BOUNDARY FEEDBACK STABILIZATION OF SOME EVOLUTIONARY PARTIAL DIFFERENTIAL EQUATIONS

A Thesis Submitted to
the Graduate School of Engineering and Sciences of
İzmir Institute of Technology
in Partial Fulfillment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by
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December 2022
İZMİR

To the memory of my father...

ACKNOWLEDGMENTS

First and foremost, I would like to express my deepest gratitude to my advisor Dr. Türker Özsarı for his invaluable guidance during my Ph.D. research. He, as a mathematician, is one of my major inspiration and motivation source throughout my Ph.D. education. I feel very lucky to study with someone who is a role model for me.

I would like to thank Dr. Ahmet Batal for his precious time and suggestions, which improved quality of this thesis. I would also like express my special thanks to Prof. Dr. Varga Kalantarov. Besides his crucial suggestions and remarks, there is no doubt his enthusiasm serves an excellent example for me as well as many young mathematicians on how an academician should be. I would like to thank the thesis committee Prof. Dr. Oğuz Yılmaz, Prof. Dr. Şirin Atılgan-Büyükaşık, Prof. Dr. Başak Karpuz, Assoc. Prof. Dr. Burcu Silindir-Yantır and Asst. Prof. Dr. Olha Ivanyshyn-Yaman for their valuable time, constructive comments and suggestions.

IZTECH, Department of Mathematics offers a great place to study as a graduate student. I would like to thank every member of our department who contributed on this, in particular to Prof. Dr. Oğuz Yılmaz, former chair of the Department of Mathematics.

I am grateful to my mother Demet for her endless love and continuous support, especially during the *second* half of my life. I am also grateful to my brothers Can and Deniz, my cousin Sarp for their support, and in particular to my dear cat (Hy)Patia. Her friendship is more than a human can offer.

During the years of my Ph.D., in particular during the covid-19 pandemic, one of the most enjoyable moments that sticks in my mind is hourlong phone conversations with Nermin Yolcu-Alp, involving discussions on well-posedness of boundary value problems. I wish all the best in her life. I would like to thank my dear friends; Sinem Benli-Göral for her friendship, Aylin Bozacı-Serdal for her friendship and more importantly for her valuable support especially during the last semester, and in particular Kıvılcım Alkan for her precious suggestions on my concerns regarding my health issues.

I gratefully acknowledge the support "2211-A National PhD Scholarship Program" that I received from The Scientific and Technological Research Council of Turkey (TÜBİTAK). This work is also partially supported by TÜBİTAK 1001 grant with the project number 122F084.

ABSTRACT

BOUNDARY FEEDBACK STABILIZATION OF SOME EVOLUTIONARY PARTIAL DIFFERENTIAL EQUATIONS

The purpose of this study is to control long time behaviour of solutions to some evolutionary partial differential equations posed on a finite interval by backstepping type controllers. At first we consider right endpoint feedback controller design problem for higher-order Schrödinger equation. The second problem is observer design problem, which has particular importance when measurement across the domain is not available. In this case, the sought after right endpoint control inputs involve state of the observer model. However, it is known that classical backstepping strategy fails for designing right endpoint controllers to higher order evolutionary equations. So regarding these controller and observer design problems, we modify the backstepping strategy in such a way that, the zero equilibrium to the associated closed-loop systems become exponentially stable. From the well-posedness point of view, this modification forces us to obtain a time-space regularity estimate which also requires to reveal some smoothing properties for some associated Cauchy problems and an initial-boundary value problem with inhomogeneous boundary conditions. As a third problem, we introduce a finite dimensional version of backstepping controller design for stabilizing infinite dimensional dissipative systems. More precisely, we design a boundary control input involving projection of the state onto a finite dimensional space, which is still capable of stabilizing zero equilibrium to the associated closed-loop system. Our approach is based on defining the backstepping transformation and introducing the associated target model in a novel way, which is inspired from the finite dimensional long time behaviour of dissipative systems. We apply our strategy in the case of reaction-diffusion equation. However, it serves only as a canonical example and our strategy can be applied to various kind of dissipative evolutionary PDEs and system of evolutionary PDEs. We also present several numerical simulations that support our theoretical results.

ÖZET

ZAMANA BAĞLI BAZI KISMİ DİFERANSİYEL DENKLEMLERİN SINIRDAN GERİ BESLEMELİ KARARLILAŞTIRILMASI

Bu çalışmanın amacı zamana bağlı ve sonlu aralıkta tanımlı kısmi diferansiyel denklemlerin çözümlerinin asimptotik davranışının geri-adım tipi kontrol ediciler ile kontrol edilmesi üzerinedir. İlk olarak yüksek mertebeden Schrödinger modeli için, uzaysal bölgenin sağ bitim noktasından etki eden kontrol edici inşası problemini ele alıyoruz. İkinci olarak ise, bilhassa uzaysal bölge boyunca ölçüm yapmanın mümkün olmadığı durumlarda öneme sahip olan gözlemci inşası problemini ele alıyoruz. Bu durumda gözlemci modelin çözümünü içeren ve modele sağ bitim noktasından etki eden kontrol ediciler inşa ediyoruz. Geri-adım yönteminin yüksek mertebeden türevler içeren modellerin kararşılaştırılmasını sağlayan ve modele sağ bitim noktasından etki eden kontrol edicilerin inşasında başarısız kaldığı bilinmektedir. Dolayısıyla kontrol ve gözlemci inşası problemlerine dair geri-adım yöntemini, bu durumun arka planında yer alan sebepleri kaldırmaya yönelik olarak ve sıfır denge çözümünün üstel kararlı kılacak bir biçimde güncelliyoruz. İyi konuşlanmışlık analizi açısından bu güncelleme, bir zaman-uzay kestirimi elde edilmesini ve bu da, analiz boyunca karşımıza çıkan çeşitli Cauchy problemlerine ve homojen olmayan sınır koşulları içeren bir başlangıç-sınır değer problemine ilişkin düzgünleştirici etkilerin ortaya çıkarılmasını gerektirmektedir. Üçüncü bir problem olarak sonsuz boyutlu disipatif sistemleri kararlı kılan, geri-adım tipi sonlu boyutlu kontrol edici inşası stratejimizi sunuyoruz. Daha kesin bir dille ifade edecek olursak, sonsuz boyutlu disipatif sistemlerin sıfır denge çözümünün kararlılığını, çözümün yalnızca sonlu boyutlu bir uzaya izdüşümünü içeren sınır tipi geri beslemeli kontrol ediciler ile sağlıyoruz. Yaklaşımımız, disipatif sistemlerin sonlu boyutlu asimptotik davranışa sahip oldukları gerçeğinden esinlenilerek özgün bir biçimde sunduğumuz hedef modele ve yine özgün bir tanımladığımız geri-adım dönüşümüne dayanmaktadır. Reaksiyon-difüzyon özelinde sunduğumuz stratejimiz, pek çok disipatif modele ve disipatif denklemler sistemine de uygulanabilirdir. Teorik çalışmalarımızın yanı sıra, elde ettiğimiz teorik bulguları doğrulayan pek çok sayısal simülasyon sunuyoruz.

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$C([0,L])$	the Banach space of all continuous functions endowed with the norm $\|\|\varphi\|\|_{C[0,L]} := \sup_{x\in[0,L]} \|\varphi(x)\|$.
$C^\infty(\Omega)$	space of functions defined on Ω that are continuously differentiable of all orders.
$L^p(0,L)$	the Banach space of all measurable functions on $(0,L)$ endowed with the norm $\|\|\varphi\|\|_{L^p(0,L)} := \left(\int_0^L \|\varphi(x)\|^p dx\right)^{\frac{1}{p}}$ for $1 \leq p < \infty$.
$L^\infty(0,L)$	the Banach space of all measurable functions on $(0,L)$ endowed with the norm $\|\|\varphi\|\|_{L^\infty(0,L)} := \operatorname{ess\,sup}_{x\in(0,L)}\|\varphi(x)\|$.
$L^2(0,L)$	the Hilbert space with the same norm defined in $L^p(0,L)$ for $p=2$ induced by the inner product $(\varphi,\psi)_2 := \int_0^L \varphi(x)\psi(x)dx$.
$W^{k,p}(0,L)$	the Banach space of all measurable functions whose weak derivatives up to the order k exists and belong to $L^p(0,L)$.
$H^k(0,L)$	the Hilbert space $W^{k,p}(0,L)$ with $p=2$.
$H_0^1(0,L)$	the space of functions which belong to $H^1(0,L)$ that vanish at the endpoints in the sense of traces.
X_T^s	the space $C([0,T];H^s(0,L)) \cap L^2(0,T;H^{s+1}(0,L))$ with $s \geq 0$.
e_j	$j-$th eigenfunction associated to the Sturm-Liouville operator under the homogeneous Dirichlet type boundary conditions.
λ_j	$j-$th eigenvalue associated to the Sturm-Liouville operator under the homogeneous Dirichlet type boundary conditions.
P_N	projection operator onto the space span$\{e_1,\dots,e_N\}$.
$\|\|\cdot\|\|_{A\to B}$	operator norm of a linear bounded operator defined on $A \to B$.
$a \lesssim b$	the inequality $a \leq cb$ where $c>0$ may depend on the fixed parameters of the problem under consideration, which is out of the interest.

CHAPTER 1

INTRODUCTION

Differential equations are mathematical models which describe several physical phenomena, underlying the behaviour of the nature from micro world to macro world. It is a mathematical relation which involves instantaneous rate of changes, so called derivatives, of some order that provides information in mathematical language for an associated event that occurs in the physical universe. In particular, one may wish to take into account the effect of spatially instantaneous rate of changes of the dynamic variable as time evolves. Such type of equations are called evolutionary partial differential equations (PDE). In order to determine the dynamic variable for all time, one usually needs to have an information at a certain moment. This information is called initial state, denoted by u_0, and typically that certain time is taken as $t = 0$, where t represents the temporal variable. Dynamical system is a pair $(X, S(t))$, which involves all possible evolutions of all initial states belonging to phase space X. That is, $(X, S(t))$ involves all of the trajectories $\{S(t)u_0 \,|\, u_0 \in X, t \geq 0\}$, where X is a Banach space endowed with the norm $\|\cdot\|_X$ and $S(t)$ denotes the evolution operator which satisfies certain properties (is given by Definition 2.1 in the next chapter). In particular, there may exist fixed point(s) of the operator $S(t)$. The trajectory associated to that fixed point is called steady state trajectory.

Since we are not living in a perfect world, there may be some physical events for which, behaviour of the state may not be as desired as time evolves. Therefore, one may aim to influence the system, i.e., to *control* the system, so as to achieve a desired goal. From mathematical point of view, this corresponds to influence trajectories lying in the associated dynamical system in a desired way, for instance asymptotically stabilizing the steady state trajectories. Under the assumption of existence of an evolution operator, we define the notion of asymptotic stability of a steady state trajectory as follows.

Definition 1.1 *Let ϕ be a steady state trajectory belongs to the dynamical system* $(X, S(t))$.

ϕ is said to be globally asymptotically stable if

$$\lim_{t\to\infty} \|S(t)u_0 - \phi\|_X = 0$$

holds for all $u_0 \in X$.

A stronger notion of asymptotically stability is exponential stability.

Definition 1.2 *Let ϕ be a steady state trajectory belongs to the dynamical system $(X, S(t))$. ϕ is said to be globally exponentially stable, if there exists a $\mu > 0$ such that*

$$\lim_{t\to\infty} (e^{\mu t}\|S(t)u_0 - \phi\|_X) = 0$$

holds for all $u_0 \in X$.

In particular, if the above definitions hold only for such u_0's that lie in a neighborhood of $\phi \in X$, then ϕ is said to be locally asymptotically (or exponentially) stable. Constructing a control input that stabilizes steady states exponentially is called exponential stabilization problem, or in short stabilization problem. Throughout the text, we will abuse the language and shortly use the word stabilization when we are refering to exponential stabilization. A stronger notion of stabilization problem is *rapid stabilization* problem, that is the case if one can construct a controller so that Definition 1.2 holds for a given prescribed value of $\mu > 0$.

There are many different approaches that have been developed to implement the control input to the differential model. One common way is to implement through the region of the evolution or a locally part of the region of the evolution. These are called internal control and locally supported control, respectively. Due to the physical nature of the problem, region of the evolution may have spatial bounds. If this is the case, then one may need to take into account the boundary effects, so called boundary conditions. In particular, if access to the interior of the region is restricted or is not available, then boundary may be the only location where one can implement the control input. Such types of controllers are referred as boundary controllers. Another common type is feedback controllers where the control action depends on the output of the evolutionary process.

These are generally prefered for eliminating errors that occur due to experimental process and steer trajectories to a desired state. Mathematically speaking, a control input that involves information about the state of the model is called feedback type control. A differential model under the influence of a non-feedback controller is called open-loop system, whereas if the controller is feedback type, then it is called closed-loop system.

1.1. Backstepping method

A powerful tool for constructing explicit boundary feedback laws that stabilizes equilibrium solutions to the evolutionary PDEs posed on a finite interval is backstepping method (Krstic and Smyshlyaev, 2008). This method was originally proposed for designing stabilizing controllers for system of ordinary differential equations (ODEs) in 1990's. In the following section, we give a brief explanation on the strategy by omitting technical details to understand origin of the word "*backstepping*" and also to understand how it is extended to PDEs (for a more technical discussion on this method for ODE systems, see e.g., (Kokotovic, 1992), (Krstic et. al., 1995)).

1.1.1. "Stepping back" to construct feedback laws for ODE systems

Suppose that we have an n dimensional ODE system with state $\mathbf{u} = [u_1\, u_2\, \cdots\, u_n]^T$, where each equation is explicit in u_i', for $i = 1, \ldots, n$. Suppose also that for each i with $1 \leq i \leq n-1$, right hand side of $i-$th equation only depends on $u_1, \ldots, u_{i+1}$ and in addition right hand side of the last equation involves $u_1, \ldots, u_n$ as well as the control input, g, which is currently unknown. Such special form of ODE system is called strict-feedback form. At first, assume that we want to stabilize zero solution of the first equation only. For a moment let us pretend the function u_2 involving in the first equation as a controller and imagine that we design it in the form $\alpha_1(u_1)$ in such a way that it cancels the terms that cause instanbilities (indeed one can do this by considering a suitable and known Lyapunov function). However, in reality, u_2 is also related with u_1 via the second equation of the

ODE system, therefore it cannot be taken as $\alpha_1(u_1)$. For this reason, this defines an error

$$w_2 := u_2 - \alpha_1(u_1) \tag{1.1}$$

together with we define $w_1 := u_1$. Consequently, we deduce $(w_1, w_2)-$system,

$$\begin{aligned} w_1' &= u_1', \\ w_2' &= u_2' - (\alpha_1(u_1))', \end{aligned} \tag{1.2}$$

where the terms on the right hand sides is replaced by considering the original model, so that the only unknowns of the system (1.2) are w_1, w_2. Now assume that we want to stabilize zero state of the $(u_1, u_2)-$system. Similar to the previous step, for a moment consider u_3, which is included in the second equation of $\mathbf{u}$ model, as a controller. Suppose it is designed by the formula $\alpha_2(u_1, u_2)$. But in reality, u_3 cannot be taken as $\alpha_2(u_1, u_2)$, because it is related with u_1 and u_2 via the third equation. Therefore this defines an error

$$w_3 := u_3 - \alpha_2(u_1, u_2).$$

Together with this present step, our new system, i.e. $(w_1, w_2, w_3)-$system, is as follows:

$$\begin{aligned} w_1' &= u_1', \\ w_2' &= u_2' - (\alpha_1(u_1))', \\ w_3' &= u_3' - (\alpha_2(u_1, u_2))'. \end{aligned}$$

Stepping iteratively *back* (where the name "backstepping" comes from), we eventually come to the last equation and obtain an explicit representation for the real controller, g. Note that this iterative process, at the end, yield another ODE system, where its state is $\mathbf{w} = [w_1\, w_2\, \cdots\, w_n]^T$. Consequently what we obtain that $\mathbf{u}-$model is transformed to

w−model

$$
\begin{aligned}
w_1' &= u_1', \\
w_2' &= u_2' - (\alpha_1(u_1))', \\
w_3' &= u_3' - (\alpha_2(u_1, u_2))', \\
&\vdots
\end{aligned}
$$

where the system involves the constructed controllers in each equation and u_i''s are replaced by considering the original **u**−model, so that the only unknowns of **w**− model are w_j's. Also the transformation is done via the errors,

$$
\begin{aligned}
w_1 &= u_1, \\
w_2 &= u_2 - \alpha_1(u_1), \\
w_3 &= u_3 - \alpha_2(u_1, u_2), \\
&\vdots
\end{aligned}
\tag{1.3}
$$

which can be viewed as change of variables. In view of this construction, following two observations are important on extending the method to PDEs.

(i) **u**−model as well as **w**−model can be considered as a semi-discrete form of a PDE, that is discretized in spatial variable and continuous in temporal variable. From this point of view, control g acts to the **u**−model from its boundary since it is included only in its last equation, whereas the designed controls, α_i's, acting to the **w**−model through interior since they are included in each of the equation except the first one.

(ii) We may see the **w**−model, as a transformation of the **u**−model by changing the variables via the system (1.3). Note that (1.3) is in lower triangular form and its continuous analog can be viewed as a Volterra type integral transformation of second kind.

1.1.2. Stabilizing backstepping controllers for PDEs

Based on the above observations, let us introduce backstepping strategy for PDEs in three steps.

(i) At first, we propose a *target* model, that corresonds to $w-$model mentioned in the previous section, so that, its zero equilibrium is readily known to be stable. Our first observation in the previous section suggests us to choose a model that involves a damping action, acting to the model through the interior of the region of the evolution. Thus, one trivial candidate for such model is a copy of our original model which additionally includes a weakly damping term on its main equation and that the boundary conditions are taken as homogeneous.

(ii) Next, we introduce so called *backstepping* transformation. Due to our second observation in the previous section, this is in general defined as a Volterra type integral transformation of second kind of the form

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy, \tag{1.4}$$

where u is the state of our original model, w is the state of our target model and k is the backstepping kernel which is currently unknown. Our task here is to find sufficient conditions on the backstepping kernel, so that, (1.4) maps our original model to the proposed target model in Step (i) pointwise. Once sufficient conditions on backstepping kernel are obtained and under those conditions its existence is proved, (1.4) becomes a relation which relates states of these two models. Without loss of generality, assuming that the control input acts through Dirichlet actuation, one takes $x = L$ on (1.4) and obtain an explicit representation for the feedback control input of the form

$$g(t) = \int_0^x k(L,y)u(y,t)dy \tag{1.5}$$

which acts to the model at $x = L$.

