x_0 e^{-\beta t}. \tag{3.43}$$

and normalizing it by total mass, we get mean position of the distribution

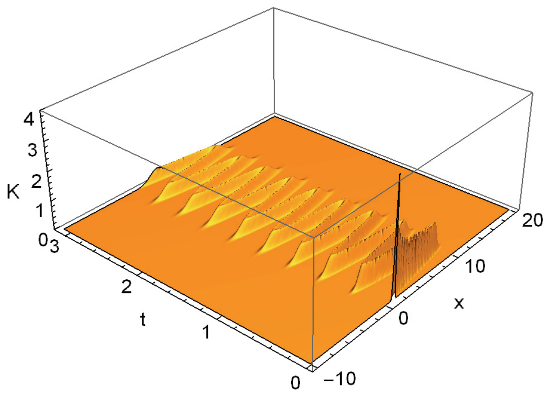
$$\langle x \rangle(t) = r_p(t) + x_0 e^{-\beta t}, \tag{3.44}$$

where $r_p(t)$ is given in (3.40).

In the case $\beta < 0$, the distribution follows the exponentially oscillating trajectory moving to the right in x-direction with decreasing amplitude. When $\beta = 0$, then it propagates just along the oscillatory path with constant amplitude. In Fig.3.1a, we plot the behavior of distribution for $\beta < 0$ and certain parameters. For the case $\beta = 0$, one can see the influence of parameter on the behavior of solution in Fig.3.1b.



(a)



(b)

Figure 3.1 Solution (3.41) with $\omega = 18, \gamma = 1,\ a_0 = 75, x_0 = 1,$ (a) $\beta = -0.7$. (b) $\beta = 0$.

3.2. Initial-Boundary Value Problems on the Half-line

In this section, we obtain analytic solutions to IBVP's on the half-line $0 < x < \infty$ for convection-diffusion-reaction equations with space and time-variable coefficients, and constructing exactly solvable models.

Results of Proposition 3.1 show that the generalized diffusion equations of the form (3.1) can be reduced to the standard heat equation by space and time transformations and the IVP on the whole real line can be solved analytically. However, more difficulties appear when we impose boundary conditions. Indeed, in the presence of certain convection and/or reaction terms, usually the fixed boundaries convert to moving boundaries. This can be explicitly seen from solution (3.4) by letting $x = 0$, and noting that the Dirichlet boundary condition, prescribed on the fixed boundary $x = 0$, converts to Dirichlet BC for the standard heat equation on the moving boundary $s_0(t) = -r_0 r_p(t)/r_1(t)$. Therefore, exactly solvable IBVP's for equation (1.1) can be constructed only for some particular cases.

First we note that, the convection and reaction terms in the diffusion type equation, generate external forces in the characteristic equation (3.3). Therefore, Eq.(3.4) shows that the displacement of the position coordinate of $\Phi(x,t)$ by the particular solution $r_p(t)$ appears due to certain convection or/and reaction terms, as expected. Moreover, according to (3.4) the value of $\Phi(x,t)$ at $x = r_p(t)$ is of the form

$$\Phi(r_p(t),t) = A(t, r_p(t)) \times \Psi(0, \tau(t)), \qquad 0 < t < T,$$

where by $A(t, r_p(t))$ we briefly denote the amplitude like function in (3.4) that depends only on time, and $\Psi(0,\tau(t))$ is the value of the standard heat solution at $\eta = 0$, which depends on $\tau(t) = r_2(t)/r_1(t)$. Then, for $r_p(t) = 0$ we have

$$\Phi(0,t) = A(t,0) \times \Psi(0, \tau(t)), \qquad 0 < t < T,$$

which suggests that IBVP's on $0 < x < \infty$ for Eq.(3.1) can be converted to IBVP's for the standard heat equation on the half-line $0 < x < \infty$, if $r_p(t) = 0$. According to the characteristic equation (3.3) this can happen in the following cases:

(i) $a(t)$- constant, $\mu(t)$-constant, $b(t) = 0$, $f(t) = 0$;

(ii) $a(t) = f(t) = 0$; (symmetric case),

(iii) $a(t)$-constant, $b(t) = \dot{\mu}(t)/\mu(t)$ and $f(t) = 0$;

(iv) $f(t) = -\dot{a}(t)$ and $b(t) = \dot{\mu}(t)/\mu(t)$.

In what follows we provide and discuss exactly solvable IBVP's for the symmetric case (ii), i.e. the case when the PDE is invariant under space inversion. Other cases can be studied in a similar way when necessary.

3.2.1. Analytical solution to the IBVP with Dirichlet boundary condition

First, we consider IBVP on the half-line with Dirichlet boundary condition. The result is formulated as follows.

Proposition 3.2 *The Dirichlet IBVP on the half-line*

$$\Phi_t = \frac{1}{2\mu(t)}\Phi_{xx} + b(t)x\Phi_x + \mu(t)[\frac{\omega^2(t)}{2}x^2 + f_0(t)]\Phi, \quad 0 < x < \infty, \quad 0 < t < T, \tag{3.45a}$$

$$\Phi(x,0) = \Phi_0(x), \quad 0 < x < \infty, \tag{3.45b}$$

$$\Phi(0,t) = D(t), \quad 0 < t < T, \tag{3.45c}$$

where the parameters $\mu(t) > 0$, $\mu(0) = 1$, $b(t), \omega(t), f_0(t)$ are given real-valued functions of time and $\Phi_0(x)$, $D(t)$ are given sufficiently smooth functions in their domains, has solution of the form

$$\begin{aligned}\Phi(x,t) \;&=\; \frac{1}{\sqrt{r_1(t)}} \times \exp\left[-\int_0^t \left(\frac{b(t')}{2} - \mu(t')f_0(t')\right)dt'\right] \times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)x^2\right] \\ &\times\; \Psi\left(\eta(x,t), \tau(t)\right),\end{aligned} \tag{3.46}$$

if $r_1(t)$, $r_2(t)$ are positive and linearly independent homogeneous solutions of the homogeneous characteristic equation

$$\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \left[\omega^2(t) + \left(\dot{b}(t) + \frac{\dot{\mu}(t)}{\mu(t)}b(t) - b^2(t)\right)\right]r = 0, \tag{3.47}$$

satisfying initial conditions $r_1(0) = 1$, $\dot{r}_1(0) = -b(0)$, $r_2(0) = 0$, $\dot{r}_2(0) = 1$ respectively, also

$$\eta(x,t) = \frac{x}{r_1(t)}, \qquad \tau(t) = \frac{r_2(t)}{r_1(t)}, \qquad 0 < t < T, \tag{3.48}$$

and $\Psi(\eta,\tau)$ is solution of the following IBVP for heat equation with Dirichlet boundary condition

$$\Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \qquad 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \tag{3.49a}$$

$$\Psi(\eta,0) = \Phi_0(\eta), \quad 0 < \eta < \infty, \tag{3.49b}$$

$$\Psi(0,\tau) = D_0(\tau), \quad 0 < \tau < \tau(T), \tag{3.49c}$$

with boundary data

$$D_0(\tau) = D(t(\tau))\sqrt{r_1(t(\tau))} \times \exp\left[\int_0^{t(\tau)} \left(\frac{b(t')}{2} - \mu(t')f_0(t')\right)dt'\right]. \tag{3.50}$$

Proof This Proposition 3.2 is a direct consequence of Proposition 3.1. Here we give an alternative proof by assuming that equation (3.45a) has solution of the form

$$\Phi(x,t) = e^{F(x,t)} \times \Psi(\eta(x,t), \tau(t)), \tag{3.51}$$

where $F(x,t) = -\rho(t)x^2/2 + \gamma(t)$, $\eta(x,t) = e^{g(t)}x$ and $\rho(t)$, $g(t), \gamma(t), \tau(t)$ are unknown parameters to be determined. For this we compute

$$\begin{aligned}
\Phi_t &= [-\frac{\dot{\rho}(t)}{2}x^2 + \dot{\gamma}(t)]e^{F(x,t)}\Psi + \dot{g}(t)e^{g(t)}xe^{F(x,t)}\Psi_\eta + \dot{\tau}(t)e^{F(x,t)}\Psi_\tau, \\
\Phi_x &= [-\rho(t)x]e^{F(x,t)}\Psi + e^{g(t)}e^{F(x,t)}\Psi_\eta, \\
\Phi_{xx} &= [-\rho(t) + \rho^2(t)x^2]e^{F(x,t)}\Psi - 2\rho(t)xe^{g(t)}e^{F(x,t)}\Psi_\eta + e^{2g(t)}e^{F(x,t)}\Psi_{\eta\eta}.
\end{aligned} \tag{3.52}$$

If we substitute these derivatives into Eq.(3.45a), then we get

$$\begin{aligned}
\dot{\tau}(t)\Psi_\tau &= \frac{e^{2g(t)}}{2\mu(t)}\Psi_{\eta\eta} - \left[\frac{\rho(t)}{\mu(t)} - b(t) + \dot{g}(t)\right]xe^{g(t)}\Psi_\eta \\
&+ \left[\left(\frac{\dot{\rho}(t)}{2} + \frac{\rho^2(t)}{2\mu(t)} - b(t)\rho(t) + \frac{\mu(t)\omega^2(t)}{2}\right)x^2 - \dot{\gamma}(t) - \frac{\rho(t)}{2\mu(t)} + \mu(t)f_0(t)\right]\Psi.
\end{aligned} \tag{3.53}$$

From (3.53), $\Psi(\eta,\tau)$ satisfies heat equation $\Psi_\tau = (1/2)\Psi_{\eta\eta}$ and with initial condition $\Psi(\eta, 0) = \Phi_0(\eta)$ from ansatz (3.51), if the auxiliary functions ρ, g, τ and γ solve the non-linear system of first order ordinary differential equations

$$
\begin{aligned}
\dot{\rho}(t) + \frac{\rho^2(t)}{\mu(t)} - 2b(t)\rho(t) + \mu(t)\omega^2(t) = 0, &\qquad \rho(0) = 0, \\
\dot{g}(t) + \frac{\rho(t)}{\mu(t)} - b(t) = 0, &\qquad g(0) = 0, \\
\dot{\tau}(t) - \frac{e^{2g(t)}}{\mu(t)} = 0, &\qquad \tau(0) = 0, \\
\dot{\gamma}(t) + \frac{\rho(t)}{2\mu(t)} - \mu(t)f_0(t) = 0, &\qquad \gamma(0) = 0.
\end{aligned} \tag{3.54}
$$

Note that when $t_0 = 0$, the first three equations of the above system (3.54) is the same as system (3.17), so does the corresponding solutions given in (3.20). It follows that, the solution to the last equation in (3.54) is

$$
\gamma(t) = -\frac{\ln(r_1(t))}{2} - \int_0^t \left(\frac{b(t')}{2} - \mu(t')f_0(t') \right) dt'. \tag{3.55}
$$

Writting the auxiliary functions back into the ansatz (3.51), gives solution (3.46). We notice that continuity of $\mu(t) > 0$ and $r_1^2(t) > 0$ for $t \in (0, T)$, imply that $\tau(t)$, $t > 0$ defined in (3.48) is strictly increasing continuous function on $(0, T)$ and thus its inverse $t(\tau)$ exists for $\tau \in (0, \tau(T))$. Then Dirichlet boundary condition (3.45c) transforms to Dirichlet boundary condition (3.49c) with $D_0(\tau)$ given in (3.50) for heat equation. Therefore IBVP (3.45) for the diffusion type equation transforms to the IBVP (3.49), which completes the proof. □

In summary, we see that the analytical solution to IBVP (3.45) is obtained in terms of solution to the second order linear homogeneous characteristic equation and heat model.

Integral Representation and Fundamental Solution **:**

Using integral representation of solution for heat IBVP with homogeneous Dirichlet BC given in (2.28), we obtain the solution to the IBVP (3.45) in integral form as

$$
\begin{aligned}
\Phi(x,t) &= \frac{1}{\sqrt{r_1(t)}} \times \exp\left[-\int_0^t \left(\frac{b(t')}{2} - \mu(t')f_0(t') \right) dt' \right] \times \exp\left[-\frac{\mu(t)}{2} \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t) \right) x^2 \right] \\
&\times \left(\int_0^\infty G_D(\eta(x,t), \xi, \tau(t)) \Phi(\xi, 0) d\xi - \int_0^{\tau(t)} K_\eta(\eta(x,t), \tau(t) - \tau') \Psi(0, \tau') d\tau' \right),
\end{aligned}
$$

provided the integrals converge for the given initial and boundary data. Here, $G_D(\eta, \xi, \tau) =$

$K(\eta-\xi,\tau)-K(\eta+\xi,\tau)$ denotes the Dirichlet heat kernel, $K_\eta(\eta,\tau)$ is the partial derivative of heat kernel wrt η, and $\eta(x,t),\ \tau(t)$ are as given in the statement of Proposition 3.1.

Fundamental solution: When we take Dirac-delta IC $\Phi_0(x) = \delta(x-x_0)$, for $0 < x < \infty,\ x_0 > 0$, and boundary condition $\Phi(0,t) = 0$, then the IBVP reduces to the heat IBVP with initial condition $\Psi(\eta,0) = \delta(\eta - x_0)$ and homogeneous Dirichlet boundary condition $\Psi(0,\tau) = 0$. Then, by using fundamental solution of heat IBVP given in (2.27), the fundamental solution to the diffusion equation is obtained as

$$K(x,x_0;t) = \frac{1}{\sqrt{2\pi r_2(t)}} \times \exp\left[-\int_0^t \left(\frac{b(t')}{2} - \mu(t')f_0(t')\right)dt'\right] \times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot r_1(t)}{r_1(t)} + b(t)\right)x^2\right]$$
$$\times\left(\exp\left[-\frac{(x-x_0r_1(t))^2}{2r_1(t)r_2(t)}\right] - \exp\left[-\frac{(x+x_0r_1(t))^2}{2r_1(t)r_2(t)}\right]\right). \tag{3.56}$$

The parameter $b(t)$ affects the amplitude and spreading of the solution. When $\omega(t) = 0$, then the Gaussian term in (3.56) vanishes as expected. Notice that the solution given in (3.28) for the IVP when $a(t) = f(t) = 0$ is different then the solution (3.56) due to the corresponding heat solution for the related problem.

In the following section we construct exactly solvable models with different initial and boundary data.

3.2.1.1. Exactly solvable convection-diffusion-reaction type models with Dirichlet boundary condition

MODEL 1 : IBVP with homogeneous boundary condition

Here, we consider a diffusion type model

$$\begin{cases} \Phi_t = \frac{1}{2}e^{-\gamma t}\Phi_{xx} + (\frac{\gamma}{2} - \Omega_b\tanh(\Omega_b t))x\Phi_x - \frac{\omega_0^2}{2}e^{\gamma t}x^2\Phi, & 0<x<\infty, \quad t>0,\\ \Phi(x,0) = \Phi_0(x), \quad 0<x<\infty, \\ \Phi(0,t) = 0, \quad t>0, \end{cases} \tag{3.57}$$

with exponentially decaying diffusion coefficient for $\gamma > 0$, positive and bounded dilation parameter $b(t) = (\gamma/2) - \Omega_b\tanh(\Omega_b t)$ with $0 \le \Omega_b \le \gamma/2$ and a reaction term with exponentially growing coefficient, depending both on time and position, where $\omega_0 > 0$. Then the corresponding characteristic equation

$$\ddot{r} + \gamma\dot{r} + (\gamma^2/4 - (\omega_0^2 + \Omega_b^2))r = 0, \qquad t > 0, \tag{3.58}$$

has positive solutions

$$r_1(t) = e^{-\gamma t/2}\cosh(\Omega t), \qquad r_2(t) = \frac{e^{-\gamma t/2}}{\Omega}\sinh(\Omega t), \quad \Omega = \sqrt{\omega_0^2 + \Omega_b^2}, \quad t > 0, \tag{3.59}$$

satisfying conditions $r_1(0) = 1$, $\dot{r}_1(0) = -\gamma/2$, $r_2(0) = 0$, $\dot{r}_2(0) = 1$. It is seen that, for given $\gamma > 0$, $0 \le \Omega_b \le \gamma/2$ and $\omega_0 > 0$, if one has $\Omega < \gamma/2$, then solutions $r_1(t)$ and $r_2(t)$ approach zero ; if $\Omega > \gamma/2$, they tend to infinity as $t \to \infty$; and for $\Omega = \gamma/2$ solutions are bounded.

Therefore, according to Proposition 3.2, the IBVP (3.57) has solution of the form

$$\Phi(x,t) = \sqrt{\frac{\cosh(\Omega_b t)}{\cosh(\Omega t)}} \times \exp\left[-\frac{e^{\gamma t}}{2}\Big(\Omega\tanh(\Omega t) - \Omega_b\tanh(\Omega_b t)\Big)x^2\right] \times \Psi(\eta(x,t),\tau(t)), \tag{3.60}$$

where the axuiliary functions

$$\eta(x,t) = \frac{e^{\gamma t/2}\,x}{\cosh(\Omega t)}, \qquad \tau(t) = \frac{\tanh(\Omega t)}{\Omega}, \qquad t > 0, \tag{3.61}$$

with inverse $t(\tau) = \tanh^{-1}(\Omega\tau)/\Omega$ for $0 < \tau < 1/\Omega$. Since $\Omega > \Omega_b$, for each $t > 0$ the Gaussian term in solution (3.60) approaches zero as $x \to \infty$. But the behavior of solution $\Phi(x,t)$ at $x = \infty$ depends also on the behavior of $\Psi(\eta(x,t),\tau(t))$ at $x = \infty$. Therefore, in what follows, we give some concrete examples vanishing at infinity.

Fundamental solution: If we take the model (3.57) with Dirac delta initial condition $\Phi(x,0) = A\delta(x - x_0)$, $0 < x < \infty$, $x_0 > 0$, $A > 0$, then according to our result (3.60), the solution becomes

$$
\begin{aligned}
K(x, x_0; t) \;&=\; A\sqrt{\frac{\Omega\cosh(\Omega_b t)}{2\pi\sinh(\Omega t)}}\times\exp\left[-\frac{e^{\gamma t}}{2}\left(\Omega\tanh(\Omega t)-\Omega_b\tanh(\Omega_b t)\right)x^2\right] \qquad (3.62)\\
&\times\;\left(\exp\left[-\frac{\Omega(x-x_0e^{-\gamma t/2}\cosh(\Omega t))^2}{e^{-\gamma t}\sinh(2\Omega t)}\right]-\exp\left[-\frac{\Omega(x+x_0e^{-\gamma t/2}\cosh(\Omega t))^2}{e^{-\gamma t}\sinh(2\Omega t)}\right]\right),
\end{aligned}
$$

which tends to zero as $t\to\infty$ for any $x > 0$ and $\Omega > \gamma/2$. Furthermore, $\lim_{t\to 0}\Phi(x, t) = A\,(\delta(x-x_0)-\delta(x+x_0)) = A\delta(x-x_0)$ where the second Delta function becomes zero since $x = -x_0$ is not in our domain $0 < x < \infty,\ x_0 > 0$. Also, since $\Omega > \Omega_b \geq 0$, $\gamma > 0$, then for $x\to\infty$ the solution approaches zero for any $t > 0$. In Fig.3.2a, we plot solution (3.62) corresponding to Dirac-delta initial distribution centered at $x = 1$ and with strength $A = 20$.

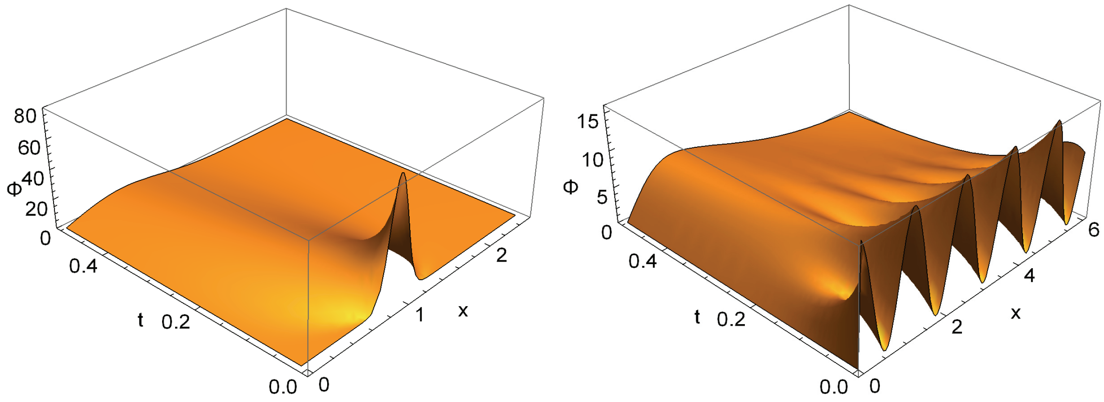
Example 3.2 *Now we consider the model (3.57) with smooth and bounded, periodic IC*

$$
\Phi(x, 0) = A\,(1+\cos(Bx)), \quad 0 < x < \infty, \qquad (3.63)
$$

where $A > 0$ is amplitude and $B\in\mathbb{R}$ is frequency. Then, we have solution (3.60) where

$$
\begin{aligned}
\Psi(\eta(x, t), \tau(t)) = &A\times Erf\left(\frac{\eta(x, t)}{\sqrt{2\tau(t)}}\right) \qquad (3.64)\\
&+\frac{A\,e^{-\frac{B^2\tau(t)}{2}}}{2}\left[e^{-iB\eta(x,t)}\,Erf\left(\frac{\eta(x, t)-iB\tau(t)}{\sqrt{2\tau(t)}}\right)+e^{iB\eta(x,t)}\,Erf\left(\frac{\eta(x, t)+iB\tau(t)}{\sqrt{2\tau(t)}}\right)\right],
\end{aligned}
$$

and $\eta(x, t)$, $\tau(t)$ are given in (3.61). For certain constant parameters the behavior of solution is shown in Fig.3.2b. We observe that, the solution is kept equal to zero on the fixed boundary $x = 0$, and near the initial time $t = 0$ the solution is oscillatory in space with frequency that can be controlled by parameter $B\in\mathbb{R}$. When time increases the amplitude of oscillations decreases approaching zero for any $x > 0$, as expected.



(a) (b)

Figure 3.2 (a) Solution (3.62) with $A = 20$, $\Omega_b = 0.5$, $\gamma = 2$, $\omega_0 = 1$, $x_0 = 1$ and $\Omega = \sqrt{5}/2$. (b) Solution with $A = 8$, $B = 5$, $\Omega_b = 0.5$, $\gamma = 1$, $\omega_0 = 0.5$ and $\Omega = 1/\sqrt{2}$.

MODEL 2 : IBVP with inhomogeneous boundary condition

Next, consider the following model with homogeneous initial condition $\Phi(x,0) = 0,\ 0 < x < \infty$, and time-dependent boundary condition $\Phi(0,t) = D(t),\ t > 0$

$$\begin{cases} \Phi_t = \frac{1}{2}e^{-\gamma t}\Phi_{xx} + \frac{\gamma}{2}x\Phi_x - \frac{\gamma^2}{8}e^{\gamma t}x^2\Phi, & 0 < x < \infty, \quad t > 0, \\ \Phi(x,0) = 0, \quad 0 < x < \infty, & \\ \Phi(0,t) = D(t), \quad t > 0. & \end{cases} \tag{3.65}$$

Then the corresponding characteristic equation becomes $\ddot{r} + \gamma\dot{r} = 0$, and the solution to the IBVP (3.65) is obtained as

$$\Phi(x,t) = \sqrt{\operatorname{sech}(\gamma t/2)} \times \exp\left[-\frac{\gamma e^{\gamma t}}{4}\tanh\left(\frac{\gamma}{2}t\right)x^2\right] \times \left(-\int_0^{\tau(t)} K_\eta(\eta(x,t),\tau(t)-\tau')D_0(\tau')d\tau'\right),$$

where boundary data

$$D_0(\tau) = \sqrt{\cosh\left(\tanh^{-1}(\gamma\tau/2)\right)} \times D\left(\frac{2}{\gamma}\tanh^{-1}(\gamma\tau/2)\right), \quad 0 < \tau < 2/\gamma, \tag{3.66}$$

and the auxilary functions

$$\eta(x,t) = \frac{e^{\gamma t/2}\, x}{\cosh(\gamma t/2)}, \qquad \tau(t) = \frac{2\tanh(\gamma t/2)}{\gamma}, \quad t > 0. \tag{3.67}$$

We note that, in general for an arbitrary boundary data $D(t)$, the expression for $D_0(\tau)$ will be complicated, which creates also difficulties in solution. However, based on exact solutions of the heat equation with zero initial data and time-dependent boundary conditions, for some special choices of $D(t)$ one can obtain exact solutions. Here, we write only two of them as follows.

Case i. Let us take $D(t) = A\sqrt{\mathrm{sech}(\gamma t/2)}$, $A > 0$. Then, the solution becomes

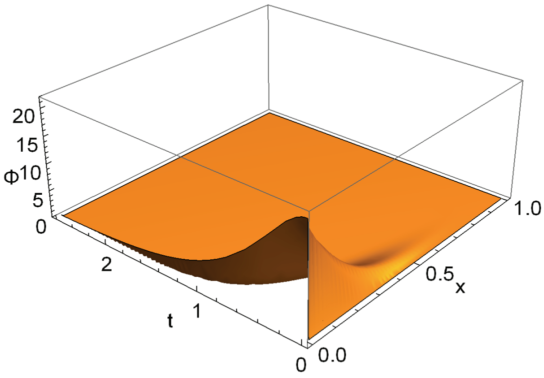
$$\Phi(x,t) = A\sqrt{\mathrm{sech}(\gamma t/2)} \times \exp\left[-\frac{\gamma e^{\gamma t}}{4}\tanh\left(\frac{\gamma}{2}t\right)x^2\right] \times \mathrm{Erfc}\left[\frac{\sqrt{\gamma}\, e^{\gamma t/2}\, x}{\sqrt{2\sinh(\gamma t)}}\right]. \tag{3.68}$$

The distribution $D(t)$ on the boundary $x = 0$ is maximum at time $t = 0$, and as γ increases the spreading of boundary data increases which causes approaching zero more rapidly. But notice that, initially while the, for instance, temperature is zero everywhere for $x \geq 0$, the boundary data is $D(0) \neq 0$ at initial time $t = 0$. So the compatibility condition is not satisfied at $(x,t) = (0,0)$. This means that there is a jump of temperature at the extremity as soon as $t > 0$. Then as time increases, this jumping temperature of which rate of decrease influenced by parameter γ will go to zero. For any $t > 0$ we have $\Phi(\infty, t) = 0$, as one can see in Fig.3.3a.

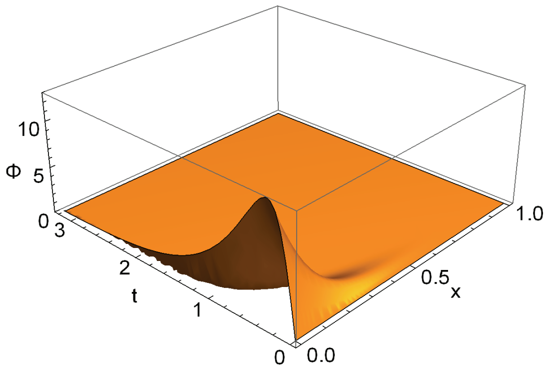
Case ii. For $D(t) = A\sqrt{\mathrm{sech}(\gamma t/2)}\tanh(\gamma t/2)$, $A > 0$, the compatibility condition is satisfied since $D(0) = 0$. So there is no jump of temperature at the extremity as soon as $t > 0$. In that case the distribution on the boundary is maximum at time $t = (2/\gamma)\tanh^{-1}(\sqrt{2/3})$ and as time increases it approaches zero smoothly. Then the solution is

$$\begin{aligned}\Phi(x,t) &= A\sqrt{\mathrm{sech}(\gamma t/2)} \times \tanh(\gamma t/2) \times \exp\left[-\frac{\gamma e^{\gamma t}}{4}\tanh\left(\frac{\gamma}{2}t\right)x^2\right] \\ &\times \left(\left(1 + \frac{\gamma e^{\gamma t} x^2}{\sinh(\gamma t)}\right)\mathrm{Erfc}\left[\frac{\sqrt{\gamma}\, e^{\gamma t/2}\, x}{\sqrt{2\sinh(\gamma t)}}\right] - \frac{\sqrt{2\gamma}\, e^{\gamma t/2}\, x}{\sqrt{\pi\sinh(\gamma t)}}\exp\left[-\frac{\gamma e^{\gamma t} x^2}{2\sinh(\gamma t)}\right]\right),\end{aligned} \tag{3.69}$$

which is localized in space and time, and for certain constant parameters its behavior is shown in Fig.3.3b.



(a)



(b)

Figure 3.3 (a) Solution (3.68) with $A = 20$, $\gamma = 7$. (b) Solution (3.69) with $A = 20$, $\gamma = 7$.

3.2.2. Analytical solution of the IBVP with Neumann boundary condition

Here, we study the IBVP with Neumann boundary condition and the results are formulated as follows.

Proposition 3.3 *The IBVP defined by equation (3.45a), initial condition (3.45b) and Neumann boundary condition imposed at $x = 0$ as follows*

$$\Phi_x(0, t) = N(t), \qquad 0 < t < T, \tag{3.70}$$

for the given smooth function $N(t)$, has solution in the form (3.46) where $\Psi(\eta, \tau)$ is the solution of IBVP defined by equation (3.49a), initial condition (3.49b) and with inhomogeneous Neumann BC

$$\Psi_\eta(0, \tau) = N_0(\tau), \qquad 0 < \tau < \tau(T), \tag{3.71}$$

where the boundary data is

$$N_0(\tau) = \sqrt{r_1^3(t(\tau))}\exp\left[\int_0^{t(\tau)}\left(\frac{b(t')}{2} - \mu(t')f_0(t')\right)dt'\right]N(t(\tau)), \quad 0 < \tau < \tau(T). \quad (3.72)$$

Proof The proof follows the similar line with proof of Proposition 3.2, but the only difference is that the inhomogeneous Neumann BC (3.70) leads to inhomogeneous Neumann boundary data (3.72). In particular, if $N(t) = 0$, then one has homogeneous Neumann BC $\Psi_\eta(0, \tau) = 0$. □

Integral Representation and Fundamental Solution **:**
According to Proposition 3.3, by using the integral representation of solution to heat IBVP (2.32) with Neumann boundary condition (3.71), we obtain solution to the IBVP defined by (3.45a), (3.45b) and (3.70) in integral form

$$\Phi(x,t) = \frac{1}{\sqrt{r_1(t)}} \times \exp\left[-\int_0^t\left(\frac{b(t')}{2} - \mu(t')f_0(t')\right)dt'\right] \times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot r_1(t)}{r_1(t)} + b(t)\right)x^2\right]$$
$$\times\left(\int_0^\infty G_N(\eta(x,t),\xi,\tau(t))\Phi(\xi,0)d\xi - \int_0^{\tau(t)} K(\eta(x,t),\tau(t)-\tau')N_0(\tau')d\tau'\right), \quad (3.73)$$

where $G_N(\eta,\xi,\tau) = K(\eta-\xi,\tau) + K(\eta+\xi,\tau)$ denotes Neumann heat kernel.

Fundamental solution **:** When the initial and boundary conditions are taken as $\Phi(x,0) = \delta(x - x_0)$, $0 < x < \infty$, $x_0 > 0$, and $\Phi_x(0,t) = 0$ respectively, then by using fundamental solution for heat IBVP with Neumann boundary condition $\Psi_\eta(0\tau) = 0$, given in (2.31), it follows that the Neumann IBVP for diffusion equation (3.45a) has fundamental solution

$$K(x,x_0;t) = \frac{1}{\sqrt{2\pi r_2(t)}} \times \exp\left[-\int_0^t\left(\frac{b(t')}{2} - \mu(t')f_0(t')\right)dt'\right] \times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot r_1(t)}{r_1(t)} + b(t)\right)x^2\right]$$
$$\times\left(\exp\left[-\frac{(x - x_0 r_1(t))^2}{2r_1(t)r_2(t)}\right] + \exp\left[-\frac{(x + x_0 r_1(t))^2}{2r_1(t)r_2(t)}\right]\right). \quad (3.74)$$

Example 3.3 *If we consider the equation in (3.57) and IC* $\Phi(x, 0) = A\delta(x - x_0)$, *defined on* $0 < x < \infty$, $x_0 > 0$, *but now we have homogeneous Neumann boundary condition* $\Phi_x(0, t) = 0$, $t > 0$, *then the fundamental solution is obtained*

$$K(x, x_0; t) = A\,\sqrt{\frac{\Omega\cosh(\Omega_b t)}{2\pi\sinh(\Omega t)}} \times \exp\left[-\frac{e^{\gamma t}}{2}\left(\Omega\tanh(\Omega t) - \Omega_b\tanh(\Omega_b t)\right)x^2\right] \qquad (3.75)$$
$$\times\left(\exp\left[-\frac{\Omega(x - x_0 e^{-\gamma t/2}\cosh(\Omega t))^2}{e^{-\gamma t}\sinh(2\Omega t)}\right] + \exp\left[-\frac{\Omega(x + x_0 e^{-\gamma t/2}\cosh(\Omega t))^2}{e^{-\gamma t}\sinh(2\Omega t)}\right]\right),$$

which in the long time limit approaches zero for any $x > 0$ if $\Omega > \gamma/2$. On the other hand, the initial condition is indeed satisfied, i.e $\lim_{t\to 0}\Phi(x, t) = A\delta(x - x_0)$. The difference of solution (3.75) from solution obtained in (3.62) is that the sign of the second exponential term in the last line, which originates from the solution of the corresponding heat problem, is negative. And as in the previous investigation, since $\Omega > \Omega_b \geq 0$, $\gamma > 0$, then for $x \to \infty$ the solution approaches zero for any $t > 0$. In Fig.3.4 we plot solution (3.75) corresponding to Dirac-delta initial data centered at $x = 1$ and with strength $A = 20$.

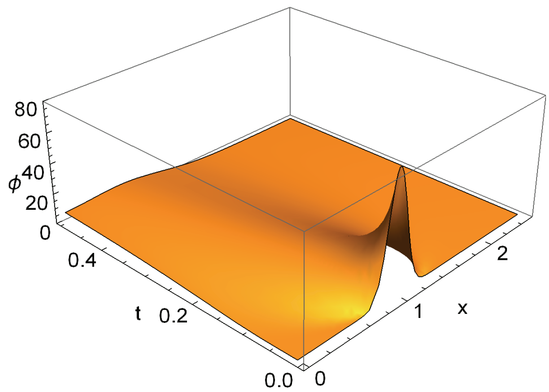


Figure 3.4 Solution (3.75) with $A = 20$, $\Omega_b = 0.5$, $\gamma = 2$, $\omega_0 = -1$, $x_0 = 1$, $\Omega = \sqrt{5}/2$.

3.2.3. Analytical solution of the IBVP with Robin boundary condition

Proposition 3.4 *The IBVP defined by defined by equation (3.45a), initial condition (3.45b) and Robin boundary condition imposed at* $x = 0$

$$\Phi_x(0,t) + \beta_1(t)\Phi(0,t) = 0, \qquad 0 < t < T, \tag{3.76}$$

where $\beta_1(t) \neq 0$ *is a given real-valued smooth function of time* $0 < t < T$, *has solution of the form (3.46) where* $\Psi(\eta, \tau)$ *is the solution of the IBVP (3.49a), (3.49b) and Robin boundary condition*

$$\Psi_\eta(0,\tau) + \beta_1(t(\tau))r_1(t(\tau))\Psi(0,\tau) = 0, \qquad 0 < \tau < \tau(T). \qquad \square \tag{3.77}$$

In that case the integral representation of solution to Robin IBVP for the diffusion type equation becomes

$$\begin{aligned}\Phi(x,t) \;=\; & \frac{1}{\sqrt{r_1(t)}} \times \exp\left[-\int_0^t \left(\frac{b(t')}{2} - \mu(t')f_0(t')\right)dt'\right] \times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)x^2\right] \\ & \times \left(\int_0^\infty G_N(\eta(x,t),\xi,\tau(t))\Phi_0(\xi)d\xi - \int_0^{\tau(t)} K(\eta(x,t),\tau(t)-\tau')Q_2(\tau')d\tau'\right),\end{aligned} \tag{3.78}$$

where the unknown function $Q_2(\tau)$ is obtained by solving the following equation

$$Q_2(\tau) = \beta_1(t(\tau))r_1(t(\tau)) \times \left(\int_0^\tau \frac{Q_2(\tau')}{\sqrt{2\pi(\tau-\tau')}})d\tau' - 2\int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\Phi_0(\xi)d\xi\right), \tag{3.79}$$

which is a second-kind Volterra type integral equation. It is seen that solving IBVP with Robin BC on the half-line for diffusion type equation requires solving second-kind Volterra integral equation which is a formidable task due to the variable coefficients.

We plan to investigate the details of the Robin IBVP in a future work.

CHAPTER 4

GENERALIZED DIFFUSION TYPE PROBLEMS WITH MOVING BOUNDARIES

In this chapter, we study initial-boundary value problems in a time-dependent semi-infinite domain $s(t) < x < \infty,\ 0 < t < T$ for the generalized diffusion type equation (1.1) with variable coefficients and Dirichlet, Neumann and Robin type boundary conditions imposed at the boundary $x = s(t)$. We provide analytical solution and present exactly solvable models.

4.1. Analytical Solution of the Dirichlet IBVP with Moving Boundary

First, we consider the mIBVP for a generalized diffusion type equation defined on $s(t) < x < \infty,\ 0 < t < T$ as follows

$$\Phi_t = \frac{1}{2\mu(t)}\Phi_{xx} - [a(t) - b(t)x]\Phi_x + \mu(t)[\frac{\omega^2(t)}{2}x^2 - f(t)x + f_0(t)]\Phi, \tag{4.1a}$$

$$\Phi(x,0) = \Phi^0(x), \quad s(0) < x < \infty, \tag{4.1b}$$

$$\Phi(s(t),t) = D(t), \quad 0 < t < T, \tag{4.1c}$$

where all time-dependent parameters are given real-valued smooth functions in their domains, $\mu(t) > 0$, $\mu(0) = 1$ and time-dependent boundary $s(t)$ is twice continuously differentiable function. We obtain analytical solution to the mIBVP under the condition that the boundary propagates according to an associated classical equation of motion determined by the time-dependent parameters of the diffusion type equation. For this, we solve the corresponding nonlinear Riccati type dynamical system, that simultaneously determines the solution of the diffusion type problem and the moving boundary $s(t)$. The results are formulated in the following proposition.