(iii) To deduce stabilization to the equilibrium of $u-$model via the result for $w-$model, one needs to obtain an inequality of the form $\|u\|_X \lesssim \|w\|_X$. This requires to prove that the backstepping transformation is invertible with a bounded inverse on a suitable functional space.

In view of this three-step strategy, backstepping method is extended to stabilizing equilibrium solutions to PDEs in early 2000's. First extensions are done for various kind of heat equations (see, e.g., (Bošković et. al., 2001), (Liu, 2003), (Smyshlyaev and Krstic, 2004)). The method is applied to other evolutionary PDEs with second order spatial derivatives such as linear Ginzburg-Landau equation (Aamo et. al., 2005), linear Schrödinger equation (Krstic et. al., 2007), unstable wave equations (Smyshlyaev et. al., 2010), (Smyshlyaev and Krstic, 2009), Fisher's equation (Yu et. al., 2014). By 2010's it was successfully applied to several higher order evolutionary PDEs that include third order dispersive terms, where a single control input acting from the one endpoint and two homogeneous boundary conditions imposed at the other end. Some of them are various kind of KdV equations (Cerpa and Coron, 2013), (Tang and Krstic, 2013), KdV-Burgers equations (Özsarı and Arabacı, 2019), linear and nonlinear higher-order Schrödinger equations (Batal et. al., 2021).

1.1.3. Observer based stabilizing controllers for PDEs

To design feedback type controllers, knowledge of the state at a certain moment is essential. However, in reality, measurement across the domain may not be available at any moment due to the physical nature of the problem. If this is the case, then constructing feedback type controllers become impossible. Nevertheless, measurement for the dynamic variable at the boundaries may still be available, say detectable through sensors placed at the boundaries. Utilizing from these partial measurements, one can derive a *state estimator*, denoted by $\hat{u}$ that estimates to the original state u asymptotically. As we detail in Section 1.3.2, derivation of $\hat{u}$ is achieved by constructing an abstract model, called observer model, that assumes $\hat{u}$ as its state variable. Note that this observer model also involves unknown functions, called observer gains. These are determined in such a way that, $\hat{u}$ approaches to u asymtotically in time. As a summary, our task here consists of two main steps:

(i) Construction of an asymptotically convergent observer model: This is done by deriving suitable observer gains.

(ii) Stabilization of zero equilibrium of observer model: This is done by constructing stabilizing boundary controllers.

Combination of these two steps, zero equilibrium to the original plant becomes stable under the influence of the controllers, involving $\hat{u}$ and which is also used to control observer model in Step (ii).

It is possible to design observers and observer based controllers for PDEs by applying the backstepping method separately to achieve the steps (i) and (ii), above. It was applied to several second order evolutionary PDEs (e.g., (Krstic et. al., 2007), (Smyshlyaev and Krstic, 2005)) and to third order evolutionary PDEs (e.g., (Batal et. al., 2021), (Marx and Cerpa, 2018), (Tang and Krstic, 2015)).

1.2. Finite dimensional long time behaviour

Researches in long time behaviour infinite dimensional dissipative systems are goes back to the pioneering work of (Foias and Prodi, 1967) on 2D Navier-Stokes equation. They proved that long time behaviour of solutions to 2D Navier-Stokes equations are determined by their projections onto a finite dimensional subspace provided that the dimension of the finite dimensional space is sufficiently large.

Definition 1.3 *Let X be a Hilbert space and P_N be a projection operator $P_N : X \to span\{e_1, \ldots, e_N\}$ where e_j is the $j-$th Fourier basis. Let u and v be two solutions of an evolutionary PDE with different initial states, u_0 and v_0, respectively. Then, $\{e_1, \ldots, e_N\}$ is said to be a set of determining modes, if*

$$\lim_{t\to\infty} \|P_N u(\cdot, t) - P_N v(\cdot, t)\|_X = 0$$

implies the following asymptotic behaviour

$$\lim_{t\to\infty} \|u(\cdot, t) - v(\cdot, t)\|_X = 0.$$

Here, the operator P_N is called determining projection operator.

In the following years, researchers introduced different kind of determining parameters such as determining nodes, determining volume elements and other kind of determining interpolant operators, determining functionals (see, e.g., (Cockburn et. al., 1997), (Foias and Temam, 1984), (Foias and Titi, 1991), (Jones and Titi, 1992a), (Jones and Titi, 1992b), (Jones and Titi, 1993)).

1.2.1. Determining operators as internal feedback controllers

The pioneering study for using determining operators as an internal feedback controllers was performed by (Azouani and Titi, 2014). They studied global stabilization of zero equilibrium to cubic reaction-diffusion equation posed on a finite interval. They considered a finite rank interpolant operator based on the local spatial averages of the form

$$I_N\varphi = \sum_{k=1}^{N} \bar{\varphi}_k \chi_{J_k}, \quad \bar{\varphi}_k = \frac{1}{|J_k|}\int_{J_k} \varphi(x)dx,$$

as an internal feedback controller, where $\cup_{k=1}^{N} J_k$ is a uniform disjoint partition of the spatial domain. They proved that for sufficiently large value of N, zero equilibrium to the associated closed-loop system becomes globally stable. Later, this idea is extended to stabilize other kind of dissipative PDEs such as various kind of dissipative wave equations (Kalantarov and Titi, 2016), complex Ginzburg-Landau equation (Kalantarova and Özsarı, 2017), original Burgers' equations and Burgers' equations with nonlocal nonlinearity (Gumus and Kalantarov, 2022), system of equations such as chevron pattern equations (Kalantarova et. al., 2021). A numerical study that illustrates this strategy for controlling various kind of dissipative PDEs such as Chafee-Infante equation and Kuramoto-Sivashinsky is performed in (Lunasin and Titi, 2017).

Remark 1.1 *Although it is out of the context of this thesis, we would like to note that the notion of determining operators as internal feedback controllers also have applications in other areas such as continuous data assimilation (see (Azouani et. al., 2014)) or in construction of a determining form that is, roughly speaking, an ODE which embeds the long time dynamics of an infinite dimensional dynamical system generated by a dissipative PDE model (see (Foias et. al., 2012) for 2D Navier-Stokes equations, (Jolly et. al., 2015)* *for damped nonlinear Schrödinger equation, (Jolly et. al., 2017) for damped nonlinear KdV equation).*

1.2.2. A short literature on other type of finite dimensional controllers

Our work on designing finite dimensional boundary controllers in Chapter 5 is inspired by the theory that lies behind the long time behaviour of infinite dimensional dissipative systems. However, we would also like to note that stabilizing equilibrium solutions to nonlinear parabolic PDEs by finite dimensional controllers are not restricted by the above literature. There are also some other pioneering works on 2D Navier-Stokes equations performed by other techniques (see (Barbu and Triggiani, 2004), (Fursikov, 2001)). The results obtained in these works and the pioneering results on finite dimensional asymptotic behavior of infinite-dimensional disspative systems that generate 2D Navier-Stokes equations, nonlinear parabolic equations, nonlinear damped wave equations and related systems of PDEs (see, e.g., (Babin and Vishik, 1992), (Foias et. al., 2001), (Ladyzhenskaya, 1991)) inspired further investigations on stabilization of 2D Navier-Stokes equations, nonlinear parabolic equations and some damped wave equations by internal and boundary finite-dimensional controllers (see (Chebotarev, 2010), (Liu et. al., 2016), (Munteanu, 2017) and references therein). A number of papers are devoted to the boundary stabilization of nonlinear parabolic equations by finite-dimensional controllers ((Barbu, 2013), (Munteanu, 2017), (Munteanu, 2019)). Another recent studies proposed in (Katz and Fridman, 2021) and (Lhachemi and Prieur, 2022) for heat equation, which rely on homogenizing the boundary conditions by changing the variables to transfer the boundary control action into the domain, and then decomposing the main equation with respect to the Fourier modes. This process eventually corresponds to stabilizing zero equilibrium to an ODE system.

1.3. Statement ot the problems

In this section, we state our problems and describe our motivation lying behind those problems.

1.3.1. Right endpoint feedback controller design for linear higher-order Schrödinger equation

Physical motivation. The higher-order nonlinear Schrödinger equation was first proposed in the studies (Kodama, 1985) and (Kodama and Hasegawa, 1987),s originally in the form

$$iu_t + \frac{1}{2}u_{xx} + |u|^2u + \epsilon i\left(\beta_1 u_{xxx} + \beta_2(|u|^2u)_x + \beta_3 u|u|_x^2\right) = 0, \quad (1.6)$$

which has been used to describe the evolution of femtosecond pulse propagation in a nonlinear optical fiber. In this equation the first term represents the evolution, second term is the group velocity dispersion, third term is self-phase modulation, fourth term is the higher order linear dispersive term, fifth term is related to self-steepening and sixth term is related to self-frequency shift due to the stimulated Raman scattering. In the absence of the last three terms, the model becomes classical nonlinear Schrödinger equation which describes slowly varying wave envelopes in a dispersive medium. It has applications in several fields of physics such as plasma physics, solid-state physics, nonlinear optics. It also describes the propagation of picosecond optical pulse in a mono-mode fiber (Xu et. al., 2002). However, for the pulses in the femtosecond regime, the nonlinear Schrödinger equation becomes inadequate and higher order nonlinear and dispersive terms become crucial. See (Agrawaal, 2013) for a detailed discussion of the higher order effects upon the propagation of an optical pulse.

From a practical point of view, stabilization of equilibrium may be necessary in order to prevent any chaotic behaviour during the transmission and propagation of optical pulses. Our study offers a practical solution to this issue because: (i) the stabilization is fast, i.e. the absorption effect is exponential and (ii) the control acts only from the boundary which is desirable when access to medium is limited.

Mathematical motivation. A well-posed third order evolutionary problem posed on a finite interval requires two boundary conditions imposed at one endpoint and one boundary condition imposed at the opposite end. Classical backstepping strategy we described in Section 1.1.2 works very well on third order evolutionary PDEs, if a single boundary control acts to the system at one endpoint and two homogeneous boundary conditions are imposed at the other end. Conversely, if a single homogeneous boundary condition is imposed at one end and control input(s) acting from the opposite end, then the standard backstepping strategy fails as we will detail this in Chapter 3. This issue was first addressed in (Cerpa and Coron, 2013) in the case of KdV equation as an open problem and then treated by (Coron and Lü, 2014). They assumed Dirichlet type homogeneous boundary conditions imposed to the problem at both end and considered a single Neumann type boundary controller acting from the right endpoint. They proved rapid stabilization of the zero equilibrium to the closed-loop system under the assumption that, length of the spatial domain L, does not belong to the following countable set

$$\left\{2\pi\sqrt{\frac{k^2+kl+l^2}{3}} : k,l \in \mathbb{Z}^+\right\}. \tag{1.7}$$

Note that this set, referred as set of *critical* length of intervals, has a particular importance in the context of mathematical control theory. More precisely, in the absence of controller(s) and if L is critical, then the linear model assumes a nontrivial time independent solution. This means that not all solutions decay to zero and therefore zero equilibrium is unstable. Moreover, it can be proved that the linear model with right endpoint controllers is not exact controllable. That is, there exists an initial state u_0 and a final state u_T at a fixed given $T > 0$ such that, no matter what the right endpoint control input is chosen, $S(T)u_0 = u_T$ does not hold (see (Lionel, 1997) for right endpoint Neumann controller case and (Glass and Guerrero, 2010) for right endpoint Dirichlet controller case). Based on this situation, it is an important task to study stabilization problem with right endpoint controllers, in particular in the case of critical length of intervals. Later (Özsarı and Batal, 2019) treated the problem in the case of critical length of intervals and proved exponential stabilization of zero equilibrium for some decay rate under the right endpoint controllers.

Problem statement. Let us introduce the linear higher-order Schöringer equation

$$\begin{cases} iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = 0, & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = u(L,t) = u_x(L,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,L), \end{cases} \tag{1.8}$$

where $\beta > 0$, $\alpha, \delta \in \mathbb{R}$. The situation we explained in the case of linear KdV equation also exists for the linear higher-order Schöringer equation. That is, there exists a set of length of intervals, given by

$$\mathcal{N} := \left\{ 2\pi\beta \sqrt{\frac{k^2 + kl + l^2}{3\beta\delta + \alpha^2}} : k, l \in \mathbb{Z}^+ \right\}, \tag{1.9}$$

for which, the model (1.8) may assume a nontrivial time independent solution (see (Ceballos et. al., 2005), (da Silva and Vasconcellos, 2011)). (e.g.: Take $\beta = 1$, $\alpha = 2$ and $\delta = 8$ with $k = 1$ and $l = 2$, we obtain $L = \pi \in \mathcal{N}$. Then the initial state $u_0(x) = 3 - e^{4ix} - 2e^{-2ix}$ does not vary in time and become a time independent solution of (1.8)). So, we are interested in constructing suitable feedback controllers $h_0(t) = h_0(u(\cdot,t))$ and $h_1(t) = h_1(u(\cdot,t))$ to make sure that zero equilibrium of the closed loop system

$$\begin{cases} iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = 0, & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = 0,\ u(L,t) = h_0(t),\ u_x(L,t) = h_1(t), & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,L), \end{cases} \tag{1.10}$$

becomes globally exponentially stable on domains either $L \in \mathcal{N}$ or $L \notin \mathcal{N}$. More precisely, we consider the problem below:

Problem 1.1 *Given $L > 0$, construct right endpoint feedback control laws $h_0(t) = h_0(u(\cdot,t))$ and $h_1(t) = h_1(u(\cdot,t))$ such that, zero equilibrium of the closed-loop system* (1.10) *is exponentially stable for some decay rate $\lambda > 0$.*

1.3.2. Observer based right endpoint controller design for linear higher-order Schrödinger equation

In this problem, we consider the case where fully measurement on the state of model (1.10) across the domain is not available. Therefore, constructing feedback type controllers become impossible. However, we suppose that first and second order boundary traces $y_1(t) = u_x(0,t)$, $y_2(t) = u_{xx}(0,t)$ are known, say detectable through boundary sensors. In order to deal with the robustness of the state, we introduce an observer model, which involves the observed boundary measurements as follows:

$$\begin{cases} i\hat{u}_t + i\beta\hat{u}_{xxx} + \alpha\hat{u}_{xx} + i\delta\hat{u}_x - p_1(x)(y_1(t) - \hat{u}_x(0,t)) & \\ -p_2(x)(y_2(t) - \hat{u}_{xx}(0,t)) = 0, & (x,t) \in (0,L) \times (0,T), \\ \hat{u}(0,t) = 0,\ \hat{u}(L,t) = h_0(t),\ \hat{u}_x(L,t) = h_1(t), & t \in (0,T), \\ \hat{u}(x,0) = \hat{u}_0(x), & x \in (0,L). \end{cases} \tag{1.11}$$

In this model p_1 and p_2 are called observer gains, and they are currently unknown. They are supposed to be obtained in such a way that $\hat{u} \to u$ uniformly and exponentially as $t \to \infty$. Feedback controllers $h_0(t) = h_0(\hat{u}(\cdot,t))$, $h_1 = h_1(\hat{u}(\cdot,t))$ will be constructed in such a way that zero equilibrium of the closed loop system (1.11) becomes exponentially stable. Note that these controllers will also be applied to the original plant (1.10). These are the steps (i) and (ii), we explained in Section 1.1.3. Combining these steps, we obtain exponential stabilization of zero equilibrium to our original plant under the controllers $h_0(t) = h_0(\hat{u}(\cdot,t))$, $h_1 = h_1(\hat{u}(\cdot,t))$.

Now applying the standard backstepping strategy to perform the steps (i) and (ii), one ends up with similar issues that we mentioned above in Section 1.3.1, due to the placement of the controllers. These will be detailed in Section 4.1. As in the case of controller design problem, this issue was first addressed in the case of KdV equation and then was solved by (Batal and Özsarı, 2019). So for the case of linear higher-order Schrödinger equation, either $L \in \mathcal{N}$ or $L \notin \mathcal{N}$, our aim is to design observer gains and observer based controllers so that, zero equilibrium of original plant becomes globally exponentially stable. More precisely, we are interested in the following problem.

Problem 1.2 *Given $L > 0$, find observer gains p_1, p_2, and right endpoint control laws $h_0(t) = h_0(\hat{u}(\cdot,t))$, $h_1(t) = h_1(\hat{u}(\cdot,t))$ that involve state of the observer model* (1.11) *such that, zero equilibrium of the closed-loop system* (1.10) *is exponentially stable for some decay rate $\lambda > 0$.*

1.3.3. Finite-dimensional backstepping type controllers involving determining operators

Motivation. Theory behind finite-dimensional long time behaviour of infinite dimensional dissipative systems and studies on controlling long time behaviour of various type of dissipative systems via finite-dimensional internal feedback controllers motivate us to ask ourselves whether we can construct a stabilizing boundary controllers involving only finitely many determining parameters. To this end, as a canonical example, we consider the reaction-diffusion model with cubic nonlinearity

$$\begin{cases} u_t - \nu u_{xx} - \alpha u + u^3 = 0, & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = 0,\ u(L,t) = h(t), & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,L). \end{cases} \tag{1.12}$$

Here $\nu, \alpha > 0$ are given constant values. $h(t) = h(u(\cdot,t))$ is a soughtafter feedback control that acts through Dirichlet actuation at $x = L$ and involves only finitely many Fourier sine modes of u. In the absence of controller and for certain values of ν, α and L, zero equilibrium may be either asymptotically stable or unstable. To be more precise, if $\nu\lambda_1 - \alpha \leq 0$, where λ_1 is the first eigenvalue to the Sturm-Liouville operator subject to the Dirichlet type homogeneous boundary conditions, (1.12) has a unique equilibrium solution and it is $u \equiv 0$. For this case, it is asymptotically stable. Conversely, if $\nu\lambda_1 - \alpha < 0$, then there exist at least two nontrivial equilibrium solutions, exactly two of which are asymptotically stable, and all solutions bifurcate from the zero equilibrium. Therefore $u \equiv 0$ is no more stable.