Proposition 4.1 *If the boundary function $s(t)$ is of the form*

$$s(t) = r_g^{\alpha}(t) \equiv \alpha_1 r_1(t) + \alpha_2 r_2(t) + r_p(t), \qquad \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2\,, \tag{4.2}$$

where $r_1(t), r_2(t)$ are positive, linearly independent homogeneous solutions and $r_p(t)$ is a particular solution of the inhomogeneous characteristic equation

$$\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \left[\omega^2(t) + \left(\dot{b}(t) + \frac{\dot{\mu}(t)}{\mu(t)}b(t) - b^2(t)\right)\right] r = \dot{a}(t) - a(t)\left[b(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right] + f(t), \tag{4.3}$$

satisfying IC's $r_1(0) = 1$, $\dot{r}_1(0) = -b(0)$, $r_2(0) = 0$, $\dot{r}_2(0) = 1$ and $r_p(0) = 0, \dot{r}_p(0) = a(0)$ respectively, then the mIBVP (4.1) has solution

$$\Phi_g^{\alpha}(x,t) = \frac{1}{\sqrt{r_1(t)}} \times \exp\left[-\int_0^t \left(\frac{b(t')}{2} - \mu(t')f_0(t')\right)\right] \times \exp\left[-\int_0^t L_g^{\alpha}(t')dt'\right] \tag{4.4}$$
$$\times\ \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)\left(x - r_g^{\alpha}(t)\right)^2\right] \times \exp\Big[-p_g^{\alpha}(t)(x - r_g^{\alpha}(t))\Big] \times \Psi\left(\eta_g^{\alpha}(x,t), \tau(t)\right),$$

where one has the Lagrangian type function for the moving boundary

$$L_g^{\alpha}(t) = \frac{\mu(t)}{2}\Big((\dot{r}_g^{\alpha}(t) + b(t)r_g^{\alpha}(t) - a(t))^2 - \omega^2(t)(r_g^{\alpha})^2(t) + 2f(t)r_g^{\alpha}(t)\Big), \tag{4.5}$$

the corresponding generalized momentum

$$p_g^{\alpha}(t) = \mu(t)\left(\dot{r}_g^{\alpha}(t) + b(t)r_g^{\alpha}(t) - a(t)\right), \tag{4.6}$$

the coordinate transformation $(x,t) \mapsto (\eta, \tau)$ as follows

$$\eta_g^{\alpha}(x,t) = \frac{x - r_g^{\alpha}(t)}{r_1(t)}, \qquad \tau(t) = \frac{r_2(t)}{r_1(t)}, \qquad 0 < t < T, \tag{4.7}$$

and $\Psi(\eta, \tau)$ is solution of the Dirichlet IBVP for the standard heat equation defined on the *half-line*

$$\Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \qquad 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \tag{4.8a}$$

$$\Psi(\eta, 0) = \Phi^0(\eta + \alpha_1)\, e^{\alpha_2 \eta}, \quad 0 < \eta < \infty, \tag{4.8b}$$

$$\Psi(0, \tau) = D_0(\tau), \quad 0 < \tau < \tau(T), \tag{4.8c}$$

with boundary data

$$D_0(\tau) = \sqrt{r_1(t(\tau))}\exp\left[\int_0^{t(\tau)}\left(\frac{b(t')}{2} - \mu(t')f_0(t') + L_g^\alpha(t')\right)dt'\right]D(t(\tau)). \tag{4.9}$$

Proof To transfom the moving boundary to the fixed one, first we define new variable $y = x - s(t)$ and denote $\Phi(x,t) = \tilde{\Phi}(y(x,t),t)$. Then performing time and space differentiations of $\Phi(x,t)$, we get

$$\Phi_t = -\dot{s}(t)\tilde{\Phi}_y + \tilde{\Phi}_t, \quad \Phi_x = \tilde{\Phi}_y, \quad \Phi_{xx} = \tilde{\Phi}_{yy}, \tag{4.10}$$

and using the variable for initial and boundary conditions

$$\Phi(x, 0) = \Phi^0(x) \quad \Longrightarrow \quad \tilde{\Phi}(y, 0) = \Phi(y + s(0), 0) = \Phi^0(y + s(0)), \tag{4.11}$$

$$\Phi(s(t), t) = D(t), \quad \Longrightarrow \quad \tilde{\Phi}(0, t) = D(t), \tag{4.12}$$

we obtain the following IBVP defined on $0 < y < \infty$ for the new function $\tilde{\Phi}(y,t)$

$$\begin{cases} \tilde{\Phi}_t = \frac{1}{2\mu(t)}\tilde{\Phi}_{yy} + \left[b(t)(y + s(t)) - a(t) + \dot{s}(t)\right]\tilde{\Phi}_y + \mu(t)\left[\frac{\omega^2(t)}{2}(y + s(t))^2 - f(t)(y + s(t)) + f_0(t)\right]\tilde{\Phi}, \\ \tilde{\Phi}(y, 0) = \Phi^0(y + s(0)), \quad 0 < y < \infty, \\ \tilde{\Phi}(0, t) = D(t), \quad 0 < t < T. \end{cases} \tag{4.13}$$

This IBVP (4.13) is defined on the half-line $0 < y < \infty$ as expected, but the equation for the new function $\tilde{\Phi}(y,t)$ is more complicated than the original one since the boundary

$s(t)$ has moved to the equation by contributing to the convection and reaction coefficients. Motivated from works mentioned in previous chapter, we assume solution

$$\tilde{\Phi}(y,t) = e^{F(y,t)} \times \Psi(\eta(y,t), \tau(t)), \tag{4.14}$$

where the functions $F(y,t)$ and $\eta(y,t)$ are of the form

$$F(y,t) = -\rho(t)y^2/2 - p(t)y + \gamma(t)/2, \qquad \eta(y,t) = e^{g(t)}y,$$

with $\rho(t)$, $g(t)$, $\tau(t)$, $p(t)$, $\gamma(t)$ to be determined. Since we have

$$\begin{aligned}
\tilde{\Phi}_t &= \left([-\frac{\dot{\rho}(t)}{2}y^2 - \dot{p}(t)y + \frac{\dot{\gamma}(t)}{2}]\Psi + \dot{g}(t)e^{g(t)}y\Psi_\eta + \dot{\tau}(t)\Psi_\tau \right) e^{F(y,t)} \\
\tilde{\Phi}_y &= \left(-[p(t) + \rho(t)y]\Psi + e^{g(t)}\Psi_\eta \right) e^{F(y,t)} \\
\tilde{\Phi}_{yy} &= \left([p^2(t) - \rho(t) + 2p(t)\rho(t)y + \rho^2(t)y^2]\Psi - 2(p(t) + \rho(t)y)e^{g(t)}\Psi_\eta + e^{2g(t)}\Psi_{\eta\eta} \right) e^{F(y,t)},
\end{aligned} \tag{4.15}$$

then substituting the derivatives (4.15) into (4.13), we obtain

$$\begin{aligned}
\dot{\tau}(t)\Psi_\tau = &\ \frac{e^{2g(t)}}{2\mu(t)}\Psi_{\eta\eta} - \left[(\dot{g}(t) + \frac{\rho(t)}{\mu(t)} - b(t))y - \dot{s}(t) - b(t)s(t) + \frac{p(t)}{\mu(t)} + a(t) \right] e^{g(t)}\Psi_\eta \\
+ &\ \left[\frac{-\dot{\gamma}(t)}{2} - \frac{\rho(t)}{2\mu(t)} + \frac{p^2(t)}{2\mu(t)} - b(t)s(t)p(t) - \dot{s}(t)p(t) + a(t)p(t) \right] \Psi \\
+ &\ \left[\left(\dot{p}(t) - \dot{s}(t)\rho(t) - b(t)s(t)\rho(t) + \frac{p(t)\rho(t)}{\mu(t)} + a(t)\rho(t) + \mu(t)\omega^2(t)s(t) - b(t)p(t) \right) y \right] \Psi \\
+ &\ \left[\left(\frac{\dot{\rho}(t)}{2} + \frac{\rho^2(t)}{2\mu(t)} + \frac{\mu(t)\omega^2(t)}{2} - b(t)\rho(t) \right) y^2 + \mu(t)\left(f_0(t) - f(t)s(t) + \frac{\omega^2(t)s^2(t)}{2} - f(t)y \right) \right] \Psi.
\end{aligned} \tag{4.16}$$

Then, the function $\Psi(\eta,\tau)$ satisfies $\Psi_\tau = (1/2)\Psi_{\eta\eta}$ and IC $\Psi(\eta,0) = \Phi^0(\eta + \alpha_1)\ e^{\alpha_2\eta}$ if the auxiliary functions and the moving boundary $s(t)$ with initial position $s(0) = \alpha_1$ and initial velocity $\dot{s}(0) = \alpha_2 + a(0) - \alpha_1 b(0)$ satisfy the following nonlinear system of six differential equations

$$\begin{aligned}
\dot{\rho}(t) + \frac{\rho^2(t)}{\mu(t)} - 2b(t)\rho(t) + \mu(t)\omega^2(t) = 0\,, \qquad \rho(0) = 0, \\
\dot{g}(t) + \frac{\rho(t)}{\mu(t)} - b(t) = 0\,, \qquad g(0) = 0, \\
\dot{\tau}(t) - \frac{e^{2g(t)}}{\mu(t)} = 0\,, \qquad \tau(0) = 0,
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
\dot{s}(t) + b(t)s(t) - \frac{p(t)}{\mu(t)} - a(t) = 0\,, \qquad s(0) = \alpha_1, \\
\dot{p}(t) - b(t)p(t) + \mu(t)\omega^2(t)s(t) - \mu(t)f(t) = 0\,, \qquad p(0) = \alpha_2, \\
\dot{\gamma}(t) + \frac{p^2(t)}{\mu(t)} - \mu(t)\omega^2(t)s^2(t) + 2\mu(t)f(t)s(t) - 2\mu(t)f_0(t) + \frac{\rho(t)}{\mu(t)} = 0, \;\; \gamma(0) = 0,
\end{aligned} \tag{4.18}$$

where α_1 and α_2 are arbitrary real constants. The first system (4.17) is same as the one in (3.17), so does the solution, given in (3.20). In system (4.18), taking time derivative of the first equation and substituting into the second one, we obtain that $s(t)$ must satisfy the differential equation

$$\ddot{s} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{s} + \left[\omega^2(t) + \left(\dot{b}(t) + \frac{\dot{\mu}(t)}{\mu(t)}b(t) - b^2(t)\right)\right] s = \dot{a}(t) - a(t)\left[b(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right] + f(t), \tag{4.19}$$

and initial conditions $s(0) = \alpha_1$, $\dot{s}(0) = \alpha_2 + a(0) - \alpha_1 b(0)$. Therefore, its solution can be written in the form $s(t) = r_g^\alpha(t) \equiv \alpha_1 r_1(t) + \alpha_2 r_2(t) + r_p(t)$, where $r_1(t), r_2(t), r_p(t)$ are as defined in the statement of Proposition 4.1. Thus, the solution to the second system is obtained as follows

$$\begin{aligned}
s(t) &= r_g^\alpha(t), \\
p(t) &= p_g^\alpha(t) \equiv \mu(t)\left(\dot{r}_g^\alpha(t) + b(t)r_g^\alpha(t) - a(t)\right), \\
\gamma(t) &= -\int_0^t \left(\frac{(p_g^\alpha)^2(t')}{\mu(t')} + \mu(t')\left[-\omega^2(t')(r_g^\alpha)^2(t') + 2f(t')r_g^\alpha(t') - 2f_0(t')\right] + b(t') + \frac{\dot{r}_1(t')}{r_1(t')}\right) dt'.
\end{aligned} \tag{4.20}$$

Notice that the above non-linear Riccati type system determines both the moving boundary and the analytical solution of the diffusion problem, in terms of solutions to the second-order linear characteristic equation (4.3). Indeed, writing the auxiliary functions into the ansatz (4.14), we get

$$\begin{aligned}\tilde{\Phi}(y,t) &= \frac{1}{\sqrt{r_1(t)}} \times \exp\left[-\int_0^t \left(\frac{b(t')}{2} - \mu(t')f_0(t')\right)\right] \times \exp\left[-\int_0^t L_g^{\alpha}(t')dt'\right] \\ &\times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)y^2\right] \times \exp\left[-p_g^{\alpha}(t)y\right] \times \Psi\left(\frac{y}{r_1(t)}, \frac{r_2(t)}{r_1(t)}\right), \qquad (4.21)\end{aligned}$$

where $L_g^{\alpha}(t)$ is as defined in (4.5) and using back substitution $y = x - r_g^{\alpha}(t)$, we get the desired solution (4.4) satisfying the IC (4.1b). Lastly, we notice that $\tau(t)$ is positive and strictly increasing for $0 < t < T$, so that $\tau = \tau(t)$, $0 < t < T$ if and only if $t = t(\tau)$, $0 < \tau < \tau(T)$. Therefore, one can easily show that solution (4.4) will satisfy the Dirichlet BC (4.1c) if $\Psi(\eta, \tau)$ satisfies the Dirichlet BC given in (4.9), which completes the proof.

□

Obviously, the characteristic equation (4.3) is directly related to the transport process described by the equation (4.1a). Its solution is of the form $r_g^{\alpha}(t) = r_h^{\alpha}(t) + r_p(t)$, where the homogeneous solution $r_h^{\alpha}(t)$ is affected by diffusion, dilation and a first-order reaction with quadratic in position coefficient, while the particular solution $r_p(t)$ appears due to convection and/or a first-order reaction with linear in position coefficient. The solution of the pure IVP for the PDE (4.1a) is also described in terms of solution of the ODE (4.3), (Atılgan Büyükaşık & Bozacı, 2021). Therefore, the special choice of the boundary as to satisfy the characteristic equation appears as a natural setting for reduction to standard model and construction of exactly solvable models.

4.1.1. Integral representation and fundamental solution

Using the integral representation of solution to the Dirichlet IBVP for heat equation (4.8) and the result of Proposition 4.1, an integral representation of the solution to mIBVP (4.1) is found as

$$\begin{aligned}\Phi_g^{\alpha}(x,t) = &\frac{1}{\sqrt{r_1(t)}} \times \exp\left[-\int_0^t \left(\frac{b(t')}{2} - \mu(t')f_0(t')\right)\right] \times \exp\left[-\int_0^t L_g^{\alpha}(t')dt'\right] \\ &\times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)\left(x - r_g^{\alpha}(t)\right)^2\right] \times \exp\left[-p_g^{\alpha}(t)(x - r_g^{\alpha}(t))\right] \qquad (4.22) \\ &\times \left(\int_0^{\infty} G_D(\eta_g^{\alpha}(x,t), \xi, \tau(t))\Phi^0(\xi + \alpha_1)e^{\alpha_2\xi}d\xi - \int_0^{\tau(t)} K_{\eta}(\eta_g^{\alpha}(x,t), \tau(t) - \tau')\Psi(0,\tau')d\tau'\right),\end{aligned}$$

whenever the given initial and boundary data guarantee its convergence.

We notice that solution properties depend on the initial data $\Phi^0(x)$, the time-dependent parameters of equation (4.1a), and the moving boundary $s(t)$. When we have $s(t) = 0$ and $a(t) = f(t) = 0$, then the mIBVP reduces to the IBVP defined on the half-line which we consider in Proposition 3.2. Therefore, the influence of the moving boundary can be seen by comparing solutions given by (4.4) and (3.46). We see that the prescribed moving boundary not only acts as displacement, but as expected it induces an additional exponential factor with linear in space argument and the Lagrangian function, $L_g^\alpha(t) \equiv L(r(t), \dot{r}(t), t)$ which affects the solution amplitude. The Lagrangian type function describes the motion of the boundary point and the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)(r(t), \dot{r}(t), t) - \frac{\partial L}{\partial r}(r(t), \dot{r}(t), t) = 0,$$

recovers the Newtonian equation of motion given by (4.3). The corresponding action functional $S_g^\alpha(t) = \int_0^t L_g^\alpha(t')dt'$ carries the properties of the boundary function and the time-dependent parameters, and influences the amplitude of the solution. For instance, if $S_g^\alpha(t)$ is positive and increasing, then the amplitude decreases in time, while if $S_g^\alpha(t)$ is oscillatory, the amplitude will also oscillate.

***Fundamental Solution*:**

Consider equation (4.1a) with shifted Dirac-delta initial condition $\Phi(x, 0) = \delta(x - x_0)$ on $\alpha_1 < x < \infty,\ x_0 > \alpha_1$, where $\alpha_1 = s(0)$, and homogeneous Dirichlet boundary data $D(t) = 0,\ t > 0$. Then it reduces to the Dirichlet problem for standard heat equation on the half-line with initial condition $\Psi(\eta, 0) = \delta(\eta - (x_0 - \alpha_1))\, e^{\alpha_2 \eta}$ and Dirichlet boundary condition $\Psi(0, \tau) = 0$. By using fundamental solution for heat problem, mIBVP for diffusion type equation with boundary $s(t) = r_g^\alpha(t)$, as described in Proposition 4.1, has fundamental solution of the form

$$\begin{aligned} K_g^\alpha(x, x_0; t) &= e^{\alpha_2(x_0-\alpha_1)} \times \frac{1}{\sqrt{2\pi r_2(t)}} \times \exp\left[-\int_0^t \left(\frac{b(t')}{2} - \mu(t') f_0(t')\right)\right] \times \exp\left[-\int_0^t L_g^\alpha(t')dt'\right] \\ &\times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)\left(x - r_g^\alpha(t)\right)^2\right] \times \exp\left[-p_g^\alpha(t)(x - r_g^\alpha(t))\right] \qquad (4.23) \\ &\times \left(\exp\left[-\frac{(x - r_g^\alpha(t) - (x_0 - \alpha_1) r_1(t))^2}{2 r_1(t) r_2(t)}\right] - \exp\left[-\frac{(x - r_g^\alpha(t) + (x_0 - \alpha_1) r_1(t))^2}{2 r_1(t) r_2(t)}\right]\right). \end{aligned}$$

Clearly, it is seen that the fundamental solution is completely determined by the time-dependent parameters of the generalized diffusion type equation and the boundary $s(t) = r_g^{\alpha}(t)$. For instance, in the absence of the reaction term with quadratic in x coefficient ($\omega(t) = 0$), for given $b(t)$ we have the relation $\dot{r}_1(t)/r_1(t) = -b(t)$, so that the Gaussian term in above solution vanishes as expected. On the other side, even when $a(t) = f(t) = 0$, there will be shifting of position coordinate by $r_h^{\alpha}(t)$ due to moving boundary, and exponential term with linear in x argument will be present in the solution.

However, in general, evaluating the integrals in (4.22) explicitly is a formidable task and requires numerical or asymptotic approaches. Due to this, we provide exact solutions corresponding only to homogeneous boundary data in next models.

4.1.2. Exactly solvable models with moving boundary

In this section, we consider three types of models as diffusion-reaction, convection-diffusion and convection-diffusion-reaction type as follows.

4.1.2.1. Model 1 : Diffusion-Reaction type mIBVP

First we study the following model

$$\begin{cases} \Phi_t = \frac{1}{2}e^{-\gamma t}\Phi_{xx} - \frac{\omega_0^2}{2}e^{\gamma t}x^2\Phi, & s(t) < x < \infty, \quad t > 0, \\ \Phi(x,0) = \Phi^0(x), \quad s(0) < x < \infty, \\ \Phi(s(t),t) = 0, \quad t > 0, \end{cases} \tag{4.24}$$

with exponentially decaying diffusion coefficient where $\mu(t) = e^{\gamma t}$, $\gamma > 0$, and reaction term with time-dependent and quadratic in x coefficient, where $\omega^2(t) = -\omega_0^2$, $\omega_0 > 0$. The corresponding characteristic equation is homogeneous

$$\ddot{r} + \gamma\dot{r} - \omega_0^2 r = 0, \quad t > 0, \tag{4.25}$$

with damping $\Gamma(t) = \gamma > 0$ due to the time-dependent diffusion coefficient. It has smooth and positive solutions increasing with time

$$r_1(t) = \frac{\omega_0}{\Omega} e^{-\gamma t/2} \cosh(\Omega t + \sigma), \qquad r_2(t) = \frac{e^{-\gamma t/2}}{\Omega} \sinh(\Omega t), \quad t > 0, \tag{4.26}$$

with

$$\Omega = \sqrt{\gamma^2/4 + \omega_0^2}\,, \qquad \sigma = \tanh^{-1}(\gamma/(2\Omega)),$$

and satisfying the initial conditions $r_1(0) = 1$, $\dot{r}_1(0) = 0$, $r_2(0) = 0$, $\dot{r}_2(0) = 1$. Since $\Omega > \gamma/2$, both $r_1(t)$ and $r_2(t)$ tend to infinity as time increases. Then, if the position of the boundary changes according to $s(t) = r_h^\alpha(t) \equiv \alpha_1 r_1(t) + \alpha_2 r_2(t)$, where $\alpha_1 = s(0)$ is the initial position and $\alpha_2 = \dot{s}(0)$ is the initial velocity, then the mIBVP (4.24) has solution

$$\begin{aligned} \Phi_h^\alpha(x,t) = &\sqrt{\frac{\Omega e^{\gamma t/2}}{\omega_0 \cosh(\Omega t + \sigma)}} \times \exp\left[-\int_0^t L_h^\alpha(t')dt'\right] \\ \times\ &\exp\left[-\frac{e^{\gamma t}}{2}\left(\Omega \tanh(\Omega t + \sigma) - \frac{\gamma}{2}\right)(x - r_h^\alpha(t))^2\right] \times \exp\left[-p_h^\alpha(t)(x - r_h^\alpha(t))\right] \\ \times\ &\Psi\left(\eta_h^\alpha(x,t), \tau(t)\right), \end{aligned} \tag{4.27}$$

where

$$L_h^\alpha(t) = \frac{e^{\gamma t}}{2}\left((\dot{r}_h^\alpha(t))^2 + \omega_0^2 (r_h^\alpha(t))^2\right), \qquad p_h^\alpha(t) = e^{\gamma t}\dot{r}_h^\alpha(t), \tag{4.28}$$

and classical action

$$S_h^\alpha(t) = \int_0^t L_h^\alpha(t')dt' = \frac{\alpha_1^2\omega_0^2\gamma - \alpha_2^2\gamma + 4\alpha_1\alpha_2\omega_0^2}{8\Omega^2}(\cosh(2\Omega t) - 1) + \frac{\alpha_2^2 + \alpha_1^2\omega_0^2}{4\Omega}\sinh(2\Omega t),$$

are smooth and increasing in time functions. Also, in (4.27) we have

$$\eta_h^\alpha(x,t) = \frac{\Omega e^{\gamma t/2}}{\omega_0 \cosh(\Omega t + \sigma)}(x - r_h^\alpha(t)), \qquad \tau(t) = \frac{\sinh(\Omega t)}{\omega_0 \cosh(\Omega t + \sigma)}\,, \tag{4.29}$$

and $\Psi(\eta, \tau)$ is solution of the following IBVP

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, & 0 < \eta < \infty, \quad 0 < \tau < 2/(\gamma + 2\Omega), \\ \Psi(\eta, 0) = \Phi^0(\eta + \alpha_1)\, e^{\alpha_2 \eta}, & 0 < \eta < \infty, \\ \Psi(0, \tau) = 0, & 0 < \tau < 2/(\gamma + 2\Omega). \end{cases} \tag{4.30}$$

In this model, since $\tau(t)$ is positive and increasing function of time, for $t > 0$, then its inverse $t = t(\tau) = 1/\Omega \tanh^{-1}(2\Omega\tau/(2 - \gamma\tau))$ is defined for $\tau < 2/(\gamma + 2\Omega)$. And we note that, since the PDE in this problem is symmetric with respect to space inversion and there is no external forcing, then by letting $\alpha_1 = 0$, $\alpha_2 = 0$ one has $r_h^\alpha(t) = 0$, $p_h^\alpha(t) = 0$, $L_h^\alpha(t) = 0$, and therefore from (4.27) one can easily recover the solution of the problem on the half-line $0 < x < \infty$. This allows us to see explicitly how the moving boundary affects the solution. Indeed, in the presence of a moving boundary we see that the amplitude of the solution is influenced by the Lagrangian $L_h^\alpha(t)$, the boundary also contributes to displacement in position by $r_h^\alpha(t)$, and momentum $p_h^\alpha(t)$ brings an additional exponential term with linear in position argument.

Fundamental solution: We consider mIBVP (4.24) with Dirac-delta data

$$\Phi^0(x) = \delta(x - x_0), \qquad \alpha_1 < x < \infty, \quad x_0 > \alpha_1. \tag{4.31}$$

In that case solution is found explicitly

$$\begin{aligned} K_h^\alpha(x, x_0; t) &= e^{\alpha_2(x_0 - \alpha_1)} \times \sqrt{\frac{\Omega e^{\gamma t/2}}{2\pi \sinh(\Omega t)}} \times \exp\left[-\int_0^t L_h^\alpha(t')dt'\right] \\ &\times \exp\left[-\frac{e^{\gamma t}}{2}\left(\Omega \tanh(\Omega t + \sigma) - \frac{\gamma}{2}\right)(x - r_h^\alpha(t))^2\right] \times \exp\left[-p_h^\alpha(t)(x - r_h^\alpha(t))\right] \\ &\times \left(\exp\left[-\frac{\left(x - r_h^\alpha(t) - (x_0 - \alpha_1) r_1(t)\right)^2}{2 r_1(t) r_2(t)}\right] - \exp\left[-\frac{\left(x - r_h^\alpha(t) + (x_0 - \alpha_1) r_1(t)\right)^2}{2 r_1(t) r_2(t)}\right]\right), \end{aligned} \tag{4.32}$$

where $\alpha_1 = s(0)$ and $\alpha_2 = \dot{s}(0)$ are initial position and velocity, respectively.

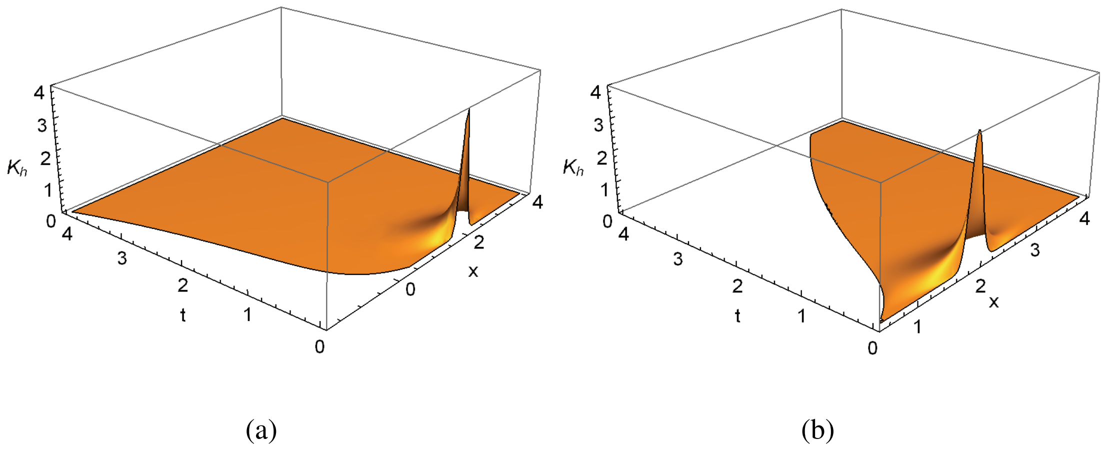


Figure 4.1 Solution (4.32) with $\alpha_1 = 0.5$, $\omega_0 = 1$, $\gamma = 3$, $x_0 = 2$ and (a) $\alpha_2 = -3.5$. (b) $\alpha_2 = 2$.

Here, since $\Omega > \gamma/2$, as $x \to \infty$ the solution approaches zero for any time $t > 0$, that is it satisfies the boundary condition $\Phi_h^\alpha(\infty, t) = 0$. In Fig.4.1a for parameters $\alpha_1 = 0.5$, $\alpha_2 < 0$ the boundary point initially located at $x = 0.5$ moves along x-axis in negative direction and with increasing speed, while in Fig.4.1b we take $\alpha_1 = 0.5$ and $\alpha_2 > 0$, and observe the boundary propagates in positive x-direction with speed, that can be controlled by parameters $\gamma > 0$ and $\omega_0 > 0$. In any case, the Dirac-delta distribution initially located at $x = 2$ smoothly spreads out and vanishes with time.

Example 4.1 *For the mIBVP (4.24), we take a family of nonnegative and oscillatory type initial data*

$$\Phi^0(x) = Ae^{-Cx}(1 + \cos(Bx)), \qquad \alpha_1 < x < \infty, \tag{4.33}$$

with amplitude $A > 0$, frequency $B \in \mathbb{R}$ and parameter $C \geq 0$ that determines the rate of decrease as $x \to \infty$. Clearly, parameter B can be used also to control the values of the position coordinate, where local minima and maxima of the initial distribution occur. In that case, we obtain a family of exact solutions given as

$$\begin{aligned} \Phi_h^\alpha(x,t) &= \sqrt{\frac{\Omega e^{\gamma t/2}}{\omega_0 \cosh(\Omega t + \sigma)}} \times \exp\left[-\int_0^t L_h^\alpha(t')dt'\right] \\ &\times \exp\left[-\frac{e^{\gamma t}}{2}\left(\Omega \tanh(\Omega t + \sigma) - \frac{\gamma}{2}\right)(x - r_h^\alpha(t))^2\right] \times \exp\left[-p_h^\alpha(t)(x - r_h^\alpha(t))\right] \\ &\times \Psi(\eta_h^\alpha(x,t), \tau(t)), \end{aligned} \tag{4.34}$$

where $\eta_h^{\alpha}(x,t)$, $\tau(t)$ are given in (4.29) and

$$\begin{aligned}\Psi(\eta,\tau) &= A\, e^{\tilde{C}^2\tau}\Big[e^{-\tilde{C}\eta}\Big(1+Erf\Big(\frac{\eta-\tilde{C}\tau}{\sqrt{2\tau}}\Big)\Big)+e^{\tilde{C}\eta}\Big(-1+Erf\Big(\frac{\eta+\tilde{C}\tau}{\sqrt{2\tau}}\Big)\Big)\Big] \\ &+ \frac{A}{4}e^{\frac{(\tilde{C}-iB)^2\tau}{2}}\Big[e^{-(\tilde{C}-iB)\eta}\,Erf\Big(\frac{\eta+(iB-\tilde{C})\tau}{\sqrt{2\tau}}\Big)-e^{(\tilde{C}-iB)\eta}\,Erf\Big(\frac{-\eta+(iB-\tilde{C})\tau}{\sqrt{2\tau}}\Big)\Big] \\ &+ \frac{A}{4}e^{\frac{(\tilde{C}+iB)^2\tau}{2}}\Big[e^{-(\tilde{C}+iB)\eta}\,Erf\Big(\frac{\eta-(iB+\tilde{C})\tau}{\sqrt{2\tau}}\Big)-e^{(\tilde{C}+iB)\eta}\,Erf\Big(\frac{-\eta-(iB+\tilde{C})\tau}{\sqrt{2\tau}}\Big)\Big],\end{aligned} \quad (4.35)$$

with $\tilde{C} = C - \alpha_2 \geq 0$. Here, parameter B controls the frequency of the oscillations in space, and parameters γ and ω_0 control the spreading rate of the solution as time increases. For certain parameters, where we take $\alpha_1 = 0$ for simplicity, the behavior of the solution is shown in Fig.4.2. The boundary initially located at $x = 0$ moves along x-axis in positive direction and solution is zero on the moving boundary for all $t > 0$, as required. The initial profile is oscillatory in space but oscillations quickly disappear and their amplitude decreases as $x \to \infty$.

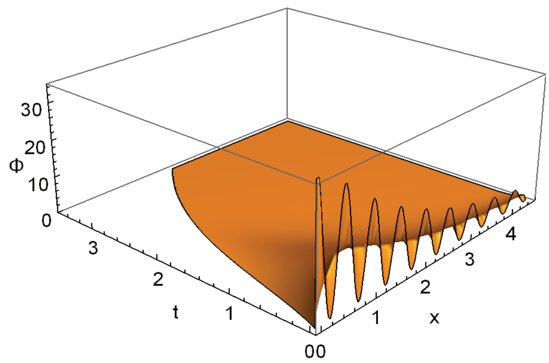


Figure 4.2 Solution (4.34) with $A = 20$, $B = 12$, $C = 0.5$, $\omega_0 = 1$, $\gamma = 1$, $\alpha_1 = 0$ and $\alpha_2 = 0.5$.

Example 4.2 *Next, we take a family of positive initial functions*

$$\Phi^0_{c,n}(x) = A\, e^{-c_2(x-c_1)}(x-c_1)^n, \qquad \alpha_1 < x < \infty, \quad c_1 \geq \alpha_1, \quad n = 0,1,2,..., \quad (4.36)$$

parametrized by $c = (c_1, c_2)$, with $A > 0$, displacement parameter c_1 and parameter $c_2 > 0$ that determines the rate of convergence for given n. Since for given n the maximum of the initial function occurs at $x = c_1 + (n/c_2)$, the parameters c_1, c_2 can be used also to control the value of the position coordinate where the maximum occurs. Then, the mIBVP (4.24) has solution of the form (4.27), where $\eta_h^\alpha(x,t)$, $\tau(t)$ are given in (4.29) and $\Psi(\eta,\tau)$ is solution of the IBVP (4.8) with IC $\Psi(\eta, 0) = A\ e^{\alpha_2\eta}\ e^{-c_2(\eta-(c_1-\alpha_1))}(\eta - (c_1 - \alpha_1))^n$, $\eta > 0$ and BC $\Psi(0,\tau) = 0$ for $0 < \tau < 2/(\gamma + 2\Omega)$. For the special choice $\alpha_2 = c_1$ and $\alpha_2 = c_2$, solution of the heat problem is of the form

$$\Psi_n(\eta,\tau) = A\int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} - e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)\xi^n d\xi, \quad n = 0, 1, 2, ... \tag{4.37}$$

For odd powers $n = 2p + 1,\ p = 0, 1, 2, ...$, solutions (4.37) become

$$\Psi_{2p+1}(\eta,\tau) = AH_{2p+1}(\eta,\tau), \qquad p = 0, 1, 2, ..., \tag{4.38}$$

where $H_{2p+1}(\eta,\tau)$ are odd Kampe de Feriet polynomials (KFP). The first few Kampe de Feriet polynomials in explicit form are $H_0 = 1,\ H_1(\eta,\tau) = \eta,\ H_2(\eta,\tau) = \eta^2+\tau,\ H_3(\eta,\tau) = \eta^3 + 3\eta\tau$. Therefore, one family of solutions to the diffusion problem is

$$\begin{aligned}\Phi_{\alpha,2p+1}(x,t) &= A\times\sqrt{\frac{\Omega e^{\gamma t/2}}{\omega_0\cosh(\Omega t+\sigma)}}\times\exp\left[-\int_0^t L_h^\alpha(t')dt'\right]\\ &\times\ \exp\left[-\frac{e^{\gamma t}}{2}\left(\Omega\tanh(\Omega t+\sigma)-\frac{\gamma}{2}\right)(x-r_h^\alpha(t))^2\right]\times\exp\left[-p_h^\alpha(t)(x-r_h^\alpha(t))\right]\\ &\times\ H_{2p+1}\left(\frac{\Omega e^{\gamma t/2}(x-r_h^\alpha(t))}{\omega_0\cosh(\Omega t+\sigma)},\frac{\sinh(\Omega t)}{\omega_0\cosh(\Omega t+\sigma)}\right).\end{aligned} \tag{4.39}$$

For $p = 0$, i.e $n = 1$, and by choosing parameters $\alpha_1 = 0$ for simplicity, $\alpha_2 = 2$, the solution (4.39) corresponding to IC $\Phi_{2,1}^0(x) = Axe^{-2x}$ explicitly becomes

$$\begin{aligned}\Phi_{\alpha,1}(x,t) &= A\times\sqrt{\frac{\Omega e^{\gamma t/2}}{\omega_0\cosh(\Omega t+\sigma)}}\times\exp\left[-\int_0^t L_h^\alpha(t')dt'\right]\\ &\times\ \exp\left[-\frac{e^{\gamma t}}{2}\left(\Omega\tanh(\Omega t+\sigma)-\frac{\gamma}{2}\right)(x-r_h^\alpha(t))^2\right]\times\exp\left[-p_h^\alpha(t)(x-r_h^\alpha(t))\right]\\ &\times\ \left(\frac{\Omega e^{\gamma t/2}(x-r_h^\alpha(t))}{\omega_0\cosh(\Omega t+\sigma)}\right).\end{aligned} \tag{4.40}$$

Indeed, it is seen that $\lim_{t\to 0}\Phi_{\alpha,1}(s(t),t) = 0$ and $\lim_{x\to 0}\Phi_{\alpha,1}(x,0) = \lim_{x\to 0}\Phi^0_{2,1}(x) = 0$. Since the compatibility condition is satisfied, there is no jump at origin and as $x \to \infty$, the initial function approaches zero. On the boundary solution is also zero for $t > 0$. For any time $t > 0$, the solution approaches zero as $x \to \infty$, since $\Omega > \gamma/2$ and all parameters γ, ω_0, α_2 are positive real numbers. In Fig.4.3a, one can see the behavior of solution.

On the other hand, we note that the even KFP's do not satisfy the Dirichlet BC $\Psi(0,\tau) = 0$. So, for even powers $n = 2p,\ p = 0, 1, 2, ...$, solutions (4.37) can be written

$$\Psi_{2p}(\eta,\tau) = A\left(\int_0^\infty \frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} - e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^{2p}d\xi\right) \equiv A\left(h^-_{2p}(\eta,\tau) - h^+_{2p}(\eta,\tau)\right), \tag{4.41}$$

where $h^-_{2p}(\eta,\tau)$ and $h^+_{2p}(\eta,\tau)$ are defined as, (Widder, 1975)

$$h^-_p(\eta,\tau) = \int_0^\infty \frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^p d\xi, \qquad h^+_p(\eta,\tau) = \int_0^\infty \frac{e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^p d\xi. \tag{4.42}$$

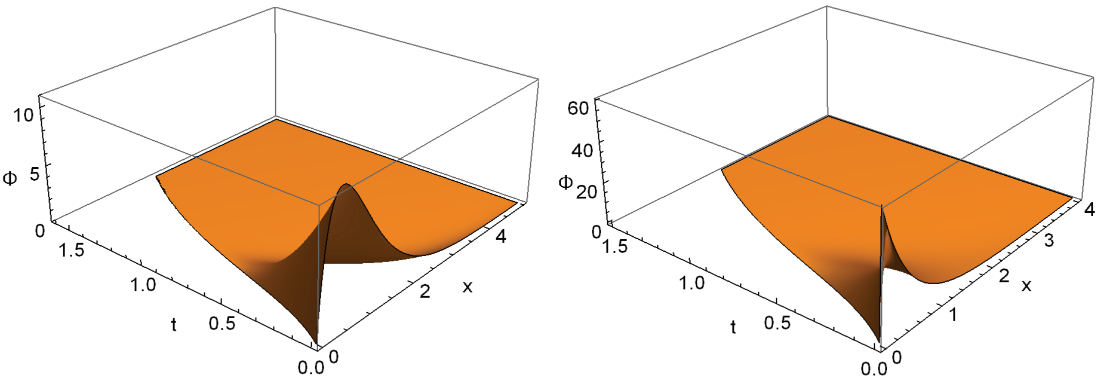
Therefore another solution family for diffusion problem corresponding to (4.41) is

$$\begin{aligned}\Phi_{\alpha,2p}(x,t) =\ & A \times \sqrt{\frac{\Omega e^{\gamma t/2}}{\omega_0\cosh(\Omega t+\sigma)}} \times \exp\left[-\int_0^t L^\alpha_h(t')dt'\right] \\ \times\ & \exp\left[-\frac{e^{\gamma t}}{2}\left(\Omega\tanh(\Omega t+\sigma) - \frac{\gamma}{2}\right)(x - r^\alpha_h(t))^2\right] \times \exp\left[-p^\alpha_h(t)(x - r^\alpha_h(t))\right] \\ \times\ & \left[h^-_{2p}\left(\frac{\Omega e^{\gamma t/2}(x - r^\alpha_h(t))}{\omega_0\cosh(\Omega t+\sigma)}, \frac{\sinh(\Omega t)}{\omega_0\cosh(\Omega t+\sigma)}\right) - h^+_{2p}\left(\frac{\Omega e^{\gamma t/2}(x - r^\alpha_h(t))}{\omega_0\cosh(\Omega t+\sigma)}, \frac{\sinh(\Omega t)}{\omega_0\cosh(\Omega t+\sigma)}\right)\right].\end{aligned} \tag{4.43}$$

For $p = 0$ and parameters $\alpha_1 = 0$, $\alpha_2 = 2$, the solution (4.43) corresponding to IC $\Phi^0_{2,0}(x) = Ae^{-2x}$ is found as

$$\begin{aligned}\Phi_{\alpha,0}(x,t) &= A \times \sqrt{\frac{\Omega e^{\gamma t/2}}{\omega_0\cosh(\Omega t+\sigma)}} \times \exp\left[-\int_0^t L^\alpha_h(t')dt'\right] \\ &\times \exp\left[-\frac{e^{\gamma t}}{2}\left(\Omega\tanh(\Omega t+\sigma) - \frac{\gamma}{2}\right)(x - r^\alpha_h(t))^2\right] \times \exp\left[-p^\alpha_h(t)(x - r^\alpha_h(t))\right] \\ &\times \mathrm{Erf}\left(\frac{\Omega e^{\gamma t/2}(x - r^\alpha_h(t))}{\sqrt{2\omega_0\cosh(\Omega t+\sigma)\sinh(\Omega t)}}\right).\end{aligned} \tag{4.44}$$

Here, since $\lim_{t\to 0} \Phi_{\alpha,0}(s(t), t) = 0$ and $\lim_{x\to 0} \Phi_{\alpha,0}(x, 0) = A$, the compatibility condition is not satisfied at point $(x, t) = (0, 0)$. This means that there is a jump of, for example, temperature at the extremity as soon as $x > 0$. Then as $x \to \infty$, the initial function goes to zero exponentially. Behavior of solution and spreading of moving boundary controlled by γ, ω is illustrated in Fig.4.3b.



(a) (b)

Figure 4.3 (a) Solution $\Phi_{\alpha,1}(x, t)$ with $A = 60$, $\omega_0 = 2$, $\gamma = 3$, $c_1 = \alpha_1 = 0$ and $c_2 = \alpha_2 = 2$. (b) Solution $\Phi_{\alpha,0}(x, t)$ with $A = 60$, $\omega_0 = 2$, $\gamma = 3$, $c_1 = \alpha_1 = 0$ and $c_2 = \alpha_2 = 2$.

4.1.2.2. Model 2 : Convection-Diffusion type mIBVP

Now, we introduce the following model

$$\begin{cases} \Phi_t = \frac{1}{2\mu(t)}\Phi_{xx} - \left((a(t) - b(t)x)\Phi\right)_x, & s(t) < x < \infty, \quad t \in (0, T), \\ \Phi(x, 0) = \Phi^0(x), \quad s(0) < x < \infty, \\ \Phi(s(t), t) = 0, \quad 0 < t < T, \end{cases} \tag{4.45}$$

with time-dependent diffusion coefficient and velocity field $v(x, t) = a(t) - b(t)x$ of the fluid (medium) flow. Here, the characteristic equation is inhomogeneous

$$\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \left(\dot{b}(t) + \frac{\dot{\mu}(t)}{\mu(t)}b(t) - b^2(t)\right)r = \dot{a}(t) - a(t)\left[b(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right], \tag{4.46}$$

with external forcing term induced by the flow velocity of the medium. According to Proposition 4.1, if the boundary satisfies the equation (4.46), that is if $s(t) \equiv r_g^\alpha(t) = \alpha_1 r_1(t) + \alpha_2 r_2(t) + r_p(t)$, then the solution of the mIBVP (4.45) is of the form

$$\begin{aligned} \Phi_g^\alpha(x,t) &= \exp\Big[-\int_0^t L_g^\alpha(t')dt'\Big] \times \exp\Big[-p_g^\alpha(t)(x - r_g^\alpha(t))\Big] \\ &\times \Big(\int_0^\infty G_D(\eta_g^\alpha(x,t),\xi,\tau(t))\Phi_0(\xi+\alpha_1)e^{\alpha_2\xi}d\xi, \end{aligned} \tag{4.47}$$

with

$$L_g^\alpha(t) = \frac{\mu(t)}{2}\Big(\big(\dot{r}_g^\alpha(t) + b(t)r_g^\alpha(t) - a(t)\big)^2\Big), \qquad p_g^\alpha(t) = \mu(t)(\dot{r}_g^\alpha(t) + b(t)r_g^\alpha(t) - a(t)).$$

Fundamental solution: When we consider mIBVP (4.45) with Dirac-delta initial condition (4.31), the fundamental solution is found explicitly

$$\begin{aligned} K_g^\alpha(x,x_0;t) &= e^{\alpha_2(x_0-\alpha_1)} \times \frac{1}{\sqrt{2\pi r_1(t)r_2(t)}} \times \exp\Big[-\int_0^t L_g^\alpha(t')dt'\Big] \times \exp\Big[-p_g^\alpha(t)(x-r_g^\alpha(t))\Big] \\ &\times \left(\exp\left[-\frac{(x - r_g^\alpha(t) - (x_0-\alpha_1)r_1(t))^2}{2r_1(t)r_2(t)}\right] - \exp\left[-\frac{(x - r_g^\alpha(t) + (x_0-\alpha_1)r_1(t))^2}{2r_1(t)r_2(t)}\right]\right). \end{aligned} \tag{4.48}$$

<u>Remark</u>: Notice that, since the velocity field is $v(x,t) = a(t) - b(t)x$, then the velocity of a fluid particle along any path $x(t)$ by definition is

$$\dot{x}(t) = a(t) - b(t)x(t). \tag{4.49}$$

By direct calculation, we see that any path $x(t)$ with velocity (4.49) satisfies the inhomogeneous characteristic equation (4.46). In this model, since $r_p(t)$ satisfies the characteristic equation (4.46) and the IC's $r_p(0) = 0,\ \dot{r}_p(0) = a(0)$, then the following holds

$$\dot{r}_p(t) = a(t) - b(t)r_p(t).$$

It follows that, the boundary $s(t)$ will move with the velocity of the fluid flow if

$$\dot{s}(t) = a(t) - b(t)s(t). \tag{4.50}$$

Since $s(t) = \alpha_1 r_1(t) + \alpha_2 r_2(t) + r_p(t)$, then the condition (4.50) becomes

$$\alpha_1\big(\dot{r}_1(t) + b(t)r_1(t)\big) + \alpha_2\big(\dot{r}_2(t) + b(t)r_2(t)\big) + \Big(\dot{r}_p(t) - \big(a(t) - b(t)r_p(t)\big)\Big) = 0. \tag{4.51}$$

In this model, we have $b(t) = -\dot{r}_1(t)/r_1(t)$ since $\omega(t) = 0$, and particular solution $r_p(t)$ satisfies $\dot{r}_p(t) = a(t) - b(t)r_p(t)$. Then the equation (4.51) will be true if $\alpha_2 = 0$, where $\alpha_2 = \dot{s}(0)$. As a result, any boundary point with position described by $s(t) = \alpha_1 r_1(t) + r_p(t)$ will move with the velocity of the fluid flow and $p_g^{\alpha}(t) = 0$, $L_g^{\alpha}(t) = 0$. Therefore, if we let $\alpha_2 = 0$, the fundamental solution (4.48) becomes

$$\begin{aligned} K_g^{\alpha_1}(x, x_0; t) &= \frac{1}{\sqrt{2\pi r_1(t) r_2(t)}} \\ &\times \left(\exp\left[-\frac{\left(x - r_g^{\alpha_1}(t) - (x_0 - \alpha_1) r_1(t)\right)^2}{2 r_1(t) r_2(t)} \right] - \exp\left[-\frac{\left(x - r_g^{\alpha_1}(t) + (x_0 - \alpha_1) r_1(t)\right)^2}{2 r_1(t) r_2(t)} \right] \right). \end{aligned} \tag{4.52}$$

Moments of the Solution Distribution

When $\alpha_2 = 0$, for the solution distribution $K_g^{\alpha_1}(x, x_0; t)$, initially the total amount is

$$\int_{x=\alpha_1}^{\infty} \Phi(x, 0)dx = \int_{x=\alpha_1}^{\infty} \delta(x - x_0)dx = 1, \quad x_0 > \alpha_1,$$

and the total amount of substance is found as

$$\begin{aligned} M_0(t) &= \int_{x=r_g^{\alpha_1}(t)}^{\infty} K_g^{\alpha_1}(x, x_0; t)dx = \frac{1}{\sqrt{2\pi r_1(t) r_2(t)}} \\ &\times \int_{x=r_g^{\alpha_1}(t)}^{\infty} \left(\exp\left[-\frac{\left(x - r_g^{\alpha_1}(t) - (x_0 - \alpha_1) r_1(t)\right)^2}{2 r_1(t) r_2(t)} \right] - \exp\left[-\frac{\left(x - r_g^{\alpha_1}(t) + (x_0 - \alpha_1) r_1(t)\right)^2}{2 r_1(t) r_2(t)} \right] \right) dx. \end{aligned} \tag{4.53}$$

As a result, we obtain

$$M_0(t) = \text{Erf}\left(\frac{x_0 - \alpha_1}{\sqrt{2\tau(t)}}\right), \qquad \tau(t) = \frac{r_2(t)}{r_1(t)}, \tag{4.54}$$

where $\text{Erf}(x)$ is the error function with $\text{Erf}(0) = 0$ and $\text{Erf}(\infty) = 1$. In that case, in general, the total amount of substance is not conserved in the spatial domain $s(t) < x < \infty$. It changes according to $\tau(t)$ as follows

(i) If $\tau(t) \to 0$ as $t \to \infty$, then $M_0(t) \to 1$. In that case, the concentration amount is conserved during time evolution.

(ii) If $\tau(t) \to \infty$, then $M_0(t) \to 0$.

(iii) If $|\tau(t)| < c$ as $t \to \infty$, for any positive real constant c, then $0 < |M_0(t)| < 1$.

So, one can say that the total substance amount is between 0 and 1.

First moment : In this case, the first moment becomes

$$\begin{aligned}
M_1(t) &= \int_{x=r_g^{\alpha_1}(t)}^{\infty} x\, K_g^{\alpha_1}(x, x_0; t)\, dx = \frac{1}{\sqrt{2\pi r_1(t) r_2(t)}} \\
&\times \int_{x=r_g^{\alpha_1}(t)}^{\infty} x \left(\exp\left[-\frac{\left(x - r_g^{\alpha_1}(t) - (x_0 - \alpha_1) r_1(t)\right)^2}{2 r_1(t) r_2(t)} \right] - \exp\left[-\frac{\left(x - r_g^{\alpha_1}(t) + (x_0 - \alpha_1) r_1(t)\right)^2}{2 r_1(t) r_2(t)} \right] \right) dx, \\
&= \frac{1}{\sqrt{\pi}} \left(\int_{\frac{-(x_0-\alpha_1)\sqrt{r_1(t)}}{\sqrt{2 r_2(t)}}}^{\infty} \left(\sqrt{2 r_1(t) r_2(t)}\, y + r_g^{\alpha_1}(t) + (x_0 - \alpha_1) r_1(t) \right) \exp\left(-y^2\right) dy \right. \\
&\left. - \int_{\frac{(x_0-\alpha_1)\sqrt{r_1(t)}}{\sqrt{2 r_2(t)}}}^{\infty} \left(\sqrt{2 r_1(t) r_2(t)}\, y + r_g^{\alpha_1}(t) - (x_0 - \alpha_1) r_1(t) \right) \exp\left(-y^2\right) dy \right).
\end{aligned}$$

Evaluating the integrals gives

$$\begin{aligned}
M_1(t) &= (x_0 - \alpha_1) r_1(t) + r_g^{\alpha_1}(t)\, \text{Erf}\left(\frac{(x_0 - \alpha_1)}{\sqrt{2\tau(t)}}\right). \\
&= (x_0 - \alpha_1) r_1(t) + r_g^{\alpha_1}(t)\, M_0(t).
\end{aligned} \tag{4.55}$$

• **Mean position** : Normalizing the first moment by total amount gives the center of the distribution

$$\langle x\rangle(t) \;=\; \frac{M_1(t)}{M_0(t)} \;=\; \frac{\displaystyle\int_{x=r_g^{\alpha_1}(t)}^{\infty} x\, K_g^{\alpha_1}(x,x_0;t)dx}{\displaystyle\int_{x=r_g^{\alpha_1}(t)}^{\infty} K_g^{\alpha_1}(x,x_0;t)dx} \;=\; r_g^{\alpha_1}(t) + \frac{(x_0-\alpha_1)r_1(t)}{\mathrm{Erf}\left(\frac{(x_0-\alpha_1)}{\sqrt{2\tau(t)}}\right)}, \qquad (4.56)$$

which depends on the behavior of $r_1(t)$, $r_2(t)$ and $r_p(t)$. We notice that in this model boundary propagates according to $s(t) = r_g^{\alpha_1}(t)$ and therefore

$$|\langle x\rangle(t) - s(t)| = \frac{(x_0-\alpha_1)r_1(t)}{\mathrm{Erf}\left(\frac{(x_0-\alpha_1)}{\sqrt{2\tau(t)}}\right)}, \qquad (4.57)$$

from which we can determine the behavior as $t \to \infty$.

Second moment : We have the second spatial moment as follows

$$\begin{aligned}
M_2(t) &= \int_{x=r_g^{\alpha_1}(t)}^{\infty} x^2\, K_g^{\alpha_1}(x,x_0;t)dx = \frac{1}{\sqrt{2\pi r_1(t)r_2(t)}} \\
&\times \int_{x=r_g^{\alpha_1}(t)}^{\infty} x^2 \left(\exp\left[-\frac{(x-r_g^{\alpha_1}(t)-(x_0-\alpha_1)r_1(t))^2}{2r_1(t)r_2(t)}\right] - \exp\left[-\frac{(x-r_g^{\alpha_1}(t)+(x_0-\alpha_1)r_1(t))^2}{2r_1(t)r_2(t)}\right]\right) dx, \\
&= \frac{1}{\sqrt{\pi}}\left(\int_{\frac{-(x_0-\alpha_1)\sqrt{r_1(t)}}{\sqrt{2r_2(t)}}}^{\infty} \left(\sqrt{2r_1(t)r_2(t)}y + r_g^{\alpha_1}(t) + (x_0-\alpha_1)r_1(t)\right)^2 \exp\left(-y^2\right)\, dy\right. \\
&\left. - \int_{\frac{(x_0-\alpha_1)\sqrt{r_1(t)}}{\sqrt{2r_2(t)}}}^{\infty} \left(\sqrt{2r_1(t)r_2(t)}y + r_g^{\alpha_1}(t) - (x_0-\alpha_1)r_1(t)\right)^2 \exp\left(-y^2\right)\, dy\right),
\end{aligned}$$

which gives

$$\begin{aligned}
M_2(t) =&\; 2\,(x_0-\alpha_1)\,r_1(t)r_g^{\alpha_1}(t) + \left((r_g^{\alpha_1}(t))^2 + (x_0-\alpha_1)^2\, r_1^2(t) + r_1(t)r_2(t)\right)\mathrm{Erf}\left(\frac{(x_0-\alpha_1)\sqrt{r_1(t)}}{\sqrt{2r_2(t)}}\right) \\
&+ \frac{(x_0-\alpha_1)\sqrt{2r_1(t)r_2(t)}\; r_1(t)}{\sqrt{\pi}} \times \exp\left[-\frac{r_1(t)(x_0-\alpha_1)^2}{2r_2(t)}\right].
\end{aligned}$$

Then normalizing the second moment by total amount of concentration, we get mean square

$$
\begin{aligned}
\langle x^2\rangle(t) &= \frac{M_2(t)}{M_0(t)} = \frac{\displaystyle\int_{x=r_g^{\alpha_1}(t)}^{\infty} x^2\ K_g^{\alpha_1}(x,x_0;t)dx}{\displaystyle\int_{x=r_g^{\alpha_1}(t)}^{\infty} K_g^{\alpha_1}(x,x_0;t)dx}, \qquad (4.58)\\
&= (r_g^{\alpha_1}(t))^2 + (x_0-\alpha_1)^2 r_1^2(t) + r_1(t)r_2(t)\\
&+ \frac{2(x_0-\alpha_1)r_1(t)r_g^{\alpha_1}(t) + (x_0-\alpha_1)\sqrt{2r_1(t)r_2(t)}r_1(t)\exp\left[-\frac{r_1(t)(x_0-\alpha_1)^2}{2r_2(t)}\right]/\sqrt{\pi}}{\mathrm{Erf}\left(\frac{(x_0-\alpha_1)\sqrt{r_1(t)}}{\sqrt{2r_2(t)}}\right)}.
\end{aligned}
$$

• **Variance** : The variance about the mean is

$$
\begin{aligned}
\mathrm{Var}(t) &= \int_{x=r_g^{\alpha_1}(t)}^{\infty} (x-\langle x\rangle)^2\ K_g^{\alpha_1}(x,x_0;t)dx = \frac{M_2(t)}{M_0(t)} - \left(\frac{M_1(t)}{M_0(t)}\right)^2,\\
&= r_1(t)r_2(t) + (x_0-\alpha_1)^2 r_1^2(t)\left(1 - \frac{1}{\left(\mathrm{Erf}\left(\frac{(x_0-\alpha_1)\sqrt{r_1(t)}}{\sqrt{2r_2(t)}}\right)\right)^2}\right)\\
&+ (x_0-\alpha_1)r_1(t)\left(\frac{\sqrt{2r_1(t)r_2(t)}\exp\left[-\frac{r_1(t)(x_0-\alpha_1)^2}{2r_2(t)}\right]}{\sqrt{\pi}\,\mathrm{Erf}\left(\frac{(x_0-\alpha_1)\sqrt{r_1(t)}}{\sqrt{2r_2(t)}}\right)}\right).
\end{aligned}
$$

Notice that only homogeneous solutions, $r_1(t)$, $r_2(t)$, of the characteristic equations affect the variance about mean.

In the following examples, we study the model (4.45) with concrete coefficients, Dirac-delta initial data and homogeneous boundary condition. Then we analyze the motion of the mean position and the moving boundary.

Example 4.3 ***(mIBVP with homogeneous Dirichlet BC)***

Now, in mIBVP (4.45), by taking the coefficients

$$
\mu(t) = 1, \qquad a(t) = a_0\cosh(\Lambda_0 t),\ \ a_0 \geq 0, \qquad b(t) = -\Lambda_0\tanh(\Lambda_0 t),\ \ \Lambda_0 > 0, \quad (4.59)
$$

we have the following inhomogeneous characteristic equation

$$\ddot{r} - \Lambda_0^2 r = 2a_0\Lambda_0 \sinh(\Lambda_0 t), \quad t > 0, \tag{4.60}$$

with solutions

$$r_1(t) = \cosh(\Lambda_0 t), \quad r_2(t) = \sinh(\Lambda_0 t)/\Lambda_0, \quad r_p(t) = a_0\, t \cosh(\Lambda_0 t).$$

Therefore, if the boundary propagates according to

$$s(t) = r_g^\alpha(t) \equiv (\alpha_1 + a_0 t)\cosh(\Lambda_0 t) + \alpha_2 \sinh(\Lambda_0 t)/\Lambda_0, \tag{4.61}$$

then the problem has analytical solution of the form

$$\begin{aligned} \Phi_g^\alpha(x,t) = {} & \frac{\sqrt{\Lambda_0}}{\sqrt{\pi \sinh(2\Lambda_0 t)}} \times \exp\Big[-\frac{\alpha_2^2}{2\Lambda_0}\tanh(\Lambda_0 t)\Big] \times \exp\Big[-\alpha_2 \mathrm{sech}(\Lambda_0 t)(x - r_g^\alpha(t)\Big] \\ \times & \int_0^\infty \Bigg(\exp\Big[-\frac{(x - r_g^\alpha(t) - \xi\cosh(\Lambda_0 t))^2}{\sinh(2\Lambda_0 t)/\Lambda_0}\Big] \\ & - \exp\Big[-\frac{(x - r_g^\alpha(t) + \xi\cosh(\Lambda_0 t))^2}{\sinh(2\Lambda_0 t)/\Lambda_0}\Big]\Bigg) \times \Phi^0(\xi + \alpha_1) e^{\alpha_2 \xi} d\xi. \end{aligned} \tag{4.62}$$

Thus, the exact form of the solution can be found if the integral converges for the given initial data $\Phi^0(x)$. Notice that since $a(t) = a_0\cosh(\Lambda_0 t)$ and $b(t) = -\Lambda_0 \tanh(\Lambda_0 t)$, it implies that $r_p(t)$ satisfies $\dot{r}_p(t) = a(t) - b(t) r_p(t)$. And we have generalized momentum $p_g^\alpha(t) = \alpha_2 \mathrm{sech}(\Lambda_0 t)$. In what follows we consider two particular cases : in the first one by choosing $\alpha_2 \neq 0$, we have boundary moves with different velocity than the velocity of the fluid flow. The latter involves the case $\alpha_2 = 0$, where the boundary moves with the velocity of the fluid flow.

Case 1 . Let $\alpha_2 \neq 0$ so that the boundary moves according to (4.61). In that case the generalized momentum and Lagrangian type function is obtained respectively,

$$p_g^{\alpha_2}(t) = \alpha_2 \mathrm{sech}(\Lambda_0 t), \qquad L_g^{\alpha_2}(t) = \frac{\alpha_2^2}{2}\mathrm{sech}^2(\Lambda_0 t).$$

Therefore, the explicit form of the fundamental solution is

$$\begin{aligned}
K_g^{\alpha}(x, x_0; t) \;=\; & e^{\alpha_2(x_0-\alpha_1)}\sqrt{\frac{\Lambda_0}{\pi \sinh(2\Lambda_0 t)}} \times \exp\Big[-\frac{\alpha_2^2}{2\Lambda_0}\tanh(\Lambda_0 t)\Big] \\
& \times \exp\Big(-\alpha_2 \mathrm{sech}(\Lambda_0 t)(x - [(\alpha_1 + a_0\, t)\cosh(\Lambda_0 t)) + \alpha_2(\sinh(\Lambda_0 t)/\Lambda_0)])\Big) \\
& \times \Bigg[\exp\Big(-\frac{(x - [(a_0\, t + x_0)\cosh(\Lambda_0 t) + \alpha_2(\sinh(\Lambda_0 t)/\Lambda_0)])^2}{\sinh(2\Lambda_0 t)/\Lambda_0}\Big) \\
& - \exp\Big(-\frac{(x - [(a_0\, t - x_0 + 2\alpha_1)\cosh(\Lambda_0 t) + \alpha_2(\sinh(\Lambda_0 t)/\Lambda_0)])^2}{\sinh(2\Lambda_0 t)/\Lambda_0}\Big)\Bigg]. \qquad (4.63)
\end{aligned}$$

We note that

$$\lim_{t\to 0} K_g^{\alpha}(x, x_0; t) = e^{-\alpha_2(x-x_0)}\left(\delta(x - x_0) - \delta(x + x_0 - 2\alpha_1)\right) = \delta(x - x_0), \qquad (4.64)$$

since $x = 2\alpha_1 - x_0$ is not in our domain, $\alpha_1 < x < \infty$. In the long time behavior the solution (4.63) approaches zero.

Here, parameters a_0 and Λ_0 affect the displacement of the solution distribution, i.e mean position, the velocity of both moving boundary and flowing medium. Moreover, Λ_0 has influence also on spreading of the distribution.

The parameter α_2 controls both the velocity of the boundary and the rate of decrease of distribution amplitude as $x \to \infty$. In this case, since $\alpha_2 \neq 0$, the boundary propagates with different velocity than the velocity of the medium. As α_2 increases, the distribution reaches the boundary in less time, then its amplitude vanishes.

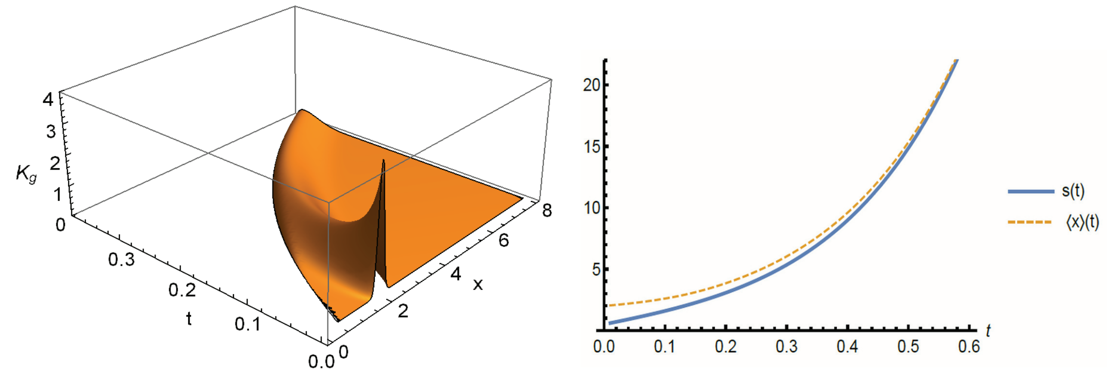
On the other side, since the fundamental distribution (4.63) includes the exponential terms which are quadratic and linear in x, finding the spatial moments is rather complicated task. Here, we provide only the total amount of mass and mean position of the distribution as follows

$$M_0(t) = \frac{1}{2}\text{Erfc}\left(\frac{\sqrt{\Lambda_0}(\alpha_1 - x_0 + \alpha_2 \tanh(\Lambda_0 t)/\Lambda_0)}{\sqrt{2\tanh(\Lambda_0 t)}}\right)$$
$$- \frac{e^{2\alpha_2(x_0-\alpha_1)}}{2}\text{Erfc}\left(\frac{\sqrt{\Lambda_0}(x_0 - \alpha_1 + \alpha_2 \tanh(\Lambda_0 t)/\Lambda_0)}{\sqrt{2\tanh(\Lambda_0 t)}}\right),$$

which is not conserved as time increases. Then the first spatial moment of distribution is found as

$$M_1(t) = \frac{1}{2}\sqrt{\frac{\sinh(2\Lambda_0 t)}{\pi\Lambda_0}} \times \left(\exp\left[-\frac{(\Lambda_0(x_0-\alpha_1)\cosh(\Lambda_0 t) - \alpha_2\sinh(\Lambda_0 t))^2}{\Lambda_0\sinh(2\Lambda_0 t)}\right]\right.$$
$$\left.- e^{2\alpha_2(x_0-\alpha_1)}\exp\left[-\frac{(\Lambda_0(x_0-\alpha_1)\cosh(\Lambda_0 t) + \alpha_2\sinh(\Lambda_0 t))^2}{\Lambda_0\sinh(2\Lambda_0 t)}\right]\right)$$
$$- \frac{\alpha_2\sinh(\Lambda_0 t)}{2\Lambda_0}\left[\text{Erfc}\left(\frac{\alpha_1 - x_0 + \alpha_2\tanh(\Lambda_0 t)/\Lambda_0}{\sqrt{2\tanh(\Lambda_0 t)/\Lambda_0}}\right)\right.$$
$$\left.- e^{2\alpha_2(x_0-\alpha_1)}\text{Erfc}\left(\frac{x_0 - \alpha_1 + \alpha_2\tanh(\Lambda_0 t)/\Lambda_0}{\sqrt{2\tanh(\Lambda_0 t)/\Lambda_0}}\right)\right]$$
$$+ \frac{1}{2}((x_0 + a_0 t)\cosh(\Lambda_0 t) + \alpha_2\sinh[\Lambda_0 t]/\Lambda_0)\text{Erfc}\left(\frac{\alpha_1 - x_0 + \alpha_2\tanh(\Lambda_0 t)/\Lambda_0}{\sqrt{2\tanh(\Lambda_0 t)/\Lambda_0}}\right)$$
$$-\frac{e^{2\alpha_2(x_0-\alpha_1)}}{2}\Big((-x_0 + 2\alpha_1 + a_0 t)\cosh(\Lambda_0 t) + \alpha_2\sinh(\Lambda_0 t)/\Lambda_0\Big)$$
$$\times\,\text{Erfc}\left(\frac{x_0 - \alpha_1 + \alpha_2\tanh(\Lambda_0 t)/\Lambda_0}{\sqrt{2\tanh(\Lambda_0 t)/\Lambda_0}}\right), \qquad (4.65)$$

and normalizing the first moment by the total amount, we get the mean position of the concentration, $\langle x\rangle(t)$. For certain parameters, one can see the behavior of solution distribution (4.63) initially located at $x = 2$ in Fig.4.4a. The time evolution for the mean position and the boundary is shown in Fig.4.4b, where the boundary moves with the velocity different than the velocity of the medium. It is seen that after certain time the center of the solution reaches to the boundary for chosen parameters, then it goes on propagating along the boundary.



(a) (b)

Figure 4.4 For the parameters $x_0 = 2,\ \Lambda_0 = 4,\ \ \alpha_1 = 0.5,\ \ a_0 = 4,\ \ \alpha_2 = 6,$ (a) the behavior of the solution distribution (4.63). (b) the motion of the mean position of the solution distribution and the boundary.

Case 2 . Now let $\alpha_2 = 0$ so that the boundary moves according to

$$s(t) = (\alpha_1 + a_0 t)\cosh(\Lambda_0 t).$$

Then we have $p_g^{\alpha}(t) = L_g^{\alpha}(t) = 0$ and obtain the fundamental solution as

$$K_g^{\alpha_1}(x, x_0; t) = \sqrt{\frac{\Lambda_0}{\pi \sinh(2\Lambda_0 t)}} \times \tag{4.66}$$
$$\times \left[\exp\left(- \frac{(x - [(a_0\, t + x_0)\cosh(\Lambda_0 t)])^2}{\sinh(2\Lambda_0 t)/\Lambda_0} \right) - \exp\left(- \frac{(x - [(a_0\, t - x_0 + 2\alpha_1)\cosh(\Lambda_0 t)])^2}{\sinh(2\Lambda_0 t)/\Lambda_0} \right) \right],$$

which approaches zero as $t \to \infty$. Fig.4.5a exhibits the behavior of fundamental distribution (4.66) for certain parameters. It is seen that the concentration distribution never reaches the boundary as time increases.

In this case, we have the total amount of substance

$$M_0(t) = \mathrm{Erf}\left(\frac{\sqrt{\Lambda_0}(x_0 - \alpha_1)}{\sqrt{2\tanh(\Lambda_0 t)}} \right),$$

which depends on time so that the total amount of substance is not conserved. Moreover we have $0 <| M_0(t) |< 1$.

The first spatial moment is found as

$$M_1(t) = (x_0 - \alpha_1)\cosh(\Lambda_0 t) + (\alpha_1 + a_0 t)\cosh(\Lambda_0 t) \times \mathrm{Erf}\left(\frac{\sqrt{\Lambda_0}(x_0 - \alpha_1)}{\sqrt{2\tanh(\Lambda_0 t)}}\right),$$

and normalizing by the total amount of substance, we get the expectation value of position given explicitly as

$$\langle x\rangle(t) = \left(\alpha_1 + a_0 t + \frac{(x_0 - \alpha_1)}{\mathrm{Erf}\left(\frac{\sqrt{\Lambda_0}(x_0-\alpha_1)}{\sqrt{2\tanh(\Lambda_0 t)}}\right)}\right)\cosh(\Lambda_0 t). \tag{4.67}$$

Therefore, the relation between mean position $\langle x\rangle(t)$ and the boundary $s(t)$ is of the form

$$\langle x\rangle(t) = \ s(t) + \frac{(x_0 - \alpha_1)\cosh(\Lambda_0 t)}{\mathrm{Erf}\left(\frac{\sqrt{\Lambda_0}(x_0-\alpha_1)}{\sqrt{2\tanh(\Lambda_0 t)}}\right)}. \tag{4.68}$$

In Fig.4.5b, we illustrate the time evolution for the center of the distribution and the moving boundary.

Then by finding the second spatial moment of the solution distribution

$$\begin{aligned} M_2(t) &= \left(\frac{\sinh(2\Lambda_0 t)}{2\Lambda_0} + ((\alpha_1 + a_0 t)^2 + (x_0 - \alpha_1)^2)\cosh^2(\Lambda_0 t)\right)\mathrm{Erf}\left(\frac{\sqrt{\Lambda_0}(x_0 - \alpha_1)}{\sqrt{2\tanh(\Lambda_0 t)}}\right) \\ &+ 2(x_0 - \alpha_1)(\alpha_1 + a_0 t)\cosh^2(\Lambda_0 t) \\ &+ \frac{(x_0 - \alpha_1)\sqrt{\sinh(2\Lambda_0 t)}\cosh(\Lambda_0 t)}{\sqrt{\Lambda_0 \pi}} \times \exp\left[-\frac{\Lambda_0(x_0 - \alpha_1)^2}{2\tanh(\Lambda_0 t)}\right], \end{aligned} \tag{4.69}$$

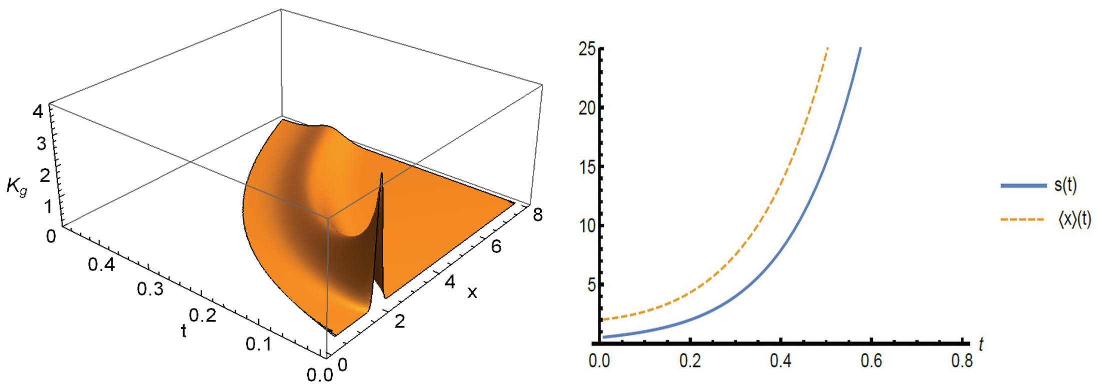
we can obtain the expectation value of square of position as

$$
\begin{aligned}
\langle x^2\rangle(t) &= \left(\frac{\sinh(2\Lambda_0 t)}{2\Lambda_0} + \left((\alpha_1 + a_0 t)^2 + (x_0-\alpha_1)^2\right)\cosh^2(\Lambda_0 t)\right) \\
&+ \left(2\,(x_0-\alpha_1)\,(\alpha_1 + a_0 t)\cosh^2(\Lambda_0 t)\right) \times \frac{1}{\mathrm{Erf}\left(\frac{\sqrt{\Lambda_0}(x_0-\alpha_1)}{\sqrt{2\tanh(\Lambda_0 t)}}\right)} \\
&+ \frac{(x_0-\alpha_1)\sqrt{\sinh(2\Lambda_0 t)}\cosh(\Lambda_0 t)}{\sqrt{\Lambda_0 \pi}} \times \exp\left[-\frac{\Lambda_0 (x_0-\alpha_1)^2}{2\tanh(\Lambda_0 t)}\right] \times \frac{1}{\mathrm{Erf}\left(\frac{\sqrt{\Lambda_0}(x_0-\alpha_1)}{\sqrt{2\tanh(\Lambda_0 t)}}\right)}.
\end{aligned}
\tag{4.70}
$$

Then, we get the variance formulated in (3.31)

$$
\begin{aligned}
Var(t) &= \frac{\sinh(2\Lambda_0 t)}{2\Lambda_0} + \left(1 - \frac{1}{\left(\mathrm{Erf}\left(\frac{\sqrt{\Lambda_0}(x_0-\alpha_1)}{\sqrt{2\tanh(\Lambda_0 t)}}\right)\right)^2}\right)(x_0-\alpha_1)^2\cosh^2(\Lambda_0 t) \\
&+ \frac{(x_0-\alpha_1)\sqrt{\sinh(2\Lambda_0 t)}\cosh(\Lambda_0 t)}{\sqrt{\Lambda_0 \pi}} \times \exp\left[-\frac{\Lambda_0 (x_0-\alpha_1)^2}{2\tanh(\Lambda_0 t)}\right] \times \frac{1}{\mathrm{Erf}\left(\frac{\sqrt{\Lambda_0}(x_0-\alpha_1)}{\sqrt{2\tanh(\Lambda_0 t)}}\right)}.
\end{aligned}
\tag{4.71}
$$

It is easy to see that as time increases, the variance goes to infinity, which means that the concentration distribution decreases in time.



(a) (b)

Figure 4.5 For the parameters $x_0 = 2$, $\Lambda_0 = 4$, $\alpha_1 = 0.5$, $a_0 = 4$, $\alpha_2 = 0$, (a) the behavior of the distribution (4.66). (b) the mean position and the boundary.

Example 4.4 ***(mIBVP with homogeneous Dirichlet BC)***

As a second example, in mIBVP (4.45) we consider

$$\mu(t) = e^{\gamma t},\ \gamma > 0, \qquad a(t) = a_0 \sin(\omega t),\ a_0 \geq 0, \qquad b(t) = \beta,\ \beta \in \mathbb{R},$$

so that we have exponentially decaying diffusion coefficient and sinusoidal velocity of the flowing fluid with frequency $\omega > 0$. *The corresponding ODE is*

$$\ddot{r} + \gamma \dot{r} + (\gamma\beta - \beta^2) r = F_0 \cos(\omega t + \theta), \qquad t > 0, \tag{4.72}$$

where we denote $F_0 = a_0 \sqrt{\omega^2 + (\beta - \gamma)^2}$ and $\theta = \arctan(\beta - \gamma/\omega)$. We note that, the sinusoidal velocity in diffusion equation has generated external periodic force in (4.72) as expected, with amplitude F_0 and phase shifting θ, both depending on frequency $\omega > 0$ and β. For the discriminant $\Delta = (\gamma - 2\beta)^2$, we have homogeneous and particular solutions respectively given as

$$\begin{aligned}
r_1(t) &= e^{-\beta t}, \\
r_2(t) &= \frac{1}{\gamma - 2\beta}\left(e^{-\beta t} - e^{-(\gamma-\beta)t}\right), \\
r_p(t) &= \frac{a_0\omega(\omega^2 + (\beta-\gamma)^2)}{\Omega} e^{-\beta t} - \frac{(\omega^2 - \gamma\beta + \beta^2)F_0}{\Omega}\cos(\omega t + \theta) \\
&+ \frac{\gamma\omega F_0}{\Omega}\sin(\omega t + \theta),
\end{aligned} \tag{4.73}$$

where $\gamma \neq 2\beta$ and $\Omega = (\omega^2 - \gamma\beta + \beta^2)^2 + (\gamma\omega)^2$. The behavior of these solutions change according to the the followings :

(i) When $\beta < 0$ and $\gamma > 0$, all solutions tends to infinity as $t \to \infty$.

(ii) If $\beta = 0$ and $\gamma > 0$, then $r_1(t)$ becomes constant function, $r_2(t) \to 1/\gamma$ and $r_p(t)$ just oscillates as time increases.

(iii) If $0 < \gamma < \beta$, then while $r_1(t)$ goes to zero, $r_2(t)$ tends to infinity as $t \to \infty$, and the particular solution keeps oscillating in time.

As before, in what follows we consider two particular cases : $\alpha_2 \neq 0$ and $\alpha_2 = 0$.

Case 1 . Assume $\alpha_2 \neq 0$ in order to have boundary moves according to

$$\begin{aligned} s(t) &= \left(\alpha_1 + \frac{\alpha_2}{\gamma - 2\beta} + \frac{a_0\omega(\omega^2 + (\beta-\gamma)^2)}{\Omega}\right)e^{-\beta t} - \frac{\alpha_2}{\gamma - 2\beta}e^{-(\gamma-\beta)t} \\ &- \frac{(\omega^2 - \gamma\beta + \beta^2)F_0}{\Omega}\cos(\omega t + \theta) + \frac{\gamma\omega F_0}{\Omega}\sin(\omega t + \theta), \end{aligned} \tag{4.74}$$

with the momentum and Lagrangian function are obtained respectively as follows

$$p_g^\alpha(t) = \alpha_2 e^{\beta t}, \qquad L_g^\alpha(t) = \frac{\alpha_2^2}{2}e^{-(\gamma-2\beta)t}. \tag{4.75}$$

Therefore, we obtain the fundamental solution

$$\begin{aligned} K_g^\alpha(x, x_0; t) = e^{\alpha_2(x_0-\alpha_1)}\sqrt{\frac{\gamma - 2\beta}{2\pi(e^{-2\beta t} - e^{-\gamma t})}} \times \exp\left[-\frac{\alpha_2^2\left(1 - e^{-(\gamma-2\beta)t}\right)}{2\gamma - 4\beta}\right] \times \exp\left[-\alpha_2 e^{\beta t}(x - r_g^\alpha(t))\right] \\ \times\left(\exp\left[-\frac{(x - r_g^\alpha(t) - (x_0 - \alpha_1)e^{-\beta t})^2}{2(e^{-2\beta t} - e^{-\gamma t})/(\gamma - 2\beta)}\right] - \exp\left[-\frac{(x - r_g^\alpha(t) + (x_0 - \alpha_1)e^{-\beta t})^2}{2(e^{-2\beta t} - e^{-\gamma t})/(\gamma - 2\beta)}\right]\right), \end{aligned} \tag{4.76}$$

which satisfies the boundary condition, $K_g^\alpha(s(t), x_0; t) = 0$, as required. And we note also that $K_g^\alpha(\infty, x_0, t) = 0$ for any $t > 0$.

For this example, the total mass is in the following form

$$\begin{aligned} M_0(t) &= \frac{1}{2}\mathrm{Erfc}\left(\frac{\alpha_1 - x_0 + \alpha_2\left(1 - e^{-(\gamma-2\beta)t}\right)/(\gamma - 2\beta)}{2\sqrt{\left(1 - e^{-(\gamma-2\beta)t}\right)/(\gamma - 2\beta)}}\right) \\ &- \frac{e^{2\alpha_2(x_0-\alpha_1)}}{2}\mathrm{Erfc}\left(\frac{x_0 - \alpha_1 + \alpha_2\left(1 - e^{-(\gamma-2\beta)t}\right)/(\gamma - 2\beta)}{\sqrt{\left(1 - e^{-(\gamma-2\beta)t}\right)/(\gamma - 2\beta)}}\right). \end{aligned}$$

Here, we notice that for the cases (i), (ii) and (iii), we have $M_0(t) \to 0$ as $t \to \infty$.

The first moment of distribution is calculated as

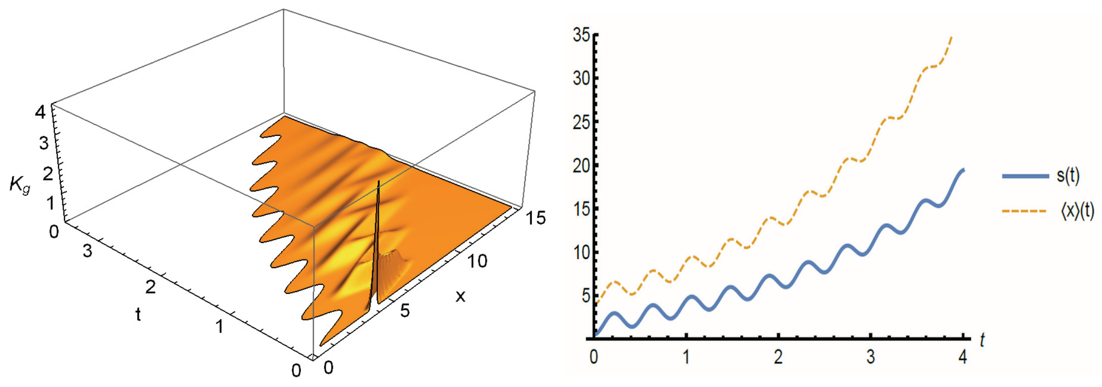
$$
\begin{aligned}
M_1(t) = & \sqrt{\frac{e^{-2\beta t} - e^{-\gamma t}}{2\pi(\gamma - 2\beta)}} \times \exp\left[-\frac{(\alpha_2\tau(t) - (x_0 - \alpha_1))^2}{2\tau(t)}\right] \\
& - e^{2\alpha_2(x_0-\alpha_1)} \sqrt{\frac{e^{-2\beta t} - e^{-\gamma t}}{2\pi(\gamma - 2\beta)}} \times \exp\left[-\frac{(\alpha_2\tau(t) + x_0 - \alpha_1)^2}{2\tau(t)}\right] \\
& - \frac{\alpha_2\left(e^{-\beta t} - e^{-(\gamma-\beta)t}\right)}{2\gamma - 4\beta} \times \text{Erfc}\left(\frac{\alpha_2\tau(t) - (x_0 - \alpha_1)}{\sqrt{2\tau(t)}}\right) \\
& + \frac{\alpha_2 e^{2\alpha_2(x_0-\alpha_1)}\left(e^{-\beta t} - e^{-(\gamma-\beta)t}\right)}{2\gamma - 4\beta} \times \text{Erfc}\left(\frac{\alpha_2\tau(t) + x_0 - \alpha_1}{\sqrt{2\tau(t)}}\right) \\
& + \frac{1}{2}(s(t) + (x_0 - \alpha_1)r_1(t))\text{Erfc}\left(\frac{\alpha_2\tau(t) - (x_0 - \alpha_1)}{\sqrt{2\tau(t)}}\right) \\
& - e^{2\alpha_2(x_0-\alpha_1)}(s(t) - (x_0 - \alpha_1)r_1(t)) \times \text{Erfc}\left(\frac{\alpha_2\tau(t) + x_0 - \alpha_1}{\sqrt{2\tau(t)}}\right),
\end{aligned}
$$

where $\tau(t) = \left(1 - e^{-(\gamma-2\beta)t}\right)/(\gamma - 2\beta)$ and $r_1(t), r_2(t), r_p(t)$ are given in (4.73).

In what follows, we illustrate how the mean position and the moving boundary change according to time and how the fundamental solution behaves for $\beta < 0$, $\beta = 0$, and $0 < \gamma < \beta$:

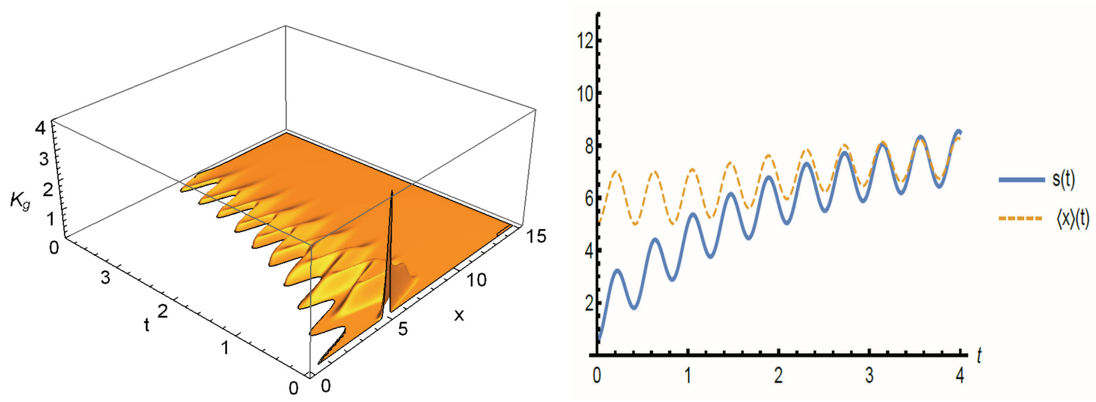
(i) In the case $\beta < 0$, the distribution $K_g^\alpha(x, x_0; t)$ given in (4.76) approaches zero with decreasing amplitude, as $t \to \infty$ for any x, see Fig.4.6a for the behavior of the solution propagating to the right in x-direction by choosing $\alpha_2 > 0$. In Fig.4.6b, we plot the time evolution for the the center of the distribution $\langle x\rangle(t)$ and the moving boundary $s(t)$. It is seen that the mean position moves away from the boundary in time.

(ii) When $\beta = 0$, it is seen in Fig.4.7a that while the solution and the moving boundary propagate to the positive x-direction for chosen parameters, they interact after a certain time. Then, the amplitude of distribution becomes zero on the boundary and the mean position keeps oscillating along the moving boundary. We illustrate the motion of both the boundary and the center of the distribution in Fig.4.7b. One can also concern different choices of parameters $a_0 \geq 0$, α_2, $\omega > 0$ to see different motion of distribution and the boundary.



(a) (b)

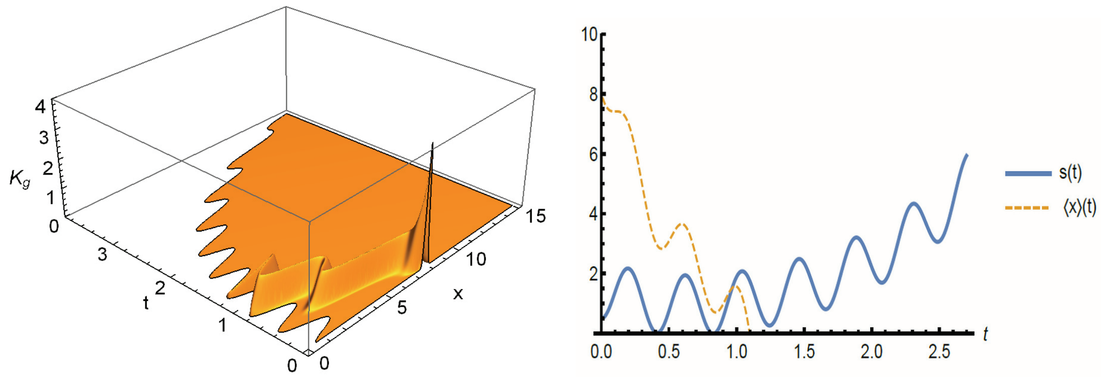
Figure 4.6 For the parameters $\alpha_1 = 0.5$, $\omega = 15$, $a_0 = 15$, $\alpha_2 = 1.5$, $x_0 = 4$, $\beta = -0.5$. $\gamma = 0.5$, (a) the behavior of the solution distribution (4.76). (b) the time evolution for the mean position of the distribution $\langle x \rangle(t)$ and the boundary $s(t)$.



(a) (b)

Figure 4.7 For the parameters $\alpha_1 = 0.5$, $\omega = 15$, $a_0 = 15$, $\alpha_2 = 3.5$, $x_0 = 5$, $\beta = 0$, $\gamma = 0.5$, (a) the behavior of the solution (4.76). (b) the mean position and the boundary.

(iii) For $0 < \gamma < \beta$, by choosing $\alpha_2 > 0$, while the center of the solution propagates to the left in x-axis for certain parameters, the oscillatory boundary moves to the right in x-axis tending to infinity and the amplitude of oscillations decreases approaching zero as $t \to \infty$, see Fig.4.8a. The behavior of the boundary and the mean position can be seen Fig.4.8b for certain parameters.



(a) (b)

Figure 4.8 For the parameters $\alpha_1 = 0.5,\ \omega = 15,\ a_0 = 15,\ \alpha_2 = 3.5,\ x_0 = 5,\ \beta = 0,$ $\gamma = 0.5$, (a) the solution (4.76). (b) mean position and the boundary.

Case 2 . Let $\alpha_2 = 0$ so that the boundary moves according to

$$s(t) = r_g^{\alpha_1}(t) = \alpha_1 r_1(t) + r_p(t).$$

Then we have $p_g^{\alpha}(t) = 0$ and $L_g^{\alpha}(t) = 0$. Thus the fundamental solution is

$$K_g^{\alpha_1}(x, x_0; t) = \sqrt{\frac{\gamma - 2\beta}{2\pi(e^{-2\beta t} - e^{-\gamma t})}} \tag{4.77}$$
$$\times \left(\exp\left[-\frac{(x - r_g^{\alpha_1}(t) - (x_0 - \alpha_1)e^{-\beta t})^2}{2(e^{-2\beta t} - e^{-\gamma t})/(\gamma - 2\beta)} \right] - \exp\left[-\frac{(x - r_g^{\alpha_1}(t) + (x_0 - \alpha_1)e^{-\beta t})^2}{2(e^{-2\beta t} - e^{-\gamma t})/(\gamma - 2\beta)} \right] \right).$$

In this case, the total amount of concentration is

$$M_0(t) = \text{Erf}\left(\frac{\sqrt{\gamma - 2\beta}\,(x_0 - \alpha_1)}{\sqrt{2(1 - e^{-(\gamma - 2\beta)t})}} \right).$$

Note that, initially total amount is $M_0(t) = 1$, then as time increases the amount of concentration is not conserved, i.e. we have $M_0(t) \to 0$ for all cases $\beta < 0, \beta = 0$ and $0 < \gamma < \beta$.

Then the first moment and the mean position of the distribution (4.77) are obtained respectively

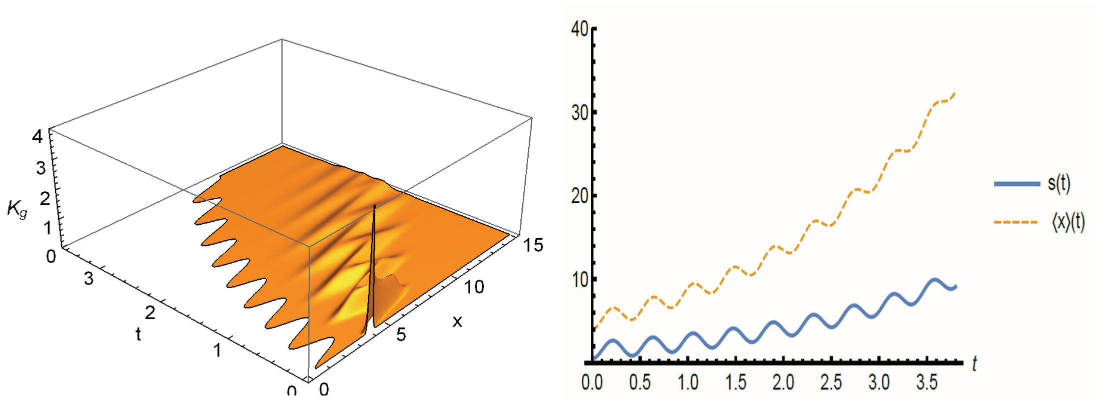
$$M_1(t) = (x_0 - \alpha_1)e^{-\beta t} + r_g^{\alpha_1}(t) \times \mathrm{Erf}\left(\frac{\sqrt{\gamma - 2\beta}\,(x_0 - \alpha_1)}{\sqrt{2(1 - e^{-(\gamma-2\beta)t})}} \right).$$

$$\langle x \rangle(t) = r_g^{\alpha_1}(t) + \frac{(x_0 - \alpha_1)e^{-\beta t}}{\mathrm{Erf}\left(\frac{\sqrt{\gamma-2\beta}\,(x_0-\alpha_1)}{\sqrt{2(1-e^{-(\gamma-2\beta)t})}} \right)}. \qquad (4.78)$$

It is seen from the equation (4.78) that the parameters β and $\gamma > 0$ have influence on determining the distance between the mean position and the boundary. Therefore by choosing $\beta < 0$, $\beta = 0$ and $\beta > 0$, we investigate the motion of the boundary and the center of the distribution as follows :

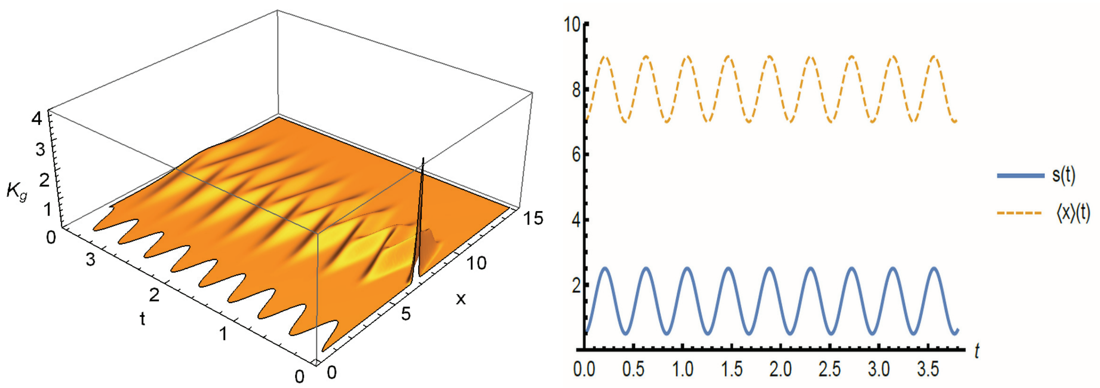
(i) For $\beta < 0$ and $\gamma > 0$, we plot the behavior of solution in Fig.4.9a. The points on the mean position and the boundary move away from each other in the positive x-direction as $t \to \infty$, see Fig.4.9b.