Remark 1.2 *Since our problem is to stabilize unstable zero equilibrium to the reaction* *diffusion model* (1.13),

$$\begin{cases} u_t - \nu u_{xx} - \alpha u + u^3 = 0, & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = u(L,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,L), \end{cases} \tag{1.13}$$

we would like to state a few words on its asymptotic behaviour by refering to the related literature. First, note that solutions of (1.13) *neither blow up in a finite time nor grow up to infinity as* $t \to \infty$*, i.e., it converges to some equilibrium solution as* $t \to \infty$ *(Matano, 1978). Therefore, to study on possible nontrivial equilibrium solutions, one can investigate the following steady state problem*

$$\begin{cases} -\phi'' - \alpha\phi + \phi^3 = 0, \quad in\ (0,L), \\ \phi(0) = \phi(L) = 0. \end{cases} \tag{1.14}$$

For a given ν, α, L*, proving existence of a nontrivial solution of* (1.14) *and finding the number of them are based on applying phase-portrait ideas. It is proved in (Chafee and Infante, 1974) that number of solutions of* (1.14) *are exactly related with the coefficients* ν, α *and the eigenvalues of the operator* $-\frac{d^2}{dx^2}$ *subject to the homogeneous Dirichlet type boundary conditions. (They actually studied the problem with a main equation* $-\phi'' - \lambda f(\phi)$ *on* $(0,\pi)$*, but the properties that they assume for* $f(\phi)$ *also holds for* $(\alpha\phi - \phi^3)$*, which makes our model is a particular case of their model.) More precisely, if*

$$\lambda_n < \frac{\alpha}{\nu} \leq \lambda_{n+1},$$

then it was proved that there are exactly $(2n+1)$ *equilirium solutions where one of them is the trivial one, two of them is asymptotically stable, and the others together with the trivial one is unstable. In particular, (Sattinger, 1971) showed existence of stable ones, denoted by* ϕ_1^+ *and* ϕ_1^-*, by a constructive method so called monotone methods. It is based on the concept of upper and lower solutions, and comparison theorems for elliptic and* *parabolic PDEs. Briefly speaking, suppose that φ and ψ are lower and upper solutions to steady state model, respectively, and introduce a specially constructed monotone, compact operator T. Consider sequences $\{T^m\varphi\}_{m=1}^{\infty}$ and $\{T^m\psi\}_{m=1}^{\infty}$, constructed by applying iteratively T to ϕ and ψ, respectively. Then he proved if $T^n\varphi \nearrow \phi$ and $T^n\psi \searrow \phi$, then the limiting function ϕ is an equilibrium solution and is asymptotically stable for time dependent problem. One can apply this idea to* (1.13) *together with* (1.14) *and conclude that a sequence initialized by any positive upper solution no matter how close it is to zero equilibrium and a sequence initialized by any positive lower solution no matter how large it is, converge to same function, that is ϕ_1^+. Asymptotic stability of ϕ_1^- can be also characterized in an analogous way. Note that from this analysis, one also reveals that zero equilibrium is unstable. For a more detailed discussion on this subject we refer the reader to (Hale, 1998), (Henry, 1981), (Robinson, 2001), in particular for comparison theorems and monotone methods (Protter and Weinberger, 1984), (Sattinger, 1973), (Smoller, 1994).*

Problem statement. Based on the above discussion, our aim is to construct a feedback law of the form

$$h(t) = h(P_N u(\cdot, t)) = \int_0^L \xi(y)\Gamma[P_N u](y, t)dy,$$

with Dirichlet actuation at $x = L$ such that, all solutions are steered asymptotically to zero. Here, Γ is a linear bounded operator on a certain L^2-based functional space, ξ is a suitable smooth function to be constructed, and P_N is the projection operator

$$P_N\varphi(x) = \sum_{j=1}^{N} e_j(x)\left(e_j(\cdot), \varphi(\cdot)\right)_2, \quad e_j(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{j\pi x}{L}\right).$$

More precisely, we want to solve the following problem:

Problem 1.3 *Let $\nu, \alpha > 0$ be such that $\nu\lambda_1 - \alpha < 0$. For a given $\gamma > 0$, can you find $N > 0$ and construct a feedback control law $h(t) = h(P_N u(\cdot, t))$ acting from the boundary that uses only the first N Fourier sine modes of u such that the zero equilibrium to the closed loop system* (1.12) *becomes exponentially stable with a prescribed decay rate γ?*

Note that the above question, called the rapid stabilization problem, asks if all solutions decays to zero equilibrium with a prescribed decay rate. Another interesting problem is to determine the minimal number of Fourier sine modes of u to gain exponential stabilization of zero equilibrium for some unprescribed and not necessarily large decay rate. More precisely, we also want to solve the following problem:

Problem 1.4 *Let $\nu, \alpha > 0$ be such that $\nu\lambda_1 - \alpha < 0$. What is the minimum value of N for which, zero equilibrium to the closed loop system* (1.12) *becomes exponentially stable for some decay rate $\gamma > 0$ when the boundary feedback is of the form $h(t) = h(P_N u(\cdot, t))$?*

We would like to emphasize that the reaction-diffusion model serves a only a canonical example but our strategy can be applied to various kind of dissipative PDEs for stabilizing their zero equilibrium.

1.4. Thesis layout

This thesis consists of six chapters including this chapter. The layout starting from Chapter 2 is as follows.

In Chapter 2, we give some definitions, introduce some tools, theorems, lemmas and well-known inequalities that we utilize throughout the thesis.

In Chapter 3, we focus on the controller design problem that we stated by Problem 1.1. First in Section 3.1, we explain how the standard backstepping strategy works on an example with a left endpoint controller and detail why it fails in the case of right endpoint controllers. Then we introduce how we modifty the method. Next, in Section 3.2, we study the stabilization problem in this order: (i) we prove existence of a smooth backstepping kernel, (ii) we prove exponential decay of solutions to target model, namely $w-$model, (iii) we state invertibility of the backstepping transformation. These eventually yields a positive answer to the Problem 1.1. Note that modification of the backstepping strategy we introduced in Section 3.1 yields several difficulties from well-posedness point of view, which is the subject of Section 3.3. Local and global well-posedness require to obtain some time-space regularity estimate, which also requires to reveal smoothing properties for associated Cauchy problems and an initial-boundary value problem with inhomogeneous boundary conditions. A detailed and delicate analysis for these problems will be carried out via transform methods by applying Fourier transform and Laplace transform, respectively. Once we derive the global well-posedness result, we summarize our well-posedness and stabilization results in Theorem 3.1 at the end of Section Section 3.3. Finally in Section 3.4, we provide a numerical algorithm and present a numerical experiment that verifies our theoretical decay result.

In Chapter 4, we focus on the observer design problem that we state in Problem 1.2. Topics in this chapter will be in the same order as in the Chapter 3. However, stabilization and well-posedness analysis is now carried out through observer model and so called *error* model, which is formed by defining its state $\tilde{u} := u - \hat{u}$ and subtracting the observer model from the original plant. In Section 4.1, we detail our strategy as well as we obtain candidates for the observer gains p_1, p_2. In Section 4.2 we prove that for such observer gains, state of the observer model indeed converges to the state of the original model exponentially, by proving that $\tilde{u}(\cdot, t) \to 0$ uniformly and exponentially in time. We also construct boundary controllers that stabilizies zero equilibrium to the observer model. Both analyses are carried out by the backstepping strategy that is the modified one in the sense that we introduced in Chapter 3. These two results imply stabilization of the zero equilibrium to the original plant along with the same controllers. Well-posedness analysis of plant-observer-error system is carried out in Section 4.3. Then, at the end of the same section, we state our main result of Chapter 4 in Theorem 4.1. Finally, in Section 4.4, we provide our numerical algorithm together with an experiment verifying our theoretical decay result.

Chapter 5 is devoted to the Problem 1.3 and Problem 1.4. We introduce a finite dimensional version of a backstepping controller design that involves determining projection operator. That is, now the designed controller involves only finitely many Fourier sine modes of the state rather that full state. First, we introduce our target model as well as the backstepping transformation in a novel way for our purposes, and then describe our motivation behind these choices. Then in Section 5.1, we precisely calculate the sufficient number of Fourier modes so that, unstable zero equilibrium to the both linear and nonlinear problem becomes stable with a prescribed exponential decay rate. In particular, in Section 5.1.3, we precisely show that the minimal number of Fourier modes to gain stabilization of zero equilibrium is exactly equal to the instability level of the problem, i.e., the number of the unstable Fourier modes. Section 5.2 is devoted to the well-posedness of the linear and nonlinear models. Finally, in Section 5.3, we present various kind of numerical simulations for linear and nonlinear models, verifying our theoretical results for both Problem 1.3 and Problem 1.4.

In Chapter 6, we state our concluding remarks, some drawbacks regarding the strategies we performed and state some problems that we are willing to study in the future.

CHAPTER 2

PRELIMINARIES

In this part, we present some definitions, useful inequalities and lemmas that we utilize throughout the thesis.

As an essential tool for proving existence and uniqueness of the solutions of a differential system, we briefly present the operator semigroup theory.

Definition 2.1 (C_0 semigroup) *Let X be a Banach space. A one parameter family $S(t)$, $0 \leq t < \infty$, of bounded linear operators from X into X is a strongly continuous semigroup of bounded linear operators on X if*

(i) $S(0) = I$,

(ii) $S(t + \tau) = S(t)S(\tau)$ *for every* $t, \tau \geq 0$,

and

(iii) $\lim_{t\searrow 0} S(t)u = u$, *for every* $u \in X$.

Definition 2.2 (Infinitesimal generator of the C_0 semigroup) *The infinitesimal generator A of a C_0 semigroup is a linear operator defined by*

$$Au := \lim_{t\searrow 0} \frac{S(t)u - u}{t}, \tag{2.1}$$

with domain $D(A)$, that consists of all $u \in X$ for which the limit (2.1) *exists.*

A characterization of the existence of an intinitesimal generator A, of a C_0 semigroup is due to following lemma which is a corollary of Lumer-Phillips Theorem (see Theorem 4.3 and Corollary 4.4 of Chapter 1 in (Pazy, 1983)).

Lemma 2.1 ((Pazy, 1983)) *Let A be a densely defined closed linear operator. If both A and its adjoint A^* with domain $D(A^*)$ are dissipative, then A is the infinitesimal generator of a C_0 semigroup of contractions on X.*

Lemma 2.1 has a particular importance on proving existence of a unique classical solution to the following abstract Cauchy problem

$$\begin{cases} u' = Au, & t > 0, \\ u(0) = \phi. \end{cases} \tag{2.2}$$

That is, $u : [0, \infty) \to X$ is a classical solution of (2.2), if for every for every $\phi \in D(A)$, u is continuous on $[0, \infty)$, differentiable on $(0, \infty)$, $u(t) := S(t)u_0 \in D(A)$ for each $t > 0$ and u satisfies (2.2).

Remark 2.1 *Due to the physical nature of the problem, in general $\phi \in D(A)$ may not be true. If this is the case, then one may need to define a solution in a general setting, i.e., in a more general space. One way to define such solutions is as follows: Let $\phi \in X$ and I be the index set. Provided that $D(A)$ is dense in X, let $\{\phi_n\}_{n\in I} \subset D(A)$ be a sequence such that $\phi_n \to \phi$ with respect to the metric induced by the norm $\|\cdot\|_X$. In particular, for each initial state ϕ_n, continuous dependence result of a solution can be obtained (e.g., by multipliers) as*

$$\sup_{0\le t<\infty} \|S(t)\phi_n\|_X \lesssim \|\phi_n\|_X. \tag{2.3}$$

Set $u_n(t) := S(t)\phi_n$ for each $t \geq 0$. To show that $\{u_n\}$ converges in $C([0, \infty); X)$, it is enough to show that $\{u_n\}$ is Cauchy in $C([0, \infty); X)$. This follows from the inequality (2.3). *Denote the limiting function as u.*

Definition 2.3 *The function $u \in C([0, \infty); X)$, constructed as in Remark 2.1 with representation $u(t) := S(t)\phi$, is called mild solution.*

Such type of solutions given in Definition 2.3 solve (2.2) in the sense of distributions.

Regarding the problem with an interior homogeneous source, say $F(u)$,

$$\begin{cases} u' = Au + F(u), & t > 0, \\ u(0) = 0. \end{cases} \tag{2.4}$$

which can be either linear or nonlinear in u, one can apply Duhamel's principle and obtain the following relation for the associated mild solution:

$$u(t) = \int_0^t S(t-\tau)F(u(\tau))d\tau. \tag{2.5}$$

Note that this relation involves u implicitly. Therefore, proving existence of a function that satisfies the relation (2.5) is equivalent to prove existence of possible fixed points of the operator

$$\Psi[\varphi] := \int_0^t S(t-\tau)F(\varphi(\tau))d\tau.$$

A characterization for a unique fixed point of operator Ψ is given by the Banach fixed point theorem (see Theorem 1 of Section 9.2 in (Evans, 1998)).

Theorem 2.1 (Banach fixed point theorem) *Let X be a Banach space and $u, v \in X$. Assume $\Psi : X \to X$ is a nonlinear mapping and suppose that*

$$||\Psi[u] - \Psi[v]||_X \leq \gamma||u - v||_X$$

for some constant $0 < \gamma < 1$, that is Ψ is a strict contraction. Then Ψ has a unique fixed point.

Throughout well-posedness analyses in each chapter, contraction property of an operator that appears via the Duhamel's principle can be achieved only for sufficiently small values of t. In other words, application of Therem 2.1 provides existence of a unique mild solution for an associated PDE model only up to a some fixed temporal value, say $T_{\max}$ called maximal time of existence. Such solutions are refered as local (in time) solutions. To extend a local solutions globally, one can apply energy estimates and multipliers to deduce that $||u||_X$ remains finite as $t \to T_{\max}^-$ for any $T_{\max}$.

Operator theoretic tools such as operator semigroup theory may become inadequate to reveal some further properties (e.g., smoothing properties as we require in Section 3.3.1) of the solution. Therefore, in some cases, one may need to consult transform methods which yields a more useful solution representation formula for an associated PDE model. One of them is the Fourier transform, in particular a useful tool for analyzing Cauchy problems.

Definition 2.4 (Fourier transform) *The Fourier transform* $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ *of a function* $f \in L^1(\mathbb{R})$ *is defined by*

$$(\mathcal{F} f)(\xi) := \int_{\mathbb{R}} f(x)e^{-i\xi x}dx. \tag{2.6}$$

Remark 2.2 *Throughout the thesis, mostly we will deal with* L^2-*functions defined on a finite interval. Therefore, to be able to apply the Fourier transform* (2.6) *while performing the well-posedness analysis, the first thing to do is to extend their domains to* $\mathbb{R}$ *provided that the extended function is still square integrable. This can be done for instance by considering a zero extension. Since an* L^2-*function on a finite interval is already integrable, the extended function belongs to* $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *and Definition 2.4 is applicable.*

A key tool that we utilize while performing well-posedness analyses is Plancherel's identity, which also provides that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $(\mathcal{F} f) \in L^2(\mathbb{R})$ and the Fourier transform defined on the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is a linear isometry into $L^2(\mathbb{R})$ up to a multiplicative constant.

Theorem 2.2 (Plancherel's identity) *Assume* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. *Then* $(\mathcal{F} f) \in L^2(\mathbb{R})$ *and*

$$\int_{\mathbb{R}} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{\mathbb{R}} |(\mathcal{F} f)(\xi)|^2 \, d\xi. \tag{2.7}$$

Obtaining a solution representation via the Fourier transform requires to recover the function f from its Fourier transform $(\mathcal{F} f)$. If $f, (\mathcal{F} f) \in L^1(\mathbb{R})$, then it can be done by the following inversion formula

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi}(\mathcal{F} f)(\xi)d\xi. \tag{2.8}$$

Remark 2.3 *In view of Definition 2.4 and Theorem 2.2, we do not know whether* $(\mathcal{F} f)$ *is integrable or not. Therefore, for a given* $(\mathcal{F} f)$*, it may not be possible to recover* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *by using the inversion formula* (2.8)*. To remedy this situation, note that it is possible to define the Fourier transform of an* L^2-*function which does not necessarily belong to* $L^1(\mathbb{R})$*. Indeed, since* $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *is dense in* $L^2(\mathbb{R})$ *and once proving that the space* $\left\{\mathcal{F} f \,|\, f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})\right\}$ *is also dense in* $L^2(\mathbb{R})$*, one can show by using density* *arguments and completeness of L^2-space that the isometry of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ can be uniquely extended to an isometry of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. That is, there exists a unique surjective continuous linear operator $\mathcal{F}^*$ from $L^2(\mathbb{R})$ onto itself such that, $(\mathcal{F}^* f) = (\mathcal{F} f)$ for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and (2.7) holds for $f \in L^2(\mathbb{R})$ (see Theorem 9.3 in (Rudin, 1987) for details). Moreover, one can prove that*

$$\left\| \int_{-\gamma}^{\gamma} e^{-i\xi x} f(x)dx - (\mathcal{F}^* f) \right\|_{L^2(\mathbb{R})} \to 0, \quad \left\| \int_{-\gamma}^{\gamma} e^{i\xi x} (\mathcal{F}^* f)(\xi) d\xi - f \right\|_{L^2(\mathbb{R})} \to 0$$

as $\gamma \to \infty$. In view of this setting, one can recover $f \in L^2(\mathbb{R})$ for a given $(\mathcal{F}^* f) \in L^2(\mathbb{R})$.

Throughout the thesis, we abuse the notation and use $\hat{(\cdot)}$ for the Fourier transform for a general L^2-function while we use the representation given in Definition 2.4.

L^2-based Sobolev spaces of an integer order m, are induced by the norm

$$\|\varphi\|_{H^m(\mathbb{R})} := \left(\sum_{k=1}^{m} \|\varphi^{(k)}\|_{L^2(\mathbb{R})} \right)^{\frac{1}{2}},$$

where φ is a measurable function and $(\cdot)^{(k)}$ denotes the $k-$th weak derivative. An equivalent representation for this norm can be characterized by virtue of the Fourier transform in the following way

$$\|\varphi\|_{H^m(\mathbb{R})} = \left(\int_{\mathbb{R}} (1 + \xi^2)^m |\hat{\varphi}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

This characterization allows us to define Sobolev spaces in any order, $m \geq 0$.

In Section 3.3.1 and Section 4.3.1, we will perform well-posedness analysis for initial-boundary value problems posed on a finite interval with inhomogeneous boundary condition(s). However, representation formula for a solution obtained by the finite interval version of the Fourier transform

$$\hat{f}(\xi) := \frac{1}{2\pi} \int_0^L f(x) e^{-i\xi x} dx \tag{2.9}$$

may not be uniformly convergent at the boundary of the spatial domain, i.e., does not become a representation formula for solution. Therefore, in this case, instead of the Fourier transform, we obtain a solution representation via the Laplace transform in the variable t and the associated well-posedness analysis will be carried out by using that representation.

Definition 2.5 (Laplace transformation) *Laplace transformation of a function f is defined by*

$$\tilde{f}(s) := \int_0^\infty e^{-st} f(t)dt. \tag{2.10}$$

If it is known that the Laplace integral converges for some s_0, then s can be chosen in the right half plane $\mathrm{Re}(s) > \mathrm{Re}(s_0)$, called domain of absolute convergence, and corresponding Laplace transform $\tilde{f}(s)$ becomes bounded in that half plane (see Theorem 3.1 and Theorem 3.2 in (Doetsch, 1974)).