(ii) When $\beta = 0$, one can see the behavior of solution distribution in Fig.4.10a. The boundary and the mean position follow the paths where the distance between them remains constant, Fig.4.10b.



(a) (b)

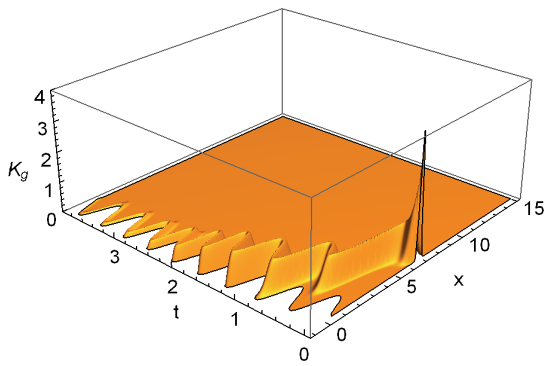
Figure 4.9 For the parameters $\alpha_1 = 0.5$ $\alpha_2 = 0$, $\omega = 15$, $a_0 = 15$, $\gamma = 0.5$, $x_0 = 4$, $\beta = -0.5$, (a) the behavior of the solution (4.77). (b) the mean position $\langle x \rangle(t)$ and the boundary $s(t)$.



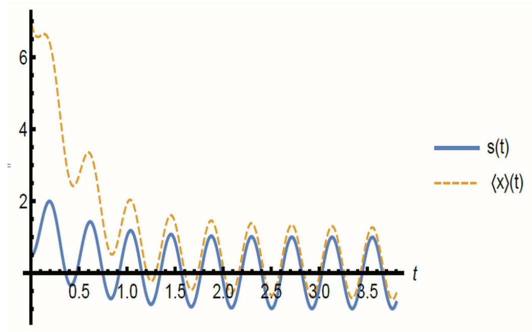
(a) (b)

Figure 4.10 For the parameters $\alpha_1 = 0.5$ $\alpha_2 = 0$, $\omega = 15$, $a_0 = 15$, $\gamma = 0.2$, $x_0 = 7$, $\beta = 0$, (a) the solution (4.77). (b) the mean position and the boundary.

(iii) If $0 < \gamma < \beta$, the behavior of distribution is shown in Fig.4.11a. As time increases, the mean position of the distribution moves to the boundary and on the boundary the amplitude of distribution decreases approaching zero. Then, the center keeps oscillating along the boundary, see Fig.4.11b.



(a)



(b)

Figure 4.11 For the parameters $\alpha_1 = 0.5 \;\; \alpha_2 = 0, \;\; \omega = 15, \;\; a_0 = 15, \;\; \gamma = 0.5, \;\; x_0 = 7,$ $\beta = 2,$ (a) the solution (4.77). (b) the mean position and the boundary.

The second moment for (4.77) is

$$M_2(t) = (x_0 - \alpha_1)e^{-\beta t} r_g^{\alpha_1}(t) + \frac{(x_0 - \alpha_1)\sqrt{2(e^{-4\beta t} - e^{-(\gamma+2\beta)t})}}{\sqrt{(\gamma - 2\beta)\pi}} \times \exp\left[-\frac{(\gamma - 2\beta)(x_0 - \alpha_1)^2}{2(1 - e^{-(\gamma-2\beta)t})}\right]$$
$$+ \left((r_g^{\alpha_1}(t))^2 + (x_0 - \alpha_1)^2 e^{-2\beta t} + \frac{e^{-2\beta t} - e^{-\gamma t}}{\gamma - 2\beta}\right) \times \mathrm{Erf}\left(\frac{\sqrt{\gamma - 2\beta}\,(x_0 - \alpha_1)}{\sqrt{2(1 - e^{-(\gamma-2\beta)t})}}\right). \qquad (4.79)$$

Then normalizing (4.79) by concentration total amount, we obtain mean square

$$\langle x^2\rangle(t) = \left((r_g^{\alpha_1}(t))^2 + (x_0 - \alpha_1)^2 e^{-2\beta t} + \frac{e^{-2\beta t} - e^{-\gamma t}}{\gamma - 2\beta}\right) + \frac{(x_0 - \alpha_1)e^{-\beta t} r_g^{\alpha_1}(t)}{\mathrm{Erf}\left(\frac{\sqrt{\gamma-2\beta}\,(x_0-\alpha_1)}{\sqrt{2(1-e^{-(\gamma-2\beta)t})}}\right)}$$
$$+ \frac{(x_0 - \alpha_1)\sqrt{2(e^{-4\beta t} - e^{-(\gamma+2\beta)t})}}{\sqrt{(\gamma - 2\beta)\pi}} \times \exp\left[-\frac{(\gamma - 2\beta)(x_0 - \alpha_1)^2}{2(1 - e^{-(\gamma-2\beta)t})}\right] \frac{1}{\mathrm{Erf}\left(\frac{\sqrt{\gamma-2\beta}\,(x_0-\alpha_1)}{\sqrt{2(1-e^{-(\gamma-2\beta)t})}}\right)}, \qquad (4.80)$$

and the variance is

$$\sigma(t) = \left((x_0-\alpha_1)^2 e^{-2\beta t} + \frac{e^{-2\beta t}-e^{-\gamma t}}{\gamma-2\beta}\right) - \frac{(x_0-\alpha_1)e^{-\beta t} r_g^{\alpha_1}(t)}{\mathrm{Erf}\left(\frac{\sqrt{\gamma-2\beta}\,(x_0-\alpha_1)}{\sqrt{2(1-e^{-(\gamma-2\beta)t})}}\right)} \tag{4.81}$$
$$+\left(\frac{\sqrt{2(e^{-4\beta t}-e^{-(\gamma+2\beta)t})}}{\sqrt{(\gamma-2\beta)\pi}} \times \exp\left[-\frac{(\gamma-2\beta)(x_0-\alpha_1)^2}{2(1-e^{-(\gamma-2\beta)t})}\right] - \frac{(x_0-\alpha_1)e^{-2\beta t}}{\mathrm{Erf}\left(\frac{\sqrt{\gamma-2\beta}\,(x_0-\alpha_1)}{\sqrt{2(1-e^{-(\gamma-2\beta)t})}}\right)}\right)\frac{(x_0-\alpha_1)}{\mathrm{Erf}\left(\frac{\sqrt{\gamma-2\beta}\,(x_0-\alpha_1)}{\sqrt{2(1-e^{-(\gamma-2\beta)t})}}\right)},$$

which tends to infinity for $\beta < 0,\ \beta = 0$ and $0 < \gamma < \beta$ as $t \to \infty$.

4.1.2.3. Model 3 : Convection-Diffusion-Reaction type mIBVP

Lastly, we introduce an exactly solvable diffusion-convection-reaction type model

$$\begin{cases} \Phi_t = \frac{1}{2}\Phi_{xx} - (\sin(\omega t) - \beta x)\Phi_x - \frac{\omega_0^2}{2}x^2\Phi, \quad s(t) < x < \infty, \quad t > 0, \\ \Phi(x,0) = \Phi^0(x), \quad s(0) < x < \infty, \\ \Phi(s(t),t) = 0, \quad t > 0, \end{cases} \tag{4.82}$$

with moving boundary, constant diffusion coefficient, time-periodic velocity of fluid flow with $a(t) = \sin(\omega t), \omega > 0$, $b(t) = \beta \geq 0$ and a reaction term with $\omega^2(t) = -\omega_0^2$, $\omega_0 > 0$. In this model, the presence of convection term creates again forced characteristic equation as follows

$$\ddot{r}(t) - \Omega^2 r(t) = F_0\cos(\omega t + \theta), \qquad t > 0, \tag{4.83}$$

where $\Omega = \sqrt{\omega_0^2 + \beta^2}$, $F_0 = \sqrt{\omega^2+\beta^2}$ and $\theta = \arctan(\beta/\omega)$. Eq.(4.83) has positive and increasing homogeneous solutions

$$r_1(t) = \frac{\omega_0}{\Omega}\cosh(\Omega t - \sigma), \qquad r_2(t) = \frac{\sinh(\Omega t)}{\Omega}, \qquad \sigma = \mathrm{arctanh}(\beta/\Omega), \tag{4.84}$$

and an oscillatory particular solution,

$$r_p(t) = \frac{1}{\Omega^2 + \omega^2}\left(\frac{\omega_0 \omega}{\Omega}\cosh(\Omega t - \sigma) - F_0 \cos(\omega t + \theta)\right), \quad t > 0. \tag{4.85}$$

Therefore, if the moving boundary is $s(t) = r_g^\alpha(t) \equiv \alpha_1 r_1(t) + \alpha_2 r_2(t) + r_p(t)$, with $s(0) = \alpha_1$, $\dot{s}(0) = \alpha_2 - \alpha_1 \beta$ being the initial position and velocity respectively, then the mIBVP (4.82) has solution found as

$$\begin{aligned} \Phi_g^\alpha(x,t) &= \sqrt{\frac{\Omega \exp(-\beta t)}{\omega_0 \cosh(\Omega t - \sigma)}} \times \exp\left[-\int_0^t L_g^\alpha(t')dt'\right] \\ &\times \exp\left[-\frac{1}{2}\Big(\Omega \tanh(\Omega t - \sigma) + \beta\Big)(x - r_g^\alpha(t))^2\right] \times \exp\left[-p_g^\alpha(t)(x - r_g^\alpha(t))\right] \\ &\times \Psi\left(\eta_g^\alpha(x,t), \tau(t)\right), \end{aligned} \tag{4.86}$$

where Lagrangian function and the generalized momentum are respectively

$$L_g^\alpha(t) = \frac{1}{2}\Big(\left(\dot{r}_g^\alpha(t) + \beta r_g^\alpha(t) - \sin(\omega t)\right)^2 + \omega_0^2 (r_g^\alpha)^2(t)\Big), \qquad p_g^\alpha(t) = \dot{r}_g^\alpha(t) + \beta r_g^\alpha(t) - \sin(\omega t).$$

Also, the coordinate transformations are

$$\eta_g^\alpha(x,t) = \frac{\Omega(x - r_g^\alpha(t))}{\omega_0 \cosh(\Omega t - \sigma)}, \qquad \tau(t) = \frac{\sinh(\Omega t)}{\omega_0 \cosh(\Omega t - \sigma)}, \tag{4.87}$$

with $t(\tau) = (1/\Omega)\tanh^{-1}(\Omega\tau/(1+\beta\tau))$ for $0 < \tau < 1/(\Omega - \beta)$, and $\Psi(\eta, \tau)$ is solution of IBVP (4.8) with BC $\Psi(0,\tau) = 0$.

In this model since $\dot{r}_p(t) \neq a(t) - b(t) r_p(t)$, the velocity of the boundary is different than the velocity of the flowing stream as expected. And the parameters are chosen so that the boundary $s(t) = r_g^\alpha(t)$ propagates forward in positive x-direction while oscillating.

Fundamental solution : The mIBVP (4.82) with Dirac-delta initial condition $\Phi^0(x) = \delta(x - x_0)$, centered at $x_0 > \alpha_1$ has fundamental solution of the form

$$
\begin{aligned}
K_g^\alpha(x, x_0; t) &= e^{\alpha_2(x_0-\alpha_1)}\sqrt{\frac{\Omega \exp(-\beta t)}{2\pi \sinh(\Omega t)}} \times \exp\left[-\int_0^t L_g^\alpha(t')dt'\right] \\
&\times \exp\left[-\frac{1}{2}\Big(\Omega \tanh(\Omega t - \sigma) + \beta\Big)(x - r_g^\alpha(t))^2\right] \times \exp\left[-p_g^\alpha(t)(x - r_g^\alpha(t))\right] \\
&\times \left(\exp\left[-\frac{(\Omega(x - r_g^\alpha(t)) - (x_0 - \alpha_1)\omega_0 \cosh(\Omega t - \sigma))^2}{2\omega_0 \cosh(\Omega t - \sigma)\sinh(\Omega t)}\right]\right. \\
&- \left.\exp\left[-\frac{(\Omega(x - r_g^\alpha(t)) + (x_0 - \alpha_1)\omega_0 \cosh(\Omega t - \sigma))^2}{2\omega_0 \cosh(\Omega t - \sigma)\sinh(\Omega t)}\right]\right), \qquad (4.88)
\end{aligned}
$$

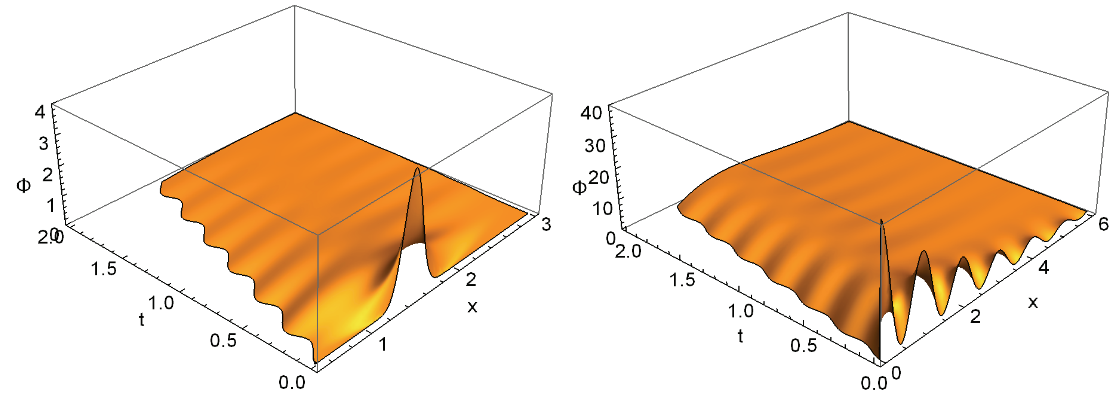
which is shown in Fig.4.12a. We observe that the oscillatory boundary motion creates solution oscillating with respect to time and this influence is felt mainly in regions close to the moving boundary. Also it is clearly seen that solution spreads in positive x-direction as time increases and as $x \to \infty$ it goes to zero, that is $K_g^\alpha(\infty, t) = 0$ for any $t > 0$.

Some concrete examples corresponding to different initial data and boundary propagating according to (4.83), are discussed in what follows.

Example 4.5 *We take a family of oscillatory initial data given by (4.33). Then, mIBVP (4.82) has family of solutions*

$$
\begin{aligned}
\Phi_g^\alpha(x, t) &= \sqrt{\frac{\Omega \exp(-\beta t)}{\omega_0 \cosh(\Omega t - \sigma)}} \times \exp\left[-\int_0^t L_g^\alpha(t')dt'\right] \\
&\times \exp\left[-\frac{1}{2}\Big(\Omega \tanh(\Omega t - \sigma) + \beta\Big)(x - r_g^\alpha(t))^2\right] \times \exp\left[-p_g^\alpha(t)(x - r_g^\alpha(t))\right] \\
&\times \Psi\left(\eta_g^\alpha(x, t), \tau(t)\right), \qquad (4.89)
\end{aligned}
$$

where $\Psi(\eta_g^\alpha(x, t), \tau(t))$ is given in (4.35) and $\eta_g^\alpha(x, t), \tau(t)$ are defined in (4.87). Since Lagrangian function is oscillatory, the classical action function $S(t)$ carries this oscillatory behavior and creates oscillating fluctuations with decreasing amplitude felt in a very short time interval and this can be controlled by the frequency $\omega > 0$ of the flowing medium. Also the solution has oscillatory behavior in space due to the initial data controlled by parameter $B \in \mathbb{R}$. In Fig.4.12b we plot the solution for certain values of the parameters. Here, it is seen that $\Phi_g^\alpha(\infty, t) = 0$ for any $t > 0$.



(a) (b)

Figure 4.12 (a) Solution (4.88) with $\omega = 20$, $\omega_0 = 0.1$, $x_0 = 1.5$, $\alpha_1 = 0.5$, $\alpha_2 = 0.4$ and $\beta = 0.01$. (b) Solution (4.89) with $A = 20$, $B = 6$, $\omega = 20$, $\omega_0 = 0.1$, $\alpha_1 = 0$, $C = \alpha_2 = 0.5$ and $\beta = 0.01$.

Example 4.6 *Next, for initial data given by (4.36) with $c_1 = \alpha_1$ and $c_2 = \alpha_2$, the mIBVP (4.82) has solutions $\Phi^{\alpha}_{g,n}(x,t)$, which for odd $n = 2p+1$, $p = 0, 1, 2, ...$ are*

$$\begin{aligned}
\Phi^{\alpha}_{g,2p+1}(x,t) &= A\sqrt{\frac{\Omega\exp(-\beta t)}{\omega_0\cosh(\Omega t-\sigma)}}\times\exp\left[-\int_0^t L^{\alpha}_g(t')dt'\right]\\
&\times \exp\left[-\frac{1}{2}\Big(\Omega\tanh(\Omega t-\sigma)+\beta\Big)(x-r^{\alpha}_g(t))^2\right]\times\exp\left[-p^{\alpha}_g(t)(x-r^{\alpha}_g(t))\right]\\
&\times H_{2p+1}\left(\frac{\Omega(x-r^{\alpha}_g(t))}{\omega_0\cosh(\Omega t-\sigma)},\frac{\sinh(\Omega t)}{\omega_0\cosh(\Omega t-\sigma)}\right).
\end{aligned} \tag{4.90}$$

When $p = 0$, we have explicit solution as

$$\Phi^{\alpha}_{g,1}(x,t) = A\sqrt{\frac{\Omega\exp(-\beta t)}{\omega_0\cosh(\Omega t-\sigma)}}\times\exp\left[-\int_0^t L^{\alpha}_g(t')dt'\right] \tag{4.91}$$

$$\times\exp\left[-\frac{1}{2}\Big(\Omega\tanh(\Omega t-\sigma)+\beta\Big)(x-r^{\alpha}_g(t))^2\right]\times\exp\left[-p^{\alpha}_g(t)(x-r^{\alpha}_g(t))\right]\times\left(\frac{\Omega(x-r^{\alpha}_g(t))}{\omega_0\cosh(\Omega t-\sigma)}\right).$$

And for even $n = 2p$ they become

$$\Phi^{\alpha}_{g,2p}(x,t) = A\sqrt{\frac{\Omega\exp(-\beta t)}{\omega_0\cosh(\Omega t-\sigma)}}\times\exp\left[-\int_0^t L^{\alpha}_g(t')dt'\right]$$
$$\times\exp\left[-\frac{1}{2}\Big(\Omega\tanh(\Omega t-\sigma)+\beta\Big)(x-r^{\alpha}_g(t))^2\right]\times\exp\left[-p^{\alpha}_g(t)(x-r^{\alpha}_g(t))\right]$$
$$\times\left[h^{-}_{2p}\left(\frac{\Omega(x-r^{\alpha}_g(t))}{\omega_0\cosh(\Omega t-\sigma)},\frac{\sinh(\Omega t)}{\omega_0\cosh(\Omega t-\sigma)}\right)-h^{+}_{2p}\left(\frac{\Omega(x-r^{\alpha}_g(t))}{\omega_0\cosh(\Omega t-\sigma)},\frac{\sinh(\Omega t)}{\omega_0\cosh(\Omega t-\sigma)}\right)\right],$$

when $p = 0$ the explicit solution is found as

$$\begin{aligned}\Phi^{\alpha}_{g,0}(x,t) &= A\sqrt{\frac{\Omega\exp(-\beta t)}{\omega_0\cosh(\Omega t-\sigma)}}\times\exp\left[-\int_0^t L^{\alpha}_g(t')dt'\right]\\ &\times \exp\left[-\frac{1}{2}\Big(\Omega\tanh(\Omega t-\sigma)+\beta\Big)(x-r^{\alpha}_g(t))^2\right]\times\exp\left[-p^{\alpha}_g(t)(x-r^{\alpha}_g(t))\right]\\ &\times Erf\left[\frac{x-r^{\alpha}_g(t)}{\sqrt{\sinh(2\Lambda_0 t)/\Lambda_0}}\right]. \end{aligned} \tag{4.92}$$

Notice that while in solution (4.91) the compatibility condition is satisfied, in solution (4.92) it is not satisfied which causes jump at point $(x,t) = (0,0)$ in the case $\alpha_1 = 0$. The behavior of the solutions (4.91) and (4.92) is shown in Fig.4.13a and Fig.4.13b, respectively. As before, we see that solution is kept equal to zero on the boundary point that moves in positive x-direction while oscillating. Here again fluctuations is felt at time close to zero and position close to the boundary. And for all $n = 0,1,2,...$ and any $t > 0$, solutions tend to zero as $x \to \infty$.

Clearly, above solutions can be used also to solve the mIBVP with generalized initial data $\Phi^0_{c,N}(x) = \sum_{n=0}^{N} A_n e^{-c_2(x-c_1)}(x-c_1)^n$, and investigate its behavior when necessary, for different choices of the coefficients $A_n \geq 0$ and $N = 1,2,3,....$

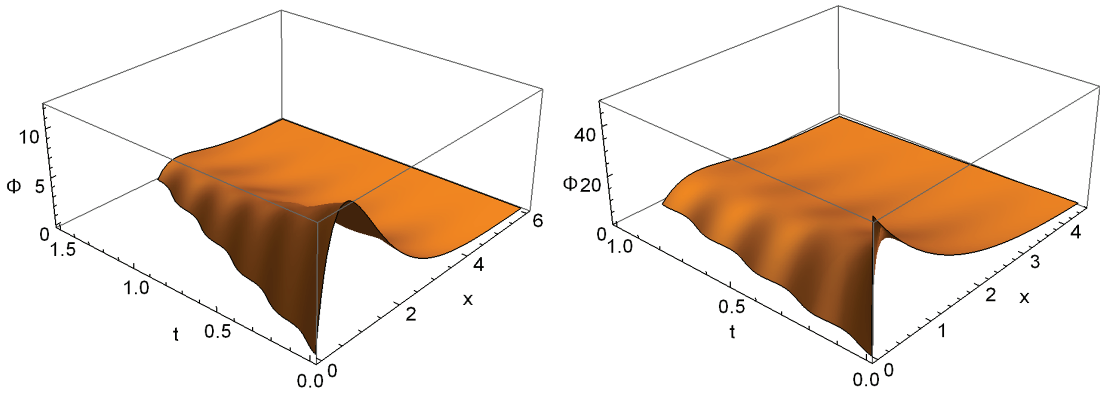


Figure 4.13 (a) Solution $\Phi^{\alpha}_{g,1}(x,t)$ with $A = 50$, $\omega = 20$, $\omega_0 = 0.1$, $\beta = 0.1$, $c_1 = \alpha_1 = 0$, $c_2 = \alpha_2 = 1.5$. (b) Solution $\Phi^{\alpha}_{g,0}(x,t)$ with $A = 50$, $\omega = 25$, $\omega_0 = 0.1$, $\beta = 0.5$, $c_1 = \alpha_1 = 0$, $c_2 = \alpha_2 = 0.7$.

4.2. Analytical Solution of the Neumann IBVP with Moving Boundary

Now we provide the solution of the mIBVP with Neumann boundary conditions. The derivation of the following proposition follows the same lines as in Proposition 4.1, however the result give an insight into to the different effect of the boundary condition and construction of exact solution.

Proposition 4.2 *The mIBVP defined on $s(t) < x < \infty$, $0 < t < T$ for the generalized diffusion type equation with initial data $\Phi^0(x)$ and inhomogeneous Neumann boundary condition imposed on the boundary $x = s(t)$ given as*

$$\Phi_t = \frac{1}{2\mu(t)}\Phi_{xx} - [a(t) - b(t)x]\Phi_x + \mu(t)[\frac{\omega^2(t)}{2}x^2 - f(t)x + f_0(t)]\Phi, \tag{4.93a}$$

$$\Phi(x,0) = \Phi^0(x), \quad s(0) < x < \infty, \tag{4.93b}$$

$$\Phi_x(s(t),t) = N(t), \quad 0 < t < T, \tag{4.93c}$$

has solution of the form (4.4), where $\Psi(\eta,\tau)$ is solution of the IBVP on the half-line with *inhomogeneous Robin type BC*

$$\Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \qquad 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \tag{4.94a}$$

$$\Psi(\eta, 0) = \Phi^0(\eta + \alpha_1)\, e^{\alpha_2 \eta}, \quad 0 < \eta < \infty, \tag{4.94b}$$

$$\Psi_\eta(0, \tau) - r_1(t(\tau)) p_g^\alpha(t(\tau))\Psi(0, \tau) = \tilde{N}(\tau), \quad 0 < \tau < \tau(T), \tag{4.94c}$$

with boundary data

$$\tilde{N}(\tau) = N(t(\tau))\sqrt{(r_1(t(\tau)))^3} \times \exp\left[\int_0^{t(\tau)} \left(\frac{b(t')}{2} - \mu(t') f_0(t') + L_g^\alpha(t')\right) dt'\right]. \quad \square$$

Proposition 4.2 shows that, the mIBVP with an inhomogeneous Neumann BC imposed on the moving boundary reduces to standard heat IBVP with an inhomogeneous Robin type BC prescribed at $x = 0$. In general, when $N(t) \neq 0$, the solution is complicated and exact solvability may not be possible. Therefore, we provide the integral representation of the solution for two particular cases:

Case 1 : $p_g^\alpha(t) \neq 0$

In this case we assume that the generalized momentum is different than zero. Then we investigate the solution to the problem corresponds to homogeneous Neumann boundary data as follows :

• When $N(t) = 0$, using the solution of the heat IBVP with homogeneous Robin BC on the half-line, we have

$$\begin{aligned}\Phi_g^\alpha(x,t) = {} & \frac{1}{\sqrt{r_1(t)}} \times \exp\left[-\int_0^t \left(\frac{b(t')}{2} - \mu(t') f_0(t')\right)\right] \times \exp\left[-\int_0^t L_g^\alpha(t') dt'\right] \\ & \times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)\left(x - r_g^\alpha(t)\right)^2\right] \times \exp\left[-p_g^\alpha(t)(x - r_g^\alpha(t))\right] \\ & \times \left(\int_0^\infty G_N(\eta_g^\alpha(x,t), \xi, \tau(t)) \Phi^0(\xi + \alpha_1) e^{\alpha_2 \xi} d\xi - \int_0^{\tau(t)} K(\eta_g^\alpha(x,t), \tau(t) - \tau') Q_1(\tau') d\tau'\right),\end{aligned} \tag{4.95}$$

where $Q_1(\tau)$ is found by solving the second-kind Volterra type integral equation

$$Q_1(\tau) = r_1(t(\tau))p_g^{\alpha}(t(\tau))\left(2\int_0^{\infty} K(\xi,\tau)\Psi(\xi,0)d\xi - \int_0^{\tau} \frac{Q_1(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau'\right). \tag{4.96}$$

Since the integral equation (4.96) has variable coefficient depending on $r_1(t)p_g^{\alpha}(t)$, evaluating it exactly is a formidable task. But for some particular cases, for instance when $r_1(t)p_g^{\alpha}(t) = c$, where $c \in \mathbb{R}$, then it is possible to solve the equation (4.96) exactly. Correspondingly, the exact analytical solution to the mIBVP (4.93) can be obtained.

Case 2 : $p_g^{\alpha}(t) = 0$

In this case we assume $p_g^{\alpha}(t) \equiv \mu(t)\left(\dot{r}_g^{\alpha}(t) + b(t)r_g^{\alpha}(t) - a(t)\right) = 0$, which is possible in the following cases:

(i) $\alpha_1 = \alpha_2 = 0$ and $\dot{r}_p(t) = a(t) - b(t)r_p(t)$, which follows $r_g^{\alpha}(t) = r_p(t)$.

(ii) $\alpha_2 = 0$, $b(t) = -\dot{r}_1(t)/r_1(t)$ and $\dot{r}_p(t) = a(t) - b(t)r_p(t)$.

If we have case (ii), then in that case the boundary is at rest with respect to fluid motion and instead of $L_g^{\alpha}(t)$, we have

$$L_g^{\alpha_1}(t) = \mu(t)f(t)r_g^{\alpha_1}(t),$$

and the corresponding heat problem has inhomogeneous Neumann BC ($N(t) \neq 0$). Therefore, for the mIBVP we have

$$\Phi_g^{\alpha_1}(x,t) = \exp\left[\int_0^t \mu(t')f_0(t')dt'\right] \times \exp\left[-\int_0^t L_g^{\alpha_1}(t')dt'\right]$$
$$\times\left(\int_0^{\infty} G_N(\eta_g^{\alpha_1}(x,t),\xi,\tau(t))\Phi^0(\xi+\alpha_1)d\xi - \int_0^{\tau(t)} K(\eta_g^{\alpha_1}(x,t),\tau(t)-\tau')\tilde{N}(\tau')d\tau'\right), \tag{4.97}$$

where $\eta_g^{\alpha_1}(x,t) = (x - r_g^{\alpha_1}(t))/r_1(t)$ and

$$\tilde{N}(\tau) = N(t(\tau))r_1(t(\tau)) \times \exp\left[\int_0^{t(\tau)} \left(L_g^{\alpha_1}(t') - \mu(t')f_0(t')\right)dt'\right]. \tag{4.98}$$

<u>Fundamental solution</u>**:** In this case if we take Dirac-delta initial distribution $\Phi(x,0) = \delta(x - x_0)$, and homogeneous Neumann boundary data $N(t) = 0$, then we get the fundamental solution

$$K_g^{\alpha_1}(x,x_0;t) = \frac{\sqrt{r_1(t)}}{\sqrt{2\pi r_2(t)}} \times \exp\left[-\int_0^t \left(L_g^{\alpha_1}(t') - \mu(t')f_0(t')\right)dt'\right]$$
$$\times\left(\exp\left[-\frac{(x - r_g^{\alpha_1}(t) - (x_0-\alpha_1)r_1(t))^2}{2r_1(t)r_2(t)}\right] + \exp\left[-\frac{(x - r_g^{\alpha_1}(t) + (x_0-\alpha_1)r_1(t))^2}{2r_1(t)r_2(t)}\right]\right). \quad (4.99)$$

Zeroth moment: The total amount of distribution (4.99) is defined as

$$M_0(t) = \int_{r_g^{\alpha_1}(t)}^{\infty} K_g^{\alpha_1}(x,x_0;t)dx =$$
$$\frac{\sqrt{r_1(t)}}{\sqrt{2\pi r_2(t)}} \times \exp\left[-\int_0^t \left(L_g^{\alpha_1}(t') - \mu(t')f_0(t')\right)dt'\right]$$
$$\times\int_{r_g^{\alpha_1}(t)}^{\infty}\left(\exp\left[-\frac{(x - r_g^{\alpha_1}(t) - (x_0-\alpha_1)r_1(t))^2}{2r_1(t)r_2(t)}\right] + \exp\left[-\frac{(x - r_g^{\alpha_1}(t) + (x_0-\alpha_1)r_1(t))^2}{2r_1(t)r_2(t)}\right]\right)dx,$$

and evaluating the integrals gives the result as follows

$$M_0(t) = r_1(t)\exp\left[-\int_0^t \left(L_g^{\alpha_1}(t') - \mu(t')f_0(t')\right)dt'\right].$$

First moment: We obtain the first moment as

$$M_1(t) = \int_{r_g^{\alpha_1}(t)}^{\infty} x\, K_g^{\alpha_1}(x,x_0;t)dx$$
$$= M_0(t)\left(r_g^{\alpha_1}(t) + (x_0-\alpha_1)r_1(t)\mathrm{Erf}\left(\frac{(x_0-\alpha_1)}{\sqrt{2\tau(t)}}\right) + \frac{\sqrt{2r_1(t)r_2(t)}}{\sqrt{\pi}} \times \exp\left[-\frac{(x_0-\alpha_1)^2 r_1(t)}{2r_2(t)}\right]\right).$$

• **Mean position** : Therefore, normalizing the first moment by total amount, we get the mean position

$$\langle x\rangle(t) = r_g^{\alpha_1}(t) + (x_0-\alpha_1)\,r_1(t)\mathrm{Erf}\left(\frac{(x_0-\alpha_1)}{\sqrt{2\tau(t)}}\right) + \frac{\sqrt{2r_1(t)r_2(t)}}{\sqrt{\pi}} \times \exp\left[-\frac{r_1(t)\,(x_0-\alpha_1)^2}{2r_2(t)}\right].$$

In what follows we consider the mIBVP with homogeneous Neumann BC for the diffusion-convection equation studied in previous section.

4.2.1. Model : Convection - Diffusion type mIBVP with Neumann BC

Consider the following convection-diffusion model

$$\begin{cases} \Phi_t = \frac{1}{2\mu(t)}\Phi_{xx} - \left((a(t) - b(t)x)\Phi\right)_x, \quad s(t) < x < \infty, \quad t \in (0, T), \\ \Phi(x, 0) = \Phi^0(x), \quad s(0) < x < \infty, \\ \Phi_x(s(t), t) = 0, \quad 0 < t < T. \end{cases} \tag{4.100}$$

If $s(t)$ moves according to

$$s(t) = r_g^{\alpha_1}(t) = \alpha_1 r_1(t) + r_p(t), \tag{4.101}$$

then for the Dirac-delta initial data $\Phi^0(x) = \delta(x - x_0),\ x_0 > \alpha_1$ we obtain the fundamental solution

$$K_g^{\alpha_1}(x, x_0; t) = \frac{1}{\sqrt{2\pi r_1(t) r_2(t)}} \times \left(\exp\left[-\frac{(x - r_g^{\alpha_1}(t) - (x_0 - \alpha_1) r_1(t))^2}{2 r_1(t) r_2(t)} \right] + \exp\left[-\frac{(x - r_g^{\alpha_1}(t) + (x_0 - \alpha_1) r_1(t))^2}{2 r_1(t) r_2(t)} \right] \right). \tag{4.102}$$

Zeroth moment : In this case, the total amount of concentration distribution is

$$M_0(t) = \int_{r_g^{\alpha_1}(t)}^{\infty} K_g^{\alpha_1}(x, x_0; t) dx = \frac{1}{\sqrt{2\pi r_1(t) r_2(t)}} \times \int_{r_g^{\alpha_1}(t)}^{\infty} \left(\exp\left[-\frac{(x - r_g^{\alpha_1}(t) - (x_0 - \alpha_1) r_1(t))^2}{2 r_1(t) r_2(t)} \right] + \exp\left[-\frac{(x - r_g^{\alpha_1}(t) + (x_0 - \alpha_1) r_1(t))^2}{2 r_1(t) r_2(t)} \right] \right) dx,$$

which gives

$$M_0(t) = 1.$$

So, the total amount of substance is conserved.

First moment : The first moment is

$$M_1(t) = \int_{r_g^{\alpha_1}(t)}^{\infty} K_g^{\alpha_1}(x, x_0; t)dx = \frac{1}{\sqrt{2\pi r_1(t) r_2(t)}}$$
$$\times \int_{r_g^{\alpha_1}(t)}^{\infty} x \left(\exp\left[-\frac{(x - r_g^{\alpha_1}(t) - (x_0 - \alpha_1) r_1(t))^2}{2r_1(t) r_2(t)} \right] + \exp\left[-\frac{(x - r_g^{\alpha_1}(t) + (x_0 - \alpha_1) r_1(t))^2}{2r_1(t) r_2(t)} \right] \right) dx.$$

As a result we get

$$M_1(t) = r_g^{\alpha_1}(t) + (x_0 - \alpha_1)\, r_1(t) \mathrm{Erf}\left(\frac{(x_0 - \alpha_1)}{\sqrt{2\tau(t)}} \right) + \frac{\sqrt{2r_1(t) r_2(t)}}{\sqrt{\pi}} \times \exp\left[-\frac{(x_0 - \alpha_1)^2}{2\tau(t)} \right].$$

• **Mean position** : Since the total mass is $M_0(t) = 1$, then the center of the distribution is

$$\langle x \rangle (t) \; = \; M_1(t).$$

Now, we discuss the model (4.100) defined by the equation having some concrete coefficients as follows.

Example 4.7 *Let consider the mIBVP (4.100) with coefficients*

$$\mu(t) = 1, \qquad a(t) = a_0 \cosh(\Lambda_0 t), \; a_0 \geq 0, \qquad b(t) = -\Lambda_0 \tanh(\Lambda_0 t), \; \Lambda_0 > 0. \quad (4.103)$$

In that case, we have characteristic equation $\ddot{r} - \Lambda_0^2 r = 2a_0 \Lambda_0 \sinh(\Lambda_0 t), \; t > 0$, with solutions satisfying required initial conditions $r_1(t) = \cosh(\Lambda_0 t)$, $r_2(t) = \sinh(\Lambda_0 t)/\Lambda_0$, $r_p(t) = a_0\, t \cosh(\Lambda_0 t)$. If the boundary propagates according to

$$s(t) = (\alpha_1 + a_0 t) \cosh(\Lambda_0 t), \qquad (4.104)$$

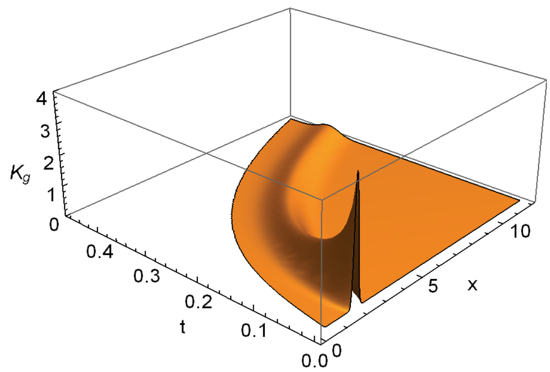
then we have fundamental solution

$$K_g^{\alpha_1}(x, x_0; t) = \sqrt{\frac{\Lambda_0}{\pi \sinh(2\Lambda_0 t)}} \times \tag{4.105}$$
$$\times \left[\exp\left(- \frac{(x - [(a_0\, t + x_0) \cosh(\Lambda_0 t)])^2}{\sinh(2\Lambda_0 t)/\Lambda_0} \right) + \exp\left(- \frac{(x - [(a_0\, t - x_0 + 2\alpha_1) \cosh(\Lambda_0 t)])^2}{\sinh(2\Lambda_0 t)/\Lambda_0} \right) \right].$$

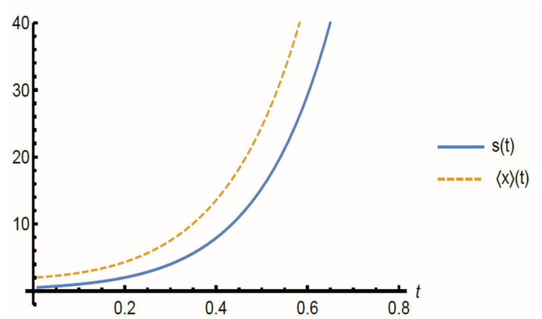
Therefore, the mean position of the solution distribution (4.105) is

$$\begin{aligned} \langle x \rangle(t) &= r_g^{\alpha_1}(t) + (x_0 - \alpha_1)\mathrm{Erf}\left(\frac{\sqrt{\Lambda_0}(x_0 - \alpha_1)}{\sqrt{2 \tanh(\Lambda_0 t)}} \right) \cosh(\Lambda_0 t) \\ &+ \sqrt{\frac{\sinh(2\Lambda_0 t)}{\pi \Lambda_0}} \times \exp\left[- \frac{\Lambda_0 (x_0 - \alpha_1)^2}{2 \tanh(\Lambda_0 t)} \right]. \end{aligned}$$

As time increases, since both the mean and boundary grow hyperbolically, they propagates as if parallel to the each other, see Fig.4.14a for the behavior of the solution distribution and Fig.4.14b for the time evolution of the boundary and the mean position of the distribution.



(a)



(b)

Figure 4.14 For the parameters $\alpha_1 = 0.5$, $\alpha_2 = 0$, $\Lambda_0 = 5$, $a_0 = 4$, $x_0 = 2$, (a) the solution $K_g^{\alpha_1}(x, x_0; t)$. (b) the center of distribution and the boundary.

Example 4.8 ***(mIBVP with homogeneous Neumann BC)***

Now consider the mIBVP (4.100) with coefficients

$$\mu(t) = e^{-\gamma t}, \;\; \gamma > 0, \qquad a(t) = a_0 \sin(\omega t), \;\; a_0 \geq 0, \qquad b(t) = \beta, \;\; \beta \in \mathbb{R}. \quad (4.106)$$

The corresponding characteristic equation and its solutions are given in (4.72) and (4.73), respectively. Therefore if the boundary moves according to (4.101), then we have

$$K_g^{\alpha_1}(x, x_0; t) = \sqrt{\frac{\gamma - 2\beta}{2\pi(e^{-2\beta t} - e^{-\gamma t})}}$$
$$\times \left(\exp\left[-\frac{(x - r_g^{\alpha_1}(t) - (x_0 - \alpha_1)e^{-\beta t})^2}{2(e^{-2\beta t} - e^{-\gamma t})/(\gamma - 2\beta)} \right] + \exp\left[-\frac{(x - r_g^{\alpha_1}(t) + (x_0 - \alpha_1)e^{-\beta t})^2}{2(e^{-2\beta t} - e^{-\gamma t})/(\gamma - 2\beta)} \right] \right). \quad (4.107)$$

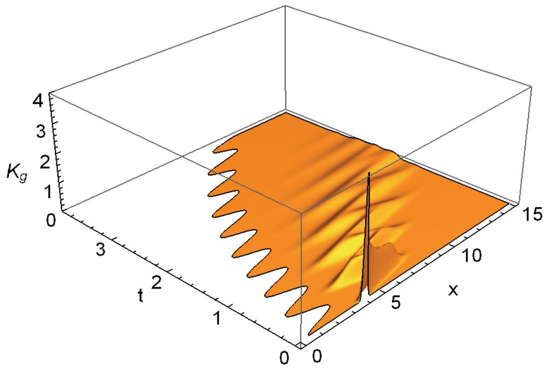
Mean position : The center of the distribution is

$$\begin{aligned} \langle x \rangle(t) &= r_g^{\alpha_1}(t) + (x_0 - \alpha_1)e^{-\beta t} \mathrm{Erf}\left(\frac{(x_0 - \alpha_1)\sqrt{\gamma - 2\beta}}{\sqrt{2(1 - e^{-(\gamma - 2\beta)t})}} \right) \\ &+ \sqrt{\frac{2(e^{-2\beta t} - e^{-\gamma t})}{\pi(\gamma - 2\beta)}} \exp\left[-\frac{(\gamma - 2\beta)(x_0 - \alpha_1)^2}{e^{-2\beta t} - e^{-\gamma t}} \right]. \end{aligned}$$

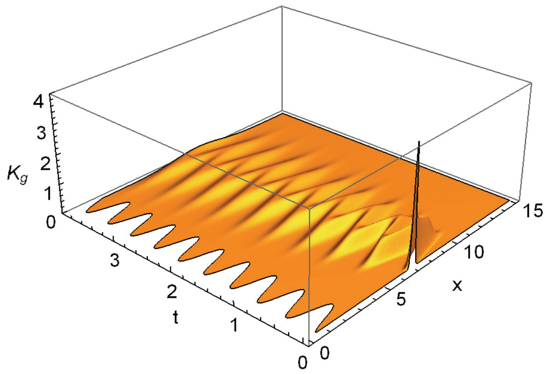
As time increses, the mean position of the solution distribution moves away from the boundary for $\beta < 0$, never approaches to boundary for $\beta = 0$.

(i) For $\beta < 0$, the trajectories of the mean position and the boundary move away from each other, see Fig.4.15a.

(ii) When $\beta = 0$, as in the Example 4.4 (Case 2), the solution propagates with decreasing amplitude and following the trajectory of mean position parallel to the boundary, see Fig.4.15b.



(a)



(b)

Figure 4.15 Solution (4.107) with $\alpha_1 = 0.5$, $\alpha_2 = 0$, $a_0 = 15$, $\omega = 15$, (a) $\gamma = 0, 5$ $\beta = -0.5$, $x_0 = 4$. (b) $\gamma = 0.2$, $\beta = 0$, $x_0 = 7$.

4.3. Analytical Solution of the Robin type IBVP with moving boundary

In this section, we provide the analytical solution to the mIBVP for generalized diffusion type equation with Robin boundary condition. The result is formulated as follows.

Proposition 4.3 *The mIBVP for the generalized diffusion type equation with initial data $\Phi^0(x)$ and homogeneous Robin type boundary condition given as*

$$\Phi_t = \frac{1}{2\mu(t)}\Phi_{xx} - [a(t) - b(t)x]\Phi_x + \mu(t)[\frac{\omega^2(t)}{2}x^2 - f(t)x + f_0(t)]\Phi, \tag{4.108a}$$

$$\Phi(x, 0) = \Phi^0(x), \quad s(0) < x < \infty, \tag{4.108b}$$

$$\Phi_x(s(t), t) + \epsilon(t)\Phi(s(t), t) = 0, \quad 0 < t < T, \tag{4.108c}$$

for $\epsilon(t) \neq 0$, has solution of the form (4.4), where $\Psi(\eta, \tau)$ is solution of the IBVP on the

$$\Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \tag{4.109a}$$
$$\Psi(\eta, 0) = \Phi^0(\eta + \alpha_1)\, e^{\alpha_2 \eta}, \quad 0 < \eta < \infty, \tag{4.109b}$$
$$\Psi_\eta(0, \tau) - r_1(t(\tau))\Big(p_g^\alpha(t(\tau)) - \epsilon(t(\tau))\Big)\Psi(0, \tau) = 0. \qquad \square \tag{4.109c}$$

In that case solution of the mIBVP (4.108) is

$$\begin{aligned}
\Phi_g^\alpha(x,t) = &\ \frac{1}{\sqrt{r_1(t)}} \times \exp\left[-\int_0^t \left(\frac{b(t')}{2} - \mu(t') f_0(t')\right)\right] \times \exp\left[-\int_0^t L_g^\alpha(t')dt'\right] \\
&\times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot r_1(t)}{r_1(t)} + b(t)\right)\left(x - r_g^\alpha(t)\right)^2\right] \times \exp\Big[-p_g^\alpha(t)(x - r_g^\alpha(t))\Big] \\
&\times \left(\int_0^\infty G_N(\eta_g^\alpha(x,t), \xi, \tau(t))\Phi^0(\xi + \alpha_1)e^{\alpha_2\xi}d\xi - \int_0^{\tau(t)} K(\eta_g^\alpha(x,t), \tau(t) - \tau')Q_2(\tau')d\tau'\right),
\end{aligned} \tag{4.110}$$

where $Q_2(\tau)$ is found by solving the second kind Volterra type integral equation

$$Q_2(\tau) = r_1(t(\tau))\Big(p_g^\alpha(t(\tau)) - \epsilon(t(\tau))\Big)\left(2\int_0^\infty K(\xi, \tau)\Psi(\xi, 0)d\xi - \int_0^\tau \frac{Q_2(\tau')}{\sqrt{2\pi(\tau - \tau')}}d\tau'\right). \tag{4.111}$$

Therefore, the mIBVP (4.108) reduces solving the integral equation. One can solve the integral equation (4.111) correspondingly the mIBVP (4.108) explicitly for some particular cases :

(*i*) $\epsilon(t) = p_g^\alpha(t)$.

(*ii*) $\epsilon(t) = p_g^\alpha(t) - c/r_1(t)$ for $c \in \mathbb{R}$.

In what follows we consider the case (*i*) and the other case will be investigated in future.

Case (i) : If $\epsilon(t) = p_g^\alpha(t)$ in boundary condition (4.108c), then the mIBVP for generalized diffusion type equation with Robin type BC reduces to IBVP for heat equation with homogeneous Neumann BC. In that case we have

$$\Phi_g^\alpha(x,t) = \frac{1}{\sqrt{r_1(t)}} \times \exp\left[-\int_0^t \left(\frac{b(t')}{2} - \mu(t')f_0(t')\right)\right] \times \exp\left[-\int_0^t L_g^\alpha(t')dt'\right]$$
$$\times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)\left(x - r_g^\alpha(t)\right)^2\right] \times \exp\left[-p_g^\alpha(t)\left(x - r_g^\alpha(t)\right)\right]$$
$$\times \left(\int_0^\infty G_N(\eta_g^\alpha(x,t), \xi, \tau(t))\Phi^0(\xi + \alpha_1)e^{\alpha_2\xi}d\xi\right). \tag{4.112}$$

If we take $\Phi(x,0) = \delta(x - x_0), \quad \alpha_1 < x < \infty, \quad x_0 > \alpha_1$, then we obtain the fundamental solution

$$K_g^\alpha(x,x_0;t) = e^{\alpha_2(x_0-\alpha_1)}\frac{1}{\sqrt{2\pi r_2(t)}} \times \exp\left[-\int_0^t \left(\frac{b(t')}{2} - \mu(t')f_0(t')\right)\right] \times \exp\left[-\int_0^t L_g^\alpha(t')dt'\right]$$
$$\times \exp\left[-\frac{\mu(t)}{2}\left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)\left(x - r_g^\alpha(t)\right)^2\right] \times \exp\left[-p_g^\alpha(t)\left(x - r_g^\alpha(t)\right)\right]$$
$$\times \left(\exp\left[-\frac{(x - r_g^\alpha(t) - (x_0-\alpha_1)r_1(t))^2}{2r_1(t)r_2(t)}\right] + \exp\left[-\frac{(x - r_g^\alpha(t) + (x_0-\alpha_1)r_1(t))^2}{2r_1(t)r_2(t)}\right]\right). \tag{4.113}$$

Therefore, we can construct an exactly solvable model for this particular case, $\epsilon(t) = p_g^\alpha(t)$, as follows.

4.3.1. Model : Convection - Diffusion type mIBVP with Robin BC

Consider the convection-diffusion model with homogeneous Robin BC

$$\begin{cases} \Phi_t = \frac{1}{2}\Phi_{xx} - a_0\sin(\omega t)\Phi_x, \quad s(t) < x < \infty, \quad t > 0, \\ \Phi(x,0) = \Phi^0(x), \quad s(0) < x < \infty, \\ \Phi_x(s(t),t) + p_g^\alpha(t)\Phi(s(t),t) = 0, \quad t > 0, \end{cases} \tag{4.114}$$

where we have constant diffusion coefficient, time-periodic flux velocity $a(t) = a_0\sin(\omega t)$, with frequency $\omega \in \mathbb{R}$, $a_0 \geq 0$. Then, the characteristic equation $\ddot{r}(t) = a_0\omega\cos(\omega t), \ \ t > 0$, has two independent positive homogeneous solutions and oscillatory particular solution

$$r_1(t) = 1, \qquad r_2(t) = t, \qquad r_p(t) = \frac{a_0}{\omega}\left(1 - \cos(\omega t)\right), \quad t > 0. \tag{4.115}$$

Therefore, if the boundary propagates according to

$$s(t) = r_g^{\alpha}(t) \equiv \alpha_1 + \alpha_2 t + \frac{a_0}{\omega}\left(1 - \cos(\omega t)\right), \tag{4.116}$$

then we obtain the solution to the mIBVP (4.114) explicitly

$$\Phi_g^{\alpha}(x,t) = \exp\left[-\frac{\alpha_2^2 t}{2}\right] \times \exp\left[-\alpha_2(x - r_g^{\alpha}(t))\right] \times \Psi\left(x - r_g^{\alpha}(t), t\right), \tag{4.117}$$

where $\Psi(\eta, \tau)$ is solution of the IBVP for the standard heat equation defined by (4.8a), (4.8b) and homogeneous Neumann BC $\Psi_\eta(0,\tau) = 0$. Since the generalized momentum is $p_g^{\alpha}(t) = \alpha_2$, it implies that the Robin BC (4.108c) has constant coefficient.

However, we notice that if we had different constant from $p_g^{\alpha}(t) = \alpha_2$ in Robin BC (4.108c), then the corresponding heat problem would have Robin type BC in the form $\Psi_\eta(0,\tau) + d_0\Psi(0,\tau) = 0$, for $d_0 \in \mathbb{R}$. And in that case constructing exactly solvable models would require solving second-kind Volterra integral equation.

Fundamental solution : Take Dirac delta IC $\Phi(x,0) = \delta(x - x_0),\ A > 0,\ \alpha_1 < x < \infty,\ x_0 > \alpha_1$. If the boundary is as in (4.116), then the fundamental solution becomes

$$K_g^{\alpha}(x, x_0; t) = e^{\alpha_2(x_0 - \alpha_1)} \times \frac{1}{\sqrt{2\pi t}} \times \exp\left[-\frac{\alpha_2^2 t}{2}\right] \times \exp\left[-\alpha_2(x - r_g^{\alpha}(t))\right] \times\left(\exp\left[-\frac{(x - r_g^{\alpha}(t) - (x_0 - \alpha_1))^2}{2t}\right] + \exp\left[-\frac{(x - r_g^{\alpha}(t) + (x_0 - \alpha_1))^2}{2t}\right]\right). \tag{4.118}$$

The evolution of the solution distribution on the boundary is found as

$$K_g^{\alpha}(s(t), x_0; t) = 2e^{\alpha_2(x_0 - \alpha_1)} \times \frac{1}{\sqrt{2\pi t}} \times \exp\left[-\frac{\alpha_2^2 t}{2}\right] \times \exp\left[-\frac{(x_0 - \alpha_1)^2}{2t}\right], \tag{4.119}$$

which is a Gaussian distribution centered at $x = x_0$ and with time-dependent amplitude. From the Robin BC in (4.114), where $p_g^\alpha(t) = \alpha_2$, one can say that the flux of the solution distribution on the boundary is proportional to the value (4.119) which means that the flux at the boundary is also Gaussian distribution.

The total amount is

$$\begin{aligned} M_0(t) &= \int_{r_g^\alpha(t)}^{\infty} K_g^\alpha(x, x_0; t)dx = \frac{1}{\sqrt{2\pi t}} \times e^{\alpha_2(x_0-\alpha_1)} \times \exp\left[-\frac{\alpha_2^2 t}{2}\right] \\ &\times \int_{r_g^\alpha(t)}^{\infty} \exp\left[-\alpha_2(x - r_g^\alpha(t))\right] \times \left(\exp\left[-\frac{(x - r_g^\alpha(t) - (x_0 - \alpha_1))^2}{2t}\right] \right. \\ &+ \left. \exp\left[-\frac{(x - r_g^\alpha(t) + (x_0 - \alpha_1))^2}{2t}\right] \right) dx, \end{aligned}$$

which gives the result

$$M_0(t) = 1 - \frac{1}{2}\mathrm{Erfc}\left[\frac{x_0 - \alpha_1 - \alpha_2 t}{\sqrt{2t}}\right] + \frac{e^{2\alpha_2(x_0-\alpha_1)}}{2} \times \mathrm{Erfc}\left[\frac{x_0 - \alpha_1 + \alpha_2 t}{\sqrt{2t}}\right].$$

(i) If $\alpha_2 = 0$, then the Robin mIBVP becomes mIBVP with homogeneous Neumann boundary condition. And also in that case the concentration amount is $M_0(t) = 1$.

(ii) When $\alpha_2 > 0$, then $M_0(t) \to 0$ as time increases, since $\mathrm{Erfc}(\infty) = 0$, $\mathrm{Erfc}(-\infty) = 2$. In the case $\alpha_2 < 0$, the amount increases but remains bounded, i.e. we have $M_0(t) \to 1 + e^{2\alpha_2(x_0-\alpha_1)}$.

Then we obtain the first spatial moment

$$\begin{aligned} M_1(t) &= r_g^\alpha(t) + \sqrt{\frac{2t}{\pi}} \exp\left[-\frac{(x_0 - \alpha_1 - \alpha_2 t)^2}{2t}\right] \\ &+ \frac{1}{2}\left(x_0 - 2\alpha_1 - 2\alpha_2 t - \frac{a_0}{\omega} + \frac{a_0}{\omega}\cos(\omega t)\right) \times \mathrm{Erfc}\left[\frac{x_0 - \alpha_1 - \alpha_2 t}{\sqrt{2t}}\right] \\ &+ (x_0 - \alpha_1 - \alpha_2 t) \times \mathrm{Erf}\left[\frac{x_0 - \alpha_1 - \alpha_2 t}{\sqrt{2t}}\right] \\ &+ \frac{1}{2}e^{2\alpha_2(x_0-\alpha_1)} \times \left(2\alpha_1 - x_0 + \frac{a_0}{\omega} - \frac{a_0}{\omega}\cos(\omega t)\right) \times \mathrm{Erfc}\left[\frac{x_0 - \alpha_1 + \alpha_2 t}{\sqrt{2t}}\right]. \end{aligned}$$

Therefore, the mean position is

$$\langle x\rangle(t) = \frac{M_1(t)}{M_0(t)}.$$

In Fig.4.16a, the behavior of solution distribution is shown for certain parameters. It is seen that solution, with pulse initially, propagates with decreasing amplitude in flowing medium, at the time it reaches to the boundary its flux on the boundary changes for a while, and then it vanishes.

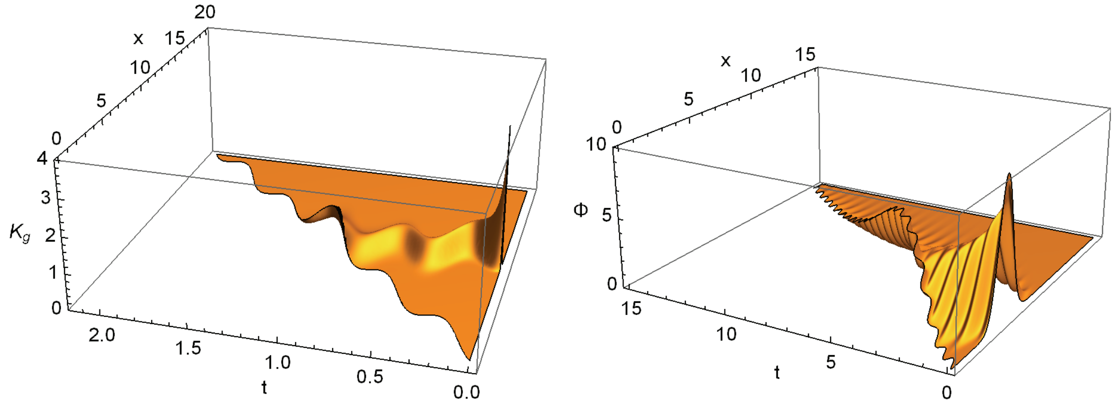
Next, we consider Gaussian initial data $\Phi^0(x) = Ae^{-(x-c)^2}$, centered at $x = c$ with maximum amplitude $A > 0,\ \alpha_1 < x < \infty,\ c > \alpha_1$. Then the corresponding heat problem has Gaussian type IC $\Psi(\eta, 0) = Ae^{\alpha_2(c-\alpha_1+\alpha_2/4)}e^{-(\eta-(c-\alpha_1+\alpha_2/2))^2},\ 0 < \eta < \infty$ centered at $\eta = c - \alpha_1 + \alpha_2/2$ with maximum amplitude $Ae^{\alpha_2(c-\alpha_1+\alpha_2/4)}$ and homogeneous Neumann BC $\Psi_\eta(0, \tau) = 0,\ \tau > 0$. Therefore, if the boundary is of the form (4.116), then we obtain exact analytical solution as follows

$$\begin{aligned}\Phi_g^\alpha(x,t) &= \frac{Ae^{\alpha_2(c-\alpha_1+\alpha_2/4)}}{2\sqrt{(2t+1)}}\exp\left[-\frac{\alpha_2^2 t}{2}\right]\times\exp\left[-\alpha_2(x-r_g^\alpha(t))\right] \qquad (4.120)\\ &\times\left(\exp\left[-\frac{\left(x-r_g^\alpha(t)+(c-\alpha_1+\alpha_2/2)\right)^2}{2t+1}\right]\times\left(1+\mathrm{Erf}\left(\frac{-x+r_g^\alpha(t)+2\,(c-\alpha_1+\alpha_2/4)\,t}{\sqrt{2\,t\,(2t+1)}}\right)\right)\right.\\ &+\left.\exp\left[-\frac{\left(x-r_g^\alpha(t)-(c-\alpha_1+\alpha_2/2)\right)^2}{2t+1}\right]\times\left(1+\mathrm{Erf}\left(\frac{x-r_g^\alpha(t)+2\,(c-\alpha_1+\alpha_2/4)\,t}{\sqrt{2\,t\,(2t+1)}}\right)\right)\right).\end{aligned}$$

The evolution of solution on the boundary $x = s(t)$ is

$$\begin{aligned}\Phi_g^\alpha(s(t),t) &= \frac{Ae^{\alpha_2(c-\alpha_1+\alpha_2/4)}}{\sqrt{(2t+1)}}\exp\left[-\frac{\alpha_2^2 t}{2}\right]\times\exp\left[-\frac{(c-\alpha_1+\alpha_2/2)^2}{2t+1}\right] \qquad (4.121)\\ &\times\left(1+\mathrm{Erf}\left(\frac{2\,(c-\alpha_1+\alpha_2/4)\,t}{\sqrt{2\,t\,(2t+1)}}\right)\right).\end{aligned}$$

Similar effect is seen also here, i.e while the solution propagates following the mean position path with decreasing amplitude in flowing medium, at the time it reaches to the boundary, its flux on the boundary changes for a while, and then it vanishes, see Fig.4.16b.



(a)

(b)

Figure 4.16 (a) Solution (4.118) with $\omega = 15$, $a_0 = 12$, $\alpha_1 = 0.5$, $\alpha_2 = 9$, $x_0 = 10$. (b) Solution (4.121) with $A = 10$, $\omega = 10$, $a_0 = 3$, $\alpha_1 = 0.5$, $\alpha_2 = 1$, $c = 5$.

CHAPTER 5

GENERALIZED BURGERS TYPE EQUATION WITH VARIABLE COEFFICIENTS

In this chapter, we study IVP and IBVP with Dirichlet boundary condition for a one dimenional generalized forced Burgers type equation of the form

$$U_t + \frac{\dot{\mu}(t)}{\mu(t)}U + UU_x = \frac{1}{2\mu(t)}U_{xx} - ((a(t) + b(t)x)U)_x - \omega^2(t)x + f(t), \tag{5.1}$$

for the field $U(x,t)$, with smooth coefficients of damping $\Gamma(t) = \dot{\mu}(t)/\mu(t)$, diffusion $D(t) = 1/2\mu(t)$, convection coefficient linear in position $v(x,t) = a(t) - b(t)x$, and $F(x,t) = -\omega^2(t)x + f(t)$ is an external forcing term linear in position variable. First we consider IVP on whole real line and obtain analytical solution in terms of solution to corresponding characteristic equation and standard Burgers (or heat) models. Then we study IBVP defined on half-line $0 < x < \infty$ with Dirichlet boundary condition. As a result, we construct exactly solvable models and discuss the influence of variable parameters.

5.1. Analytical Solution of the Initial Value Problem on the Whole Line

In this section, we derive the solution of the IVP defined on $x \in (-\infty, \infty)$ and $t \in (t_0, \infty)$, $t_0 > 0$, for the generalized Burgers type equation

$$\begin{cases} U_t + \frac{\dot{\mu}(t)}{\mu(t)}U + UU_x = \frac{1}{2\mu(t)}U_{xx} - ((a(t) + b(t)x)U)_x - \omega^2(t)x + f(t), & x \in \mathbb{R}, \quad t > t_0 > 0, \\ U(x, t_0) = U_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{5.2}$$

where coefficients $\mu(t) > 0$, $\omega^2(t) > 0$, $a(t)$, $b(t)$ and $f(t)$ are given real-valued smooth functions depending on time and initial data $U_0(x)$ at time $t = t_0$ is given smooth and bounded function of x. The result is formulated as follows.

Proposition 5.1 *If $r_1(t)$, $r_2(t)$ are two independent homogeneous solutions and $r_p(t)$ is a particular solution of characteristic equation*

$$\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \left[\omega^2(t) + \left(\dot{b}(t) + \frac{\dot{\mu}(t)}{\mu(t)}b(t) - b^2(t)\right)\right] r = \dot{a}(t) - a(t)\left[b(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right] + f(t), \quad (5.3)$$

satisfying initial conditions $r_1(t_0) = r_0 \neq 0$, $\dot{r}_1(t_0) = -b(t_0)r_0$, $r_2(t_0) = 0$, $\dot{r}_2(t_0) = r_0/\mu(t_0)$ and $r_p(t_0) = 0$, $\dot{r}_p(t_0) = a(t_0)$ respectively, then IVP (5.2) has solution in the forms:

$$(a)\ \ U(x,t) = -\frac{r_1(t_0)}{\mu(t)r_1(t)}\frac{\Psi_\eta(\eta_p(x,t),\tau(t))}{\Psi(\eta_p(x,t),\tau(t))} + \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)(x - r_p(t)) + \frac{p_p(t)}{\mu(t)}, \quad (5.4)$$

where the generalized momentum

$$p_p(t) = \mu(t)\left(\dot{r}_p(t) + b(t)r_p(t) - a(t)\right), \quad (5.5)$$

coordinate transformation $(x,t) \to (\eta,\tau)$

$$\eta_p(x,t) = \frac{r_1(t_0)}{r_1(t)}(x - r_p(t)), \qquad \tau(t) = \frac{r_2(t)}{r_1(t)}, \qquad t > t_0, \quad (5.6)$$

and $\Psi(\eta,\tau)$ satisfies the IVP on whole line for the heat equation

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, & -\infty < \eta < \infty, \quad \tau > 0, \\ \Psi(\eta,0) = \exp\left[-\mu(t_0)\int^\eta U(x,t_0)dx\right], & -\infty < \eta < \infty. \end{cases} \quad (5.7)$$

$$(b)\ \ U(x,t) = \frac{r_1(t_0)}{\mu(t)r_1(t)}V(\eta(x,t),\tau(t)) + \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)(x - r_p(t)) + \frac{p_p(t)}{\mu(t)}, \quad (5.8)$$

where $V(\eta,\tau)$ satisfies the IVP for the standard Burgers equation

$$\begin{cases} V_\tau + VV_\eta = \frac{1}{2}V_{\eta\eta}, & -\infty < \eta < \infty, \quad \tau > 0, \\ V(\eta, 0) = \mu(t_0)U(\eta, t_0), & -\infty < \eta < \infty. \qquad \square \end{cases} \tag{5.9}$$

Proof (a) Here, it is straightforward to verify that Proposition 5.1 is a direct consequence of Proposition 3.1, that is by generalized Cole-Hopf transform

$$U(x,t) = -\frac{\Phi_x(x,t)}{\mu(t)\Phi(x,t)}, \tag{5.10}$$

the generalized Burgers equation in (5.2) is reduced to generalized diffusion type equation in (3.2) and the initial condition in (5.2) directly transforms to initial condition $\Phi(x,t_0) = \exp\left[-\mu(t_0)\int^x U_0(x')dx'\right]$. Therefore applying generalized Cole-Hopf transform for the solution obtained in (3.2), we get the desired result (5.4).

Part (b) follows from part (a) and Cole-Hopf transformation

$$V(\eta,\tau) = -\frac{\Psi_\eta(\eta,\tau)}{\Psi(\eta,\tau)}. \tag{5.11}$$

$\square$

Thus we show that the solution to the IVP (5.2) is obtained in terms of solutions of forced characteristics equation and standard heat or Burgers model.

Moreover, using symmetries of standard Burgers equation such as space and time translation, scaling and Galilean transform, we can create new solutions from given ones. In particular, if $V(\eta,\tau)$ is solution of BE (5.9), using translation in space and Galilean invariance given in (2.3.2.1), we can find two parametric, an infinite family of solutions

$$V_\alpha(\eta,\tau) = \alpha_2 + V(\eta - (\alpha_1 + \alpha_2\tau), \tau), \quad \eta \in \mathbb{R}, \quad \tau > 0,$$

parameterized by $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. Then, according to Proposition 5.1, the generalized Burgers equation (5.1) will have family of solutions as in the following form

$$\begin{aligned} U_\alpha(x,t) &= \frac{1}{\mu(t)r_1(t)}\Big[\alpha_2 + V\Big(\frac{1}{r_1(t)}(x - (\alpha_1 r_1(t) + \alpha_2 r_2(t) + r_p(t)), \frac{r_2(t)}{r_1(t)}\Big)\Big] \\ &+ \Big(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\Big)(x - r_p(t)) + \frac{p_p(t)}{\mu(t)}, \end{aligned} \tag{5.12}$$

where $r_1(t)$, $r_2(t)$ and $r_p(t)$ are solutions of Eq.(3.3) satisfying the prescribed initial conditions and without loss of generality we take $r_0 = 1$. By using Wronskian of homogeneous solutions to the characteristics equation (3.3),

$$W(r_1(t), r_2(t)) = \dot{r}_2(t)r_1(t) - \dot{r}_1(t)r_2(t) = \frac{1}{\mu(t)},$$

we have equivalent form for the solution (5.12) as

$$\begin{aligned} U_g^\alpha(x,t) &= \frac{1}{\mu(t)r_1(t)}V\Big(\frac{1}{r_1(t)}(x - r_g^\alpha(x,t)), \frac{r_2(t)}{r_1(t)}\Big) \\ &+ \Big(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\Big)(x - r_g^\alpha(t)) + \frac{p_g^\alpha(t)}{\mu(t)}, \end{aligned} \tag{5.13}$$

where we denote

$$r_g^\alpha(t) \equiv \alpha_1 r_1(t) + \alpha_2 r_2(t) + r_p(t) = r_h^\alpha(t) + r_p(t), \qquad (\alpha_1, \alpha_2) \in \mathbb{R}^2, \tag{5.14}$$

and the generalized momentum

$$p_g^\alpha(t) = \dot{r}_g^\alpha(t) + b(t)r_g^\alpha(t) - a(t),$$

and $U_g^\alpha(x,t)$ for the generalized Burgers solution obtained in terms of (5.14).

On the unforced characteristic and Burgers equation

As a particular case to have homogeneous characteristic equation, consider the Burgers equation (5.1) without forcing terms, $\omega(t) = f(t) = 0$, and shifting term, $a(t) = 0$,

$$U_t + \frac{\dot{\mu}(t)}{\mu(t)}U + UU_x = \frac{1}{2\mu(t)}U_{xx} + b(t)(xU)_x, \qquad -\infty < x < \infty, \quad t > t_0. \tag{5.15}$$

Thus, we have corresponding homogeneous characteristic equation as

$$\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \left(\dot{b}(t) + \frac{\dot{\mu}(t)}{\mu(t)}b(t) - b^2(t)\right)r = 0, \tag{5.16}$$

and solution of BE (5.15) becomes

$$U_h^\alpha(x,t) = \frac{1}{\mu(t)r_1(t)}\ V\Big(x - r_h^\alpha(t), \frac{r_2(t)}{r_1(t)}\Big) + \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)(x - r_h^\alpha(x,t)) + \dot{r}_h^\alpha(t) + b(t)r_h^\alpha(t),$$

or equivalently

$$U_h^\alpha(x,t) = \frac{1}{\mu(t)r_1(t)}\Big(\alpha_2 + V\Big(x - r_h^\alpha(t), \frac{r_2(t)}{r_1(t)}\Big)\Big) + \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)x. \tag{5.17}$$

Notice that, in general, the solution (5.17) is unbounded at $x = \pm\infty$. However, in many applications we are interested in bounded solutions that are localized in space or approaching finite values as $x \to \pm\infty$. As mentioned in Chapter 3, we are allowed to replace the original $b(t)$ by special choice, $b(t) = -\dot{r}_1(t)/r_1(t)$. In that case the solution of Burgers equation (5.15) becomes

$$U_h^\alpha(x,t) = \frac{1}{\mu(t)r_1(t)}\Big(\alpha_2 + V\Big(\frac{1}{r_1(t)}(x - r_h^\alpha(t)), \frac{r_2(t)}{r_1(t)}\Big)\Big). \tag{5.18}$$

Now, how solutions of the form (5.18) behave depends on $V(\eta,\tau)$ as shown in what follows.

5.1.1. Special solutions of the generalized Burgers model

In this part, we write and investigate some special solutions such as single and multiple shocks, triangular waves, N-shaped waves and rational type solutions of the generalized Burgers equation.

(a) Generalized single shocks:

As well known, the standard BE (5.9) has shock traveling wave solution (2.70). By Proposition 5.1, it follows that the generalized Burgers equation (5.1) subject to initial condition

$$U_g^\alpha(x, t_0) = (\alpha_2 - A\tanh[A(x-\alpha_1)])/\mu(t_0), \qquad (\alpha_1, \alpha_2) \in \mathbb{R}^2, \tag{5.19}$$

has generalized single shock type solution of the form

$$U_g^\alpha(x,t) = \frac{-A}{\mu(t)r_1(t)}\tanh\left[\frac{A}{r_1(t)}\left(x - r_g^\alpha(t)\right)\right] + \left[\frac{\dot r_1(t)}{r_1(t)} + b(t)\right](x - r_g^\alpha(t)) + \frac{p_g^\alpha(t)}{\mu(t)}. \tag{5.20}$$

Here, the shock amplitude $A(t) = A/\mu(t)r_1(t)$ and steepness $B(t) = A/r_1(t)$ depend on time, and $x = r_g^\alpha(t)$ describes the motion of the "center" of the wave profile explicitly written as (5.14), where $v = \dot r_g^\alpha(t)$ gives its velocity. In solution (5.20), the term which is linear in position x contributes to the total wave amplitude and due to it, in general solution is unbounded as $x \to \pm\infty$. However, if we consider the special homogeneous BE (5.15) with $b(t) = -\dot r_1(t)/r_1(t)$, then it has a single shock solution of the form

$$U_h^\alpha(x,t) = \frac{1}{\mu(t)r_1(t)}\left(\alpha_2 - A\tanh\left[\frac{A}{r_1(t)}\left(x - r_h^\alpha(t)\right)\right]\right), \tag{5.21}$$

satisfying time-dependent boundary conditions

$$U_h^\alpha(-\infty,t) = \frac{1}{\mu(t)r_1(t)}(\alpha_2 + A), \quad U_h^\alpha(+\infty,t) = \frac{1}{\mu(t)r_1(t)}(\alpha_2 - A), \quad t \geq 0, \tag{5.22}$$

whose behavior at large times is determined by $\mu(t)r_1(t)$.

(b) Generalized multi-shocks:

By using the standard Burgers multi shock traveling wave solution given in (2.76), we can obtain generalized multi-shock wave solutions for the generalized Burgers equation (5.1)

$$\begin{aligned} U_g^{\alpha(k)}(x,t) &= \frac{2}{\mu(t)r_1(t)}\left[\frac{\alpha_2^{(1)}\exp[p_1(\eta(x,t),\tau(t))] + ... + \alpha_2^{(k)}\exp[p_k(\eta(x,t),\tau(t))]}{\exp[p_1(\eta(x,t),\tau(t))] + ... + \exp[p_k(\eta(x,t),\tau(t))]}\right] \\ &+ \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)(x - r_p(t)) + \left[\dot{r}_p(t) + b(t)r_p(t) - a(t)\right], \end{aligned} \tag{5.23}$$

where $p_i(\eta(x,t),\tau(t))$ are polynomial functions defined by

$$p_i(\eta(x,t),\tau(t)) = -\frac{2\alpha_2^{(i)}}{r_1(t)}\left(x - r_{g,\alpha}^{(i)}(t)\right), \qquad r_{g,\alpha}^{(i)}(t) = \alpha_1^{(i)}r_1(t) + \alpha_2^{(i)}r_2(t) + r_p(t), \tag{5.24}$$

for each $i = 1, 2, ..., k$. Again, for suitably chosen parameters one can produce generalized multi-shock traveling waves, as we show and discuss in next section for certain models.

(c) Generalized triangular waves: It is known that the standard Burgers equation has family of triangular type wave solution given in (2.72). Then the generalized Burgers equation (5.1) has generalized triangular wave solutions

$$\begin{aligned} U_g^{\alpha}(x,t) &= \frac{1}{\mu(t)\sqrt{2\pi r_1(t)r_2(t)}}\left(\frac{(e^R - 1)\exp\left[-\frac{(x-r_g^{\alpha}(t))^2}{2r_1(t)r_2(t)}\right]}{1 + \frac{1}{2}(e^R - 1)\,\mathrm{Erfc}\left[\frac{(x-r_g^{\alpha}(t))}{\sqrt{2r_1(t)r_2(t)}}\right]}\right) \\ &+ \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)(x - r_g^{\alpha}(t)) + \frac{p_g^{\alpha}(t)}{\mu(t)}, \end{aligned} \tag{5.25}$$

when $\sigma(t) \equiv r_1(t)r_2(t) > 0$. Here, amplitude and width of the wave profile depend on $\sigma(t)$, displacement of position is given by $x = r_g^{\alpha}(t)$, and in general the wave is not localized in space. However, for the unforced BE (5.15), if $b(t) = -\dot{r}_1(t)/r_1(t)$, then solution becomes

$$U_h^{\alpha}(x,t) = \frac{\alpha_2}{\mu(t)r_1(t)} + \frac{1}{\mu(t)\sqrt{2\pi\sigma(t)}}\left(\frac{(e^R - 1)\exp\left[-\frac{(x-r_h^{\alpha}(t))^2}{2\sigma(t)}\right]}{1 + \frac{1}{2}(e^R - 1)\,\mathrm{Erfc}\left[\frac{(x-r_h^{\alpha}(t))}{\sqrt{2\sigma(t)}}\right]}\right), \tag{5.26}$$

which is a wave packet whose amplitude and width again depend on $\sigma(t)$ but position is displaced in time according to $x = r^h_\alpha(t)$, and boundary conditions become

$$U^\alpha_h(\pm\infty, t) = \frac{\alpha_2}{\mu(t) r_1(t)}, \qquad t \geq 0. \tag{5.27}$$

Clearly, if $\alpha_2 = 0$ in (5.26), then the wave packet, which is Gaussian like for small R, is positive and smooth for all $x \in \mathbb{R}$, localized with $U_\alpha(\pm\infty, t) = 0$ and moves with velocity $v = \alpha_1 \dot{r}_1(t)$. If in addition $\mu(t) = 1$, then solution belongs to $L_1(\mathbb{R})$ and can be normalized, with norm

$$\int_{-\infty}^{\infty} |U^\alpha_h(x, t)| dx = R. \tag{5.28}$$

(d) Generalized N-shaped waves: Using N-shaped similarity solutions of standard BE given in (2.75), then the corresponding family of generalized N-shaped traveling wave solutions become

$$\begin{aligned} U^\alpha_g(x, t) &= \left(\frac{x - r^\alpha_g(t)}{\mu(t) r_1(t) r_2(t)}\right)\left(\frac{\sqrt{\frac{c\, r_1(t)}{r_2(t)}} \exp\left[-\frac{(x - r^\alpha_g(t))^2}{2 r_1(t)\, r_2(t)}\right]}{1 + \sqrt{\frac{c\, r_1(t)}{r_2(t)}} \exp\left[-\frac{(x - r^\alpha_g(t))^2}{2 r_1(t)\, r_2(t)}\right]}\right) \\ &+ \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)(x - r^\alpha_g(t)) + \frac{p^\alpha_g(t)}{\mu(t)}, \end{aligned} \tag{5.29}$$

where $r^\alpha_g(t)$ is again given by (5.14), and amplitude and width of the wave profile is controlled by $\sigma(t) = r_1(t) r_2(t)$. If we consider Burgers equation (5.15) with $b(t) = -\dot{r}_1(t)/r_1(t)$, then we get solution

$$U^\alpha_h(x, t) = \frac{\alpha_2}{\mu(t) r_1(t)} + \left(\frac{x - r^\alpha_h(t)}{\mu(t) \sigma(t)}\right)\left(\frac{\sqrt{\frac{c\, r_1(t)}{r_2(t)}} \exp\left[-\frac{(x - r^\alpha_h(t))^2}{2\sigma(t)}\right]}{1 + \sqrt{\frac{c\, r_1(t)}{r_2(t)}} \exp\left[-\frac{(x - r^\alpha_h(t))^2}{2\sigma(t)}\right]}\right), \tag{5.30}$$

which satisfies boundary conditions (5.27). If $\alpha_2 = 0$ in (5.30), then the wave packet is localized in space with $U_\alpha(\pm\infty, t) = 0$, its amplitude and width depend on $\sigma(t)$ and it has moving zero whose position changes with time according to $x = \alpha_1 r_1(t)$. Here, for $\mu(t) = 1$, $L_1(\mathbb{R})$ norm depends on time

$$\int_{-\infty}^{\infty} |U_h^{\alpha}(x,t)| dx = 2\ln\left(1 + \sqrt{\frac{c\, r_1(t)}{r_2(t)}} \times \exp\left[-\frac{(r_h^{\alpha}(t))^2}{2\sigma(t)}\right]\right). \tag{5.31}$$

(e) Generalized rational type solutions: We know that the standard Burgers equation has family of rational solution given in (2.76). Then, it follows that the generalized Burgers equation (5.1) with initial condition

$$U_m(x,t_0) = \frac{1}{\mu(t_0)}\left(\alpha_2 - \frac{m}{x-\alpha_1}\right), \qquad m = 1,2,\ldots, \qquad (\alpha_1,\alpha_2) \in \mathbb{R}^2, \tag{5.32}$$

has family of rational type solution

$${}_{\alpha}U_m(x,t) = -\frac{m}{\mu(t) r_1(t)}\left[\frac{H_{m-1}(\frac{x-r_g^{\alpha}(t)}{r_1(t)}, \frac{r_2(t)}{r_1(t)})}{H_m(\frac{x-r_g^{\alpha}(t)}{r_1(t)}, \frac{r_2(t)}{r_1(t)})}\right] + \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)(x - r_g^{\alpha}(t)) + \frac{p_g^{\alpha}(t)}{\mu(t)}. \tag{5.33}$$

We notice that ${}_{\alpha}U_m(x,t)$ has singularities at points where

$$H_m\left(\frac{x-r_g^{\alpha}(t)}{r_1(t)}, \frac{r_2(t)}{r_1(t)}\right) = 0.$$

Then using the relation between KFP and Hermite polynomials, (2.13), the position of the pole singularities is obtained

$$x = r_g^{\alpha}(t) - i\xi_m^{(l)}\sqrt{2r_1(t)r_2(t)}, \quad t > t_0. \tag{5.34}$$

Only for the case $m = 2p + 1$ and $p = 0, 1, 2, ...$, the generalized rational solution (5.33), that is only ${}_{\alpha}U_{2p+1}(x,t)$ has a moving singularity on the real domain, and its position is described by $x = r_g^{\alpha}(t)$. It shows that the generalized Burgers solutions may have singularities propagating in time according to a Newtonian equation of motion.

In particular, for the BE (5.15) with $b(t) = -\dot{r}_1(t)/r_1(t)$, the solution becomes

$$ {}_{\alpha}U_m(x,t) = \frac{1}{\mu(t)r_1(t)}\left(\alpha_2 - m\left[\frac{H_{m-1}\left(\frac{x-r_h^\alpha(t)}{r_1(t)},\frac{r_2(t)}{r_1(t)}\right)}{H_m\left(\frac{x-r_h^\alpha(t)}{r_1(t)},\frac{r_2(t)}{r_1(t)}\right)}\right]\right), \tag{5.35} $$

which at $x = \pm\infty$ satisfies (5.27), and for odd m, position of the pole type singularity is described by $x = r_h^\alpha(t)$. Finally, we recall that the standard BE has solutions (2.79), then the corresponding solutions of the generalized BE will take the form

$$ \begin{aligned} {}_{\alpha}U_m(x,t) &= -\frac{1}{\mu(t)r_1(t)}\left[\frac{\sum_{m=1}^{N} m\, a_{m-1}H_{m-1}\left(\frac{x-r_g^\alpha(t)}{r_1(t)},\frac{r_2(t)}{r_1(t)}\right)}{\sum_{m=0}^{N} a_m H_m\left(\frac{x-r_g^\alpha}{r_1(t)},\frac{r_2(t)}{r_1(t)}\right)}\right] \\ &+ \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)(x - r_g^\alpha(t)) + \frac{p_g^\alpha(t)}{\mu(t)}. \end{aligned} \tag{5.36} $$

Depending on the constant coefficients a_m, these solutions may or may not have singularities on the real domain, and an analysis can be done when necessary.

5.1.2. Exactly solvable generalized Burgers models

In this section we introduce some exactly solvable Burgers type models with external terms in order to illustrate certain aspects of the general results. In Model 1, we consider Burgers type model including only a dilation external term with special time-variable coefficient. In Model 2, in addition to dilation term we take also a constant damping term and a decreasing with time diffusion coefficient. In Model 3, we study simultaneously the effects of dilation and time-periodic forcing.

MODEL 1 : Constant diffusion and variable convection coefficient

Consider the generalized Burgers type equation

$$ U_t + UU_x = \frac{1}{2}U_{xx} - \Lambda_0\tanh(\Lambda_0 t)\,(xU)_x\,, \tag{5.37} $$

with $\mu(t) = 1$, dilation coefficient $b(t) = -\Lambda_0 \tanh(\Lambda_0 t)$, $\Lambda_0 > 0$. The corresponding characteristic equation is $\ddot{r}(t) - \Lambda_0^2 r = 0$, and solutions satisfying required initial conditions are $r_1(t) = \cosh(\Lambda_0 t)$, $r_2(t) = \sinh(\Lambda_0 t)/\Lambda_0$, $t \geq 0$, which are increasing functions of time such that $r_1, r_2 \to \infty$ as $t \to \infty$. Since $b(t) = -\dot{r}_1(t)/r_1(t)$, then according to our results, the equation (5.37) has solution

$$U_h^\alpha(x,t) = \operatorname{sech}(\Lambda_0 t)\Big[\alpha_2 + V(\operatorname{sech}(\Lambda_0 t)(x - r_h^\alpha(t)), \tanh(\Lambda_0 t)/\Lambda_0)\Big], \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2,$$

where $V(\eta, \tau)$ is a solution of standard BE and displacement of position is given by

$$r_\alpha^h(t) = \alpha_1 \cosh(\Lambda_0 t) + \alpha_2 \sinh(\Lambda_0 t)/\Lambda_0, \qquad \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2, \quad t \geq 0. \tag{5.38}$$

Clearly, time behavior of $r_\alpha^h(t)$ can be controlled by parameters $\alpha_1 = r_\alpha^h(0)$ and $\alpha_2 = \dot{r}_\alpha^h(0)$, which are the initial position and velocity respectively. For example, when $\alpha_1, \alpha_2 > 0$, then $r_\alpha^h(t) \to +\infty$ as $t \to \infty$. If $\alpha_1, \alpha_2 < 0$, then $r_\alpha^h(t) \to -\infty$ and if $\alpha_2 = -\Lambda_0 \alpha_1$, then $r_\alpha^h(t) \to 0$ as $t \to \infty$. In what follows, we discuss some special solutions.

Example 5.1 ***Shock type traveling waves***

Burgers model (5.37) with IC (5.19) has shock type solution

$$U_h^\alpha(x,t) = \operatorname{sech}(\Lambda_0 t)\Big(\alpha_2 - A \tanh\left[A \operatorname{sech}(\Lambda_0 t)\,(x - r_h^\alpha(t))\right]\Big), \tag{5.39}$$

which is a traveling wave satisfying boundary conditions

$$U_h^\alpha(-\infty, t) = \operatorname{sech}(\Lambda_0 t)(\alpha_2 + A), \qquad U_h^\alpha(+\infty, t) = \operatorname{sech}(\Lambda_0 t)(\alpha_2 - A), \quad t \geq 0,$$

and its center propagates according to $x = r_h^\alpha(t)$ given by (5.38). Due to (5.38), if $\alpha_1, \alpha_2 > 0$, then shock profile will move away in positive $x-$direction, with speed $v_\alpha = \dot{r}_h^\alpha(t)$, as we show in Fig.5.1a. If $\alpha_1, \alpha_2 < 0$, then "center" of the shock profile will go away from the origin in negative x-direction, and using (5.38) one can work other possibilities. Here, since $\operatorname{sech}(\Lambda_0 t) \to 0$ as time increases, then both amplitude and steepness of the shock profile decrease during the propagation, and it is not difficult to see that $U_h^\alpha(x,t) \to 0$ as $t \to \infty$, for $x \in \mathbb{R}$.