For a given $\tilde{f}$, f can be recovered by the inverse of the Laplace transformation uniquely up to a measure zero.

Definition 2.6 (Inverse Laplace transformation) *The inverse Laplace transformation of $\tilde{f}$ is given by the Bromwich integral*

$$f(t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{st} \tilde{f}(s)ds, \tag{2.11}$$

where r is a real value so that the vertical path $(r - i\infty, r + i\infty)$ lies in the domain of absolute convergence.

Throughout well-posedness analysis, we sometimes require that initial and boundary conditions are compatible in the following sense.

Definition 2.7 (Compatibility) *Let $\phi \in H^3(0, L)$, $\psi \in H^1(0, T)$. We say that the couple (ϕ, ψ) satisfies compatibility conditions, if*

$$\phi(0) = 0, \quad \phi(L) = 0, \quad \phi'(L) = \psi(0). \tag{2.12}$$

Sometimes we require higher compatibility in the following sense.

Definition 2.8 (Higher compatibility) *Let $\phi \in H^6(0, L)$, $\psi \in H^2(0, T)$. We say that the* *couple (ϕ, ψ) satisfies higher compatibility conditions, if it satisfies compatibility conditions* (2.12) *and, also satisfies*

$$(-\beta\tilde{w}_0''' + i\alpha\tilde{w}_0'' - \delta\tilde{w}_0' - r\tilde{w}_0)(x) = 0, \quad x = 0, L, \tag{2.13}$$

and

$$(-\beta\tilde{w}_0''' + i\alpha\tilde{w}_0'' - \delta\tilde{w}_0' - r\tilde{w}_0)'(L) = \psi'(0). \tag{2.14}$$

Following inequalities and theorems will be useful while carrying out the energy estimates and deriving uniform bounds.

Theorem 2.3 (Lebesgue Dominated Convergence Theorem) *Let $\{f_n\}_{n\in I}$ be a sequence of measurable functions on $\mathbb{R}$ such that $f_n \to f$ pointwise almost everywhere. If there exists a function $g \in L^1(\mathbb{R})$ such that $|f_n| \leq |g|$ holds for all $n \in I$, then $f \in L^1(\mathbb{R})$ and* $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} f(x)dx$.

Cauchy-Schwarz inequality: For all $u, v \in L^2(0, L)$,

$$\left| \int_0^L u(x)v(x)dx \right| \leq \|u\|_{L^2(0,L)} \|v\|_{L^2(0,L)}.$$

Young's inequality with ε: For each $a, b > 0$ and $\varepsilon > 0$,

$$ab \leq \frac{\varepsilon a^p}{p} + \frac{b^q}{\varepsilon^{p/q} q},$$

where $p, q > 0$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. In particular, if $p = q = 2$, then the corresponding inequality is referred as Cauchy's inequality with ε.

Poincaré inequality: For all $u \in \left\{\varphi \in H^1(0, L) \,|\, \varphi(0) = \varphi(L) = 0 \text{ or } \int_0^L \varphi(x)dx = 0\right\}$,

$$\|u\|_{L^2(0,L)}^2 \leq \lambda_1^{-1} \|u'\|_{L^2(0,L)}^2.$$

Poincaré type inequality: For all $u \in H_0^1(0, L)$,

$$\|u - P_N u\|_{L^2(0,L)}^2 \leq \lambda_{N+1}^{-1} \|u'\|_{L^2(0,L)}^2, \quad \lambda_{N+1} = (N+1)^2 \lambda_1.$$

Gagliardo-Nirenberg inequality: Let $u \in H_0^1(0, L)$, $p \geq 2$, $\alpha = \frac{1}{2} - \frac{1}{p}$. Then

$$\|u\|_{L^p(0,L)} \leq c \|u'\|^{\alpha} \|u\|^{1-\alpha},$$

where $c > 0$ depends on p, α and L.

Gronwall's inequality: Let $u(t)$ and $\alpha(t)$ be nonnegative continuous functions for $t \geq 0$ that satisfy the inegral inequality

$$u(t) \leq C + \int_0^t \alpha(\tau) u(\tau) d\tau, \quad \forall t \geq 0,$$

where $C \geq 0$. Then

$$u(t) \leq C e^{\int_0^t \alpha(\tau) d\tau}, \quad \forall t \geq 0.$$

CHAPTER 3

CONTROLLER DESIGN FOR HIGHER-ORDER SCHRÖDINGER EQUATION

In this chapter, our purpose is to design feedback type boundary controllers h_0, h_1 for the linear higher-order Schrödinger equation

$$\begin{cases} iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = 0, & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = 0, \quad u(L,t) = h_0(t),\ u_x(L,t) = h_1(t), & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,L), \end{cases} \tag{3.1}$$

which are placed at the right endpoint of the spatial interval. The designed controllers ensure that the zero equilibrium of the closed-loop system (3.1) become globally exponentially stable either in the case of critical or noncritical length of intervals. As we detailed in Section 1.3.1, location of the controller(s) has importance on application of backstepping method to higer-order Schrödinger equation. So in Section 3.1, we explain how the situation differs from mathematical point of view, if we place the controller(s) at different endpoints. In Section 3.2 we show stabilization of zero equilibrium under the designed right endpoint controllers h_0, h_1, and in Section 3.3 we prove that the closed-loop system (3.1) under the influence of these controllers, is well-posed in the Hadamard sense. Finally, in Section 3.4, we present a numerical experiment which verifies our theoretical results.

The results in this chapter were published in a part of our study (Özsarı and Yılmaz, 2022).

3.1. Backstepping method

Three step strategy of backstepping method we described in Section 1.1.2 works well for third order evolutionary equations, if a single boundary feedback control is located at one endpoint of the interval and number of homogeneous boundary conditions located at opposite endpoint are two. Conversely, if there is a single homogeneous boundary condition imposed at one endpoint and control input(s) acting from the other end, the situation becomes different from the mathematical point of view as we explain below in Section 3.1.1 and Section 3.1.2. Without loss of generality, in the first situation control input acting to the system at left end so we name this case as *left endpoint controller*, whereas the latter case is called *right endpoint controller(s)*.

3.1.1. Left endpoint controller design

Let us first consider the case with a single feedback control input g_0, acting to the system through Dirichlet actuation at $x = 0$ and two boundary conditions imposed at the other end are homogeneous. To this end, consider the closed-loop system

$$\begin{cases} iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = 0, & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = g_0(t), \quad u(L,t) = u_x(L,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,L), \end{cases} \tag{3.2}$$

and apply the three step backstepping strategy we described in Section 1.1.2.

(i) Consider the so called target model

$$\begin{cases} iw_t + i\beta w_{xxx} + \alpha w_{xx} + i\delta w_x + irw = 0, & (x,t) \in (0,L) \times (0,T), \\ w(0,t) = w(L,t) = w_x(L,t) = 0, & t \in (0,T), \\ w(x,0) = w_0(x), & x \in (0,L), \end{cases} \tag{3.3}$$

where w_0 will be specified below. Observe that the main equation of (3.3) involves a weakly damping term irw, with $r > 0$, and the boundary conditions are homogeneous. Applying multipliers, it is a straightforward task to show that for $w_0 \in L^2(0, L)$, then solution w of (3.3) satisfies the following decay estimate

$$\|w(\cdot, t)\|_{L^2(0,L)} \le e^{-rt}\|w_0\|_{L^2(0,L)}, \quad t \ge 0. \tag{3.4}$$

In other words, the zero equilibrium of (3.3) is globally exponentially stable.

(ii) Now consider the backstepping transformation

$$w(x,t) = u(x,t) - \int_x^L k(x,y)u(y,t)dy, \quad x \in [0, L]. \tag{3.5}$$

After some calculations, one can show that if k satisfies the following boundary value problem

$$\begin{cases} \beta\left(k_{xxx} + k_{yyy}\right) - i\alpha\left(k_{xx} - k_{yy}\right) + \delta\left(k_x + k_y\right) + rk = 0, \\ k(x,x) = k(x,L) = 0, \\ k_x(x,x) = \frac{r}{3\beta}(L - x), \end{cases} \tag{3.6}$$

posed on the triangular region $\{(x,y) \in \mathbb{R}^2 \mid x \in (0,L), y \in (x,L)\}$, then the transformation (3.5) maps the plant (3.2) successfully to the target model (3.3), and intial state of (3.3) is given by

$$w_0(x) := u_0(x) - \int_0^x k(x,y)u_0(y)dy.$$

Existence of a smooth backstepping kernel k can be established by analyzing the model (3.6). Moreover, since $w(0,t) = 0$, we take $x = 0$ on (3.5) see that control input is of the form

$$g_0(t) = \int_0^L k(L,y)u(y,t)dy. \tag{3.7}$$

(iii) Finally, one can also prove that the backstepping transformation is invertible on

$L^2(0, L) \to L^2(0, L)$ with a bounded inverse. This implies existence of a constant $c_k > 0$ depends on k such that, $\|u(\cdot, t)\|_{L^2(0,L)} \le c_k \|w(\cdot, t)\|_{L^2(0,L)}$. Combining this with (3.4), one concludes that zero equilibrium of (3.2) is globally exponentially stable along with the controller (3.7). Moreover, since $r > 0$ is arbitrary, decay rate of solutions to zero equilibrium can be made as rapid as desired.

See Figure 3.1 for a graphical illustration of backstepping method.

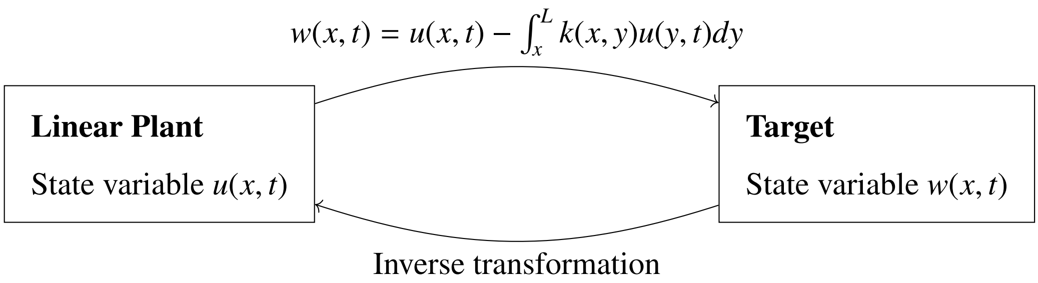

Figure 3.1. Classical backstepping scheme for controller design.

The backstepping strategy we summerized above works successfully on designing a left endpoint boundary controller that globally stabilizes zero equilibrium. Details and related results are published in our study (Batal et. al., 2021). However, if we locate the control inputs at the right endpoint and consider the same target model (3.3), then the above approach fails in the second stage as we detail below.

3.1.2. Issues with right endpoint controllers

Let us recall the plant under the influence of right endpoint controllers:

$$\begin{cases} iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = 0, & (x, t) \in (0, L) \times (0, T), \\ u(0, t) = 0, \quad u(L, t) = h_0(t),\ u_x(L, t) = h_1(t), & t \in (0, T), \\ u(x, 0) = u_0(x), & x \in (0, L). \end{cases} \tag{3.8}$$

Consider the same target model we introduced in (3.3) and consider also the backstepping transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy, \quad x \in [0,L]. \tag{3.9}$$

We find that if k satisfies the following boundary value problem

$$\begin{cases} \beta(k_{xxx} + k_{yyy}) - i\alpha(k_{xx} - k_{yy}) + \delta(k_x + k_y) + rk = 0, & (x,y) \in \Delta_{x,y}, \\ k(x,x) = k_y(x,0) = k(x,0) = 0, & x \in [0,L], \\ k_x(x,x) = \frac{rx}{3\beta}, & x \in [0,L], \end{cases} \tag{3.10}$$

where $\Delta_{x,y} := \left\{(x,y) \in \mathbb{R}^2 \,|\, y \in (0,x), x \in (0,L)\right\}$, then the backstepping transformation (3.9) maps the plant (3.8) to target model (3.3) (see Appendix A.1 for detailed calculations). To solve (3.10), we change variables as $\bar{x} = x - y$, $\bar{y} = y$, and write $G(\bar{x},\bar{y}) = k(x,y)$. Then (3.10) transforms into

$$\begin{cases} \beta(3G_{\bar{x}\bar{x}\bar{y}} - 3G_{\bar{x}\bar{y}\bar{y}} + G_{\bar{y}\bar{y}\bar{y}}) + i\alpha(G_{\bar{y}\bar{y}} - 2G_{\bar{x}\bar{y}}) + \delta G_{\bar{y}} + rG = 0, & (\bar{x},\bar{y}) \in \Delta_{\bar{x},\bar{y}}, \\ G(0,\bar{y}) = G_{\bar{y}}(\bar{x},0) = G(\bar{x},0) = 0, & \bar{x} \in [0,L], \\ G_{\bar{x}}(0,\bar{y}) = \dfrac{r\bar{y}}{3\beta}, & \bar{x} \in [0,L], \end{cases} \tag{3.11}$$

where $\Delta_{\bar{x},\bar{y}} := \left\{(\bar{x},\bar{y}) \in \mathbb{R}^2 \,|\, \bar{y} \in (0, L-\bar{x}), \bar{x} \in (0,L)\right\}$. Figure 3.2 shows the transformation of the triangular region $\Delta_{x,y}$ onto $\Delta_{\bar{x},\bar{y}}$.

Note that the boundary value problem (3.11) is overdetermined due to the mismatch between the boundary conditions $G_{\bar{y}}(\bar{x},0) = 0$ and $G_{\bar{x}}(0,\bar{y}) = \frac{r\bar{y}}{3\beta}$ in the sense that second order mixed derivatives are not compatible:

$$G_{\bar{y}\bar{x}}(0,0) = 0, \quad G_{\bar{x}\bar{y}}(0,0) = \frac{r}{3\beta}. \tag{3.12}$$

This mismatch implies that the boundary value problem (3.11) cannot have a smooth so-

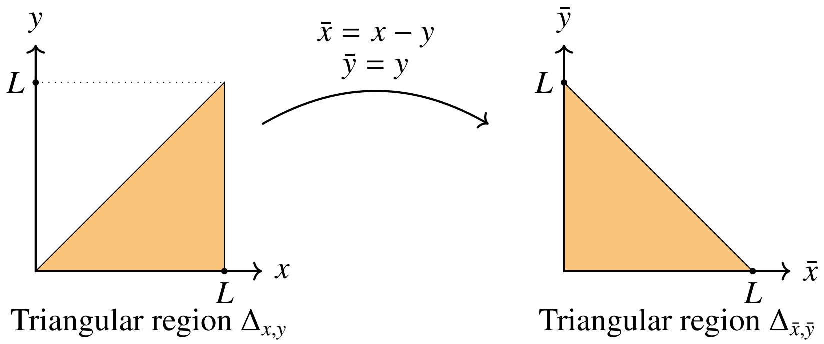

Figure 3.2. Triangular regions.

lution. This is crucial since it allows us to go forward and backward between the original plant and the target model pointwise. Therefore, lack of the smooth solution prevents us to apply the standard backstepping strategy with right endpoint controllers.

In the next section, we show how we modify the backstepping method so as to remedy this issue.

3.1.3. Modification of backstepping kernel model and the target model

Our strategy relies on correcting the imperfect backstepping kernel model (3.11) in such a way that, the boundary conditions of the corrected version are no more incompatible at the origin of $\bar{x}\bar{y}-$plane. Thus, one guarantees existence of a smooth backstepping kernel.

Observe that if $r > 0$ is sufficiently small, then the mismatch in (3.12) is also small. So provided that $r > 0$ is sufficiently small and dropping one of the boundary conditions that causes the mismatch, one can hope that solution of the updated backstepping kernel model will yield a good enough kernel for our purposes. To this end, we disregard the boundary condition $G_{\bar{y}}(\bar{x},0)=0$ from (3.11) and introduce the corrected version as

$$\begin{cases} \beta(3G^*_{\bar{x}\bar{x}\bar{y}} - 3G^*_{\bar{y}\bar{y}\bar{x}} + G^*_{\bar{y}\bar{y}\bar{y}}) + i\alpha(G^*_{\bar{y}\bar{y}} - 2G^*_{\bar{y}\bar{x}}) + \delta G^*_{\bar{y}} + rG^* = 0, & (x,y)\in\Delta_{\bar{x},\bar{y}}, \\ G^*(0,\bar{y}) = G^*(\bar{x},0) = 0, & x\in[0,L], \\ G^*_{\bar{x}}(0,\bar{y}) = \dfrac{r\bar{y}}{3\beta}, & x\in[0,L]. \end{cases} \tag{3.13}$$

Setting $k^*(x,y)=G^*(\bar{x},\bar{y})$, we deduce that k^* is the sought after solution of

$$\begin{cases} \beta(k^*_{xxx} + k^*_{yyy}) - i\alpha(k^*_{xx} - k^*_{yy}) + \delta(k^*_x + k^*_y) + rk^* = 0, & (x,y)\in\Delta_{x,y}, \\ k^*(x,x) = k^*(x,0) = 0, & x\in[0,L], \\ k^*_x(x,x) = \frac{rx}{3\beta}, & x\in[0,L]. \end{cases} \tag{3.14}$$

Based on the above discussion, we use the following backstepping transformation

$$w^*(x,t) = u(x,t) - \int_0^x k^*(x,y)u(y,t)dy, \tag{3.15}$$

with a kernel function k^* that solves (3.14). Here w^* denotes the state of the associated modified target model due to the modification of backstepping kernel model. Indeed throughout the calculations in (A.5), disregarding the condition yields and extra term $i\beta k_y(x,0)u_x(0,t)$, which is now involved in the main equation of the associated target model. Note that differentiating both sides of (3.15) with respect to x, evaluating it at $x=0$ and using the boundary condition $k^*(x,x)=0$, one can see that $u_x(0,t)=w^*_x(0,t)$. Therefore the main equation of the modified target model involves an additional trace term $i\beta k^*_y(x,0)w^*_x(0,t)$ on its right hand side as we give below

$$\begin{cases} iw^*_t + i\beta w^*_{xxx} + \alpha w^*_{xx} + i\delta w^*_x + irw^* = i\beta k^*_y(x,0)w^*_x(0,t), & (x,t)\in(0,L)\times(0,T), \\ w^*(0,t) = w^*(L,t) = w^*_x(L,t) = 0, & t\in(0,T), \\ w^*(x,0) = w^*_0(x) := u_0(x) - \int_0^x k^*(x,y)u_0(y)dy, & x\in(0,L). \end{cases} \tag{3.16}$$

Graphical illustration of the updated backstepping scheme is given in Figure 3.3.

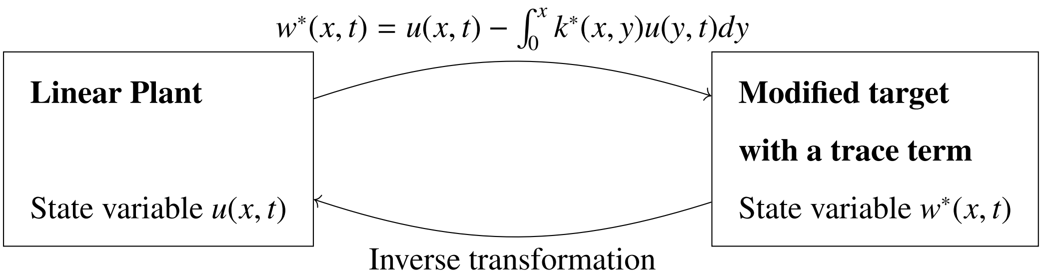

Figure 3.3. Modified backstepping scheme for controller design.

Based on the above strategy, feedback controllers take the following forms:

$$h_0(t) = \int_0^L k^*(L,y)u(y,t)dy, \quad h_1(t) = \int_0^L k_x^*(L,y)u(y,t)dy. \tag{3.17}$$

As we will see in Section 3.2, in the presence of these controllers, trivial equilibrium of the closed-loop system (3.8) becomes globally exponentially stable. The drawback here is that one loses rapid stabilization in comparison with the problem using a left endpoint controller.

Throughout the following sections, for the sake of easy readability of the text, we drop the superscript notation $(\cdot)^*$ and simply write k, w but refering to their modified versions.

3.2. Exponential stabilization of zero equilibrium

Stabilization of the zero equilibrium to the plant (3.8) consists of three steps. First, we prove existence of a smooth backstepping kernel which is the subject of Section 3.2.1. Smoothness of the backstepping kernel also implies that, for a given $u_0 \in L^2(0,L)$, associated initial state w_0 of the modified target model (3.16), defined by

$$w_0(x) := u_0(x) + \int_0^x k(x,y)w_0(y)dy,$$

is also in $L^2(0, L)$. So, in Section 3.2.2, we prove that if $w_0 \in L^2(0, L)$, then the corresponding solution of the target model (3.16) decays exponentially to zero equilibrium with respect to L^2-metric. As a next and final step in Section 3.2.3, we state that the backstepping transformation is invertible with a bounded inverse on $H^\ell(0, L) \to H^\ell(0, L)$, $\ell \geq 0$.

3.2.1. Existence of a smooth backstepping kernel

Let us rewrite the main equation of (3.13) as

$$G_{\bar{x}\bar{x}\bar{y}} = DG := \frac{1}{3\beta}\left[\beta(3G_{\bar{y}\bar{y}\bar{x}} - G_{\bar{y}\bar{y}\bar{y}}) - i\alpha(G_{\bar{y}\bar{y}} - 2G_{\bar{y}\bar{x}}) - \delta G_{\bar{y}} - rG\right]. \tag{3.18}$$

Integrating (3.18) in the first variable and using boundary condition $G_{\bar{x}}(0, \bar{y}) = \frac{r\bar{y}}{3\beta}$, we obtain

$$G_{\bar{x}\bar{y}}(\bar{x}, \bar{y}) = \frac{r}{3\beta} + \int_0^{\bar{x}} [DG](\xi, \bar{y})d\xi.$$

Integrating once again in the first variable and using boundary condition $G(0, \bar{y}) = 0$ we get

$$G_{\bar{y}}(\bar{x}, \bar{y}) = \frac{r\bar{x}}{3\beta} + \int_0^{\bar{x}}\int_0^{\omega} [DG](\xi, t)d\xi d\omega.$$

Finally, integrating in the second variable and using boundary condition $G(s, 0) = 0$, we obtain that G solves

$$G(\bar{x}, \bar{y}) = \frac{r\bar{x}\bar{y}}{3\beta} + \int_0^{\bar{y}}\int_0^{\bar{x}}\int_0^{\omega} [DG](\xi, \eta)d\xi d\omega d\eta. \tag{3.19}$$

Thus, solution of the boundary value problem (3.13) can be constructed by applying a successive approximation method to the integral equation (3.19).

Lemma 3.1 *There exists a $C^\infty-$function G such that G solves the integral equation* (3.19).

Proof Let P be defined by

$$(Pf)(\bar{x}, \bar{y}) := \int_0^{\bar{y}}\int_0^{\bar{x}}\int_0^{\omega} [Df](\xi, \eta)d\xi d\omega d\eta. \tag{3.20}$$

Then we express (3.19) as

$$G(\bar{x},\bar{y}) = \frac{r}{3\beta}\bar{x}\bar{y} + PG(\bar{x},\bar{y}). \tag{3.21}$$

Define $G^0 \equiv 0$, $G^1(\bar{x},\bar{y}) = -\frac{r}{3\beta}\bar{x}\bar{y}$ and $G^{n+1} = G^1 + PG^n$, $n \in \mathbb{Z}^+$. Then we have

$$G^{n+1} - G^n = P(G^n - G^{n-1}), \quad n \in \mathbb{Z}^+. \tag{3.22}$$

To prove existence of a smooth solution of (3.21), it is enough to show that the sequence $\{G^n\}_{n\in\mathbb{Z}^+}$ and its partial derivatives of any order are Cauchy with respect to the supremum norm. To this end, define $H^0(\bar{x},\bar{y}) = \bar{x}\bar{y}$, $H^n = \frac{3\beta}{r}\left(G^{n+1} - G^n\right)$. Then by (3.22), $H^{n+1} = PH^n$ and for $j > i$,

$$G^j - G^i = \sum_{n=i}^{j-1}(G^{n+1} - G^n) = \frac{r}{3\beta}\sum_{n=i}^{j-1} H^n. \tag{3.23}$$

We see from (3.23) that the sequence $\{G^n\}_{n\in\mathbb{Z}^+}$ (and its partial derivatives) is Cauchy with respect to the supremum norm if and only if the series at the right hand side of (3.23) (and the series formed by H_n's partial derivatives of any order) is convergent with respect to the same norm. To this end, let us express P as sum of six operators

$$P = P_{1,-1} + P_{2,-2} + P_{2,-1} + P_{1,0} + P_{2,0} + P_{2,1},$$

where

$$\begin{aligned}
P_{1,-1}f &:= \int_0^{\bar{y}}\int_0^{\bar{x}}\int_0^{\omega} f_{\bar{y}\bar{y}\bar{x}}(\xi,\eta)d\xi d\omega d\eta,\\
P_{2,-2}f &:= -\frac{1}{3}\int_0^{\bar{y}}\int_0^{\bar{x}}\int_0^{\omega} f_{\bar{y}\bar{y}\bar{y}}(\xi,\eta)d\xi d\omega d\eta,\\
P_{2,-1}f &:= -\frac{i\alpha}{3\beta}\int_0^{\bar{y}}\int_0^{\bar{x}}\int_0^{\omega} f_{\bar{y}\bar{y}}(\xi,\eta)d\xi d\omega d\eta,
\end{aligned}$$

$$
\begin{aligned}
P_{1,0}f &:= \frac{2i\alpha}{3\beta}\int_0^{\bar{y}}\int_0^{\bar{x}}\int_0^{\omega} f_{\bar{y}\bar{x}}(\xi,\eta)d\xi d\omega d\eta,\\
P_{2,0}f &:= -\frac{\delta}{3\beta}\int_0^{\bar{y}}\int_0^{\bar{x}}\int_0^{\omega} f_{\bar{y}}(\xi,\eta)d\xi d\omega d\eta,\\
P_{2,1}f &:= -\frac{r}{3\beta}\int_0^{\bar{y}}\int_0^{\bar{x}}\int_0^{\omega} f(\xi,\eta)d\xi d\omega d\eta.
\end{aligned}
$$

Then

$$
\begin{aligned}
H^n = P^n H^0 &= (P_{1,-1} + P_{2,-2} + P_{2,-1} + P_{1,0} + P_{2,0} + P_{2,1})^n st\\
&= \sum_{r=1}^{6^n} R_{r,n} st,
\end{aligned}
\tag{3.24}
$$

where

$$
R_{r,n} := P_{i_{r,n},j_{r,n}} P_{i_{r,n-1},j_{r,n-1}} \cdots P_{i_{r,1},j_{r,1}}, \quad i_{r,q} \in \{1,2\}, \quad j_{r,q} \in \{-2,-1,0,1\},
$$

for $1 \le q \le n$. Observe that for positive integers m and nonnegative integers k,

$$
P_{1,-1}\bar{x}^m\bar{y}^k = c_{1,-1}\bar{x}^{m+1}\bar{y}^{k-1}, \qquad c_{1,-1} = \begin{cases} 0, & k \le 0,\\ \dfrac{k}{m+1}, & \text{else,}\end{cases} \tag{3.25}
$$

$$
P_{2,-2}\bar{x}^m\bar{y}^k = c_{2,-2}\bar{x}^{m+2}\bar{y}^{k-2}, \qquad c_{2,-2} = \begin{cases} 0, & k \le 1,\\ -\dfrac{k(k-1)}{3(m+1)(m+2)}, & \text{else,}\end{cases} \tag{3.26}
$$

$$
P_{2,-1}\bar{x}^m\bar{y}^k = c_{2,-1}\bar{x}^{m+2}\bar{y}^{k-1}, \qquad c_{2,-1} = \begin{cases} 0, & k \le 0,\\ -\dfrac{i\alpha k}{3\beta(m+1)(m+2)}, & \text{else,}\end{cases} \tag{3.27}
$$

$$
P_{1,0}\bar{x}^m\bar{y}^k = c_{1,0}\bar{x}^{m+1}\bar{y}^{k}, \qquad c_{1,0} = \frac{2i\alpha}{3\beta(m+1)}, \tag{3.28}
$$

$$
P_{2,0}\bar{x}^m\bar{y}^k = c_{2,0}\bar{x}^{m+2}\bar{y}^{k}, \qquad c_{2,0} = -\frac{\delta}{3\beta(m+1)(m+2)}, \tag{3.29}
$$

$$
P_{2,1}\bar{x}^m\bar{y}^k = c_{2,1}\bar{x}^{m+2}\bar{y}^{k+1}, \qquad c_{2,1} = -\frac{r}{3\beta(m+1)(m+2)(k+1)}, \tag{3.30}
$$

Let $\sigma = \sigma(r) \equiv \sum_{q=1}^{n} j_{r,q}$. Then from (3.25)-(3.30), for each n and r,

$$R_{r,n}st = \begin{cases} 0, & \text{if } \sigma \leq -1, \\ C_{r,n}\bar{x}^{\gamma}\bar{y}^{\sigma+1}, & \text{if } \sigma > -1, \end{cases} \tag{3.31}$$

where $n+1 \leq \gamma \leq 2n+1$ and $C_{r,n}$ is a constant which only depends on r and n. Let $M = \max\{1, \frac{\alpha}{\beta}, \frac{\delta}{\beta}, \frac{r}{\beta}\}$. We claim that for each r and n,

$$|C_{r,n}| \leq \frac{M^n}{(n+1)!(\sigma+1)!}. \tag{3.32}$$

Taking $m = 1$, $k = 1$ in (3.25)-(3.30), we see that (3.32) holds for $n = 1$. Suppose it holds for $n = \ell - 1$ and for all $r \in \{1, 2, \ldots, 6^{\ell-1}\}$. Then for $n = \ell$ and $r^* \in \{1, 2, \ldots, 6^{\ell}\}$, using (3.25)-(3.31), we get

$$R_{r^*,\ell}st = P_{i,j}R_{r,\ell-1}st = C_{r,\ell-1}P_{i,j}\bar{x}^{\gamma}\bar{y}^{\sigma+1} = C_{r,\ell-1}c_{i,j}\bar{x}^{\gamma^*}\bar{y}^{\sigma^*+1}$$

for some $i \in \{1, 2\}$, $j \in \{-2, -1, 0, 1\}$ and $r \in \{1, 2, \ldots, 6^{\ell-1}\}$, where γ^* is either $(\gamma + 1)$ or $(\gamma + 2)$, $\sigma^* = \sigma + j$. By the induction assumption,

$$C_{r,\ell-1} \leq \frac{M^{\ell-1}}{\ell!(\sigma+1)!}.$$

Moreover using (3.25)-(3.30) and the fact that $\gamma \geq \ell$ we see that $|c_{i,j}| \leq M\frac{\sigma+1}{\ell+1}$ for $j = -1, -2$, $|c_{i,0}| < \frac{M}{\ell+1}$, and $|c_{i,1}| < \frac{M}{(\sigma+2)(\ell+1)}$. Hence for each $i \in \{1, 2\}$ and $j \in \{-2, -1, 0, 1\}$ we obtain

$$|C_{r^*,\ell}| = |C_{r,(\ell-1)}c_{i,j}| \leq \frac{M^{\ell}}{(\ell+1)!(\sigma+j+1)!} = \frac{M^{\ell}}{(\ell+1)!(\sigma^*+1)!},$$

which proves that the claim holds for $n = \ell$ as well. Using (3.24), (3.31), (3.32) and the fact that $(\bar{x},\bar{y}) \in \Delta_{\bar{x},\bar{y}}$, we obtain

$$\|H^n\|_{C(\overline{\Delta_{\bar{x},\bar{y}}})} \leq \frac{6^n M^n L^{3n+2}}{(n+1)!}. \tag{3.33}$$

This shows that $\sum_{n=1}^{\infty} \|H^n\|_{C(\overline{\Delta_{\bar{x},\bar{y}}})}$ is convergent. On the other hand since H^n is a linear combination of 6^n monomials of the form $\bar{x}^{\gamma}\bar{y}^{\sigma+1}$ with $\gamma \leq 2n+1$ and $\sigma \leq n$, any partial derivative $\partial_{\bar{x}}^a \partial_{\bar{y}}^b H^n$ of H^n is absolutely less than

$$\|\partial_{\bar{x}}^a \partial_{\bar{y}}^b H^n\|_{C(\overline{\Delta_{\bar{x},\bar{y}}})} \leq \frac{(2n+1)^a (n+1)^b 6^n M^n L^{3n+2-a-b}}{(n+1)!}, \tag{3.34}$$

which shows that $\sum_{n=1}^{\infty} \|\partial_{\bar{x}}^a \partial_{\bar{y}}^b H^n\|_{C(\overline{\Delta_{\bar{x},\bar{y}}})}$ is convergent as well. □

3.2.2. Exponential decay of solutions of target model to zero equilibrium

In this part, we prove that for $w_0 \in L^2(0,L)$ zero equilibrium of the target model (3.16) is exponentially stable. The proof relies on multipliers and be done formally. However, calculations can be justified rigorously by using the higher regularity results proved in Section 3.3.4 and the classical density argument.

Proposition 3.1 *Let $\beta > 0$, $\alpha, \delta \in \mathbb{R}$, k be a smooth backstepping kernel that solves* (3.6). *Suppose that $w_0 \in L^2(0,L)$. Then for sufficiently small $r > 0$, zero equilibrium of the target model* (3.16) *is expoenentially stable. Moreover for such values of r, it is true that*

$$\lambda := \beta\left(\frac{r}{\beta} - \frac{\|k_y(\cdot,0;r)\|_{L^2(0,L)}^2}{2}\right) > 0$$

and solution w of (3.16) *satisfies the following decay estimate*

$$\|w(\cdot,t)\|_{L^2(0,L)} \leq \|w_0\|_{L^2(0,L)} e^{-\lambda t}, \quad \forall t \geq 0.$$

Proof Let us multiply the main equation of (3.16) in $L^2(0,L)$ by $2w$

$$2\text{Im}\int_0^L iw_t\overline{w}dx + 2\text{Im}\int_0^L i\beta w_{xxx}\overline{w}dx + 2\text{Im}\int_0^L \alpha w_{xx}\overline{w}dx + 2\text{Im}\int_0^L i\delta w_x\overline{w}dx + 2r\,\|w(\cdot,t)\|^2_{L^2(0,L)} = 2\beta\text{Re}\left(w_x(0,t)\int_0^L k_y(x,0)\overline{w}(x,t)dx\right). \quad (3.35)$$

The first term at the left hand side of (3.35) can be written as

$$2\text{Im}\int_0^L iw_t\overline{w}dx = 2\text{Re}\int_0^L w_t\overline{w}dx = \frac{d}{dt}\|w(\cdot,t)\|^2_{L^2(0,L)}. \quad (3.36)$$

The second term can be integrated by parts in x, and then using boundary conditions we have

$$2\text{Im}\int_0^L i\beta w_{xxx}\overline{w}dx = -2\text{Re}\int_0^L \beta w_{xx}\overline{w}_x dx = \beta|w_x(0,t)|^2. \quad (3.37)$$

The third term, via integration by parts in x, vanishes

$$2\text{Im}\int_0^L \alpha w_{xx}\overline{w}dx = -2\text{Im}\int_0^L \alpha|w_x|^2dx = 0. \quad (3.38)$$

The fourth term again vanishes since

$$2\text{Im}\int_0^L i\delta w_x\overline{w}dx = \delta|w(x,t)|^2\Big|_0^L = 0. \quad (3.39)$$

Using Cauchy's inequality and then the Cauchy-Schwarz inequality, the term at the right hand side of (3.35) can be estimated as

$$\begin{aligned}2\beta\text{Re}\int_0^L k_y(x,0)w_x(0,t)\overline{w}(x,t)dx &\le 2\beta\left(\frac{1}{2}|w_x(0,t)|^2 + \left(\int_0^L |k_y(x,0)||\overline{w}(x,t)|dx\right)^2\right)\\ &\le \beta|w_x(0,t)|^2 + \beta\|k_y(\cdot,0)\|^2_{L^2(0,L)}\|w(\cdot,t)\|^2_{L^2(0,L)}.\end{aligned} \quad (3.40)$$

Combining (3.36)-(3.40), it follows from (3.35) that

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2(0,L)} + 2\beta\left(\frac{r}{\beta} - \frac{\|k_y(\cdot,0)\|^2_{L^2(0,L)}}{2}\right)\|w(\cdot,t)\|^2_{L^2(0,L)} \leq 0. \tag{3.41}$$

Solving this differential inequality, we get

$$\|w(\cdot,t)\|_{L^2(0,L)} \leq \|w_0\|_{L^2(0,L)}e^{-\lambda t}, \quad \lambda := \beta\left(\frac{r}{\beta} - \frac{\|k_y(\cdot,0;r)\|^2_{L^2(0,L)}}{2}\right). \tag{3.42}$$

In order for the inequality (3.42) to be a decay estimate, we need to show that λ remains positive at least for some values of $r > 0$. In the remaining part of the proof, we show that this is true provided that $r > 0$ is sufficiently small.