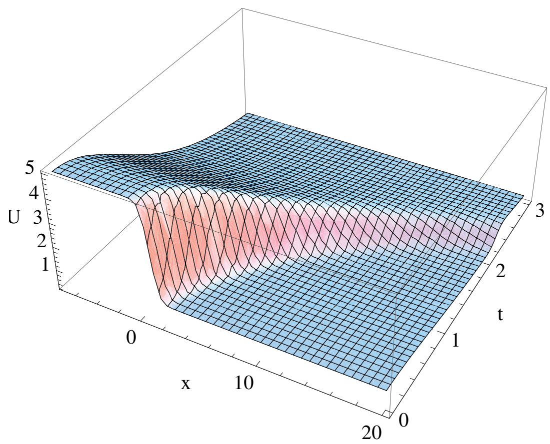
In that model, interaction of two-shocks is given by

$$
\begin{aligned}
U_h^\alpha(x,t) &= 2\operatorname{sech}(\Lambda_0 t) \qquad (5.40)\\
&\times \left[\frac{\alpha_2^{(1)}\exp[p_1(\eta(x,t),\tau(t))] + \alpha_2^{(2)}\exp[p_2(\eta(x,t),\tau(t))] + \alpha_2^{(3)}\exp[p_3(\eta(x,t),\tau(t))]}{\exp[p_1(\eta(x,t),\tau(t))] + \exp[p_2(\eta(x,t),\tau(t))] + \exp[p_3(\eta(x,t),\tau(t))]}\right],
\end{aligned}
$$

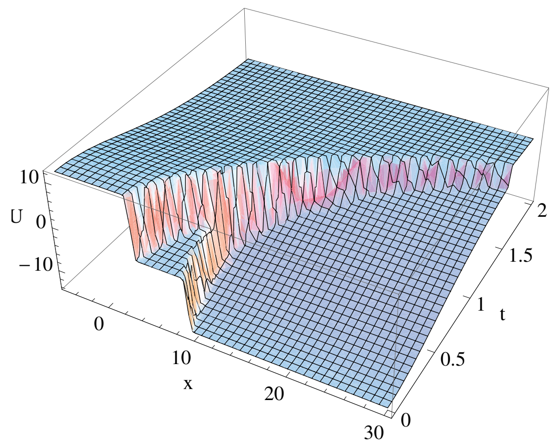
where

$$
p_i(\eta(x,t),\tau(t)) = -2\alpha_2^{(i)}\operatorname{sech}(\Lambda_0 t)\left(x - [\alpha_1^{(i)}\cosh(\Lambda_0 t) + \frac{\alpha_2^{(i)}}{\Lambda_0}\sinh(\Lambda_0 t)]\right), \quad i = 1,2,3,
$$

and in Fig.5.1b we illustrate confluence of two shocks.



(a)



(b)

Figure 5.1 (a) Single shock solution (5.39) with $\alpha = (1,3)$, $A = 2$, $\Lambda_0 = 1$, (b) Two-shock solution (5.40) with $\alpha^{(1)} = (1,6)$, $\alpha^{(2)} = (16,-1)$, $\alpha^{(3)} = (10,-7)$, $\Lambda_0 = 1.5$.

Example 5.2 ***Triangular waves***

Burgers model (5.37) has triangular wave solutions of the form

$$U_h^\alpha(x,t) = \alpha_2 \mathrm{sech}(\Lambda_0 t) + \frac{(e^R - 1)\exp\left[-\frac{(x-r_h^\alpha(t))^2}{2\sigma(t)}\right]}{\sqrt{2\pi\sigma(t)}\left(1 + \frac{1}{2}(e^R - 1)\,\mathrm{Erfc}\left[\frac{x-r_h^\alpha(t)}{\sqrt{2\sigma(t)}}\right]\right)}, \tag{5.41}$$

where parameter

$$\sigma(t) = \sinh(2\Lambda_0 t)/2\Lambda_0, \tag{5.42}$$

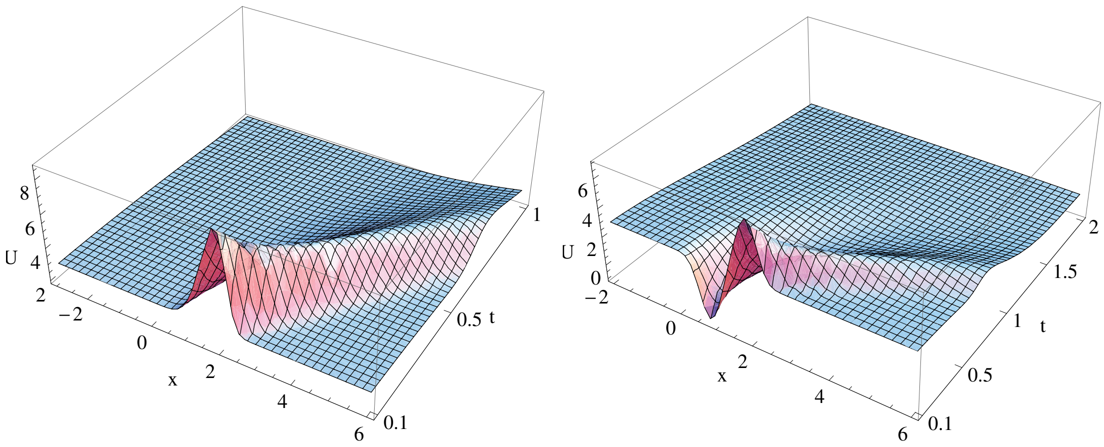
shows that the wave packet is spreading in space and $r_\alpha^h(t)$ given by (5.38) describes the displacement of the wave profile during the evolution process, as one can see in Fig.5.2a. In particular, if $\alpha_2 = 0$, then $U_\alpha(\pm\infty, t) = 0$ and we have smooth and localized in space wave packet which is normalizable.

Example 5.3 ***N-shaped waves***

The N-shaped wave solution of Burgers model (5.37) is

$$U_h^\alpha(x,t) = \alpha_2 \mathrm{sech}(\Lambda_0 t) + \frac{(x - r_h^\alpha(t))\sqrt{c\,\Lambda_0 \coth(\Lambda_0 t)}\,\exp\left[-\frac{(x-r_h^\alpha(t))^2}{2\sigma(t)}\right]}{\sigma(t)\left(1 + \sqrt{c\,\Lambda_0 \coth(\Lambda_0 t)}\exp\left[-\frac{(x-r_h^\alpha(t))^2}{2\sigma(t)}\right]\right)}, \tag{5.43}$$

where $\sigma(t)$ given by (5.42) controls the spreading of the wave packet and $r_h^\alpha(t)$ given by (5.38) describes shifting of its position, as illustrated in Fig.5.2b.



(a) (b)

Figure 5.2 (a) Triangular wave solution (5.41) with $\alpha = (1, 3)$, $R = 5$, $\Lambda_0 = 1$. (b) N-shaped wave solution (5.43) with $\alpha = (1, 3)$, $c = 15$, $\Lambda_0 = 1$.

Example 5.4 ***Rational type solutions***

The generalized Burgers equation (5.37) with IC (5.32), has solution

$$ {}_{\alpha}U_m(x,t) = \text{sech}(\Lambda_0 t)\left[\alpha_2 - m\frac{H_{m-1}(\text{sech}(\Lambda_0 t)(x - r_\alpha^h(t)), \frac{\tanh(\Lambda_0 t)}{\Lambda_0})}{H_m(\text{sech}(\Lambda_0 t)(x - r_\alpha^h(t)), \frac{\tanh(\Lambda_0 t)}{\Lambda_0})}\right], \quad (5.44) $$

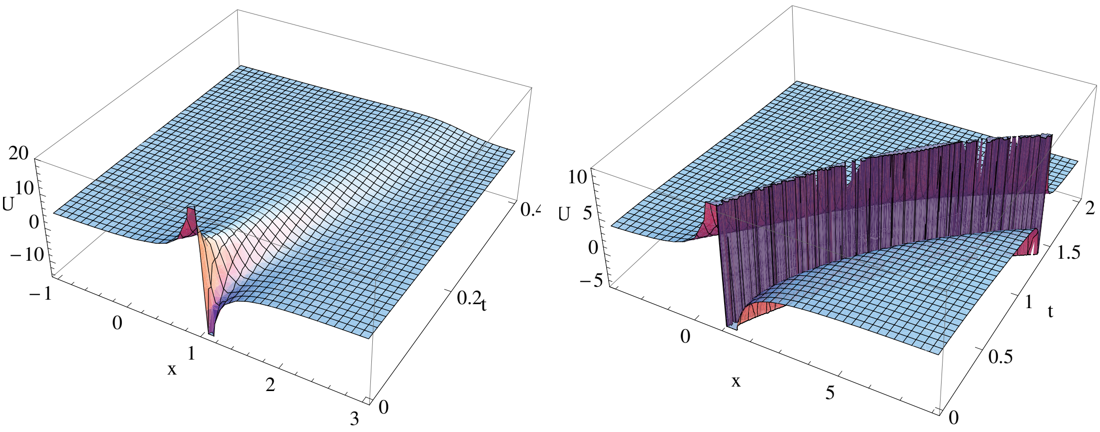
where $\alpha \in \mathbb{R}^2$, and $m = 1, 2, ...$ As an example, we write this solution for even $m = 2$,

$$ {}_{\alpha}U_2(x,t) = \alpha_2 \text{sech}(\Lambda_0 t) - \frac{2\Lambda_0(x - r_\alpha^h(t))}{\Lambda_0(x - r_\alpha^h(t))^2 + \sinh(\Lambda_0 t)\cosh(\Lambda_0 t)}, \quad (5.45) $$

and observe that for $t = 0$ it has singularity located at $x = \alpha_1$, but at later times $t > 0$ it is smooth for all $x \in \mathbb{R}$, see Fig.5.3a. On the other side for odd $m = 3$, explicit form of the solution is

$$ {}_{\alpha}U_3(x,t) = \alpha_2 \text{sech}(\Lambda_0 t) - \frac{3\Lambda_0(x - r_\alpha^h(t))^2 + 3\sinh(\Lambda_0 t)\cosh(\Lambda_0 t)}{(x - r_\alpha^h(t))\left[\Lambda_0(x - r_\alpha^h(t))^2 + 3\sinh(\Lambda_0 t)\cosh(\Lambda_0 t)\right]}, \quad (5.46) $$

and it has moving singularity whose position is described by $x = r_\alpha^h(t)$ given by (5.38), and its behavior depends on the parameters α_1, α_2, as discussed before. For $\alpha_1, \alpha_2 > 0$, as one can see in Fig.5.3b, the singularity initially located at $x = 1$, moves away in positive $x-$direction when time increases.



(a) (b)

Figure 5.3 (a) Rational type solution (5.45) with $\alpha = (1, 3)$, $\Lambda_0 = 1$. (b) Rational type solution (5.46) with $\alpha = (1, 3)$, $\Lambda_0 = 1$.

MODEL 2 : Variable diffusion and convection coefficients

Now, we consider a generalized Burgers equation

$$U_t + \gamma U + UU_x = \frac{e^{-\gamma t}}{2} U_{xx} + (\gamma/2 - \Omega \tanh(\Omega))(xU)_x, \tag{5.47}$$

with constant damping $\Gamma(t) = \gamma > 0$, exponentially decreasing diffusion coefficient $D(t) = e^{-\gamma t}/2$ and $b(t) = \gamma/2 - \Omega \tanh(\Omega)$, where $\Omega = \sqrt{\gamma^2/4 - \Lambda_0^2}$, $|\Lambda_0| < \gamma/2$. The corresponding characteristic equation is $\ddot{r}(t) + \gamma \dot{r}(t) + \Lambda_0^2 r(t) = 0$, with solutions

$$r_1(t) = \cosh(\Omega\, t) e^{-\frac{\gamma}{2}t}, \quad r_2(t) = \frac{\sinh(\Omega\, t)}{\Omega} e^{-\frac{\gamma}{2}t}, \quad t \geq 0, \tag{5.48}$$

such that $r_1(t), r_2(t) \to 0$ as $t \to \infty$. Since again dilatation coefficient satisfies $b(t) = -\dot{r}_1(t)/r_1(t)$, then Burgers equation (5.47) has solution of the form

$$U_h^\alpha(x,t) = \operatorname{sech}(\Omega\, t) e^{-\frac{\gamma}{2}t} \left[\alpha_2 + V\left(\operatorname{sech}(\Omega\, t) e^{\frac{\gamma}{2}t}(x - r_h^\alpha(t)), \frac{\tanh(\Omega\, t)}{\Omega} \right) \right], \qquad \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2,$$

where

$$r_h^{\alpha}(t) = (\alpha_1 \cosh(\Omega t) + \alpha_2 \sinh(\Omega t)/\Omega) e^{-\gamma t/2}, \quad t \geq 0. \tag{5.49}$$

In that model, we note that the long-time behavior of $r_h^{\alpha}(t)$ is not influenced by the initial parameters, since $r_h^{\alpha}(t) \to 0$ as $t \to \infty$ for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. Some particular solutions are as follows.

Example 5.5 ***Shock type traveling waves***

BE (5.47) has shock wave solution

$$U_h^{\alpha}(x,t) = \operatorname{sech}(\Omega t) e^{-\frac{\gamma}{2}t} \left(\alpha_2 - A \tanh\left[A \operatorname{sech}(\Omega t) e^{\frac{\gamma}{2}t}\,(x - r_h^{\alpha}(t))\right]\right), \tag{5.50}$$

satisfying boundary conditions

$$U_h^{\alpha}(-\infty,t) = \operatorname{sech}(\Omega t) e^{-\frac{\gamma}{2}t}(\alpha_2 + A), \quad U_h^{\alpha}(+\infty,t) = \operatorname{sech}(\Omega t) e^{-\frac{\gamma}{2}t}(\alpha_2 - A), \quad t \geq 0,$$

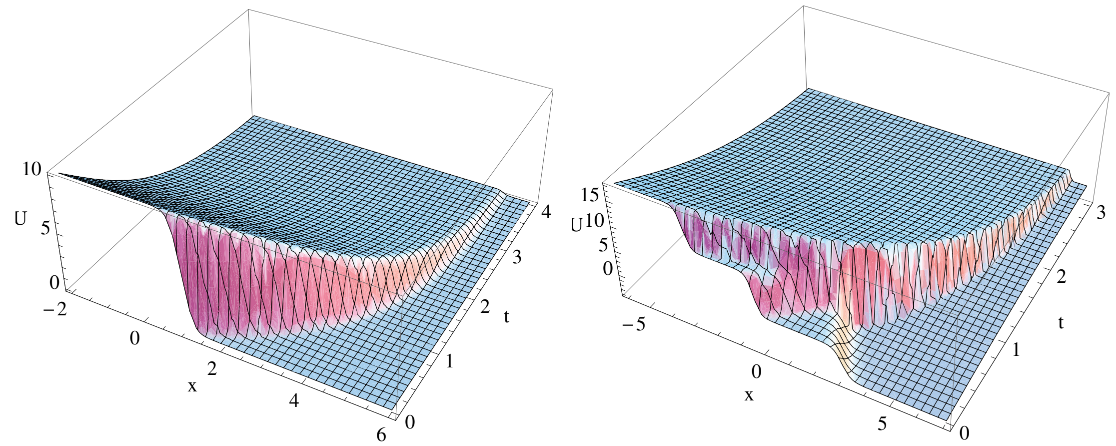
which tend to zero as time increases. In that model, center moves according to $x = r_h^{\alpha}(t)$ given by (5.49), so that when time increases it approaches $x = 0$ for every $\alpha = (\alpha_1, \alpha_2)$. Steepness of the shock profile increases with time according to $\operatorname{sech}(\Omega t) e^{\gamma t/2}$, but shock amplitude is proportional to $\operatorname{sech}(\Omega t) e^{-\gamma t/2}$, so that it decreases and tends to zero as $t \to \infty$. Clearly, parameters γ and Λ_0 can be used to control the steepness and shock amplitude of the wave packet.One can see the behavior of shock traveling wave solution in Fig.5.4a. Next, as an example, we write also the three-shock wave solution to equation (5.47)

$$U_h^{\alpha}(x,t) = \frac{2e^{-\frac{\gamma}{2}t}}{\cosh(\Omega t)} \left[\frac{\alpha_2^{(1)} \exp[p_1(\eta(x,t), \tau(t))] + ... + \alpha_2^{(4)} \exp[p_4(\eta(x,t), \tau(t))]}{\exp[p_1(\eta(x,t), \tau(t))] + ... + \exp[p_4(\eta(x,t), \tau(t))]} \right], \tag{5.51}$$

where

$$p_i(\eta(x,t),\tau(t)) = -2\alpha_2^{(i)}\mathrm{sech}(\Omega t)e^{\frac{\gamma}{2}t}\left(x - [\alpha_1^{(i)}\cosh(\Omega t) + \frac{\alpha_2^{(i)}}{\Omega}\sinh(\Omega t)]e^{-\frac{\gamma}{2}t}\right), \quad i = 1,2,3,4,$$

and in Fig.5.4b we plot it for certain parameters.



(a) (b)

Figure 5.4 (a) Single shock solution (5.50) with $\alpha = (1,5)$, $A = 5$, $\Lambda_0 = 0.1$, $\gamma = 1$. (b) Three-shock wave solution (5.51) with $\alpha^{(1)} = (-3,1)$, $\alpha^{(2)} = (6,-2)$, $\alpha^{(3)} = (-1,5)$, $\alpha^{(4)} = (-2,9)$, $\Lambda_0 = 0.1$, $\gamma = 1$.

Example 5.6 ***Triangular waves***

Generalized BE (5.47) has triangular wave solution

$$U_h^\alpha(x,t) = \alpha_2\mathrm{sech}(\Omega\, t)e^{-\frac{\gamma}{2}t} + \frac{(e^R - 1)\exp\left[-(x - r_h^\alpha(t))^2/2\sigma(t)\right]}{\sqrt{2\pi\sigma(t)}e^{\gamma t}\left(1 + \frac{1}{2}(e^R - 1)\,\mathrm{Erfc}\left[(x - r_h^\alpha(t))/\sqrt{2\sigma(t)}\right]\right)} \tag{5.52}$$

where parameter

$$\sigma(t) = e^{-\gamma t}\sinh(2\Omega t)/2\Omega, \tag{5.53}$$

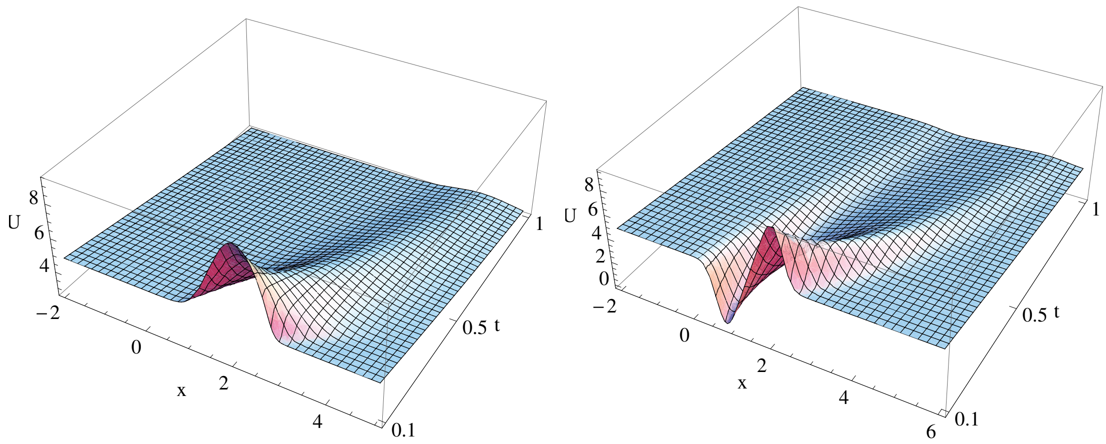
is increasing in time for any $\gamma > 0$ and $\Lambda_0 \in \mathbb{R}$, so that the wave packet is spreading during the propagation, its amplitude decreases and displacement $r_h^\alpha(t)$ is given by (5.49), see Fig.5.5a.

Example 5.7 ***N-shaped waves***

The N-shaped wave solution of BE (5.47) is

$$U_h^\alpha(x,t) = \alpha_2 \operatorname{sech}(\Omega\, t) e^{-\frac{\gamma}{2}t} + \frac{\operatorname{sech}(\Omega\, t)(x - r_h^\alpha(t))\,\sqrt{c\,\Omega\,\coth(\Omega t)}\exp\left[-(x - r_h^\alpha(t))^2/2\sigma(t)\right]}{\sigma(t) e^{\gamma t}\left(1 + \sqrt{c\,\Omega\,\coth(\Omega t)}\exp[-(x - r_\alpha^h(t))^2/2\sigma(t)]\right)} \quad (5.54)$$

where spreading of the wave profile again depends on $\sigma(t)$ given by (5.53) and $r_h^\alpha(t)$ is given by (5.49).



(a) (b)

Figure 5.5 (a) Triangular traveling wave solution (5.52) with $\alpha = (1, 5)$, $R = 10$, $\Lambda_0 = 0.1$, $\gamma = 1$. (b) N-shaped wave solution (5.54) with $\alpha = (1, 5)$, $c = 100$, $\Lambda_0 = 0.1$, $\gamma = 1$.

Example 5.8 ***Rational type solutions***

Burgers equation (5.47) has solution

$${}_\alpha U_m(x,t) = \operatorname{sech}(\Omega\, t) e^{-\frac{\gamma}{2}t}\left[\alpha_2 - m\frac{H_{m-1}(\operatorname{sech}(\Omega\, t) e^{\frac{\gamma}{2}t}(x - r_h^\alpha(t)), \tanh(\Omega t)/\Omega)}{H_m(\operatorname{sech}(\Omega\, t) e^{\frac{\gamma}{2}t}(x - r_h^\alpha(t)), \tanh(\Omega t)/\Omega)}\right], \quad m = 1, 2, \quad (5.55)$$

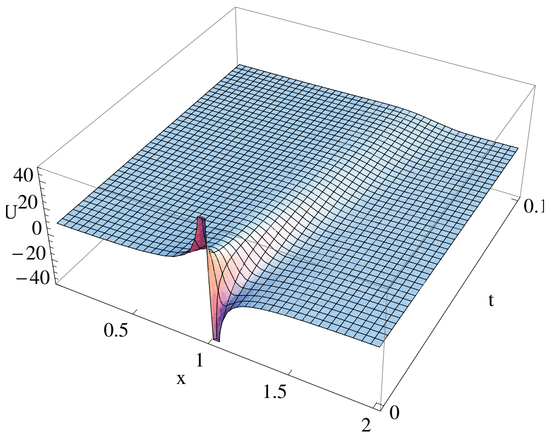
For $m = 2$ solution is of the form

$$
{}_{\alpha}U_2(x,t) \;=\; \alpha_2 \mathrm{sech}(\Omega\, t) e^{-\frac{\gamma}{2}t} - \frac{2\,\Omega\,(x - r_h^{\alpha}(t))}{\Omega e^{\gamma t}(x - r_h^{\alpha}(t))^2 + \sinh(\Omega t)\cosh(\Omega t)}, \qquad (5.56)
$$

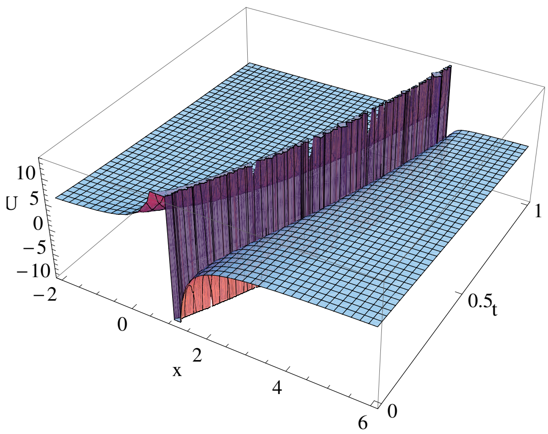
which is smooth for $t > 0$, $x \in \mathbb{R}$ and ${}_{\alpha}U_2(x,t) \to 0$ as $t \to \infty$, for any $x \in \mathbb{R}$, see Fig.5.6a. For $m = 3$ solution is explicitly found as

$$
{}_{\alpha}U_3(x,t) \;=\; \alpha_2 \mathrm{sech}(\Omega\, t) e^{-\frac{\gamma}{2}t} - \frac{3\,\Omega\,(x - r_h^{\alpha}(t))^2 + 3e^{-\gamma t}\sinh(\Omega t)\cosh(\Omega t)}{(x - r_h^{\alpha}(t))\left[\Omega\, e^{\gamma t}(x - r_h^{\alpha}(t))^2 + 3\sinh(\Omega t)\cosh(\Omega t)\right]} \qquad (5.57)
$$

and one can easily see that it has singularity moving according to $x = r_h^{\alpha}(t)$, given by (5.49). In that model, due to (5.49), independently of how parameters α_1, α_2 are chosen, as $t \to \infty$ the singularity will eventually approach the origin $x = 0$, see Fig.5.6b.



(a)



(b)

Figure 5.6 (a) Rational type solution (5.56) with $\alpha = (1,5)$, $\gamma = 1$, $\Lambda_0 = 0.1$. (b) Rational type solution (5.57) with $\alpha = (1,5)$, $\gamma = 1$, $\Lambda_0 = 0.1$.

MODEL 3 : Variable coefficient convection and forcing term

The last generalized Burgers model that we discuss is

$$
U_t + UU_x = \frac{1}{2}U_{xx} - \omega_0 \tanh(\omega_0 t)(xU)_x - \omega_0^2 x + F_0 \sin(\omega t), \qquad (5.58)
$$

where $b(t) = -\omega_0 \tanh(\omega_0 t)$, $\omega^2(t) = \omega_0^2$, $\omega_0 \in \mathbb{R}$, and we take sinusoidal forcing term $f(t) = F_0 \sin(\omega t)$, with frequency $\omega \in \mathbb{R}$. The characteristic equation is $\ddot{r}(t) = F_0 \sin(\omega t)$ with two independent homogeneous solutions and a particular solution, respectively

$$r_1(t) = 1, \qquad r_2(t) = t, \qquad r_p(t) = \frac{F_0}{\omega^2}(\omega t - \sin(\omega t)), \quad t \geq 0. \tag{5.59}$$

Then, Burgers equation (5.58) has solution

$$U_g\alpha(x,t) = \alpha_2 + V(x - r_\alpha(t), t) - \omega_0 \tanh(\omega_0 t)x + \frac{F_0}{\omega}(1 - \cos(\omega t)), \qquad \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2,$$

where

$$r_g^\alpha(t) = \alpha_1 + \alpha_2 t \; + \frac{F_0}{\omega^2}(\omega t - \sin(\omega t)), \quad t \geq 0. \tag{5.60}$$

Example 5.9 ***Shock traveling waves***

Burgers model (5.58) has solution

$$U_g^\alpha(x,t) = \alpha_2 - A \tanh\left[A\left(x - r_g^\alpha(t)\right)\right] - \omega_0 \tanh(\omega_0 t)x + \frac{F_0}{\omega}(1 - \cos(\omega t)), \tag{5.61}$$

where center of the shock profile moves according to $x = r_g^\alpha(t)$ given by (5.60), and can be controlled by the initial parameters α_1, α_2, and F_0 and ω. For example, if $\alpha_2 = -F_0/\omega$, then the center will just oscillate around the initial position $x = \alpha_1$ with frequency ω. Otherwise position of the center could oscillate while propagating, or could propagate with alternating speed. In any case, the amplitude of the shock profile and its steepness does not change during the evolution, since the relation between the coefficients $\omega^2(t) + \dot{b}(t) - b^2(t) = 0$, implies $r_1(t) = 1$. On the other side, the total wave amplitude $U_g^\alpha(x,t)$ itself is unbounded at $x = \pm\infty$ when $\omega_0 \neq 0$, due to the linear in position x term, and there is a global time oscillation of the wave due to a sinusoidal forcing, see Fig.5.7a.

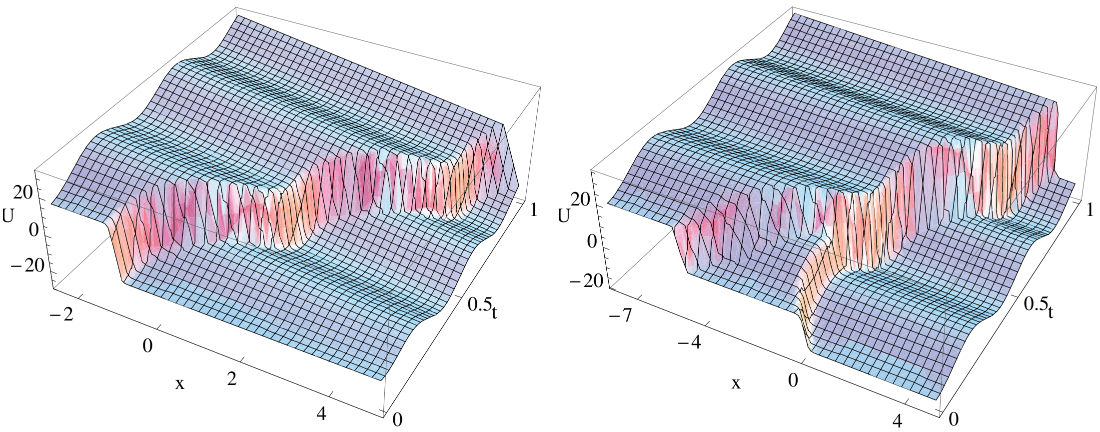
Two-shocks wave is of the form

$$
\begin{aligned}
U_g^{\alpha}(x,t) \;=\; & 2\frac{\alpha_2^{(1)}\exp[p_1(\eta(x,t),\tau(t))] + \alpha_2^{(2)}\exp[p_2(\eta(x,t),\tau(t))] + \alpha_2^{(3)}\exp[p_3(\eta(x,t),\tau(t))]}{\exp[p_1(\eta(x,t),\tau(t))] + \exp[p_2(\eta(x,t),\tau(t))] + \exp[p_3(\eta(x,t),\tau(t))]} \\
& - \;\omega_0\tanh(\omega_0 t)x + \frac{F_0}{\omega}(1-\cos(\omega t)), \qquad (5.62)
\end{aligned}
$$

where

$$
p_i(\eta(x,t),\tau(t)) = -2\alpha_2^{(i)}\left(x - [\alpha_1^{(i)} + \alpha_2^{(i)}t + \frac{F_0}{\omega^2}(\omega t - \sin(\omega t))]\right), \quad i = 1,2,3. \quad (5.63)
$$

and we plot it in Fig.5.7b.



(a) (b)

Figure 5.7 (a) Single shock solution (5.61) with $\alpha = (-1,1)$, $A = 15$, $\omega_0 = 3$, $\omega = 15$, $F_0 = 75$. (b) Two-shock wave solution (5.62) with $\alpha^{(1)} = (0,-8)$, $\alpha^{(2)} = (5,0)$, $\alpha^{(3)} = (-5,10)$, $\omega = 15$, $F_0 = 75$, $\omega_0 = 1$.

Example 5.10 ***Triangular waves***

Burgers Eq.(5.58) has generalized triangular wave solution

$$\begin{aligned} U_g^\alpha(x,t) &= \alpha_2 + \frac{1}{\sqrt{2\pi t}}\left(\frac{(e^R-1)\exp\left[-\frac{(x-r_g^\alpha(t))^2}{2t}\right]}{1+\frac{(e^R-1)}{2}\mathrm{Erfc}[\frac{x-r_g^\alpha(t)}{\sqrt{2t}}]}\right) - \omega_0\tanh(\omega_0 t)x \\ &+ \frac{F_0}{\omega}(1-\cos(\omega t)), \end{aligned} \tag{5.64}$$

where $\sigma(t) = 2t$ is the spreading parameter and displacement $r_g^\alpha(t)$ is given by (5.60). As one can see in Fig.5.8a, due to periodic forces the global wave amplitude shows oscillatory behavior in time, while the amplitude of the wave profile decreases. For $\omega_0 \neq 0$ the wave packet is unbounded at $x = \pm\infty$.

Example 5.11 ***N-shaped waves***

N-shaped wave solution for generalized forced Burgers model (5.58) is of the form

$$\begin{aligned} U_g^\alpha(x,t) &= \alpha_2 + \left(\frac{x-r_\alpha(t)}{t}\right)\left(\frac{\sqrt{\frac{c}{t}}\exp\left[-\frac{(x-r_g^\alpha(t))^2}{2t}\right]}{1+\sqrt{\frac{c}{t}}\exp\left[-\frac{(x-r_g^\alpha(t))^2}{2t}\right]}\right) \\ &- \omega_0\tanh(\omega_0 t)x + \frac{F_0}{\omega}(1-\cos(\omega t)). \end{aligned} \tag{5.65}$$

where again $\sigma(t) = 2t$ and $r_g^\alpha(t)$ is given by (5.60).

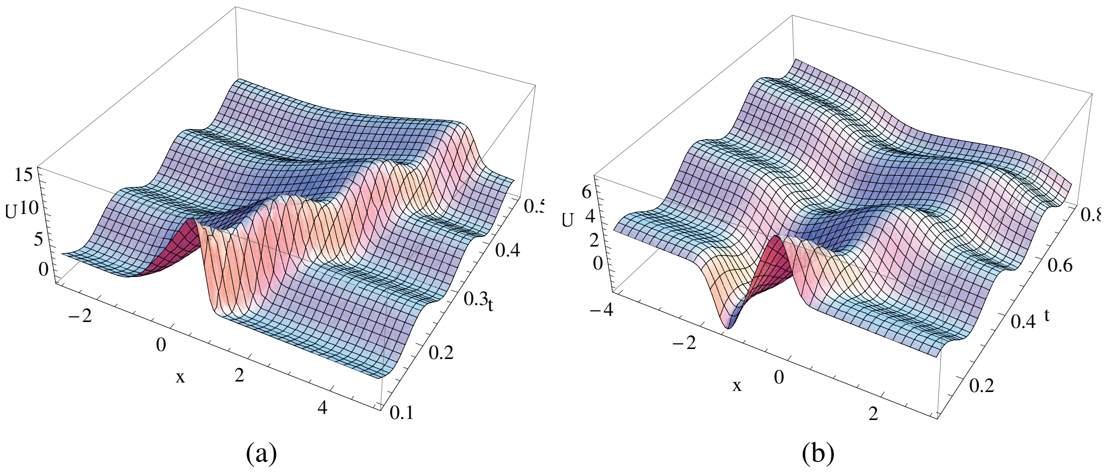


Figure 5.8 (a) Triangular wave solution (5.64) with $\alpha = (-1, 1)$, $R = 15$, $\omega = 45$, $F_0 = 75$, $\omega_0 = 1$. (b) N-shaped wave solution (5.65) with $\alpha = (-1, 1)$, $c = 50, \omega = 25$, $F_0 = 25$, $\omega_0 = 1$.

Example 5.12 ***Rational type solutions***

Generalized Burgers model (5.58) has rational type solutions

$$ {}_{\alpha}U_m(x,t) = \alpha_2 - m\frac{H_{m-1}(x - r_g^{\alpha}(t), t)}{H_m(x - r_g^{\alpha}(t), t)} - \omega_0 \tanh(\omega_0 t)x + \frac{F_0}{\omega}(1 - \cos(\omega t)), \quad (5.66) $$

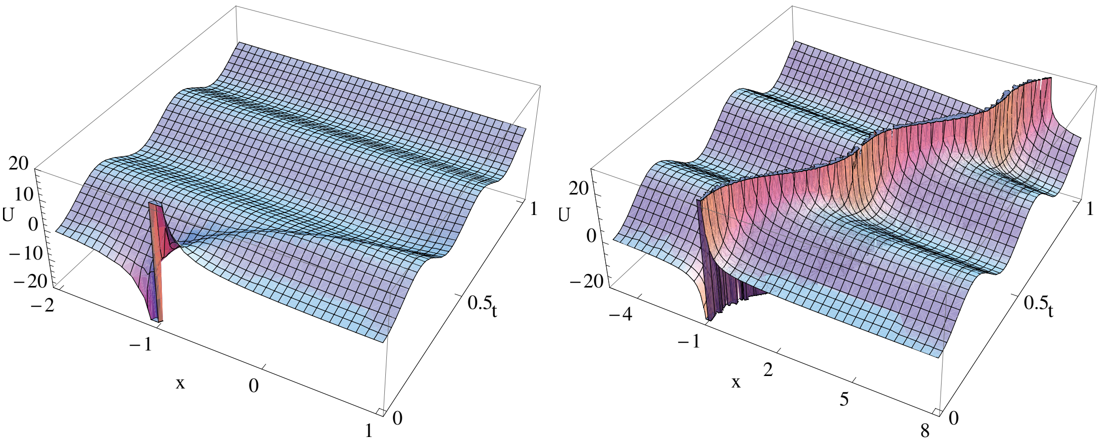
where $r_g^{\alpha}(t)$ is given by (5.60). For $m = 2$ solution is explicitly found as

$$ {}_{\alpha}U_2(x,t) = \alpha_2 - \frac{2(x - r_g^{\alpha}(t))}{(x - r_g^{\alpha}(t))^2 + t} - \omega_0 \tanh(\omega_0 t)x + \frac{F_0}{\omega}(1 - \cos(\omega t)), \quad (5.67) $$

which is smooth except at $t = 0$, for all $x \in \mathbb{R}$, it is unbounded at $x = \pm\infty$, and due to the sinusoidal forcing term there is a time oscillation of the total wave amplitude, as illustrated in Fig.5.9a. For $m = 3$ we have

$$ {}_{\alpha}U_3(x,t) = \alpha_2 - \frac{3(x - r_g^{\alpha}(t))^2 + 3t}{(x - r_g^{\alpha}(t))[(x - r_{\alpha}(t))^2 + 3t]} - \omega_0 \tanh(\omega_0 t)x + \frac{F_0}{\omega}(1 - \cos(\omega t)), (5.68) $$

from which we see that solution is unbounded at $x = \pm\infty$ and total wave amplitude periodically oscillates with time, except at singularity points where it becomes infinite. In that model, position of the singularity is given by $x = \alpha_1 + \alpha_2 t \; + (F_0/\omega^2)(\omega t - \sin(\omega t))$ showing that in general singularity oscillates while propagating along the $x-$axis, as shown in Fig.5.9b. Clearly, its exact motion can be controlled by the initial position and velocity parameters α_1, α_2, by the strength F_0 of the time-periodic force and its frequency ω.



(a) (b)

Figure 5.9 (a) Rational type wave solution ${}_{\alpha}U_2(x,t)$ with $\alpha = (-1,1)$, $\omega = 15$, $F_0 = 55$, $\omega_0 = 1$. (b) Rational type wave solution ${}_{\alpha}U_3(x,t)$ with $\alpha = (-1,1)$, $\omega = 15$, $F_0 = 100$, $\omega_0 = 1$.

5.2. Analytical Solution of the IBVP with Dirichlet Boundary Condition on the Half-line

In (A. Büyükaşık & Bozacı, 2019), we studied IBVP with Dirichlet boundary condition on the half-line for the forced Burgers equation of the form $U_t + (\dot{\mu}(t)/\mu(t))U + UU_x = \frac{1}{2\mu(t)}U_{xx} - \omega^2(t)x$. We obtained solution in terms of solution to the second order ordinary differential equation and a second-kind singular Volterra type integral equation. According to the general results, we introduced some different Burgers type models with specific damping, diffusion and forcing coefficients and constructed classes of exactly solvable models. We see that the Burgers problems with smooth time-dependent boundary data and an initial profile with pole type singularity have exact solutions with moving singularity. For each model we provided the solutions explicitly and described the dynamical properties of the singularities depending on the time-variable coefficients and the given initial and boundary data.

Now, as an extension, in this thesis we study IBVP with Dirichlet boundary condition for the generalized Burgers equation of the form (5.1). In previous section, we see that the generalized Burgers equation (5.1) is linearized to the generalized diffusion type equation of the form (3.1). Due to the difficulties of the IBVP's on the half-line for the generalized diffusion type equations mentioned in Chapter 3, the IBVP on the half-line for the generalized Burgers type equation also has difficulties, which causes from the inhomogeneity of the characteristic equation. So, in this section, we provide a solution to the following Dirichlet IBVP for the BE of the form (5.15)

$$U_t + \frac{\dot{\mu}(t)}{\mu(t)}U + UU_x = \frac{1}{2\mu(t)}U_{xx} + b(t)(xU)_x - \omega^2(t)x, \quad 0 < x < \infty, \quad 0 < t < T, \tag{5.69a}$$

$$U(x,0) = U_0(x), \quad 0 < x < \infty, \tag{5.69b}$$

$$U(0,t) = D(t), \quad 0 < t < T, \tag{5.69c}$$

where the parameters $\mu(t) > 0,\ \mu(0) = 1,\ b(t), \omega(t)$ are given real-valued smooth functions of time, initial data $U_0(x)$ and boundary data $D(t)$ are given sufficiently smooth functions in their domains. The result is summarized as follows.

Proposition 5.2 *The IBVP (5.69) has solution in the form*

$$U(x,t) = \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)x \tag{5.70}$$
$$- \left(\frac{1}{\mu(t)r_1(t)}\right)\frac{\int_0^\infty \partial_\eta G_N(\eta(x,t),\xi;\tau(t))F_0(\xi)d\xi - \int_0^{\tau(t)} \partial_\eta K(\eta(x,t),\tau(t)-\tau')Q(\tau')d\tau'}{\int_0^\infty G_N(\eta(x,t),\xi;\tau(t))F_0(\xi)d\xi - \int_0^{\tau(t)} K(\eta(x,t),\tau(t)-\tau')Q(\tau')d\tau'},$$

where $\eta(x,t), \tau(t)$ and $r_1(t), r_2(t)$ are all as defined in Proposition 3.45, $G_N(\eta,\xi,\tau), K(\eta,\tau)$ denote the Neumann heat kernel and heat kernel respectively, and

$$F_0(\xi) = \exp\Big(-\int^\xi U_0(x)dx\Big), \tag{5.71}$$

and the function $Q(\tau)$ is obtained by solving the second-kind Volterra integral equation

$$Q(\tau) = r_1(t(\tau))\mu(t(\tau))U(0,t(\tau))\Big(\int_0^{\tau} \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau' - 2\int_0^{\infty} K(\xi,\tau)F_0(\xi)d\xi\Big).$$

Proof The proof can be done by reducing the IBVP (5.69) to the IBVP for simpler PDE's and by applying linearization procedure as follows :

First approach : Motivated from previous works, when $b(t) = 0$, first we show that the IBVP (5.69) has solution of the form

$$U(x,t) = \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)x + \frac{1}{\mu(t)r_1(t)}V(\eta(x,t),\tau(t)), \tag{5.72}$$

where $V(\eta,\tau)$ satisfies the IBVP for the standard Burgers equation with Dirichlet BC

$$\begin{cases} V_\tau + VV_\eta = \frac{1}{2}V_{\eta\eta}, \qquad 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \\ V(\eta,0) = U(\eta,0), \quad 0 < \eta < \infty, \\ V(0,\tau) = \mu(t(\tau))r_1(t(\tau))U(0,t(\tau)), \quad 0 < \tau < \tau(T). \end{cases} \tag{5.73}$$

Indeed, in (A. Büyükaşık & Pashaev, 2013) it was found that the forced Burgers equation (FBE) with specific time-variable coefficients in (5.69a) has solution of the form (5.72), where the functions $\eta(x,t)$, and $\tau(t)$ are as defined in (3.7). Then, using (5.72) the initial condition $U(x,0)$ gives the initial condition $V(\eta,0) = U(\eta,0) \equiv F(\eta)$. On the other hand, we notice that continuity of $\mu(t) > 0$ and $r_1^2(t) > 0$ for $t \in [0,T)$, imply that $\tau(t)$ defined in (3.7) is strictly increasing continuous function on $[0,T)$ and thus its inverse $t(\tau)$ exists for $\tau \in [0,\tau(T))$. Then, Dirichlet boundary condition $U(0,t) = D(t)$ transforms to Dirichlet boundary condition in (5.73), and IBVP (5.69) for the FBE transforms to the IBVP (5.73).