To this end, let us denote $\tilde{\alpha} = \frac{\alpha}{\beta}$, $\tilde{\delta} = \frac{\delta}{\beta}$, $\tilde{r} = \frac{r}{\beta}$ and let $M = \max\{1, \tilde{\alpha}, \tilde{\delta}, \tilde{r}\}$. Differentiating (3.23) with respect to $\bar{y}$, taking $i = 0$ and passing to limit as $n \to \infty$, we obtain

$$G_{\bar{y}}(\bar{x},\bar{y}) = \frac{\tilde{r}}{3}\sum_{n=0}^{\infty} H^n_{\bar{y}}(\bar{x},\bar{y}). \tag{3.43}$$

Indeed we can do this because we proved in Lemma 3.1 that the series formed by H^n's and formed by their partial derivatives of any order are absolutely convergent. Now let $\tilde{r} < 1$, so M is independent of r. Then from (3.34) with $a = 0$, $b = 1$, we can write

$$\|G_{\bar{y}}\|_{C^\infty(\overline{\Delta_{\bar{x},\bar{y}}})} \leq \frac{\tilde{r}c_{L,\tilde{\alpha},\tilde{\delta}}}{3},$$

where $c_{L,\tilde{\alpha},\tilde{\delta}}$ is a constant depending on L, $\tilde{\alpha}$, $\tilde{\delta}$ but independent of r. Using $k_y(x,0) = G_{\bar{y}}(\bar{x},0)$ we have

$$\|k_y(\cdot,0)\|^2_{L^2(0,L)} \leq L\|k_y(\cdot,0)\|^2_{C^\infty([0,L])} \leq L\|G_{\bar{y}}(\cdot,0)\|^2_{C^\infty([0,L])} \leq L\|G_{\bar{y}}\|^2_{C^\infty(\overline{\Delta_{\bar{x},\bar{y}}})} \leq \frac{L\tilde{r}^2c^2_{L,\tilde{\alpha},\tilde{\delta}}}{9}.$$

Therefore, we obtain

$$\lambda = 2\beta\left(\tilde{r} - \frac{\|k_y(\cdot,0)\|^2_{L^2(0,L)}}{2}\right) = 2\beta\tilde{r}^2\left(\frac{1}{\tilde{r}} - \frac{\|k_y(\cdot,0)\|^2_{L^2(0,L)}}{2\tilde{r}^2}\right)$$
$$\geq 2\beta\tilde{r}^2\left(\frac{1}{\tilde{r}} - \frac{Lc^2_{L,\tilde{\alpha},\tilde{\delta}}}{18}\right) > 0, \quad (3.44)$$

provided that r is sufficiently small, which finishes the proof. □

3.2.3. Invertibility of the backstepping transformation

To show that the exponential stabilization result we obtained for the model (3.16) also holds for the original plant (3.8) along with the feedback controllers (3.17), we need to show that the backstepping transformation is invertible and its inverse is bounded on $L^2(0,L)$. Below we state this in a more general setting, since it will be useful later in Chapter 4.

Let $\eta = \eta(x,y)$ be a smooth function on $\Delta_{x,y}$ and $\Upsilon_\eta : H^l(0,L) \to H^l(0,L)$, $l \geq 0$ be the integral operator defined by

$$(\Upsilon_\eta\varphi)(x) := \int_0^x \eta(x,y)\varphi(y)dy.$$

Lemma 3.2 *$(I-\Upsilon_\eta)$ is invertible with a bounded inverse on $H^l(0,L) \to H^l(0,L)$ for $l \geq 0$. Moreover, $(I-\Upsilon_\eta)^{-1}$ can be written as $(I+\Phi)$, where Φ is a bounded, linear operator from $L^2(0,L)$ into $H^l(0,L)$ for $l = 0,1,2$ and from $H^{l-2}(0,L)$ into $H^l(0,L)$ for $l > 2$.*

Proof Considering the volume of the thesis, we omit the proof but it can be done in a similar way to those in (Liu, 2003) and (Özsarı and Batal, 2019). □

Now using the backstepping transformation and the fact that $k = k(x,y)$ is smooth on $\Delta_{x,y}$, we have

$$\begin{aligned}\|w_0\|_{L^2(0,L)} &\leq \|u_0\|_{L^2(0,L)} + \left\|\int_0^{\cdot} k(\cdot,t)u_0(y)dy\right\|_{L^2(0,L)} \\ &\leq \left(1 + \|k\|_{L^2(\Delta_{x,y})}\right)\|u_0\|_{L^2(0,L)}.\end{aligned} \quad (3.45)$$

Moreover, using the invertibility of the backstepping transformation given by Lemma 3.2 and the decay estimate for solutions to the target model we obtained in Proposition 3.1, we also have

$$\begin{aligned}\|u(\cdot,t)\|_{L^2(0,L)} &\le \|(I-\Upsilon_k)^{-1}\|_{L^2(0,L)\to L^2(0,L)}\|w(\cdot,t)\|_{L^2(0,L)} \\ &\le \|(I-\Upsilon_k)^{-1}\|_{L^2(0,L)\to L^2(0,L)}e^{-\lambda t}\|w_0\|_{L^2(0,L)}.\end{aligned} \tag{3.46}$$

Combining (3.45) and (3.46), we get

$$\|u(\cdot,t)\|_{L^2(0,L)} \le c_k\|u_0\|_{L^2(0,L)}e^{-\lambda t}, \quad \forall t \ge 0, \tag{3.47}$$

where $c_k = \|(I-\Upsilon_k)^{-1}\|_{L^2(0,L)\to L^2(0,L)}\left(1 + \|k\|_{L^2(\Delta_{x,y})}\right)$ is a nonnegative constant that depends on k and is independent of u_0. Hence, if $u_0 \in L^2(0,L)$ and the boundary controllers be as in (3.17), then zero equilibrium of the closed-loop system (3.8) becomes exponentially stable. Note that (3.47) also provides continuous dependence of u on the initial and boundary data.

3.3. Well-posedness

In this section we prove that the closed-loop system (3.8) is well-posed in the Hadamard sense. We perform our analysis through the target model (3.16). First, in Section 3.3.1, we use operator semigroup theory to prove existence of a unique mild solution. Then, in the same section we obtain some auxiliary estimates, which are essential throughout the rest of the section. Next in Section 3.3.2, we prove local existence and uniqueness of the target model (3.16) and in Section 3.3.3 extend the unique local solution globally. Calculations at these sections can be justified by a density argument and by a higher regularity result, which we provide in Section 3.3.4. Finally, we use the invertibility result Lemma 3.2 of the backstepping transformation to provide that the results we obtain for the target model (3.16) is also true for the original plant (3.8).

3.3.1. A-priori estimates

Consider the abstract Cauchy problem

$$\begin{cases} \dfrac{dw}{dt} = Aw, \quad t \in (0,T), \\ w(0) = w_0, \end{cases} \tag{3.48}$$

where

$$A\varphi := -\beta\varphi''' + i\alpha\varphi'' - \delta\varphi' \tag{3.49}$$

with domain

$$D(A) = \left\{\varphi \in H^3(0,L) \,|\, \varphi(0) = \varphi(L) = \varphi'(L) = 0\right\}. \tag{3.50}$$

Lemma 3.3 *A generates a strongly continuous semigroup of contractions on* $L^2(0,L)$.

Proof It is clear that $D(A)$ is dense in $L^2(0,L)$. Closedness of A can be proved similarly as in Lemma 5.2 in (Batal et. al., 2021). For the sake of completeness of the proof, we briefly show how to do it. To this end, let $\{\varphi_n\}_{n\in I} \subset D(A)$ be such that $\varphi_n \to \varphi$ and $A\varphi_n \to v$ strongly in $L^2(0,L)$, where I is the index set. Once showing that $\left\{\varphi_n^{(j)}\right\}$ is bounded in $L^2(0,L)$ for each $j = 1,2,3$, one can extract a subsequence that converges weakly in $L^2(0,L)$. But then, one can also show that its unique weak limit in $L^2(0,L)$ is nothing but the weak derivatives of φ up to order three, i.e., $\varphi \in H^3(0,L)$ and also $A\varphi_n \to A\phi$. From uniqueness of the weak limit, the latter implies that $A\phi = v$. Moreover, since $C^1([0,L])$ is continuously embedded in $H^3(0,L)$, pointwise limits of $\{\varphi_n\}$ and $\{\varphi_n'\}$ exist, and $\varphi(0) = \varphi(L) = \varphi'(L) = 0$. Hence $\varphi \in D(A)$ and A is closed.

Now applying similar arguments to those in (3.36)-(3.39), one can show that A is dissipative, i.e.,

$$\mathrm{Re}(A\varphi,\varphi)_2 = \mathrm{Re}\left\{\int_0^L \left(-\beta\varphi''' + i\alpha\varphi'' - \delta\varphi'\right)(x)\overline{\varphi}(x)dx\right\} \le 0,$$

and its adjoint $A^*\varphi := \beta\varphi''' - i\alpha\varphi'' + \delta\varphi'$ with domain

$$D(A^*) = \left\{\varphi \in H^3(0,L) \,|\, \varphi(0) = \varphi(L) = \varphi'(0) = 0\right\}$$

is also dissipative, i.e.,

$$\text{Re}(\varphi, A^*\varphi)_2 = \text{Re}\left\{\int_0^L \varphi(x)\overline{(\beta\varphi''' - i\alpha\varphi'' + \delta\varphi')}(x)dx\right\} \leq 0.$$

As a conclusion of Lemma 2.1, A is the infinitesimal generator of a C_0-semigroup of contractions, $\{S(t)\}_{t\geq 0}$, on $L^2(0, L)$. □

In view of Lemma 3.3 and application of operator semigroup theory, we conclude that if $w_0 \in L^2(0, L)$, then (3.48) has a unique mild solution, which is expressed in the form

$$w(t; w_0) = S(t)w_0$$

and belongs to the space $C([0, T]; L^2(0, L))$.

Now let us turn our attention to the target model (3.16). Define the space

$$Y_T := \left\{\varphi \in X_T^0 \,|\, \varphi_x \in C([0, L]; L^2(0, T))\right\}$$

endowed with the norm

$$\|\varphi\|_{Y_T} = \left(\|\varphi\|^2_{C([0,T];L^2(0,L))} + \|\varphi\|^2_{L^2(0,T;H^1(0,L))} + \|\varphi_x\|^2_{C([0,L];L^2(0,T))}\right)^{\frac{1}{2}}.$$

Let us express (3.16) in the abstract operator theoretic form as

$$\begin{cases} \dfrac{dw}{dt} = Aw + F(w), \quad t \in (0, T), \\ w(0) = w_0, \end{cases} \tag{3.51}$$

where $F(w(\cdot, t)) := -irw(\cdot, t) + i\beta k_y(\cdot, 0)w_x(0, t)$. By Duhamel's principle, solution w of (3.51), if it exists, satisfies the following implicit relation

$$w(t; w_0) = S(t)w_0 + \int_0^t S(t - \tau)F(w(\tau; w_0))d\tau. \tag{3.52}$$

To prove existence of a function $w \in Y_T$ that satisfies (3.52), it suffices to prove that the operator

$$\Psi[w_0, \varphi(t)] := S(t)w_0 + \int_0^t S(t-\tau)F(\varphi(\tau))d\tau \tag{3.53}$$

has a fixed point at least up to for some $t = T' > 0$. In view of Banach fixed point theorem, it is enough to show that Ψ maps Y_T into itself and Ψ is strict contraction. To verify these, as we will see in Section 3.3.2, we need to prove following type of estimates

$$\|S(t)w_0\|_{Y_T} \lesssim \|w_0\|_{L^2(0,L)}, \qquad \left\|\int_0^t S(t-s)F\varphi(s)ds\right\|_{Y_T} \lesssim \|\varphi\|_{Y_T}. \tag{3.54}$$

To derive these estimates, one needs to show that mappings

$$t \in [0,T] \to \|\Psi[w_0, \varphi(t)]\|_{L^2(0,L)} \qquad \text{is continuous in } t, \tag{3.55}$$

$$t \in (0,T) \to \|\Psi[w_0, \varphi(t)]\|_{H^1(0,L)} \qquad \text{is bounded on } L^2(0,T), \tag{3.56}$$

and the mapping

$$x \in [0,L] \to \|\partial_x \Psi[w_0(x), \varphi]\|_{L^2(0,T)} \qquad \text{is continuous in } x. \tag{3.57}$$

In what follows, we consider the following auxiliary model

$$\begin{cases} iq_t + i\beta q_{xxx} + \alpha q_{xx} + i\delta q_x = f, & (x,t) \in (0,L) \times (0,T), \\ q(0,t) = q(L,t) = q_x(L,t) = 0, & t \in (0,T), \\ q(x,0) = \phi(x), & x \in (0,L), \end{cases} \tag{3.58}$$

where $f \in L^1(0,T; L^2(0,L))$ is inhomogeneous source, $\phi \in L^2(0,L)$ and regarding the $q-$model, we derive a-priori estimates to prove the validity of (3.55)-(3.57). Considering the volume of the proofs, we organize the text as follows: In Section 3.3.1.1 we derive (3.55)-(3.56) and in Section 3.3.1.2 we derive (3.57).

3.3.1.1. Space-time estimates

In this section, we show that (3.55)-(3.56) hold true by proving the following lemma.

Lemma 3.4 *Let* $T > 0$, $\phi \in L^2(0,L)$ *and* $f \in L^1(0,T;L^2(0,L))$. *Then solution* q *of* (3.58) *satisfies*

(i) $\|q\|_{C([0,T];L^2(0,L))} + \|q_x(0,\cdot)\|_{L^2(0,T)} \lesssim \|\phi\|_{L^2(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))}$,

(ii) $\|q\|_{L^2(0,T;H^1(0,L))} \lesssim (1+\sqrt{T})\left(\|\phi\|_{L^2(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))}\right)$.

Proof

(i) Multiplying the main equation of (3.58) by $2\overline{q}$, integrating over $[0,T]\times[0,L]$ and applying similar arguments to those in (3.36)-(3.39), we get

$$\|q(\cdot,t)\|^2_{L^2(0,L)} + \beta\|q_x(0,\cdot)\|^2_{L^2(0,T)} \le \|\phi\|^2_{L^2(0,L)} + 2\int_0^T\int_0^L |f(x,t)||q(x,t)|dxdt. \quad (3.59)$$

Using Cauchy-Schwarz and Cauchy's inequality with $\varepsilon > 0$, second summand at the right hand side of (3.59) can be estimated as

$$2\int_0^T\int_0^L |f(x,t)||q(x,t)|dxdt \le \varepsilon \sup_{t\in[0,T]}\|q(\cdot,t)\|^2_{L^2(0,L)} + c_\varepsilon\|f\|^2_{L^1(0,T;L^2(0,L))}$$

which, combining by (3.59) yields

$$\|q(\cdot,t)\|^2_{L^2(0,L)} + (\beta-\varepsilon)\|q_x(0,\cdot)\|^2_{L^2(0,T)} \le \|\phi\|^2_{L^2(0,L)} + c_\varepsilon\|f\|^2_{L^1(0,T;L^2(0,L))}.$$

Right hand side is independent of t. So passing to supremum on both sides over $[0,T]$ and choosing $\varepsilon > 0$ sufficiently small so that $\beta-\varepsilon > 0$ finishes proof of (i).

(ii) We multiply the main equation of (3.58) in $L^2(0,L)$ by $2xq$ and take the imaginary parts of both sides to get

$$2\text{Im}\int_0^L ixq_t\overline{q}dx + 2\text{Im}\int_0^L i\beta xq_{xxx}\overline{q}dx + 2\text{Im}\int_0^L \alpha xq_{xx}\overline{q}dx$$
$$+ 2\text{Im}\int_0^L i\delta xq_x\overline{q}dx = 2\text{Im}\int_0^L xf(x,t)\overline{q}(x,t)dx. \quad (3.60)$$

The first term at the left hand side of (3.60) can be written as

$$2\text{Im}\int_0^L ixq_t\overline{q}dx = \frac{d}{dt}\int_0^L x|q|^2dx. \quad (3.61)$$

Integrating the second and third terms by parts and using the boundary conditions, we have

$$\begin{aligned}2\text{Im}\int_0^L i\beta xq_{xxx}\overline{q}dx &= 2\beta\left(-\int_0^L q_{xx}\overline{q}dx - \int_0^L xq_{xx}\overline{q}_xdx\right)\\ &= 2\beta\left(\int_0^L |q_x|^2dx - \frac{1}{2}\int_0^L x\frac{d}{dx}|q_x|^2dx\right)\\ &= 2\beta\left(\|q_x(\cdot,t)\|^2_{L^2(0,L)}dx + \frac{1}{2}\int_0^L |q_x|^2dx\right)\\ &= 3\beta\|q_x(\cdot,t)\|^2_{L^2(0,L)}\end{aligned} \quad (3.62)$$

and

$$\begin{aligned}2\text{Im}\int_0^L \alpha xq_{xx}\overline{q}dx &= 2\alpha\left(-\text{Im}\int_0^L q_x\overline{q}dx - \text{Im}\int_0^L x|q_x|^2dx\right)\\ &\geq -\varepsilon\|q_x(\cdot,t)\|^2_{L^2(0,L)} - c_{\alpha,\varepsilon}\|q(\cdot,t)\|^2_{L^2(0,L)},\end{aligned} \quad (3.63)$$

where we used Cauchy's inequality with $\varepsilon > 0$ on the last line of (3.63). Fourth term, again via integration by parts give us

$$2\text{Im}\int_0^L i\delta xq_x\overline{q}dx = 2\delta\int_0^L x\frac{d}{dx}|q|^2dx = -\delta\|q(\cdot,t)\|^2_{L^2(0,L)}. \quad (3.64)$$

Using Cauchy-Schwarz inequality, the term at the right hand side of (3.60) is bounded from above as

$$2\int_0^L |f(x,t)||q(x,t)|dx \le 2\|q(\cdot,t)\|_{L^2(0,L)}\|f(\cdot,t)\|_{L^2(0,L))}. \tag{3.65}$$

Consequently, combining (3.61)-(3.65), it follows from (3.60) that

$$\frac{d}{dt}\int_0^L x|q|^2dx + (3\beta-\varepsilon)\|q_x(\cdot,t)\|^2_{L^2(0,L)} \\ \le c_{\alpha,\delta,\varepsilon_1}\|q(\cdot,t)\|^2_{L^2(0,L)} + 2\|q(\cdot,t)\|_{L^2(0,L)}\|f(\cdot,t)\|_{L^2(0,L)}. \tag{3.66}$$

Now we integrate both sides of (3.66) in t over $[0,T]$ and write

$$\int_0^L x|q|^2dx + (3\beta-\varepsilon)\int_0^T \|q_x(\cdot,t)\|^2_{L^2(0,L)}dt \le \int_0^L x|\phi|^2dx \\ + c_{\alpha,\delta,\varepsilon}\int_0^T \|q(\cdot,t)\|^2_{L^2(0,L)}dt + 2\int_0^T \|q(\cdot,t)\|_{L^2(0,L)}\|f(\cdot,t)\|_{L^2(0,L)}dt. \tag{3.67}$$

Using Cauchy's inequality and then employing our first result in (i), we can estimate the last term at the right hand side of (3.67) as

$$2\int_0^T \|q(\cdot,t)\|_{L^2(0,L)}\|f(\cdot,t)\|_{L^2(0,L)}dt \le \sup_{t\in[0,T]}\|q(\cdot,t)\|^2_{L^2(0,L)} + \left(\int_0^T \|f(\cdot,t)\|_{L^2(0,L)}dt\right)^2 \\ \lesssim \|\phi\|^2_{L^2(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))}.$$

Finally, choosing ε sufficiently small that guarantees $3\beta-\varepsilon>0$ and then applying Poincaré inequality for the second term on the left hand side of (3.67), we deduce (ii).

□

3.3.1.2. Time-space estimates and smoothing properties

In this section, we show that (3.57) holds true by proving that solution q of (3.58) enjoys the following time-space estimates

$$\sup_{x\in[0,L]} \|\partial_x[S(t)\phi](x)\|_{L^2(0,T)} \lesssim (1+\sqrt{T})\|\phi\|_{L^2(0,L)} \tag{3.68}$$

and

$$\sup_{x\in[0,L]} \left\|\partial_x\left[\int_0^{\cdot} S(\cdot-\tau)f(x,\tau)d\tau\right](x)\right\|_{L^2(0,T)} \lesssim (1+\sqrt{T})\|f\|_{L^1(0,T;L^2(0,L))}. \tag{3.69}$$

The property (3.68) is referred as the sharp Kato smoothing (see, e.g., (Bona et. al., 2002), (Bona et. al., 2003), (Kato, 1983), (Kenig et. al., 1991a), (Kenig et. al., 1991b) and references therein, for a detailed discussion).

To derive the estimates (3.68)-(3.69), we apply decompose-and-reunify algorithm for the $q-$model (3.58). That is, we decompose the $q-$model

(i) to a Cauchy problem with zero interior source

$$\begin{cases} iv_t + i\beta v_{xxx} + \alpha v_{xx} + i\delta v_x = 0, & (x,t)\in\mathbb{R}\times(0,T),\\ v(x,0)=\phi^*(x), & x\in\mathbb{R}, \end{cases} \tag{3.70}$$

where ϕ^* is the zero extension of ϕ on $\mathbb{R}$,

(ii) to a Cauchy problem with zero initial datum, zero interior source and an inhomogeneous interior source

$$\begin{cases} iy_t + i\beta y_{xxx} + \alpha y_{xx} + i\delta y_x = f^*, & (x,t)\in\mathbb{R}\times(0,T),\\ y(x,0)=0, & x\in\mathbb{R}, \end{cases} \tag{3.71}$$

where f^* is zero spatial extension of f on $\mathbb{R}$,

(iii) and to an initial-boundary value problem with zero initial datum and inhomogeneous boundary data as

$$\begin{cases} iz_t + i\beta z_{xxx} + \alpha z_{xx} + i\delta z_x = 0, & (x,t) \in (0,L) \times (0,T), \\ z(0,t) = \psi_1(t),\ z(L,t) = \psi_2(t),\ z_x(L,t) = \psi_3(t), & t \in (0,T), \\ z(x,0) = 0, & x \in (0,L), \end{cases} \tag{3.72}$$

where $\psi_1(t) = -v(0,t) - y(0,t)$, $\psi_2(t) = -v(L,t) - y(L,t)$ and $\psi_3(t) = -v_x(L,t) - y_x(L,t)$.

Then, by reunifying the models (3.70)-(3.72), solution q of (3.58) can be represented as

$$\begin{aligned} q(\cdot,t) &= (v(\cdot,t) + y(\cdot,t))\Big|_{[0,L]} + z(\cdot,t) \\ &= \left(S_{\mathbb{R}}(t)\phi^* + \int_0^t S_{\mathbb{R}}(t-\tau) f^*(\cdot,\tau)d\tau \right)\Big|_{[0,L]} + z(\cdot,t). \end{aligned}$$

It turns out that, in order to obtain the estimates (3.68) and (3.69), we need to prove similar smoothing properties of the solutions of each of the models (3.70)-(3.72). This is the subject of the rest of this section.

Lemma 3.5 *Let $T > 0$ and $\phi^* \in L^2(\mathbb{R})$. Then, solution v of* (3.70) *satisfies*

$$\sup_{x \in [0,L]} \|\partial_x v(x,\cdot)\|_{L^2(0,T)} \lesssim (1 + \sqrt{T})\|\phi^*\|_{L^2(0,L)}. \tag{3.73}$$

Proof Applying Fourier transform in the spatial variable for the model (3.70), solving the resulting ODE in $\hat{v} = \hat{v}(\cdot,t)$ and then recovering v by employing inverse Fourier transform, we can express v as

$$v(x,t) = S_{\mathbb{R}}(t)\phi^*(x) := \int_{-\infty}^{\infty} e^{ix\xi} e^{i\omega(\xi)t} \hat{\phi}^*(\xi) d\xi, \tag{3.74}$$

where $\hat{\phi}^*$ is the Fourier transform of ϕ^* and $\omega(\xi) := \beta\xi^3 - \alpha\xi^2 - \delta\xi$. We pick a smooth cut-off function

$$\theta(\xi) = \begin{cases} 1, & a \le \xi \le b, \\ \text{smooth}, & a - \varepsilon < \xi < a \text{ or } b < \xi < b + \varepsilon, \\ 0, & \xi \le a - \varepsilon \text{ or } \xi \ge b + \varepsilon, \end{cases}$$

where $\varepsilon > 0$ is arbitrary, $|\theta| \le 1$ and, a and b will be chosen below in a suitable manner. Now, we decompose v as

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{\infty} e^{ix\xi} e^{i\omega(\xi)t} \theta(\xi)\hat{\phi}^*(\xi) d\xi + \int_{-\infty}^{\infty} e^{ix\xi} e^{i\omega(\xi)t} (1 - \theta(\xi))\hat{\phi}^*(\xi) d\xi \\ &=: v_1(x,t) + v_2(x,t). \end{aligned} \tag{3.75}$$

Using Cauchy-Schwarz inequality for v_1, then Plancherel's theorem and since θ is a compactly supported function, we get

$$\begin{aligned} |\partial_x v_1(x,t)| &= \left| \int_{-\infty}^{\infty} i\xi e^{ix\xi} e^{i\omega(\xi)t} \theta(\xi) \hat{\phi}^*(\xi) d\xi \right| \\ &= \left(\int_{a-\varepsilon}^{b+\varepsilon} |i\xi|^2 |\theta(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |\hat{\phi}^*(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &\lesssim \|\phi^*\|_{L^2(\mathbb{R})}, \end{aligned} \tag{3.76}$$

which implies

$$\|\partial_x v_1(x,\cdot)\|^2_{L^2(0,T)} \lesssim T \|\phi^*\|^2_{L^2(\mathbb{R})}. \tag{3.77}$$

Applying similar arguments, we can also get

$$\|\partial_t^j v_1(x,\cdot)\|^2_{L^2(0,T)} \lesssim T \|\phi^*\|^2_{L^2(\mathbb{R})} \tag{3.78}$$

for $j = 0, 1$, where the constant of the inequality depends on j and θ (estimate (3.78) is crucial and will be used later at the end of this section).

Next, let us consider the second term in (3.75) and rewrite it as

$$v_2 = \int_{-\infty}^{a} \cdot + \int_{b}^{\infty} \cdot =: v_{2-} + v_{2+}. \tag{3.79}$$

Let us change the variable as

$$\begin{aligned} \tau &= \omega_-(\xi) = \omega(\xi) : (-\infty, a] \to (-\infty, \omega(a)], \\ \tau &= \omega_+(\xi) = \omega(\xi) : [b, \infty) \to [\omega(b), \infty). \end{aligned} \tag{3.80}$$

As we will see below, a suitable choice of the support of the cut-off function, i.e. the values a and b, ensures that these transformations given by (3.80) are injective, hence their inverses exist. Let us denote the their inverses as

$$\begin{aligned} \xi &= \omega_-^{-1}(\tau) =: \xi_-(\tau), \\ \xi &= \omega_+^{-1}(\tau) =: \xi_+(\tau), \end{aligned}$$

respectively. Also, the same choices on a and b provide that the differential

$$d\xi = \frac{1}{3\beta\xi_{\mp}^2(\tau) - 2\alpha\xi_{\mp}(\tau) - \delta} d\tau,$$

due to the change of variable do not cause a singularity. Depending on the sign of $\alpha^2 + 3\beta\delta$, we have three different cases:

(a) Let $\alpha^2 + 3\beta\delta > 0$. Then, any choice $a < \frac{\alpha - \sqrt{\alpha^2 + 3\beta\delta}}{3\beta}$ and $b > \frac{\alpha + \sqrt{\alpha^2 + 3\beta\delta}}{3\beta}$ provides that the mapping is injective and integrals are proper on $(-\infty, a] \cup [b, \infty)$.

(b) Let $\alpha^2 + 3\beta\delta = 0$. Then, for any choices of $a < \frac{\alpha}{3\beta} < b$ the mapping is injective and integrals are proper on $(-\infty, a] \cup [b, \infty)$.

(c) Let $\alpha^2 + 3\beta\delta < 0$. Then the mapping is injective and integrals are proper for all choices of $a < b$.

For each case of $(\alpha^2 + 3\beta\delta)$, assume that we choose appropriate values of a and b. Then, applying the change of variables (3.80), $\partial_x v_2$ becomes

$$\begin{aligned}\partial_x v_2(x,t) =& \partial_x v_{2-}(x,t) + \partial_x v_{2+}(x,t)\\ =& \int_{-\infty}^{\omega(a)} i\xi_-(\tau) e^{i\xi_-(\tau)x} e^{i\tau t}(1-\theta(\xi_-(\tau)))\frac{\hat{\phi}^*(\xi_-(\tau))}{3\beta\xi_-^2(\tau)-2\alpha\xi_-(\tau)-\delta}d\tau\\ &+ \int_{\omega(b)}^{\infty} i\xi_+(\tau) e^{i\xi_+(\tau)x} e^{i\tau t}(1-\theta(\xi_+(\tau)))\frac{\hat{\phi}^*(\xi_+(\tau))}{3\beta\xi_+^2(\tau)-2\alpha\xi_+(\tau)-\delta}d\tau.\end{aligned}$$

Let us consider $\partial_x v_{2-}(x,\cdot)$ first. Observe that the function

$$\begin{cases} i\xi_-(\tau) e^{i\xi_-(\tau)x}(1-\theta(\xi_-(\tau)))\dfrac{\hat{\phi}^*(\xi_-(\tau))}{3\beta\xi_-^2(\tau)-2\alpha\xi_-(\tau)-\delta}, & \tau\in(-\infty,\omega(a)],\\ 0, & \text{elsewhere,}\end{cases}$$

is the Fourier transform of $\partial_x v_{2-}(x,\cdot)$ with respect to its second component. Thus,

$$\begin{aligned}\|\partial_x v_{2-}(x,\cdot)\|^2_{L^2(0,T)} \le& \|\partial_x v_{2-}(x,\cdot)\|^2_{L^2(\mathbb{R})}\\ =& \int_{-\infty}^{\infty} |\widehat{\partial_x v_{2-}}(x,\tau)|^2 d\tau\\ =& \int_{-\infty}^{\omega(a)} \left| i\xi_-(\tau) e^{i\xi_-(\tau)x}(1-\theta(\xi_-(\tau)))\frac{\hat{\phi}^*(\xi_-(\tau))}{3\beta\xi_-^2(\tau)-2\alpha\xi_-(\tau)-\delta}\right|^2 d\tau\\ \le& \int_{-\infty}^{\omega(a)} \xi_-^2(\tau)\frac{\hat{\phi}^*(\xi_-(\tau))|^2}{\left(3\beta\xi_-^2(\tau)-2\alpha\xi_-(\tau)-\delta\right)^2}d\tau.\end{aligned} \tag{3.81}$$

Changing variables back as $\tau = \omega_-(\xi)$, it follows from (3.81) that

$$\begin{aligned}\|\partial_x v_{2-}(x,\cdot)\|^2_{L^2(0,T)} \le& \int_{-\infty}^{a} \xi^2 \frac{|\hat{\phi}^*(\xi)|^2}{(3\beta\xi^2-2\alpha\xi-\delta)^2}\left(3\beta\xi^2-2\alpha\xi-\delta\right)d\xi\\ \lesssim& \int_{-\infty}^{a} \xi^2\frac{|\hat{\phi}^*(\xi)|^2}{3\beta\xi^2-2\alpha\xi-\delta}d\xi \simeq \int_{-\infty}^{a}|\hat{\phi}^*(\xi)|^2 d\xi\\ \le& \|\phi^*\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{3.82}$$

$\|\partial_x v_{2+}(x,\cdot)\|^2_{L^2(0,T)} \lesssim \|\phi^*\|^2_{L^2(\mathbb{R})}$ can be shown similarly. Hence

$$\|\partial_x v_2(x,\cdot)\|_{L^2(0,T)} \lesssim \|\phi^*\|^2_{L^2(\mathbb{R})}.$$

Combining this result with (3.77), we get

$$\|\partial_x v(x,\cdot)\|^2_{L^2(0,T)} \lesssim (1+\sqrt{T})\|\phi^*\|_{L^2(\mathbb{R})}. \tag{3.83}$$

We also have the continuity of the mapping $x \to \|\partial_x v(x,\cdot)\|_{L^2(0,T)}$. To show this, let $\{x_n\} \subset \mathbb{R}$ be a sequence that converges to $x \in \mathbb{R}$. Applying similar arguments to those in (3.76), we can write

$$\begin{aligned}\|v_1(x,\cdot)-v_1(x_n,\cdot)\|_{L^2(0,T)} &\leq \left(\int_0^T \left(\int_{a-\varepsilon}^{b+\varepsilon} |i\xi|^2 \left(e^{ix\xi}-e^{ix_n\xi}\right)|\theta(\xi)|^2 d\xi\right)^2 dt\right)^{\frac{1}{2}} \\ &\lesssim \sqrt{T}\|\phi^*\|_{L^2(\mathbb{R})} < \infty.\end{aligned} \tag{3.84}$$

Furthermore, similar calculations to those in (3.81)-(3.82), we can also write

$$\begin{aligned}&\|v_2(x,\cdot)-v_2(x_n,\cdot)\|_{L^2(0,T)} \\ \leq &\left(\int_{-\infty}^{\omega(a)} \left| i\xi_-(\tau)\left(e^{i\xi_-(\tau)x}-e^{i\xi_-(\tau)x_n}\right)(1-\theta(\xi_-(\tau)))\frac{\hat{\phi}^*(\xi_-(\tau))}{3\beta\xi_-^2(\tau)-2\alpha\xi_-(\tau)-\delta}\right|^2 d\tau\right)^{\frac{1}{2}} \\ \leq &\|\phi^*\|_{L^2(\mathbb{R})} < \infty.\end{aligned} \tag{3.85}$$

Combining (3.84) and (3.85), we have $\|v(x,\cdot)-v(x_n,\cdot)\|_{L^2(0,T)} < \infty$. Hence, by the dominated convergence theorem we also have

$$\|v_1(x,\cdot)-v_1(x_n,\cdot)\|_{L^2(0,T)} \to 0, \quad \text{as } n \to \infty, \tag{3.86}$$

which yields that the mapping $x \to \|\partial_x v(x,\cdot)\|_{L^2(0,T)}$ is continuous. Note that the right hand side of (3.83) is independent of x. So passing to supremum on both sides in x over $[0, L]$ finishes the proof. $\square$

Lemma 3.6 *Let $T > 0$ and $f^* \in L^1(0, T; L^2(\mathbb{R}))$. Then, solution y of* (3.71) *satisfies*

$$\sup_{x\in[0,L]} \|\partial_x y(x,\cdot)\|_{L^2(0,T)} \lesssim (1 + \sqrt{T})\|f^*\|_{L^1(0,T;L^2(0,L))}. \tag{3.87}$$

Proof By Duhamel's principle, we can represent y as

$$y(x,t) = \int_0^t S_{\mathbb{R}}(t-\tau)f^*(x,\tau)d\tau, \tag{3.88}$$

where $S_{\mathbb{R}}$ is given by (3.74). By differentiating (3.88) with respect to its first component, we can write

$$\begin{aligned} |\partial_x y(x,t)| &= \left|\partial_x \left[\int_0^t S(t-\tau)f(x,\tau)d\tau\right]\right| \\ &\leq \int_0^t \left|\partial_x \left[S(t-\tau)f^*(x,\tau)\right]\right| d\tau \\ &\leq \int_0^T \left|\partial_x \left[S(t-\tau)f^*(x,\tau)\right]\right| d\tau. \end{aligned} \tag{3.89}$$

Taking L^2-norm of both sides in $t \in [0, T]$ and using the result in Lemma 3.5, we get

$$\begin{aligned} \|\partial_x y(x,\cdot)\|_{L^2(0,T)} &\leq \left\|\int_0^T \left|\partial_x \left[S(\cdot-\tau)f^*(x,\tau)\right]\right| d\tau\right\|_{L^2(0,T)} \\ &\leq \int_0^T \left\|\partial_x \left[S(\cdot-\tau)f^*(x,\tau)\right]\right\|_{L^2(0,T)} d\tau \\ &\lesssim (1+\sqrt{T})\int_0^T \|f^*(\cdot,\tau)\|_{L^2(\mathbb{R})} d\tau = (1+\sqrt{T})\|f^*\|_{L^1(0,T;L^2(\mathbb{R}))}. \end{aligned} \tag{3.90}$$

Application of dominated convergence theorem as in (3.84)-(3.86) provides the continuity of the mapping $x \to \|\partial_x y(x,\cdot)\|_{L^2(0,T)}$. Now the last line of (3.90) is independent of x. So passing to supremum in x over $[0, L]$ finishes the proof. $\square$

Now let us turn our attention to the inhomogeneous boundary value problem (3.72) with the boundary conditions ψ_m, $m = 1, 2, 3$. We prove the following result.