Second, using Cole-Hopf transform $V = -\Psi_\eta/\Psi$ it is not difficult to show that the IBVP (5.69) has solution of the form

$$U(x,t) = \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)x - \frac{1}{\mu(t)r_1(t)}\frac{\Psi_\eta(\eta(x,t),\tau(t))}{\Psi(\eta(x,t),\tau(t))}, \tag{5.74}$$

where $\Psi(\eta, \tau)$ satisfies the IBVP for the standard heat equation with Robin BC

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \qquad 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \\ \Psi(\eta, 0) = \exp\left[-\int^\eta U(x, 0)dx\right], \quad 0 < \eta < \infty, \\ \Psi_\eta(0, \tau) + [\mu(t(\tau))r_1(t(\tau))U(0, t(\tau))]\,\Psi(0, \tau) = 0, \quad 0 < \tau < \tau(T). \end{cases} \tag{5.75}$$

Thus, solving the IBVP (5.69) for FBE reduces to the problem of solving IBVP (5.75) for heat equation with Robin BC. Formally, we can write solution of the heat IBVP (5.75) using two approaches: the Neumann boundary approach and Dirichlet boundary approach, (Cannon, 1984) and (Rodin, 1970). (For Dirichlet boundary approach one can assume temporary we know $\Psi(0, \tau) = H(\tau)$.) Here, we use the Neumann boundary approach and assume temporary we know $\Psi_\eta(0, \tau) = Q(\tau)$. Then, the following IBVP with Neumann BC

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \qquad 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \\ \Psi(\eta, 0) = \exp\left[-\int^\eta U(x, 0)dx\right], \quad 0 < \eta < \infty, \\ \Psi_\eta(0, \tau) = Q(\tau), \quad 0 < \tau < \tau(T), \end{cases} \tag{5.76}$$

has solution

$$\Psi(\eta, \tau) = \int_0^\infty G_N(\eta, \xi; \tau)F_0(\xi)d\xi - \int_0^\tau K(\eta, \tau - \tau')Q(\tau')d\tau', \tag{5.77}$$

where $F_0(\xi) = \Psi(\xi, 0)$. It follows that

$$\Psi(0, \tau) = 2\int_0^\infty K(\xi, \tau)F_0(\xi)d\xi - \int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau - \tau')}}d\tau'.$$

Substituting $\Psi(0, \tau)$ and $\Psi_\eta(0, \tau) = Q(\tau)$ into the Robin BC of (5.75) gives,

$$Q(\tau) = r_1(t(\tau))\mu(t(\tau))U(0, t(\tau))\Big(\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau - \tau')}}d\tau' - 2\int_0^\infty K(\xi, \tau)F_0(\xi)d\xi\Big),$$

which is exactly the second-kind singular Volterra type integral equation for the unknown function $Q(\tau)$. Then, with $Q(\tau)$ determined by this integral equation the function (5.77) is solution of the heat problem (5.75). Therefore, Cole-Hopf transform $V = -\Psi_\eta/\Psi$ gives the solution of the IBVP (5.73), that is

$$V(\eta,\tau) = -\frac{\int_0^\infty \partial_\eta G_N(\eta,\xi,\tau)F_0(\xi)d\xi - \int_0^\tau \partial_\eta K(\eta,\tau-\tau')Q(\tau')d\tau'}{\int_0^\infty G_N(\eta,\xi,\tau)F_0(\xi)d\xi - \int_0^\tau K(\eta,\tau-\tau')Q(\tau')d\tau'},$$

which substituted back in (5.72) gives the solution (5.70) of the IBVP (5.69). □

Second approach : On the other side, by generalized Cole-Hopf transformation

$$U(x,t) = -\frac{\Phi_x(x,t)}{\mu(t)\Phi(x,t)}, \tag{5.78}$$

the generalized BE (5.69a) is linearized to the equation in (3.45) and initial condition (5.69b) directly transforms to IC $\Phi(x,0) = \exp\left[-\int^x U_0(x')dx'\right]$. And we notice that continuity of $\mu(t) > 0$ and $r_1^2(t) > 0$ for $t \in [0,T)$, imply that $\tau(t)$ defined in (3.7) is strictly increasing continuous function on $[0,T)$ and thus its inverse $t(\tau)$ exists for $\tau \in [0,\tau(T))$. Then, by transformation (5.78), Dirichlet boundary condition (5.69c) transforms to Robin type boundary condition for generalized diffusion equation

$$\Phi_x(0,t) + \mu(t)D(t)\Phi(0,t) = 0. \tag{5.79}$$

Therefore, according to Proposition (3.4), the IBVP (3.76) with initial condition $\Phi(x,0) = \exp\left[-\int^x U_0(x')dx'\right]$ and Robin type BC (5.79) reduces to IBVP (3.77) with IC $\Psi(\eta,0) = \exp\left[-\int^\eta U_0(x')dx'\right]$ and Robin BC $\Psi_\eta(0,\tau) + r_1(t(\tau))\mu(t(\tau)))D(t(\tau))\Psi(0,\tau) = 0$ for the heat equation. So using the integral representation of solution for the heat IBVP with Robin BC and applying transformation (5.78), we get the desired solution (5.70). □

5.2.1. Exactly Solvable Models

First, we discuss an exactly solvable model for $b(t) = 0$. For the other models with $b(t) = 0$, one can see (A. Büyükaşık & Bozacı, 2019).

Forced Burgers Model with Damping and Time-varying Diffusion Coefficient We study the IBVP for a forced Burgers equation,

$$\begin{cases} U_t + \gamma U + UU_x = \frac{1}{2}e^{-\gamma t}U_{xx} - \frac{\gamma^2}{4}x, & 0 < x < \infty, \quad t > 0, \\ U(x,0) = U_0(x), \quad 0 < x < \infty, \\ U(0,t) = D(t), \quad t > 0, \end{cases} \tag{5.80}$$

with constant damping $\Gamma(t) = \gamma > 0$, exponentially decaying diffusion coefficient $1/2\mu(t) = (1/2)e^{-\gamma t}$, and $\omega^2(t) = -(\gamma^2/4)$. We notice that, when parameter $\gamma > 0$ increases damping and forcing coefficients become larger, while diffusion coefficient gets smaller and goes to zero when $t \to \infty$. The corresponding linear characteristic equation is

$$\ddot{r}(t) + \gamma\dot{r}(t) + \frac{\gamma^2}{4}r(t) = 0, \quad t \geq 0, \tag{5.81}$$

and it has two independent solutions satisfying the prescribed IC's

$$r_1(t) = e^{-\frac{\gamma t}{2}}\left(1 + \frac{\gamma}{2}t\right), \quad r_2(t) = te^{-\gamma t/2}, \qquad t \geq 0. \tag{5.82}$$

In this model, both solutions are positive and approaching zero when $t \to \infty$. Then Burgers problem (5.80) has solution of the form

$$U(x,t) = -\left(\frac{\gamma}{2}\right)^2\left(\frac{t}{1+\frac{\gamma}{2}t}x\right) - \left(\frac{e^{-\gamma t/2}}{1+\frac{\gamma}{2}t}\right)\frac{\Psi_\eta(\eta(x,t),\tau(t))}{\Psi(\eta(x,t),\tau(t))},$$

where

$$\eta(x,t) = \frac{e^{\gamma t/2}}{(1+\gamma t/2)}x, \qquad \tau(t) = \frac{t}{(1+\gamma t/2)}, \quad t \geq 0, \tag{5.83}$$

$\tau(t)$ being positive, strictly increasing, bounded above with inverse $t(\tau) = \tau/(1-\gamma\tau/2)$ for $t \geq 0$, and $\Psi(\eta,\tau)$ satisfies the IBVP for the heat equation

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \qquad 0 < \eta < \infty, \quad 0 < \tau < 2/\gamma, \\ \Psi(\eta,0) = \exp\left[-\int^\eta U(x,0)dx\right], \quad 0 < \eta < \infty, \\ \left[(\frac{2}{2-\gamma\tau})e^{\left(\frac{\gamma\tau}{2-\gamma\tau}\right)}U(0,t(\tau))\right]\Psi(0,\tau) + \Psi_\eta(0,\tau) = 0, \quad 0 < \tau < 2/\gamma. \end{cases} \tag{5.84}$$

Solution of the heat problem is formally of the form

$$\Psi(\eta,\tau) = \int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)e^{-\int^\xi U_0(x)dx}d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}Q(\tau')d\tau',$$

where $Q(\tau)$ is found by solving the integral equation

$$Q(\tau) = \left[\frac{2\,e^{\frac{\gamma\tau}{2-\gamma\tau}}}{(2-\gamma\tau)}U(0,t(\tau))\right]\left(\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau' - 2\int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}e^{-\int^\xi U_0(x)dx}d\xi\right).$$

Clearly, for arbitrary initial and boundary data solving the integral equation may require numeric or asymmtotic approaches. But for some special choices of initial and boundary datas it is possible to obtain exact solution to the integral equation correspondingly to the IBVP (5.80). In this model we are interested in two special boundary conditions : the first one is when $U(0,t(\tau)) = 0$, which leads to homogeneous Neumann boundary condition for the heat problem, and second one is when the boundary condition is chosen so that $U(0,t(\tau)) = D_0(2-\gamma\tau)/(2e^{\frac{\gamma\tau}{2-\gamma\tau}})$.

Example 5.13 *Let the Burgers problem (5.80) with rational initial condition and homogeneous Dirichlet boundary condition be given*

$$\begin{cases} U_t + \gamma U + UU_x = \frac{1}{2}e^{-\gamma t}U_{xx} - \frac{\gamma^2}{4}x, & 0 < x < \infty, \quad t > 0, \\ U(x,0) = \frac{-m}{x}, \quad 0 < x < \infty, \quad m = 0,1,2,... \\ U(0,t) = 0, \quad t > 0. \end{cases} \tag{5.85}$$

It reduces (without loss of generality for a suitable integration constant) to an IBVP for the heat equation, with homogeneous Neumann BC

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, & 0 < \eta < \infty, \quad 0 < \tau < 2/\gamma, \\ \Psi(\eta,0) = \eta^m, \quad 0 < \eta < \infty, \quad m = 0,1,2,..., \\ \Psi_\eta(0,\tau) = 0, \quad 0 < \tau < 2/\gamma, \end{cases} \tag{5.86}$$

whose solutions depending on m, can be easily expressed in terms of the functions

$$h_p^-(\eta,\tau) = \int_0^\infty \frac{e^{-\frac{(\eta-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^p d\xi, \tag{5.87}$$

$$h_p^+(\eta,\tau) = \int_0^\infty \frac{e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\xi^p d\xi. \tag{5.88}$$

These functions are well-known solutions of the heat equation for $-\infty < \eta < \infty$, and are positive for $0 < \eta < \infty$, $\tau > 0$, see for example (Widder, 1975), (Rosenbloom & Widder, 1958), (Jeffreys, 1988), (Sachdev, 1987). Then, the heat problem (5.86) and hence the Burgers IBVP have the following solutions:

(a) For $m = 2p$, $p = 0,1,2,...$, solution of the heat problem is the even Kampe de Feriet polynomial $\Psi_{2p}(\eta,\tau) = H_{2p}(\eta,\tau)$, and therefore solution of the Burgers problem (5.85) is of the form

$$U_{2p}(x,t) = -\left(\frac{\gamma}{2}\right)^2\left(\frac{t}{1+\gamma t/2}x\right) - \left[\frac{e^{-\gamma t/2}}{1+\gamma t/2}\right]\frac{2pH_{2p-1}\left(\frac{e^{\gamma t/2}}{1+\gamma t/2}x, \frac{t}{1+\gamma t/2}\right)}{H_{2p}\left(\frac{e^{\gamma t/2}}{1+\gamma t/2}x, \frac{t}{1+\gamma t/2}\right)}. \tag{5.89}$$

Note that the odd KFP's defined by $H_{2p+1}(\eta,\tau) = h^-_{2p+1}(\eta,\tau) - h^+_{2p+1}(\eta,\tau)$ do not satisfy the Neumann BC $\Psi_\eta(0,\tau) = 0$, and therefore are not solutions of IBVP (5.85).

(b) For $m = 2p+1$, $p = 0, 1, 2, ...$, solution of the heat problem can be written as

$$\Psi_{2p+1}(\eta,\tau) = h^-_{2p+1}(\eta,\tau) + h^+_{2p+1}(\eta,\tau), \tag{5.90}$$

and the corresponding solution of the Burgers problem (5.85) becomes

$$\begin{aligned} U_{2p+1}(x,t) &= -\left(\frac{\gamma}{2}\right)^2 \left(\frac{t}{1+\gamma t/2} x\right) \\ &- (2p+1)\left[\frac{e^{-\gamma t/2}}{1+\gamma t/2}\right]\left[\frac{h^-_{2p}(\eta(x,t),\tau(t)) - h^+_{2p}(\eta(x,t),\tau(t))}{h^-_{2p+1}(\eta(x,t),\tau(t)) + h^+_{2p+1}(\eta(x,t),\tau(t))}\right], \end{aligned} \tag{5.91}$$

where $\eta(x,t)$ and $\tau(t)$ are as defined in (5.83) and we used that $\partial_\eta[h^-_p(\eta,\tau) + h^+_p(\eta,\tau)] = p[h^-_{p-1}(\eta,\tau) - h^+_{p-1}(\eta,\tau)]$, for each $p = 1, 2, ...$ These solutions are also smooth for $x > 0$, $t > 0$.

In this model, for $x > 0$ we have

$$\lim_{t\to\infty} \eta(x,t) = \infty, \quad \lim_{t\to\infty} \tau(t) = 2/\gamma,$$

and the long-time behavior of Burgers solution $U(x,t)$ is described by

$$\lim_{t\to\infty} U(x,t) = -\gamma x/2,$$

where the limiting function $U^*(x) = -\gamma x/2$ satisfies the equation $\gamma U + UU_x = -\gamma^2 x/4$ on the interval $0 < x < \infty$ with boundary condition $U(0) = 0$.

Example 5.14 *Now we study the IBVP*

$$\begin{cases} U_t + \gamma U + UU_x = \frac{1}{2}e^{-\gamma t}U_{xx} - \frac{\gamma^2}{4}x, & 0 < x < \infty, \quad t > 0, \\ U(x,0) = -\frac{(2m+1)[x-2m/D_0]}{x[x-(2m+1)/D_0]}, & 0 < x < \infty, \\ U(0,t) = D_0e^{-\gamma t/2}/(1+\gamma t/2), & t > 0, \end{cases} \tag{5.92}$$

where $D_0 > 0$ is a constant parameter, and $m = 0, 1, 2, ...$ The initial profile has simple zero at $x = 2m/D_0$, and pole type singularity at $x = (2m+1)/D_0$, for $x > 0$. The boundary data has changed according to the time-variable coefficients, but we still have $U(0,t) \to 0$ as $t \to \infty$. Here, parameter D_0 can be used to control the relation between the initial and boundary data. That is, when the strength D_0 of the BC increases, the initial singularity becomes closer to the boundary $x = 0$, and conversely, when parameter D_0 is small and close to zero the initial singularity is away from the boundary $x = 0$.

Burgers problem (5.92) reduces to the heat problem reduce to heat problems with polynomial type initial data and Robin BC as follows

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, & 0 < \eta < \infty, \quad 0 < \tau < 2/\gamma, \\ \Psi(\eta,0) = \left(\eta^{2m+1} - \frac{(2m+1)}{D_0}\eta^{2m}\right), & 0 < \eta < \infty, \\ \Psi_\eta(0,\tau) + D_0\Psi(0,\tau) = 0, & 0 < \tau < 2/\gamma. \end{cases} \tag{5.93}$$

Solution of (5.93), according to (5.77), is given by $\Psi(\eta,\tau) \equiv \Psi_m(\eta,\tau)$, where

$$\begin{aligned} \Psi_m(\eta,\tau) &= \int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)\left(\xi^{2m+1} - \frac{(2m+1)}{D_0}\xi^{2m}\right)d\xi \\ &- \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}Q_m(\tau')d\tau', \end{aligned} \tag{5.94}$$

and $Q_m(\tau)$ is solution of the following Abel integral equation of the second kind which is a special form of Volterra integral equation of second-kind.

Using that, the integral equation has solution $Q_m(\tau) = 1.3.5...(2m+1)\tau^m$, and substituting it into (C.2), as expected solution of the heat problem (5.93) becomes

$$\Psi_m(\eta,\tau) = H_{2m+1}(\eta,\tau) - \frac{(2m+1)}{D_0} H_{2m}(\eta,\tau), \tag{5.95}$$

which is a special linear superposition of two Kampe de Feriet polynomials.

For $m = 0, 1, 2$ solutions (5.95) are explicitly written as

$$\begin{aligned}
\Psi_0(\eta,\tau) &= H_1(\eta,\tau) - \frac{1}{D_0} H_0(\eta,\tau) = \eta - 1/D_0, \\
\Psi_1(\eta,\tau) &= H_3(\eta,\tau) - \frac{3}{D_0} H_2(\eta,\tau) = (\eta^3 + 3\eta\tau) - \frac{3}{D_0}\left(\eta^2 + \tau\right), \\
\Psi_2(\eta,\tau) &= H_5(\eta,\tau) - \frac{5}{D_0} H_4(\eta,\tau) = (\eta^5 + 10\eta^3\tau + 15\eta\tau^2) - \frac{5}{D_0}\left(\eta^4 + 6\eta^2\tau + 3\tau^2\right).
\end{aligned}$$

Note that IC in (5.93) has only one simple real zero $\eta = (2m+1)/D_0$ for $D_0 > 0$, $\eta > 0$ and each $m = 0, 1, 2, ...$ Then, the corresponding heat solution (5.95) has a zero for $\eta > 0$, $\tau > 0$, which propagates along the semiline $0 < \eta < \infty$ during the evolution process and its position can be described by a continuous function $\eta = \chi_m(\tau)$, satisfying $\chi_m(0) = (2m+1)/D_0$ and

$$\Psi_m(\chi_m(\tau),\tau) = 0, \quad m = 0, 1, 2, ... \tag{5.96}$$

The corresponding solution $U(x,t) \equiv U_m(x,t)$ for the Burgers IBVP (5.92) becomes

$$U_m(x,t) = -\left(\frac{\gamma}{2}\right)^2 \left(\frac{t}{1+\gamma t/2} x\right) - \left(\frac{e^{-\gamma t/2}}{1+\gamma t/2}\right) \frac{\partial_\eta \Psi_m(\eta(x,t),\tau(t))}{\Psi_m(\eta(x,t),\tau(t))},$$

where $\Psi_m(\eta,\tau)$ is given by (5.95) and $\eta(x,t)$, $\tau(t)$ are as defined in (5.83). According to (6.41) heat function $\Psi_m(\eta,\tau)$ has moving zero $\eta = \chi_m(\tau)$, and it follows that for $x > 0$, $t > 0$, Burgers solution $U_m(x,t)$ has moving singularity described by

$$x_m(t) = (1+\gamma t/2) e^{-\gamma t/2} \chi_m\left(\frac{t}{1+\gamma t/2}\right), \quad t > 0. \tag{5.97}$$

Here, we have $x_m(t) \to 0$ as $t \to \infty$, showing that singularity initially located at $x = (2m+1)/D_0$ approaches the boundary $x = 0$, when time increases and by changing parameter $\gamma > 0$, one can control the speed at which singularity approaches the boundary. More precisely, when γ becomes larger, diffusion coefficient and boundary data go faster to zero and singularity goes faster to $x = 0$.

As an example, we investigate Burgers problem (5.92) for $m = 0, 1$, For $m = 0$, the initial profile is $U_0(x, 0) = -1/(x - 1/D_0)$, $x > 0$ with discontinuity at $x = 1/D_0$, and we have smooth boundary condition $U_0(0, t) = D_0 e^{-\gamma t/2}/(1 + \gamma t/2)$, $t > 0$. Since initial and boundary data are compatible, at time $t = 0$ the value of U_0 at point $x = 0$ can be fixed to be a constant $D_0 > 0$, and boundary data shows that at later times the value of U_0 at point $x = 0$ will smoothly decrease and approach zero. Under these conditions Burgers solution becomes

$$U_0(x,t) = -\left(\frac{\gamma}{2}\right)^2 \left(\frac{t}{1+\gamma t/2} x\right) - \frac{D_0 e^{-\gamma t/2}}{D_0 e^{\gamma t/2} x - (1+\gamma t/2)}. \tag{5.98}$$

Here, the singularity motion is described by the monotone decreasing function $x = (1/D_0)(1+\gamma t/2)e^{-\gamma t/2}$, and it shows that the singularity initially located at $x = 1/D_0$ will move along the semiline $0 < x < \infty$ continuously approaching the boundary point $x = 0$, when time increases. This behavior can be seen in Fig.5.10a for $D_0 = 0.5$ and $\gamma = 2$.

Similarly, for $m = 1$, we have $U_1(x, 0) = -3(x - 2/D_0)/x(x - 3/D_0)$, $x > 0$, which is discontinuous at $x = 3/D_0$, and boundary data is same as for case $m = 0$. Then, Burgers solution becomes

$$\begin{aligned} U_1(x,t) &= -\left(\frac{\left(\frac{\gamma}{2}\right)^2 t}{1+\gamma t/2} x\right) \\ &- \frac{3D_0 e^{\gamma t/2} x^2 - 6(1+\frac{\gamma t}{2})x + 3D_0(1+\frac{\gamma t}{2})e^{-\gamma t/2} t}{D_0 e^{3\gamma t/2} x^3 - 3(1+\frac{\gamma t}{2})e^{\gamma t} x^2 + 3D_0(1+\frac{\gamma t}{2}) t e^{\gamma t/2} x - 3(1+\frac{\gamma t}{2})^2 t}, \end{aligned} \tag{5.99}$$

and its behavior is illustrated in Fig.5.10b for $D_0 = 0.5$ and $\gamma = 2$. For these parameters, solution $U_1(x, t)$ has moving singularity described by $x_1(t) = (1+t)e^{-t}\chi_1\left(t/(1+t)\right)$, $t > 0$, where $\chi_1(\tau)$ is the zeros of the first KFP, $\Psi_1(\eta, \tau)$, at $\eta = \tau(t)$.

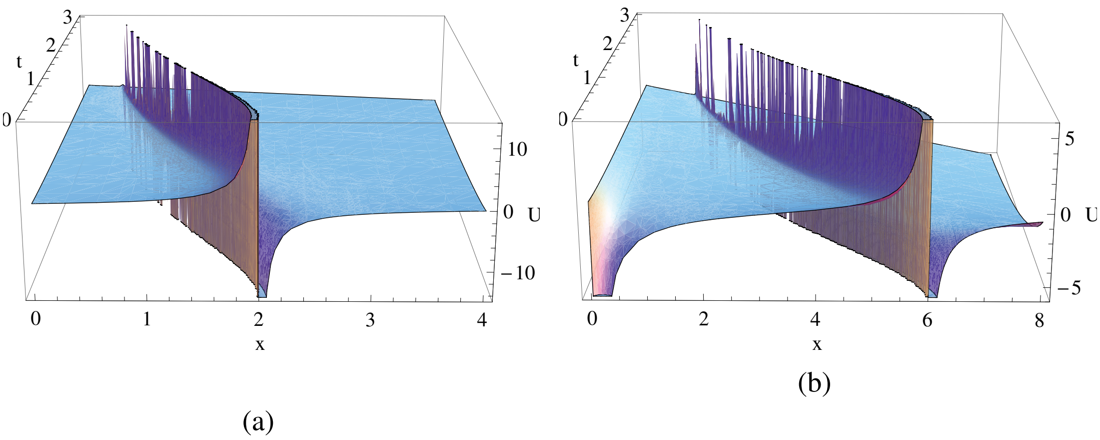


Figure 5.10 Example 5.14 (a) Solution $U_0(x,t)$ given by (5.98) with $D_0 = 0.5$ and $\gamma = 2$. (b) Solution $U_1(x,t)$ given by (5.99) with $D_0 = 0.5$ and $\gamma = 2$.

CHAPTER 6

GENERALIZED BURGERS TYPE EQUATIONS WITH MOVING BOUNDARIES

In this chapter, we study an initial-boundary value problem with moving boundary (mIBVP) on a time-dependent domain $s(t) < x < \infty,\ 0 < t < T$ for a one dimensional generalized Burgers type equation of the form

$$U_t + \frac{\dot{\mu}(t)}{\mu(t)}U + UU_x = \frac{1}{2\mu(t)}U_{xx} - ((a(t) - b(t)x)U)_x + F(x,t), \tag{6.1}$$

for the field $U(x,t)$, with smooth coefficients of damping $\Gamma(t) = \dot{\mu}(t)/\mu(t)$, diffusion $D(t) = 1/2\mu(t)$, linear convection term additional to nonlinear one and $F(x,t) = -\omega^2(t)x + f(t)$ is an external forcing term linear in position variable. We obtain analytical solution and present classes of exactly solvable models.

6.1. Analytical solution of the Dirichlet IBVP with moving boundary

Consider the generalized Burgers type equation on a time-varying domain $s(t) < x < \infty,\ 0 < t < T$, with Dirichlet boundary condition imposed at $x = s(t)$,

$$U_t + \frac{\dot{\mu}(t)}{\mu(t)}U + UU_x = \frac{1}{2\mu(t)}U_{xx} - ((a(t) - b(t)x)U)_x - \omega^2(t)x + f(t), \tag{6.2a}$$

$$U(x,0) = U_0(x), \quad s(0) < x < \infty, \tag{6.2b}$$

$$U(s(t),t) = D(t), \quad 0 < t < T, \tag{6.2c}$$

where all coefficients are given real-valued sufficiently smooth functions in their domains, $\mu(t) > 0,\ \mu(0) = 1$ and time-dependent boundary $s(t)$ is a twice differentiable function. The main result reducing to the analytically solvable standard models is summarized as follows.

Proposition 6.1 *If the boundary function $s(t)$ is of the form*

$$s(t) = r_g^\alpha(t) \equiv \alpha_1 r_1(t) + \alpha_2 r_2(t) + r_p(t), \quad \alpha \equiv (\alpha_1, \alpha_2) \in \mathbb{R}^2, \tag{6.3}$$

where $r_1(t), r_2(t)$ are positive, linearly independent homogeneous solutions and $r_p(t)$ is a particular solution of the inhomogeneous characteristic equation

$$\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \left[\omega^2(t) + \left(\dot{b}(t) + \frac{\dot{\mu}(t)}{\mu(t)}b(t) - b^2(t)\right)\right] r = \dot{a}(t) - a(t)\left[b(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right] + f(t), \tag{6.4}$$

satisfying IC's $r_1(0) = 1$, $\dot{r}_1(0) = -b(0)$, $r_2(0) = 0$, $\dot{r}_2(0) = 1$ and $r_p(0) = 0, \dot{r}_p(0) = a(0)$ respectively, then the solution to the mIBVP (6.2) is obtained in the following two forms :

$$(a)\ \ U_g^\alpha(x,t) = \frac{1}{\mu(t)r_1(t)} V(\eta_g^\alpha(x,t), \tau(t)) + \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)(x - r_g^\alpha(t)) + \frac{p_g^\alpha(t)}{\mu(t)}, \tag{6.5}$$

where $p_g^\alpha(t)$ and $\eta_g^\alpha(x,t)$, $\tau(t)$ are given in the statement of the Proposition 4.1, and $V(\eta, \tau)$ is solution of the IBVP for standard Burgers equation with inhomogeneous Dirichlet boundary condition on the half-line

$$V_\tau + VV_\eta = \frac{1}{2}V_{\eta\eta}, \qquad 0 < \eta < \infty, \quad 0 < \tau < \tau(T), \tag{6.6a}$$

$$V(\eta, 0) = U_0(\eta + \alpha_1) - \alpha_2, \quad 0 < \eta < \infty, \tag{6.6b}$$

$$V(0, \tau) = q_1(\tau), \quad 0 < \tau < \tau(T). \tag{6.6c}$$

where the boundary data

$$q_1(\tau) = \mu(t(\tau))r_1(t(\tau))\left[D(t(\tau)) - \frac{p_g^\alpha(t)}{\mu(t)}\right]. \tag{6.7}$$

$$(b)\ \ U_g^\alpha(x,t) = -\frac{1}{\mu(t)r_1(t)}\frac{\Psi_\eta(\eta_g^\alpha(x,t), \tau(t))}{\Psi(\eta_g^\alpha(x,t), \tau(t))} + \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)(x - r_g^\alpha(t)) + \frac{p_g^\alpha(t)}{\mu(t)}, \tag{6.8}$$

where $\Psi(\eta,\tau)$ is solution of the IBVP for the standard heat equation with Robin boundary condition on the half line

$$\Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \qquad 0<\eta<\infty, \quad 0<\tau<\tau(T), \tag{6.9a}$$

$$\Psi(\eta,0) = \exp\left[-\int^{\eta} U_0(x+\alpha_1)dx + \alpha_2\eta\right], \quad 0<\eta<\infty, \tag{6.9b}$$

$$\Psi_\eta(0,\tau) + q_1(\tau)\Psi(0,\tau) = 0, \quad 0<\tau<\tau(T), \tag{6.9c}$$

where $q_1(\tau)$ is as defined in (6.7).

Proof The part (a) of Proposition 6.1 can be proved by using an ansatz and part (b) can be obtained by using a generalized Cole-Hopf transformation. In both methods, the results are related to each other.

Proof (a) : Here, our aim is to transform the IBVP defined on time-dependent domain to the IBVP on fixed domain. So, we define new variable $y = x - s(t)$ and denote $U(x,t) = \tilde{U}(y,t)$. Then performing time and space differentiations

$$U_t = -\dot{s}(t)\tilde{U}_y + \tilde{U}_t, \quad U_x = \tilde{U}_y, \quad U_{xx} = \tilde{U}_{yy}, \tag{6.10}$$

we get the following IBVP defined on the half line, $0<y<\infty$, $0<t<T$, for the new function $\tilde{U}(y,t)$

$$\begin{cases} \tilde{U}_t + \frac{\dot{\mu}(t)}{\mu(t)}\tilde{U} + \tilde{U}\tilde{U}_y = \frac{1}{2\mu(t)}\tilde{U}_{yy} + \Big(\big(b(t)s(t) + \dot{s}(t) - a(t) + b(t)y\big)\tilde{U}\Big)_y - \omega^2(t)y - \omega^2(t)s(t) + f(t), \\ \tilde{U}(y,0) = U_0(y+s(0)), \qquad 0<y<\infty, \\ \tilde{U}(0,t) = D(t), \qquad 0<t<T. \end{cases} \tag{6.11}$$

Motivated from previous works, let assume the solution is of the form

$$\tilde{U}(y,t) = \frac{1}{\mu(t)}\left[\rho(t)y + p(t) + g(t)V(\eta(x,t),\tau(t))\right]. \tag{6.12}$$

where $\eta(y,t) = g(t)y$ with $\rho(t), p(t), g(t)$ and $\tau(t)$ to be determined. Taking y and t derivatives we have

$$\tilde{U}_t = \left(\frac{\dot{\rho}(t)}{\mu(t)} - \frac{\dot{\mu}(t)\rho(t)}{\mu^2(t)}\right) y - \frac{\dot{\mu}(t)p(t)}{\mu^2(t)} + \frac{\dot{p}(t)}{\mu(t)} + \frac{\dot{g}(t)g(t)}{\mu(t)} y V_\eta + \frac{g(t)\dot{\tau}(t)}{\mu(t)} V_\tau + \left(\frac{\dot{g}(t)}{\mu(t)} - \frac{\dot{\mu}(t)g(t)}{\mu^2(t)}\right) V,$$

$$\tilde{U}_y = \frac{\rho(t)}{\mu(t)} + \frac{g^2(t)}{\mu(t)} V_\eta,$$

$$\tilde{U}_{yy} = \frac{g^3(t)}{\mu(t)} V_{\eta\eta}. \tag{6.13}$$

Then substituting all these derivatives into equation (6.11), we obtain

$$\begin{aligned} \dot{\tau}(t)V_\tau + \frac{g^2(t)}{\mu(t)} V V_\eta = {} & \frac{g^2(t)}{2\mu(t)} V_{\eta\eta} - \left(\frac{g^2(t)p(t)}{\mu^2(t)} - \frac{b(t)s(t)g^2(t)}{\mu(t)} - \frac{\dot{s}(t)g^2(t)}{\mu(t)} + \frac{a(t)g^2(t)}{\mu(t)}\right) V_\eta \\ & - \left(\frac{\dot{g}(t)g(t)}{\mu(t)} - \frac{g^2(t)b(t)}{\mu(t)} + \frac{g^2(t)\rho(t)}{\mu^2(t)}\right) y V_\eta - \left(\frac{\rho(t)g(t)}{\mu^2(t)} - \frac{b(t)g(t)}{\mu(t)} + \frac{\dot{g}(t)}{\mu(t)}\right) V \\ & - \frac{\dot{p}(t)}{\mu(t)} - \frac{\rho(t)p(t)}{\mu^2(t)} - \frac{a(t)\rho(t)}{\mu(t)} + \frac{b(t)p(t)}{\mu(t)} + \frac{b(t)s(t)\rho(t)}{\mu(t)} + \frac{\dot{s}(t)\rho(t)}{\mu(t)} \\ & + \left(\frac{2b(t)\rho(t)}{\mu(t)} - \frac{\rho^2(t)}{\mu^2(t)} - \omega^2(t) - \frac{\dot{\rho}(t)}{\mu(t)}\right) y - \omega^2(t)s(t) + f(t). \end{aligned} \tag{6.14}$$

Thus, the function $V(\eta, \tau)$ satisfies $V_\tau + VV_\eta = 1/2V_{\eta\eta}$ and initial condition $V(\eta, 0) = \tilde{U}_0(\eta) - \alpha_2$, if the auxiliary functions and the boundary function $s(t)$ with initial position $s(0) = \alpha_1$ and initial velocity $\dot{s}(0) = \alpha_2 - \alpha_1 b(0) + a(0)$ satisfy the following nonlinear system of five differential equations

$$\begin{aligned} \dot{\rho}(t) + \frac{\rho^2(t)}{\mu(t)} - 2b(t)\rho(t) + \mu(t)\omega^2(t) &= 0, \qquad \rho(0) = 0, \\ \dot{g}(t) + \left(\frac{\rho(t)}{\mu(t)} - b(t)\right) g &= 0, \qquad g(0) = 1, \\ \dot{\tau}(t) - \frac{g^2(t)}{\mu(t)} &= 0, \qquad \tau(0) = 0, \end{aligned} \tag{6.15}$$

$$\begin{aligned} \dot{s}(t) + b(t)s(t) - p(t)/\mu(t) - a(t) &= 0, \qquad s(0) = \alpha_1, \\ \dot{p}(t) - b(t)p(t) + \mu(t)\omega^2(t)s(t) - \mu(t)f(t) &= 0, \qquad \alpha(0) = \alpha_2, \end{aligned} \tag{6.16}$$

for arbitrary real constants α_2, α_2. The systems (6.15) and (6.16) have solutions given in (3.20) where $g(t) = 1/r_1(t)$ and (4.20) respectively. When we write all auxiliary functions into ansatz (6.12), we get

$$\tilde{U}(y,t) = \frac{1}{\mu(t)r_1(t)}V(\eta(y,t),\tau(t)) + \left(\frac{\dot{r}_1(t)}{r_1(t)} + b(t)\right)y + \frac{p_g^\alpha(t)}{\mu(t)}. \tag{6.17}$$

Then, by using back substitution $y = x - r_g^\alpha(t)$, we obtain the desired solution (6.5) satisfying the initial condition (6.2b). And we notice that $\tau(t)$ is positive and strictly increasing for $0 < t < T$, so that $\tau = \tau(t)$, $0 < t < T$ if and only if $t = t(\tau), 0 < \tau < \tau(T)$. Therefore, the solution (6.5) will satisfy the Dirichlet BC (6.2c) if $V(\eta,\tau)$ satisfies the Dirichlet boundary condition (6.6c), which completes the proof for part (a). And by Cole-Hopf transformation

$$V(\eta,\tau) = -\frac{\Psi_\eta(\eta,\tau)}{\Psi(\eta,\tau)}, \tag{6.18}$$

part (b) of Proposition 6.1 can be easily obtained. □

Proof (b) *:* The formulation (b) can be proved by generalized Cole-Hopf transformation

$$U(x,t) = -\frac{1}{\mu(t)}\frac{\Phi_x(x,t)}{\Phi(x,t)}. \tag{6.19}$$

Then the Dirichlet IBVP (6.2) with moving boundary is linearized to the Robin type IBVP with moving boundary for generalized diffusion type equation mentioned in Chapter 4 as follows

$$\begin{cases} \Phi_t = \frac{1}{2\mu(t)}\Phi_{xx} - [a(t) - b(t)x]\Phi_x + \mu(t)[\frac{\omega^2(t)}{2}x^2 - f(t)x + f_0(t)]\Phi, & s(t) < x < \infty,\ 0 < t < T, \\ \Phi(x,0) = \exp\left[-\int^x U(x',0)dx'\right], & s(0) < x < \infty, \\ \Phi_x(s(t),t) + D(t)\Phi(s(t),t) = 0, & 0 < t < T. \end{cases} \tag{6.20}$$

By using the solution representation for the problem (6.20) obtained in Chapter 4, and applying transformation (6.19), the desired solution (6.48) can be obtained. □

Therefore in two approaches, the solution of the mIBVP for the generalized Burgers equation is obtained in terms of solution to the characteristic equation and a Burgers (or heat) model. □

***Integral Representation of the Solution* :**

The IBVP (6.9) with Robin boundary condition on the half line for the heat equation has integral representation as follows

$$\Psi(\eta,\tau) = \int_0^\infty G_N(\eta,\xi,\tau)\Psi(\xi,0)d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}Q(\tau')d\tau', \tag{6.21}$$

where $Q(\tau)$ is found by solving second kind Volterra integral equation

$$Q(\tau) = q_1(\tau)\Big(\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau' - 2\int_0^\infty G_N(0,\xi,\tau)\Psi(\xi,0)d\xi\Big), \tag{6.22}$$

with

$$q_1(\tau) = \mu(t(\tau))r_1(t(\tau))\left[D(t(\tau)) - \frac{p_g^\alpha(t)}{\mu(t)}\right]. \tag{6.23}$$

For some specific boundary data $D(t)$, it is possible to solve second kind Volterra integral equation explicitly. Therefore according to Proposition (6.1), we obtain an integral representation of the solution to the mIBVP (6.2) in the form

$$\begin{aligned} U_g^\alpha(x,t) &= \left(\frac{\dot r_1(t)}{r_1(t)} + b(t)\right)(x - r_g^\alpha(t)) + \frac{p_g^\alpha(t)}{\mu(t)} \\ &- \frac{1}{\mu(t)r_1(t)}\left(\frac{\int_0^\infty \partial_\eta G_N(\eta(x,t),\xi,\tau(t))e^{-\int^\xi U_0(x+\alpha_1)dx+\alpha_2\eta}d\xi - \int_0^\tau K_\eta(\eta(x,t),\tau(t)-\tau')Q(\tau')d\tau'}{\int_0^\infty G_N(\eta(x,t),\xi,\tau(t))e^{-\int^\xi U_0(x+\alpha_1)dx+\alpha_2\eta}d\xi - \int_0^{\tau(t)} K(\eta(x,t),\tau(t)-\tau')Q(\tau')d\tau'}\right). \end{aligned} \tag{6.24}$$

We notice that solutions of the half-line IBVP's appear as a direct consequence of Proposition 6.1. For this, by letting $\alpha_1 = \alpha_2 = 0$ and $r_p(t) = 0$, which can happen in the following cases:

$i)$ $a(t) = f(t) = 0$,

$ii)$ $a(t)$-constant, $b(t) = \dot{\mu}(t)/\mu(t)$ and $f(t) = 0$,

$iii)$ $\dot{a}(t) = -f(t)$ and $b(t) = \dot{\mu}(t)/\mu(t)$,

one can recover solutions of the corresponding IBVP's on the half-line $0 < x < \infty$.

6.1.1. Standard Burgers equation with forcing term

Here we present exactly solvable models reducing to the heat IBVP on half line and discuss some examples corresponding to different initial and boundary data imposed at $x = s(t)$ propagating according to (6.3).

Consider the following Burgers model with time-dependent oscillatory forcing term $f(t) = F_0 \sin(\omega t)$, with frequency $\omega > 0$, and strength $F_0 > 0$

$$\begin{cases} U_t + UU_x = \frac{1}{2}U_{xx} + F_0 \sin(\omega t), \quad s(t) < x < \infty, \quad t > 0, \\ U(x,0) = U_0(x), \quad s(0) < x < \infty, \\ U(s(t),t) = D(t), \quad t > 0. \end{cases} \tag{6.25}$$

The corresponding characteristic equation $\ddot{r}(t) = F_0 \sin(\omega t)$ has two independent homogeneous solutions and a particular solution, respectively

$$r_1(t) = 1, \qquad r_2(t) = t, \qquad r_p(t) = \frac{F_0}{\omega^2}(\omega t - \sin(\omega t)), \quad t \geq 0. \tag{6.26}$$

Then we have the moving boundary $s(t)$ as

$$s(t) \equiv r_g^{\alpha}(t) = \alpha_1 + \alpha_2 t + \frac{F_0}{\omega^2}(\omega t - \sin(\omega t)), \quad (\alpha_1, \alpha_2) \in \mathbb{R}^2. \tag{6.27}$$

• If $\alpha_2 > F_0/\omega$, then the time-varying boundary $s(t)$ moves to the positive x-direction, otherwise it moves to the left in x-axis.

• When $\alpha_2 = -F_0/\omega$, then the boundary will just oscillate parallel to the $x = \alpha_1$ axis where $\alpha_1 = s(0)$ is the initial position.

• If $\alpha_2 < 0$ and $|\alpha_2| > F_0/\omega$, then the boundary moves to the negative x-direction.

Also we notice that since $\mu(t) = 1$ and $a(t) = b(t) = 0$, then generalized momentum is

$$p_g^\alpha(t) = \dot{s}(t) = \left(\alpha_2 + \frac{F_0}{\omega}\right) - \frac{F_0}{\omega}\cos(\omega t).$$

Then according to our result, if $s(t)$ is of the form (6.27), then Burgers mIBVP (6.25) has solution

$$U_g^\alpha(x,t) = \dot{s}(t) - \frac{\Psi_\eta\left(\eta_g^\alpha(x,t), t\right)}{\Psi\left(\eta_g^\alpha(x,t), t\right)}, \qquad \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2, \tag{6.28}$$

where $\eta_g^\alpha(x,t) = x - r_g^\alpha(t)$, $\tau(t) = t$ and $\Psi(\eta, \tau)$ satisfies the following Robin type IBVP defined on the half line for heat equation

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \qquad 0 < \eta < \infty, \quad \tau > 0, \\ \Psi(\eta, 0) = \exp\left[-\int^\eta U_0(x+\alpha_1)dx + \alpha_2\eta\right], \qquad 0 < \eta < \infty, \\ \Psi_\eta(0,\tau) + \left(D(\tau) - \dot{s}(\tau)\right)\Psi(0,\tau) = 0. \qquad \tau > 0. \end{cases} \tag{6.29}$$

The integral representation of solution to the IBVP (6.29) is given in (6.21) where $Q(\tau)$ is found by solving the second-kind Volterra integral equation

$$Q(\tau) = \left(D(\tau) - \dot{s}(\tau)\right) \times \left(\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau - \tau')}}d\tau' - 2\int_0^\infty \left(\frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)e^{-\int^\xi U_0(x+\alpha_1)dx}e^{\alpha_2\xi}d\xi\right). \tag{6.30}$$

It is possible to find explicit solution to the integral equation (6.30) for some specific boundary data $D(t)$ as in the following cases :

Case (i) : If $D(t) = \dot{s}(t)$, then the corresponding heat IBVP (6.29) will have homogeneous Neumann boundary condition, i.e $\Psi_\eta(0,\tau) = 0$. Then using solution to the Neumann IBVP defined on the half line for the heat equation, we obtain the corresponding solution to the mIBVP (6.25).

Case (ii) : When the boundary data is chosen such a way that,

$$D(t) = D_0 + \dot{s}(t),$$

where $D_0 > 0$ is a constant parameter, then the integral equation will be constant coefficient second-kind Volterra integral equation and the corresponding heat problem will have Robin BC with constant coefficient which is possible to solve for some specific initial datas. Then, using the solution to the heat IBVP with Robin BC, the corresponding solution to the problem with moving boundary (6.25) is obtained.

One can also consider different cases to solve the integral equation and the related problem explicitly.

In what follows, we discuss some examples with specific initial and boundary conditions mentioned in the cases (i) and (ii).

Example 6.1 *Consider the Burgers model with family of rational type continuous initial data that is parametrized by* $c = (c_1, c_2) \in \mathbb{R}^2$, *and inhomogeneous boundary condition imposed at* $x = s(t)$ *given by*

$$U_t + UU_x = \frac{1}{2}U_{xx} + F_0 \sin(\omega t), \qquad s(t) < x < \infty, \quad t > 0, \tag{6.31a}$$

$$U(x,0) = c_2 - \frac{P'_{2N}(x - c_1)}{P_{2N}(x - c_1)}, \qquad c_1 \geq \alpha_1, \quad c_2 \geq 0, \quad s(0) < x < \infty, \tag{6.31b}$$

$$U(s(t),t) = \dot{s}(t), \quad t > 0, \tag{6.31c}$$

where the parameter c_1 controls shifting in space, $c_2 \geq 0$ in (6.31b) just affects the amplitude of the initial data, $P_{2N}(x)$ is a linear superposition of even Kampe de Feriet polynomials at $t = 0$, and $P'_{2N}(x)$ denotes its derivatives with respect to x, defined by respectively

$$P_{2N}(x) = \sum_{n=0}^{N} H_{2n}(x,0) = \sum_{n=0}^{N} x^{2n}, \qquad P'_{2N}(x) = \sum_{n=1}^{N} 2nH_{2n-1}(x,0),$$

where we used that $\partial_\eta H_n(\eta,\tau) = nH_{n-1}(\eta,\tau)$, for all $n = 1, 2, ..., N$.

As $x \to \infty$, the initial condition is localized in space, i.e $U(\infty,0) = c_2$.

Notice that for parameter $c = \alpha$, i.e $c_1 = \alpha_1$, $c_2 = \alpha_2$ where $\alpha_1 = s(0)$ and $\dot{s}(0) = \alpha_2$, the initial and boundary data is compatible at $x = s(0)$ and $t = 0$, i.e at $(x,t) = (s(0), 0)$.

Therefore, if moving boundary $s(t)$ is

$$s(t) = \alpha_1 + \alpha_2 t + \frac{F_0}{\omega^2}(\omega t - \sin(\omega t)),$$

then the class of mIBVP (6.31) has solution (6.28) where $\Psi(\eta, \tau)$ is solution of the IBVP for heat equation with Kampe de Feriet initial data and homogeneous Neumann BC for heat equation on the half line

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \quad 0 < \eta < \infty, \quad \tau > 0, \\ \Psi(\eta, 0) = \sum_{n=0}^{N} H_{2n}(x, 0), \quad n = 0, 1, 2, ..., N, \quad 0 < \eta < \infty, \\ \Psi_\eta(0, \tau) = 0, \quad \tau > 0. \end{cases} \tag{6.32}$$

It is easy to obtain that the problem (6.32) has solution $\Psi_N(\eta, \tau) \equiv \sum_{n=0}^{N} H_{2n}(\eta, \tau)$, for $n = 0, 1, 2, ..., N$, where $H_{2n}(\eta, \tau)$ are even Kampe de Feriet polynomials. Then, the solution to the problem (6.31) becomes

$$U_{g,N}^{\alpha}(x,t) = \left(\alpha_2 + \frac{F_0}{\omega}\right) - \frac{F_0}{\omega}\cos(\omega t) - \frac{\sum_{n=1}^{N} 2n\, H_{2n-1}(x - r_g^{\alpha}(t), t)}{\sum_{n=0}^{N} H_{2n}(x - r_g^{\alpha}(t), t)}. \tag{6.33}$$

Due to the periodic forces, the global wave amplitude has oscillatory behavior.

For $N = 2$, the Burgers model (6.31) subject to continuous initial condition

$$U(x, 0) = \alpha_2 - \frac{2(x - \alpha_1) + 4(x - \alpha_1)^3}{1 + (x - \alpha_1)^2 + (x - \alpha_1)^4},$$

has solution explicitly

$$
\begin{aligned}
U^{\alpha}_{g,2}(x,t) &= \left(\alpha_2 + \frac{F_0}{\omega}\right) - \frac{F_0}{\omega}\cos(\omega t) \\
&- \frac{4(x - r^{\alpha}_g(t))^3 + 12t(x - r^{\alpha}_g(t)) + 2(x - r^{\alpha}_g(t))}{(x - r^{\alpha}_g(t))^4 + (x - r^{\alpha}_g(t))^2 + 6t(x - r^{\alpha}_g(t))^2 + 3t^2 + t + 1}. \qquad (6.34)
\end{aligned}
$$

In Fig.6.1, we illustrate the behavior of the solution (6.34) for certain parameters which controls the boundary propagation to the positive x-direction, i.e the case $\alpha_2 > F_0/\omega$.

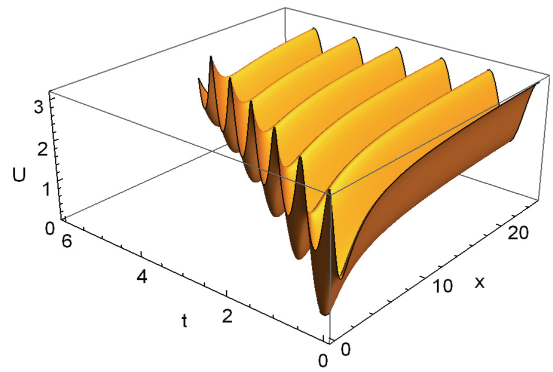


Figure 6.1 Solution (6.34) with $\alpha_1 = 0,\ \alpha_2 = 2,\ \omega = 6,\ F_0 = 5$.

Example 6.2 *Now, we study the Burgers model with family of rational type singular initial data and periodic boundary data with freguency* $\omega > 0$ *and strength* $F_0 > 0$

$$
\begin{cases}
U_t + UU_x = \frac{1}{2}U_{xx} + F_0\sin(\omega t), & s(t) < x < \infty, \quad t > 0, \\
U(x,0) = c_2 - \frac{(2n+1)H_{2n}(x-c_1,0) - \frac{2n(2n+1)}{D_0}H_{2n-1}(x-c_1,0)}{H_{2n+1}(x-c_1) - \frac{2n+1}{D_0}H_{2n}(x-c_1,0)}, & n = 0,1,2,..., \quad s(0) < x < \infty, \\
U(s(t),t) = D_0 + \dot{s}(t), \quad t > 0,
\end{cases} \qquad (6.35)
$$

where $H_n(x,0) = x^n$ is the Kampe de Feriet polynomials at $t = 0$, c_1 is shifting in space c_2 affects the amplitude which are parametrized by $c = (c_1, c_2)$, and the parameter $D_0 > 0$.

It is seen that since $D_0 > 0$, the initial data has simple zero at

$$
x = c_1 + \frac{2n}{D_0}, \quad n = 0,1,2,...,
$$

and pole type singularity at

$$x = c_1 + \frac{2n+1}{D_0}, \quad n = 0, 1, 2, ...$$

For each $n = 0, 1, 2, ...$ we have the same boundary condition imposed at $x = s(t)$. It is seen that the parameter D_0 can be used to control both the place of the initial singularity and strength of the boundary data.

Therefore, for the parameter $c = \alpha$, that is $c_1 = \alpha_1,\ c_2 = \alpha_2$, if the boundary $s(t)$ is of the form (6.27), then mIBVP (6.35) has solution given in (6.28) where $\Psi(\eta, \tau)$ is solution for the heat equation with polynomial type initial data and Robin type boundary condition on the half line

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, & 0 < \eta < \infty, \quad \tau > 0, \\ \Psi(\eta, 0) = \eta^{2n+1} - \frac{2n+1}{D_0}\eta^{2n}, & 0 < \eta < \infty, \\ \Psi_\eta(0, \tau) + D_0\Psi(0, \tau) = 0, & \tau > 0. \end{cases} \tag{6.36}$$

It is known that the integral representation of solution to the problem (6.36) is

$$\Psi_n(\eta, \tau) = \int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}} \right) \left(\xi^{2n+1} - \frac{2n+1}{D_0}\xi^{2n} \right) d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}} Q(\tau')d\tau', \tag{6.37}$$

where $Q(\tau)$ is found by solving second kind Volterra integral equation

$$Q(\tau) = D_0\left(\int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau' - 2\int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} \left(\xi^{2n+1} - \frac{2n+1}{D_0}\xi^{2n} \right) d\xi \right). \tag{6.38}$$

We obtain the solution to the second-kind Volterra integral equation as

$$Q(\tau) = 1.3.5...(2n+1)\tau^n. \tag{6.39}$$

By substituting $Q(\tau)$ into solution (6.37) we get

$$\Psi_n(\eta,\tau) = H_{2n+1}(\eta,\tau) - \frac{2n+1}{D_0} H_{2n}(\eta,\tau). \tag{6.40}$$

Since $D_0 > 0$, the heat solution (6.40) has a zero for $\eta > 0, \tau > 0$, which propagates along the semiline $0 < \eta < \infty$ during the evolution process and its position can be described by a continuous function $\eta = \chi_n(\tau)$, satisfying $\chi_n(0) = (2n+1)/D_0$ and

$$\Psi_n(\chi_n(\tau),\tau) = 0, \quad n = 0, 1, 2, ... \tag{6.41}$$

Then the corresponding solution to the mIBVP (6.35) becomes

$$\begin{aligned} U^{\alpha}_{g,n}(x,t) &= \alpha_2 + \frac{F_0}{\omega} - \frac{F_0}{\omega}\cos(\omega t) \\ &- \frac{(2n+1)H_{2n}(x - r^{\alpha}_g(t), t) - \frac{2n(2n+1)}{D_0} H_{2n-1}(x - r^{\alpha}_g(t), t)}{H_{2n+1}(x - r^{\alpha}_g(t), t) - \frac{2n+1}{D_0} H_{2n}(x - r^{\alpha}_g(t), t)}, \end{aligned} \tag{6.42}$$

which has discontinuity of infinite type at points where $\Psi_n(\eta(x,t),\tau(t)) = 0$. It follows that $U^{\alpha}_{g,n}(x,t)$ has moving singularity for $x > s(t),\ t > 0$, whose time-evolution is described by the function $x = X_n(t)$, where

$$X_n(t) = \chi_n(t) + r^{\alpha}_g(t), \qquad t > 0. \tag{6.43}$$

This relation shows that the distances between singularity curve and moving boundary can be controlled by the function $\chi_n(t)$. If $\chi_n(t) \to 0$ as $t \to \infty$, then singularity curve approaches moving boundary $x = s(t)$.

For $n = 1$, the Burgers problem has solution explicitly

$$\begin{aligned} U^{\alpha}_{g,1}(x,t) &= \alpha_2 + \frac{F_0}{\omega} - \frac{F_0}{\omega}\cos(\omega t) \\ &- \frac{3(x - r^{\alpha}_g(t))^2 - \frac{6}{D_0}(x - r^{\alpha}_g(t)) + 3t}{(x - r^{\alpha}_g(t))^3 - \frac{3}{D_0}(x - r^{\alpha}_g(t))^2 + 3t(x - r^{\alpha}_g(t)) - \frac{3}{D_0}t}, \end{aligned} \tag{6.44}$$

corresponding to the initial condition

$$U(x,0) = \alpha_2 - \frac{3(x - \alpha_1 - \frac{2}{D_0})}{(x - \alpha_1)(x - \alpha_1 - \frac{3}{D_0})},$$

which is discontinuous at

$$x = \alpha_1 + \frac{3}{D_0}. \tag{6.45}$$

For certain paramaters, we illustrate the motion of the moving boundary and the behavior of solution with moving singularity, see Fig.6.2

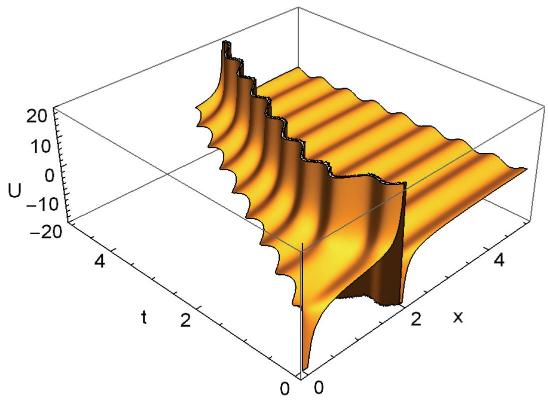


Figure 6.2 Solution (6.44) with $\alpha_1 = 0$, $\alpha_2 = -0.5$, $\omega = 10$, $F_0 = 10$, $D_0 = 1.5$.

6.1.2. Burgers equation with space and time-dependent convection term

Next, we study the following Burgers model defined by equation with space and time-dependent convection term, initial data $U_0(x)$ and homogeneous Dirichlet boundary condition imposed at $x = s(t)$

$$\begin{cases} U_t + \frac{\dot{\mu}(t)}{\mu(t)}U + UU_x = \frac{1}{2\mu(t)}U_{xx} - ((a(t) - b(t)x)U)_x, \quad s(t) < x < \infty, \quad t > 0, \\ U(x,0) = U_0(x), \quad s(0) < x < \infty, \\ U(s(t),t) = 0, \quad t > 0. \end{cases} \tag{6.46}$$

Then the corresponding the characteristic equation is

$$\ddot{r} + \frac{\dot{\mu}(t)}{\mu(t)}\dot{r} + \left(\dot{b}(t) + \frac{\dot{\mu}(t)}{\mu(t)}b(t) - b^2(t)\right) r = \dot{a}(t) - a(t)\left[b(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right]. \tag{6.47}$$

According to Proposition 6.1, if the boundary is of the form $s(t) \equiv r_g^{\alpha}(t) = \alpha_1 r_1(t) + \alpha_2 r_2(t) + r_p(t)$, then the solution of the mIBVP (6.46) is

$$U_g^{\alpha}(x,t) = \frac{p_g^{\alpha}(t)}{\mu(t)} - \frac{1}{\mu(t)r_1(t)}\frac{\Psi_\eta(\eta_g^{\alpha}(x,t),\tau(t))}{\Psi(\eta_g^{\alpha}(x,t),\tau(t))}, \tag{6.48}$$

where $\Psi(\eta,\tau)$ is solution to the IBVP with homogeneous Neumann BC for the heat equation.

Example 6.3 *Now we consider the Burgers equation in (6.46) with coefficients*

$$\mu(t) = 1, \qquad a(t) = a_0\cosh(\Lambda_0 t), \ \ a_0 \geq 0, \qquad b(t) = -\Lambda_0\tanh(\Lambda_0 t), \ \ \Lambda_0 > 0,$$

and family of rational type singular initial condition

$$U(x,0) = c_2 - \frac{(2n+1)H_{2n}(x-c_1,0) - \frac{2n(2n+1)}{D_0}H_{2n-1}(x-c_1,0)}{H_{2n+1}(x-c_1) - \frac{2n+1}{D_0}H_{2n}(x-c_1,0)}, \quad n = 0,1,2,..., \quad s(0) < x < \infty,$$

which has simple zero at

$$x = c_1 + \frac{2n}{D_0}, \quad n = 0,1,2,...,$$

and pole type singularity at

$$x = c_1 + \frac{2n+1}{D_0}, \quad n = 0, 1, 2, ...$$

The corresponding characteristic equation

$$\ddot{r} - \Lambda_0^2 r = 2a_0\Lambda_0 \sinh(\Lambda_0 t), \quad t > 0, \tag{6.49}$$

has solutions $r_1(t) = \cosh(\Lambda_0 t)$, $r_2(t) = \sinh(\Lambda_0 t)/\Lambda_0$, and $r_p(t) = a_0\, t \cosh(\Lambda_0 t)$.

The generalized momentum is found as $p_g^\alpha(t) = \alpha_2 \mathrm{sech}(\Lambda_0 t)$. Therefore, for the parameter $c = \alpha$, if $s(t)$ is

$$s(t) \equiv r_g^\alpha(t) = (\alpha_1 + a_0 t)\cosh(\Lambda_0 t) + \alpha_2 \sinh(\Lambda_0 t)/\Lambda_0, \tag{6.50}$$

with initial position $s(0) = \alpha_1$ and initial velocity $\dot{s}(0) = \alpha_2 + a_0$, then the solution of Burgers mIBVP is obtained as

$$U_g^\alpha(x,t) = \mathrm{sech}(\Lambda_0 t)\left(\alpha_2 - \frac{\Psi_\eta\left(\eta_g^\alpha(x,t), t\right)}{\Psi\left(\eta_g^\alpha(x,t), t\right)}\right), \tag{6.51}$$

where $\eta_g^\alpha(x,t) = \mathrm{sech}(\Lambda_0 t)(x - r_g^\alpha(t))$, $\tau(t) = \tanh(\Lambda_0 t)/\Lambda_0$ with inverse $t(\tau) = \tanh^{-1}(\Lambda_0\tau)/\Lambda_0$ and $\Psi(\eta,\tau)$ is solution for the heat IBVP with polynomial type initial data and Robin boundary condition on the half line

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, \quad 0 < \eta < \infty, \quad 0 < \tau < 1/\Lambda_0, \\ \Psi(\eta, 0) = \eta^{2n+1} - \frac{2n+1}{D_0}\eta^{2n}, \quad 0 < \eta < \infty, \\ \Psi_\eta(0,\tau) - \alpha_2\Psi(0,\tau) = 0, \quad 0 < \tau < 1/\Lambda_0. \end{cases} \tag{6.52}$$

The integral representation of solution to the problem (6.52) is

$$\Psi(\eta,\tau) = \int_0^\infty \left(\frac{e^{-\frac{(\eta-\xi)^2}{2\tau}} + e^{-\frac{(\eta+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}}\right)\left(\xi^{2n+1} - \frac{2n+1}{D_0}\xi^{2n}\right)d\xi - \int_0^\tau \frac{e^{-\frac{\eta^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau-\tau')}}Q(\tau')d\tau', \quad (6.53)$$

where $Q(\tau)$ is found by solving second kind Volterra integral equation

$$Q(\tau) = \alpha_2\left(2\int_0^\infty \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\left(\xi^{2n+1} - \frac{2n+1}{D_0}\xi^{2n}\right)d\xi - \int_0^\tau \frac{Q(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau'\right). \quad (6.54)$$

If we choose $\alpha_2 = -D_0$, i.e we fix the parameter as $\alpha_2 < 0$, then the solution to the second kind Volterra integral equation becomes $Q(\tau) = 1.3.5...(2n+1)\tau^n$. By substituting $Q(\tau)$ into solution (6.53) we obtain

$$\Psi_n(\eta,\tau) = H_{2n+1}(\eta,\tau) - \frac{2n+1}{|\alpha_2|}H_{2n}(\eta,\tau). \quad (6.55)$$

Therefore the corresponding solution to the Burgers mIBVP is

$$\begin{aligned} U^\alpha_{g,n}(x,t) &= \operatorname{sech}(\Lambda_0 t) \quad (6.56)\\ &\times \left(\alpha_2 - \frac{(2n+1)H_{2n}\left(\frac{x-r^\alpha_g(t)}{\cosh(\Lambda_0 t)}, \frac{\tanh(\Lambda_0 t)}{\Lambda_0}\right) - \frac{2n(2n+1)}{|\alpha_2|}H_{2n-1}\left(\frac{x-r^\alpha_g(t)}{\cosh(\Lambda_0 t)}, \frac{\tanh(\Lambda_0 t)}{\Lambda_0}\right)}{H_{2n+1}\left(\frac{x-r^\alpha_g(t)}{\cosh(\Lambda_0 t)}, \frac{\tanh(\Lambda_0 t)}{\Lambda_0}\right) - \frac{2n+1}{|\alpha_2|}H_{2n}\left(\frac{x-r^\alpha_g(t)}{\cosh(\Lambda_0 t)}, \frac{\tanh(\Lambda_0 t)}{\Lambda_0}\right)}\right). \end{aligned}$$

Notice that the parameter α_2 controls the initial singularity and velocity of the moving boundary. So, for large $|\alpha_2|$, the points on the singularity path approach the boundary. And in the case $\alpha_2 < 0$, at zeros of heat solution (6.55) defined by $\eta = \chi_n(\tau)$, satisfying $\chi_n(0) = (2n+1)/|\alpha_2|$, the corresponding Burgers solution (6.56) has time-dependent singularity described by

$$X_n(t) = \cosh(\Lambda_0 t)\chi_n\left(\frac{\tanh(\Lambda_0 t)}{\Lambda_0}\right) + r^\alpha_g(t), \quad t > 0. \quad (6.57)$$

For $n = 1$, we have solution explicitly

$$U^{\alpha}_{g,1}(x,t) = \alpha_2 \text{sech}(\Lambda_0 t) \tag{6.58}$$
$$-\frac{3\alpha_2\Lambda_0(x-r_g^{\alpha}(t))^2 + 6\Lambda_0\cosh(\Lambda_0 t)(x-r_g^{\alpha}(t)) + 3\alpha_2\sinh(\Lambda_0 t)\cosh(\Lambda_0 t)}{\alpha_2\Lambda_0(x-r_g^{\alpha}(t))^3 + 3\Lambda_0\cosh(\Lambda_0 t)(x-r_g^{\alpha}(t))^2 + 3\alpha_2\sinh(\Lambda_0 t)\cosh(\Lambda_0 t)(x-r_g^{\alpha}(t)) + 3\cosh^2(\Lambda_0 t)\sinh(\Lambda_0 t)},$$

which has singularity at $x \equiv x_1(t)$, as follows

$$X_1(t) = r_g^{\alpha}(t) + \sqrt[3]{\sigma(t)} + \frac{\Lambda_0\cosh^2(\Lambda_0 t) - \alpha_2^2\sinh(\Lambda_0 t)\cosh(\Lambda_0 t)}{\alpha_2^2\,\Lambda_0\,\sqrt[3]{\sigma(t)}} - \frac{\cosh(\Lambda_0 t)}{\alpha_2}, \tag{6.59}$$

where

$$\sigma(t) = \sqrt{\frac{\sinh^3(\Lambda_0 t)\cosh^3(\Lambda_0 t)}{\Lambda_0^3} + \frac{3\sinh(\Lambda_0 t)\cosh^5(\Lambda_0 t)}{\alpha_2^4\,\Lambda_0} - \frac{3\sinh^2(\Lambda_0 t)\cosh^4(\Lambda_0 t)}{\alpha_2^2\,\Lambda_0^2}} - \frac{\cosh^3(\Lambda_0 t)}{\alpha_2^3}.$$

Here, the parameter Λ_0 controls the velocity and the spreading of the singularity curve. Therefore, by choosing $\alpha_2 < 0$, we investigate the influence of parameters on singularity curve and moving boundary motion.

In Fig.6.3a and Fig.6.3b, by choosing positive parameter $a_0 > 0$, one can see the motion of both moving boundary and singularity curve. In Fig.6.3a, they propagates to the positive direction in space as time increases. However, in Fig.6.3b, one can see that small changing in a_0 changes the direction of the boundary and the spreading of the moving singularity.

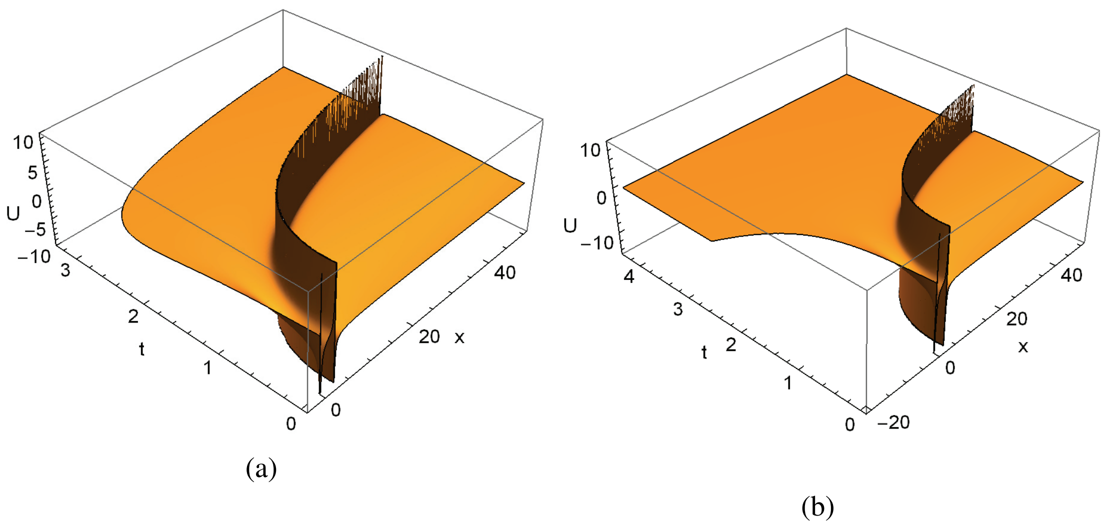


Figure 6.3 Solution (6.58) with $\alpha_1 = c_1 = 0,\ \alpha_2 = c_2 = -1,\ \Lambda_0 = 2$, and (a) $a_0 = 0.2$. (b) $a_0 = 0.1$.

CHAPTER 7

CONCLUSION

We considered the most general one-dimensional generalized diffusion type equation with time variable coefficients that can be written as a linear combination of generators of the finite dimensional $su(1,1)$ and Heisenberg-Weyl Lie algebras. First, we studied IVP on whole real-line and obtained analytical solution in terms of solutions to the characteristic equation and the standard heat model by using Wei-Norman Lie algebraic approach for finding the evolution operator of the associated diffusion type equation. We were able to obtain exact form of the evolution operator in terms of two linearly independent homogeneous solutions and a particular solution of the corresponding classical equation of motion. Then we discussed an initial-boundary value problems with Dirichlet, Neumann and Robin BC's on half-line. Later, we introduced an IBVP for a generalized diffusion type equation defined on a domain with a time-dependent boundary $s(t) < x < \infty,\ 0 < t < T$. We showed that if the boundary moves according to an associated classical equation of motion determined by the time-dependent parameters, then we obtain analytical solution in terms of the heat problem on the half-line. For this, we solved a non-linear Riccati type dynamical system, that simultaneously determines the solution of the diffusion type problem and the moving boundary. This allowed us to construct exactly solvable mIBVP's with Dirichlet, Neumann and Robin BC's imposed at boundaries evolving according to an associated Newtonian equation of motion. For each model we derived integral representations and fundamental solutions, explicitly showing how the moving boundary affects the evolution process. And we also discussed the mean position of the solution distribution, the influence of the moving boundaries and the variable parameters.

More precisely, the first exactly solvable model that we introduced, can represent reaction-diffusion processes in a motionless fluid or heat conduction in solids, where usually convection is not included (Carslaw & Jaeger, 1959). In that case, moving boundaries can describe for instance progressive freezing or solute redistribution during the solidification of liquids in semi-infinite regions. We studied the model with an exponentially decaying diffusion coefficient and exponentially growing time-varying first-order reaction, usually a chemical reaction in which the rate of reaction is proportional to the concentration of the reacting substance. Concrete examples corresponding to different initial data and homogeneous boundary conditions imposed on the moving boundary were constructed.

Second, we discussed explicit solutions to a convection-diffusion type model that can be used to study the transport of some quantity by diffusion and convection processes. It can be seen also as Fokker-Planck type problem with varying drift and absorbing boundary conditions imposed on a moving boundary. Usually, the drift coefficient controls the external forces acting on the particles, and the diffusion coefficient affects the fluctuations. As known, such models are widely used to describe the effect of fluctuations in macroscopic systems, but in general, solving them is a difficult task (Ho, 2013). Absorbing boundary conditions can be used also when chemical reactions on the boundary occur and due to this molecules are absorbed or chemically changed (Enzo Orlandini). Solution satisfying the variable parametric FP equation and vanishing on the associated classical equation of motion is interpreted as probability density.

Lastly, the convection-diffusion-reaction type model includes a convective term that significantly influences the prescribed moving boundary and the dynamics of the system. It can describe again the distribution of temperature or concentration of a substance that is kept equal to zero for all times on a boundary that changes with time, but in that case, we have an inhomogeneous and semi-infinite media with an unsteady flow. Here, the sinusoidal convection term was considered which contributed to the time-oscillating boundary, together with the boundary, it creates a solution oscillating with respect to time which is felt in areas close to the boundary. As in previous models, some examples with different initial data were studied and the dynamic properties were investigated.

Due to the generality of the results presented in this work, we note that the number of exactly solvable models can be enlarged to include many other interesting cases and applications in physics, biology, and chemical phenomena. This study is under preparation for submission to a scientific journal.

Then, we considered a one-dimensional generalized forced Burgers type equation with variable coefficients. We obtained analytical solution to the IVP on whole real line by using a Cole-Hopf transform and the Wei-Norman Lie algebraic approach for finding the evolution operator of the associated linear diffusion type equation. This allowed us to find exact solutions to the generalized Burgers equation, explicitly expressed in terms of solutions to a standard Burgers (or heat) model and the characteristic ODE. Using the translation and Galilean symmetries of the standard Burgers equation, we constructed families of some particular solutions such as generalized shocks and multi-shocks, triangular and N-shaped traveling waves corresponding to initial data parameterized by $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. By construction, a common property of all generalized nonlinear waves is that during the evolution process their wave profile or "center" follows the characteristic equation. Moreover, we showed that for some special choices of the time-variable coefficients and external terms it is possible to construct smooth and localized in space solutions. In this sense, positive and localized wave packets of the generalized Burgers model can be seen as cumulative distributions or probability density functions and for example as nonlinear analogs of coherent states in quantum mechanics. Here, we introduced also generalized rational type solutions with pole type singularities following a Newtonian type equation and we obtained particular solutions whose singularity oscillates with time while propagating along the $x-$axis. As known propagation of pole singularities corresponds formally to the motion of one-dimensional particles interacting via two-body potentials (Choodnovsky, 1977), (Calogero, 1978) and for a recent discussion on Burgers singularities one can see also (Atılgan Büyükaşık & Bozacı, 2021), and references therein.

In this work, from a large possibility of exactly solvable models, we have tried to choose some simple examples that are able to reflect certain interesting properties of the solutions. In all these models we discussed how the main characteristics of the wave motion such as position, velocity, steepness, width, and dissipation depend on the time-variable coefficients and how their dynamics can be controlled by the free parameters of the initial data. Results of this work are published in (Atılgan Büyükaşık & Bozacı, 2021).

Then, we investigated an IBVP for a variable parametric inhomogeneous Burgers equation defined on the half-line $0 < x < \infty$ for $t_0 < t < T$ and satisfying smooth Dirichlet boundary condition imposed at $x = 0$. We determined its solution in terms of a second-order homogeneous characteristic ODE and a second kind Volterra type integral equation with a weakly singular kernel. Since the associated ODE and the integral equations are linear but with time-variable coefficients, they rarely admit exact solutions.

As an application of our general results, we introduced three exactly solvable Burgers type models with different time-variable coefficients. The Burgers problems with smooth time-dependent boundary data and an initial profile with pole type singularity have exact solutions with moving singularity. For each model we provide the solutions explicitly and describe the dynamical properties of the singularities depending on the time-variable coefficients and the given initial and boundary data. The results of this work are published in (A. Büyükaşık & Bozacı, 2019).

Finally, we introduced an IBVP for a generalized Burgers type equation with time variable coefficients, which was defined on a domain with a time-dependent boundary $s(t) < x < \infty$, $0 < t < T$. We proved that if the moving boundary $s(t)$ is written as a linear combination of two linearly independent homogenous and a particular solution of the corresponding characteristic equation parameterized by the initial position and velocity, then the problem can be solved analytically in terms of solution to the characteristic equation and standard heat or Burgers model. We presented the integral representation of Burgers mIBVP which requires to solve Volterra integral equation of second-kind. To show general aspects of our results, firstly we considered mIBVP for standard Burgers equation with oscillatory time-dependent forcing term. For some special choices of initial and boundary conditions, we discussed the behavior of the solution and the motion of the boundary. Then, we studied unforced Burgers model with space and time-dependent convection term and examined the influence of parameters which creates moving singularities in the solution for rational type singular initial data and homogeneous boundary condition.

One can enlarge the class of exact solutions for the characteristic equation that can be obtained for the different type of special functions, by the so-called Sturm-Liouville problem, such as Hermite, Laguerre and Jacobi type orthogonal polynomials, (A. Büyükaşık, Pashaev & Ulaş-Tigrak, 2009).

We will consider other exactly solvable Burgers type models with variable coefficients and moving boundaries and discuss the behavior of solution by examining application fields.

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APPENDIX A

DIRAC-DELTA DISTRIBUTION

The Dirac-delta function $\delta(x)$ is not really a "function". It is a mathematical entity called a "*distribution*" which is well defined only when it appears under an integral sign. It has the following defining properties:

$$\delta(x-a) = \begin{cases} \infty & if \;\; x = a \\ 0 & if \;\; x \neq a, \end{cases} \tag{A.1}$$

with

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a). \tag{A.2}$$

There are many properties of the delta distribution which follow from the definition (A.1). Some of these are:

1. $\int_{-\infty}^{\infty} \delta(x-a)dx = 1.$
2. $\delta(x-a) = \delta(a-x).$
3. $\delta(ax) = \frac{1}{|a|}\delta(x), \quad a \neq 0.$
4. $f(x)\delta(x-a) = f(a)\delta(x-a).$
5. $\int_{-\infty}^{\infty} \delta(x-y)\delta(y-a)dy = \delta(x-a).$
6. $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk = \delta(x).$

APPENDIX B

EXACT SOLUTIONS TO THE PROBLEMS FOR DIFFUSION-CONVECTION-REACTION EQUATION WITH CONSTANTS COEFFICIENTS

In this Appendix, we give the solutions to the IVP and IBVP's with Dirichlet, Neumann and Robin type boundary conditions obtained in (Carslaw & Jaeger, 1959), (Polyanin, 2002) and references there in, for both convection-diffusion equation and convection-diffusion-reaction equation with constant coefficients.

B.1. Convection-diffusion equation with constant coefficients

Consider the equation

$$\Phi_t = \nu\Phi_{xx} + c\Phi_x. \tag{B.1}$$

An Initial Value Problem on Whole Real Line

The following IVP for convection-diffusion equation with constant coefficients

$$\begin{cases} \Phi_t = \nu\Phi_{xx} + c\Phi_x, \quad x \in \mathbb{R}, \quad t > 0, \\ \Phi(x, 0) = f(x), \quad x \in \mathbb{R}, \end{cases} \tag{B.2}$$

has solution to the problem (B.2) is

$$\Phi(x, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{(x - \xi + ct)^2}{4\nu t}\right] f(\xi)d\xi. \tag{B.3}$$

<u>*Fundamental solution*</u> : When the initial condition is taken Dirac-delta distribution, $\Phi(x, 0) = \delta(x - x_0)$, then the fundamental solution is

$$K(x, x_0; t) = \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{(x - x_0 + ct)^2}{4\nu t}\right]. \tag{B.4}$$

An Initial and Boundary Value Problems on the Half-line

1) Dirichlet IBVP : Consider the following IBVP with Dirichlet BC

$$\begin{cases} \Phi_t = \nu\Phi_{xx} + c\Phi_x, \quad 0 < x < \infty, \quad t > 0, \\ \Phi(x,0) = f(x), \quad 0 < x < \infty, \\ \Phi(0,t) = D(t), \quad t > 0, \end{cases} \tag{B.5}$$

where $f(x)$, $D(t)$ are given sufficiently smooth functions in their domains. The solution to the problem (B.5) is

$$\begin{aligned} \Phi(x,t) = & \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{c^2 t}{4\nu}\right] \\ & \times \left(\int_0^\infty \exp\left[-\frac{c(x-\xi)}{2\nu}\right] \times \left(\exp\left[-\frac{(x-\xi)^2}{4\nu t}\right] - \exp\left[-\frac{(x+\xi)^2}{4\nu t}\right]\right) f(\xi) d\xi \right) \\ & + \nu \int_0^t \frac{x}{4\sqrt{\pi\nu(t-t')^3}} \exp\left[-\frac{(x+c(t-t'))^2}{4\nu(t-t')}\right] D(t')dt'. \end{aligned}$$

<u>*Fundamental solution*</u> : When we take Dirac-delta initial condition,$\Phi(x,0) = \delta(x - x_0)$ and homogeneous boundary data $D(t) = 0$, then we get fundamental solution

$$K(x, x_0; t) = \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{c^2 t}{4\nu} - \frac{c(x-x_0)}{2\nu}\right] \times \left(\exp\left[-\frac{(x-x_0)^2}{4\nu t}\right] - \exp\left[-\frac{(x+x_0)^2}{4\nu t}\right]\right).$$

2) Neumann IBVP : The IBVP with Neumann boundary condition

$$\begin{cases} \Phi_t = \nu\Phi_{xx} + c\Phi_x, \quad 0 < x < \infty, \quad t > 0, \\ \Phi(x,0) = f(x), \quad 0 < x < \infty, \\ \Phi_x(0,t) = g(t), \quad t > 0, \end{cases} \tag{B.6}$$

where $f(x)$, $g(t)$ are given sufficiently smooth functions in their domains, has solution

$$\begin{aligned} \Phi(x,t) = & \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{c^2 t}{4\nu}\right] \\ & \times \left(\int_0^\infty \exp\left[-\frac{c(x-\xi)}{2\nu}\right] \times \left(\exp\left[-\frac{(x-\xi)^2}{4\nu t}\right] + \exp\left[-\frac{(x+\xi)^2}{4\nu t}\right]\right) f(\xi) d\xi \right) \\ & - \nu \int_0^t \frac{1}{\sqrt{\pi\nu(t-t')}} \exp\left[-\frac{(x+c(t-t'))^2}{4\nu(t-t')}\right] g(t')dt'. \end{aligned} \tag{B.7}$$

<u>*Fundamental solution*</u> : When we take Dirac-delta initial condition, $\Phi(x,0) = \delta(x - x_0)$ and homogeneous boundary data $g(t) = 0$, then we get fundamental solution

$$K(x, x_0; t) = \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{c^2 t}{4\nu} - \frac{c(x-x_0)}{2\nu}\right] \times \left(\exp\left[-\frac{(x-x_0)^2}{4\nu t}\right] + \exp\left[-\frac{(x+x_0)^2}{4\nu t}\right]\right).$$

3) Robin type IBVP : The IBVP with homogeneous Robin type boundary condition

$$\begin{cases} \Phi_t = \nu\Phi_{xx} + c\Phi_x, \quad 0 < x < \infty, \quad t > 0, \\ \Phi(x,0) = f(x), \quad 0 < x < \infty, \\ \Phi_x(0,t) - k\Phi(0,t) = g(t), \quad t > 0, \quad k \in \mathbb{R}, \end{cases} \tag{B.8}$$

has solution (B.7), where

$$G(x,\xi,t) = \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{c(x-\xi)}{2\nu} - \frac{c^2 t}{4\nu}\right]$$
$$\times \left(\exp\left[-\frac{(x-\xi)^2}{4\nu t}\right] + \exp\left[-\frac{(x+\xi)^2}{4\nu t}\right] - 2s\int_0^\infty \exp\left[-\frac{(x+\xi+y)^2}{4\nu t} - sy\right] dy\right),$$

where $s = k + c/(2\nu)$.

B.2. Convection-diffusion-reaction equation with constant coefficients

Now, we give solutions to the IVP and IBVP's for the convection-diffusion-reaction equation with constant coefficients of the form

$$\Phi_t = \nu\Phi_{xx} + c\Phi_x + r\Phi. \tag{B.9}$$

An Initial Value Problem on Whole Real Line

The IVP on whole real line

$$\begin{cases} \Phi_t = \nu\Phi_{xx} + c\Phi_x + r\Phi, \quad x \in \mathbb{R}, \quad t > 0, \\ \Phi(x,0) = f(x), \quad x \in \mathbb{R}, \end{cases} \tag{B.10}$$

has solution

$$\Phi(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{(x-\xi+ct)^2}{4\nu t} + rt\right] f(\xi) d\xi. \tag{B.11}$$

<u>*Fundamental solution*</u> : When the initial condition is taken Dirac-delta distribution, $\Phi(x,0) = \delta(x-x_0)$, then the fundamental solution is

$$K(x,x_0;t) = \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{(x-x_0+ct)^2}{4\nu t} + rt\right]. \tag{B.12}$$

An Initial and Boundary Value Problems on the Half-line

1) Dirichlet IBVP : The IBVP with Dirichlet BC

$$\begin{cases} \Phi_t = \nu\Phi_{xx} + c\Phi_x + r\Phi, \quad 0 < x < \infty, \quad t > 0, \\ \Phi(x,0) = f(x), \quad 0 < x < \infty, \\ \Phi(0,t) = D(t), \quad t > 0, \end{cases} \tag{B.13}$$

where $f(x)$, $D(t)$ are given sufficiently smooth functions in their domains, has solution

$$\begin{aligned} \Phi(x,t) = & \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{c^2 t}{4\nu} + rt\right] \\ & \times \left(\int_0^\infty \exp\left[-\frac{c(x-\xi)}{2\nu}\right] \times \left(\exp\left[-\frac{(x-\xi)^2}{4\nu t}\right] - \exp\left[-\frac{(x+\xi)^2}{4\nu t}\right] \right) f(\xi) d\xi \right) \\ & + \nu \int_0^t \frac{x}{4\sqrt{\pi\nu(t-t')^3}} \exp\left[-\frac{(x+c(t-t'))^2}{4\nu(t-t')} + r(t-t')\right] D(t') dt'. \end{aligned}$$

<u>*Fundamental solution*</u> : When we take Dirac-delta initial condition,$\Phi(x,0) = \delta(x - x_0)$ and homogeneous boundary data $D(t) = 0$, then we get fundamental solution

$$K(x,x_0;t) = \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{c^2 t}{4\nu} - \frac{c(x-x_0)}{2\nu} + rt\right] \times \left(\exp\left[-\frac{(x-x_0)^2}{4\nu t}\right] - \exp\left[-\frac{(x+x_0)^2}{4\nu t}\right] \right).$$

2) Neumann IBVP : The IBVP with Neumann boundary condition

$$\begin{cases} \Phi_t = \nu\Phi_{xx} + c\Phi_x + r\Phi, \quad 0 < x < \infty, \quad t > 0, \\ \Phi(x,0) = f(x), \quad 0 < x < \infty, \\ \Phi_x(0,t) = g(t), \quad t > 0, \end{cases} \tag{B.14}$$

has solution

$$\begin{aligned} \Phi(x,t) = & \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{c^2 t}{4\nu} + rt\right] \\ & \times \left(\int_0^\infty \exp\left[-\frac{c(x-\xi)}{2\nu}\right] \times \left(\exp\left[-\frac{(x-\xi)^2}{4\nu t}\right] + \exp\left[-\frac{(x+\xi)^2}{4\nu t}\right] \right) f(\xi) d\xi \right) \\ & - \nu \int_0^t \frac{1}{\sqrt{\pi\nu(t-t')}} \exp\left[-\frac{(x+c(t-t'))^2}{4\nu(t-t')} + r(t-t')\right] g(t') dt'. \end{aligned} \tag{B.15}$$

<u>*Fundamental solution*</u> : When we take Dirac-delta initial condition, $\Phi(x,0) = \delta(x - x_0)$ and homogeneous boundary data $g(t) = 0$, then we get fundamental solution

$$K(x,x_0;t) = \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{c^2 t}{4\nu} - \frac{c(x-x_0)}{2\nu} + rt\right] \times \left(\exp\left[-\frac{(x-x_0)^2}{4\nu t}\right] + \exp\left[-\frac{(x+x_0)^2}{4\nu t}\right] \right).$$

3) Robin type IBVP : The IBVP with homogeneous Robin type BC

$$\begin{cases} \Phi_t = \nu\Phi_{xx} + c\Phi_x + r\Phi, \quad 0 < x < \infty, \quad t > 0, \\ \Phi(x,0) = f(x), \quad 0 < x < \infty, \\ \Phi_x(0,t) - k\Phi(0,t) = g(t), \quad t > 0, \quad k \in \mathbb{R}, \end{cases} \tag{B.16}$$

has solution (B.15), where

$$G(x,\xi,t) = \frac{1}{\sqrt{4\pi\nu t}} \exp\left[-\frac{c(x-\xi)}{2\nu} - \frac{c^2 t}{4\nu} + rt\right]$$
$$\times \left(\exp\left[-\frac{(x-\xi)^2}{4\nu t}\right] + \exp\left[-\frac{(x+\xi)^2}{4\nu t}\right] - 2s\int_0^\infty \exp\left[-\frac{(x+\xi+y)^2}{4\nu t} - sy\right] dy\right),$$

where $s = k + c/(2\nu)$.

APPENDIX C

INITIAL-BOUNDARY VALUE PROBLEM WITH MOVING BOUNDARY FOR STANDARD HEAT EQUATION

The IBVP with moving boundary defined on semi-infinite time-dependent domain $s(t) < x < \infty,\ 0 < t < T$ for the standard heat equation

$$\begin{cases} \Phi_t = \frac{1}{2}\Phi_{xx}, & s(t) < x < \infty, \quad 0 < t < T, \\ \Phi(x,0) = \Phi_0(x), & s(0) < x < \infty, \\ \Phi(s(t),t) = D(t), & 0 < t < T, \end{cases} \tag{C.1}$$

has solution of the form

$$\Phi(x,t) \quad = \quad \Psi(\eta(x,t),\tau(t)) \times \exp\left[-\alpha_2(x-(\alpha_1+\alpha_2 t))\right] \times \exp\left[-\frac{\alpha_2^2}{2}t\right],$$

if moving boundary

$$s(t) = \alpha_1 + \alpha_2 t,$$

and $\eta(x,t) = x - (\alpha_1 + \alpha_2 t)$, $\tau(t) = t$ and Ψ is solution of the standard heat equation with Dirichlet BC imposed at $\eta = 0$ on half line

$$\begin{cases} \Psi_\tau = \frac{1}{2}\Psi_{\eta\eta}, & 0 < \eta < \infty, \quad 0 < \tau < \tau(T)., \\ \Psi(\eta,0) = \Phi_0(\eta+\alpha_1)\, e^{\alpha_2\eta}, & \eta > 0, \\ \Psi(0,\tau) = D(\tau)\exp\left[\alpha_2^2\, \tau/2\right], & 0 < \tau < \tau(T), \end{cases} \tag{C.2}$$

with $\alpha_1 = s(0)$ is the initial position, $\alpha_2 = \dot{s}(0)$ is the initial velocity of the boundary.

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