Lemma 3.7 *Let $T > 0$ and $(\psi_1, \psi_2, \psi_3) \in H^{1/3}(0,T) \times H^{1/3}(0,T) \times L^2(0,T)$. Then, solution z of* (3.72) *satisfies*

$$\sup_{x\in[0,L]} \|\partial_x z(x,\cdot)\|_{L^2(0,T)} \lesssim \|\psi_1\|_{H^{1/3}(0,T)} + \|\psi_2\|_{H^{1/3}(0,T)} + \|\psi_3\|_{L^2(0,T)}. \tag{3.91}$$

Remark 3.1 *Note that the boundary conditions imposed to $z-$model* (3.72) *are not arbitrary functions, and are given by*

$$\psi_1(t) = -v(0,t) - y(0,t),\ \psi_2(t) = -v(L,t) - y(L,t),\ \textit{and}\ \psi_3(t) = -v_x(L,t) - y_x(L,t),$$

i.e., in terms of the solutions of $v-$model (3.70) *and $y-$model* (3.71)*. Recall that if the initial state ϕ^* of the $v-$model belongs to $L^2(\mathbb{R})$ and the inhomogeneous source f^* of the $y-$model belongs to $L^1(0,T;L^2(\mathbb{R}))$, then $v_x(L,t)$ and $y_x(L,t)$ make sense in $L^2(0,T)$ by Lemma 3.5 and Lemma 3.6, respectively. On the other hand, our calculations through* (3.127)-(3.132) *show that, $v(x,\cdot), y(x,\cdot)$ also make sense in $H^{1/3}(0,T)$ for $x = 0$ and $x = L$. Hence the assumption of Lemma 3.7 is indeed true.*

Remark 3.2 *A classical approach to investigate initial-boundary value problems with inhomogeneous boundary data is to apply a change of variable so that, the boundary effects of the original model take place as an interior source in the transformed model. For instance, considering the following change*

$$g(x,t) = \left(1 - \frac{x}{L}\right)\psi_1(t) + \frac{x}{L}\psi_2(t) + \left(x - \frac{x^2}{L}\right)(-\psi_1(t) + \psi_2(t) - \psi_3(t)),$$

and defining $\zeta := z - g$, one can derive that the $\zeta-$model given below

$$\begin{cases} i\zeta_t + i\beta\zeta_{xxx} + \alpha\zeta_{xx} + i\delta\zeta_x = -ig_t - \alpha g_{xx} - i\delta g_x, & (x,t) \in (0,L)\times(0,T), \\ \zeta(0,t) = \zeta(L,t) = \zeta_x(L,t) = 0, & t \in (0,T), \\ \zeta(x,0) = -g(x,0), & x \in (0,L), \end{cases} \tag{3.92}$$

has homogeneous boundary conditions. Then, analysis for $z-$model (3.72) *can be carried out via analyzing the $\zeta-$model by applying the arguments in Section 3.3.1.1. To guarantee that a mild solution for the model* (3.92) *exists, right hand side of the main equation of* (3.92) *needs to belong $L^1(0,T;L^2(0,T))$. That is, regarding the boundary conditions of the $z-$model* (3.72)*, it is required that*

$$\psi_1' = -\frac{d}{dt}\left(v(0,t) + y(0,t)\right), \quad \psi_2' = -\frac{d}{dt}\left(v(L,t) + y(L,t)\right) \tag{3.93}$$

and

$$\psi_3' = -\frac{d}{dt}\left(v_x(L,t) + y_x(L,t)\right) \tag{3.94}$$

make sense in $L^1(0,T)$. However, we do not have such result. Instead, applying Laplace transform method makes the Lemma 3.7 true, where by Remark 3.1 its assumptions are also true, which is necessary for us to deduce the time estimates (3.68) *and* (3.69).

To establish Lemma 3.7, first we express solution of the $z-$model in a suitable form in order to perform the analysis. In view of Remark 3.2, we perform this by applying the Laplace transformation

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t)dt, \tag{3.95}$$

for the $z-$model (3.72) in the temporal variable. For convenience, we extend the boundary conditions ψ_m, $m = 1, 2, 3$ (still denoted by same notations), from $(0,T)$ to $\mathbb{R}$, providing their smoothness, i.e., regarding their extensions, $\psi_m \in H^{1/3}(\mathbb{R})$ for $m = 1, 2$ and $\psi_3 \in L^2(\mathbb{R})$ holds.

Let us apply the Laplace transform for the model (3.72). Then, we obtain the following one parameter family of third-order boundary value problems

$$\begin{cases} is\tilde{z}(x,s) + i\beta\tilde{z}_{xxx}(x,s) + \alpha\tilde{z}_{xx}(x,s) + i\delta\tilde{z}_x(x,s) = 0, & (x,s) \in (0,L)\times\mathbb{C}, \\ \tilde{z}(0,s) = \tilde{\psi}_1(s), \tilde{z}(L,s) = \tilde{\psi}_2(s), \tilde{z}_x(L,s) = \tilde{\psi}_3(s), & s \in \mathbb{C}. \end{cases} \tag{3.96}$$

Here s belongs to a suitable set in the complex plane which will be specified below. Using the Bromwich integral, z can be represented as

$$z(x,t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\tilde{z}(x,s)ds, \tag{3.97}$$

where the vertical integration path $(r - i\infty, r + i\infty)$ in the complex plane belongs to the region of absolute convergence. That is, r is chosen so that, all possible singularities of $\tilde{z}$ lie on the left side of the vertical line $(r - i\infty, r + i\infty)$. Note that for sufficiently large values of r, the characteristic equation,

$$s + \beta\lambda^3 - i\alpha\lambda^2 + \delta\lambda = 0, \tag{3.98}$$

associated to (3.96) has distinct roots. In fact, there exists only finitely many values of s for which (3.98) assumes double or possibly triple roots (see Appendix B for details). We can classify these cases depending on the the quantity $\alpha^2 + 3\beta\delta$. Let $\lambda_j = \lambda_j(s)$, $j = 1, 2, 3$, denote the roots of (3.98) and assume that $\lambda_2(s) = \lambda_3(s)$ for some $s \in \mathbb{C}$. Then direct calculations given in Appendix B yield the following cases.

(i) If $\alpha^2 + 3\beta\delta > 0$, then there exists only two possible values of s and these values belong to the imaginary axis.

(ii) If $\alpha^2 + 3\beta\delta = 0$, then we have one and only one possible value of s and this value belongs to the imaginary axis. Note that for this value of s, we have $\lambda_1(s) = \lambda_2(s) = \lambda_3(s)$.

(iii) If $\alpha^2 + 3\beta\delta < 0$, then there exists only two possible values of s and these values are symmetric with respect to the imaginary axis.

Consequently, for sufficiently large r, it is guaranteed that (3.98) has distinct roots on the line $\mathrm{Re}(s) = r$, hence solution of (3.96) can be expressed of the form

$$\tilde{z}(x,s) = \sum_{j=1}^{3} c_j(s)e^{\lambda_j(s)x}, \tag{3.99}$$

where the column vector $(c_1(s), c_2(s), c_3(s))^T$ is the solution of the linear system

$$\begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1(s)L} & e^{\lambda_2(s)L} & e^{\lambda_3(s)L} \\ \lambda_1(s)e^{\lambda_1(s)L} & \lambda_2(s)e^{\lambda_2(s)L} & \lambda_3(s)e^{\lambda_3(s)L} \end{pmatrix} \begin{pmatrix} c_1(s) \\ c_2(s) \\ c_3(s) \end{pmatrix} = \begin{pmatrix} \tilde{\psi}_1(s) \\ \tilde{\psi}_2(s) \\ \tilde{\psi}_3(s) \end{pmatrix}. \tag{3.100}$$

Applying Cramer's rule, c_j's can be obtained as $c_j = \frac{\Delta_j}{\Delta}$, where Δ is the determinant of the coefficient matrix in (3.100) and Δ_j's are determinants of the matrices formed by replacing the $j-$th column of the coefficient matrix by the column vector $(\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3)^T$. Thus z can be written of the form

$$z(x,t) = \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_j(s)}{\Delta(s)}e^{\lambda_j(s)x}ds. \tag{3.101}$$

For convenience, let us rewrite $z = z[\psi_1, \psi_2, \psi_3]$ as $z \equiv \sum_{m=1}^{3} z_m$, where z_m solves the same problem with boundary data $\psi_j \equiv 0$ if $j \neq m$, $j = 1, 2, 3$. Thus z_m's can be expressed as

$$z_m(x,t) = \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{\psi}_m(s)ds, \quad m = 1, 2, 3. \tag{3.102}$$

Here $\Delta_{j,m}$'s, $m = 1, 2, 3$, are obtained from Δ_j, where $\tilde{\psi}_j$ is replaced by 1 for $j = m$ and $\tilde{\psi}_j$'s are replaced by 0 for each $j \neq m$.

To change the integration path in (3.102) by a more convenient one, one needs to investigate possible zeros of $\Delta(s)$ in the complex plane. These points occur not only due to the double or possibly triple roots of (3.98), but may also occur due to the eigenvalues of the operator A defined in (3.49) with domain $D(A)$ defined in (3.50). Recall that the operator A satisfies

$$\mathrm{Re}(A\varphi, \varphi)_2 = \mathrm{Re}\left\{\int_0^L (-\beta\varphi''' + i\alpha\varphi'' - \delta\varphi')(x)\overline{\varphi}(x)dx\right\} = -\frac{\beta|\varphi'(0)|^2}{2} \leq 0.$$

Thus, all eigenvalues of A lie on the left complex half plane or possibly on the imaginary axis. The latter situation occurs only if $\varphi'(0) = 0$, i.e., the problem

$$\begin{cases} -\beta\varphi''' + i\alpha\varphi'' - \delta\varphi' = \lambda\varphi, & \text{in } (0, L), \\ \varphi(0) = \varphi(L) = \varphi'(0) = \varphi'(L), \end{cases} \tag{3.103}$$

has nontrivial solutions for some pure imaginary λ. From the characteristic equation

$$-\beta p^3 + i\alpha p^2 - \delta p = \lambda$$

associated to the eigenvalue problem (3.103) together with the boundary conditions, one can obtain that the roots p_j, $j = 1, 2, 3$, must be distinct, i.e., $\varphi(x) = \sum_{j=1}^{3} d_1 e^{p_j x}$. Employing the boundary conditions and after some calculations, one can see that p_j's must satisfy

$$e^{p_1 L} = e^{p_2 L} = e^{p_3 L}$$

(see Proposition 2 in (Glass and Guerrero, 2010) for similar calculations). Therefore, we have

$$p_2 - p_1 = \frac{2k\pi i}{L},$$
$$p_3 - p_2 = \frac{2l\pi i}{L},$$

where without loss of generality, upon relabeling p_j's we can assume that $k, l \in \mathbb{Z}^+$. Using also $p_1 + p_2 + p_3 = \frac{i\alpha}{\beta}$, we get

$$p_1 = \frac{i\alpha}{3\beta} + \frac{2(-2k - l)\pi i}{3L},$$
$$p_2 = \frac{i\alpha}{3\beta} + \frac{2(k - l)\pi i}{3L},$$
$$p_3 = \frac{i\alpha}{3\beta} + \frac{2(k + 2l)\pi i}{3L}.$$

Substituting these into $p_1p_2 + p_1p_3 + p_2p_3 = \frac{\delta}{\beta}$ and after some calculations, we obtain

$$\frac{\delta}{\beta} = -\frac{\alpha^2}{3\beta^2} + \frac{4\pi^2(k^2+kl+l^2)}{3L^2}$$

or equivalently

$$\alpha^2 + 3\beta\delta = \frac{4\pi^2\beta^2(k^2+kl+l^2)}{L^2}. \tag{3.104}$$

Consequently, depending on the value of $(\alpha^2 + 3\beta\delta)$ and the interval length L, it is possible to obtain a nontrivial solution of (3.103). This implies the existence of possible eigenvalues of the operator A lying on the imaginary axis.

(i) Let $\alpha^2 + 3\beta\delta > 0$. Then, choosing L from the set of critical lengths defined in (1.9) implies that (3.104) holds, i.e., existence of pure imaginary eigenvalues. Now from the equation $p_1p_2p_3 = -\frac{\lambda}{\beta}$ and using $L = 2\pi\beta\sqrt{\frac{k^2+kl+l^2}{\alpha^2+3\beta\delta}}$, one can obtain after some calculations that

$$\lambda = \frac{i}{27\beta^2}\left[\alpha^3 - 3\alpha(\alpha^2+3\beta\delta) + 2(\alpha^2+3\beta\delta)^{3/2}H(k,l)\right] \tag{3.105}$$

where

$$H(k,l) = \frac{(-2k-l)(k-l)(k+2l)}{2(k^2+kl+l^2)^{\frac{3}{2}}}. \tag{3.106}$$

It is not difficult to see that $-1 < H(k,l) < 1$ for $k,l \in \mathbb{Z}^+$. Thus, from (3.105), we deduce that $\mathrm{Im}(\lambda) \in \left(\mathrm{Im}s_2^+, \mathrm{Im}s_1^+\right)$, where

$$s_1^+ := \frac{i}{27\beta^2}\left[\alpha^3 - 3\alpha(\alpha^2+3\beta\delta) + 2(\alpha^2+3\beta\delta)^{3/2}\right]$$

and

$$s_2^+ := \frac{i}{27\beta^2}\left[\alpha^3 - 3\alpha(\alpha^2+3\beta\delta) - 2(\alpha^2+3\beta\delta)^{3/2}\right].$$

On the other hand s_1^+ and s_2^+ are also the points where (3.96) assumes double root (see Appendix B for detailed calculations), hence zeros of $\Delta(s)$. This fact together with the location of the possible pure imaginary eigenvalues imply that all possible singular points of (3.101) belonging to the imaginary axis lie in the closed interval $\left[\mathrm{Im}(s_2^+), \mathrm{Im}(s_1^+)\right]$. Thus, we can deform the vertical integration path of (3.101) by shifting the parts $\left\{s \,|\, \mathrm{Im}(s) > \mathrm{Im}(s_1^+) + \rho\right\}$ and $\left\{s \,|\, \mathrm{Im}(s) < \mathrm{Im}(s_2^+) - \rho\right\}$ to $C_1^+ := \left(s_1^+ + i\rho, i\infty\right)$ and $C_3^+ := \left(-i\infty, s_2^+ - i\rho\right)$, respectively, with $\rho > 0$ is fixed and sufficiently small, whereas we shift the rest of the integration path up to ρ units from the imaginary axis, by avoiding the points s_1^+, s_2^+ by a quarter-circular arcs to the upper-right and lower-right, respectively, denoted by C_2^+. See Figure 3.4 below.

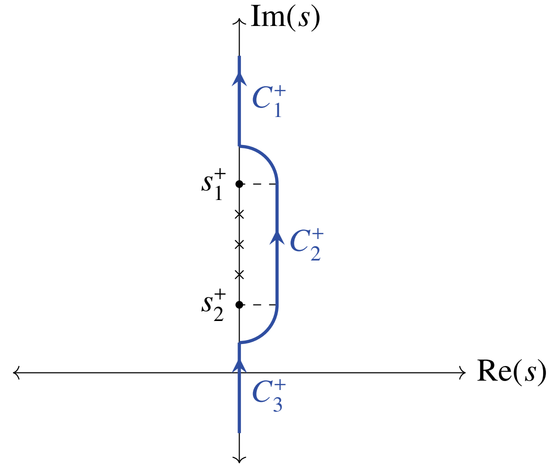

Figure 3.4. Integration path for the case $\alpha^2 + 3\beta\delta > 0$.

With this new setting of integration path, we express (3.102) as

$$
\begin{aligned}
z_m(x,t) =& \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{C_1^+} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{\psi}_m(s)ds \\
&+ \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{C_2^+} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{\psi}_m(s)ds \\
&+ \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{C_3^+} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{\psi}_m(s)ds,
\end{aligned}
\tag{3.107}
$$

for each $m = 1, 2, 3$. Now we change the variable in the first and the third integral as $s = i\omega(\xi) = i(\beta\xi^3 - \alpha\xi^2 - \delta\xi)$. Note that, for $\alpha^2 + 3\beta\delta > 0$, the function $\omega(\xi)$ has one local maximum and one local minimum. After some calculations, we find that the larger inverse image of s_1^+ and the smaller inverse image of s_2^+ as

$$\xi_1^+ := \frac{\alpha + 2\sqrt{\alpha^2 + 3\beta\delta}}{3\beta}, \quad \xi_2^+ := \frac{\alpha - 2\sqrt{\alpha^2 + 3\beta\delta}}{3\beta},$$

respectively (see Figure 3.5). Thus, the mapping $s = i\omega(\xi)$ is injective for $s \in C_1^+$ and $s \in C_3^+$, and their inverse images are $\left(\xi_1^+ + \eta_1^+, \infty\right)$ and $\left(-\infty, \xi_2^+ - \eta_2^+\right)$, respectively for $\eta_1^+, \eta_2^+ > 0$, that depend on ρ. Consequently, (3.107) becomes

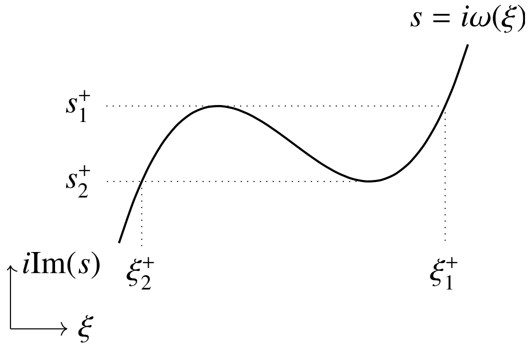

Figure 3.5. Plot of transformation $s = i\omega(\xi)$ when $\alpha^2 + 3\beta\delta > 0$.

$$\begin{aligned} z_m(x,t) =& \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{\xi_1^+ + \eta_1^+}^{\infty} e^{i\omega(\xi)t}\frac{\Delta_{j,m}^{\dagger}(\xi)}{\Delta^{\dagger}(\xi)}e^{\lambda_j^{\dagger}(\xi)x}(3\beta\xi^2 - 2\alpha\xi - \delta)\tilde{\psi}_m^{\dagger}(\xi)d\xi \\ &+ \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{C_2^+} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{\psi}_m(s)ds \\ &+ \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{-\infty}^{\xi_2^+ - \eta_2^+} e^{i\omega(\xi)t}\frac{\Delta_{j,m}^{\dagger}(\xi)}{\Delta^{\dagger}(\xi)}e^{\lambda_j^{\dagger}(\xi)x}(3\beta\xi^2 - 2\alpha\xi - \delta)\tilde{\psi}_m^{\dagger}(\xi)d\xi \\ =:& z_{m,1}^+(x,t) + z_{m,2}^+(x,t) + z_{m,3}^+(x,t), \end{aligned} \tag{3.108}$$

where † notation stands for the transformed functions in the new variable. Note also that, with this new variable, we have the following explicit representation for the roots of the characteristic equation (3.98):

$$\begin{aligned} \lambda_1^\dagger(\xi) &= i\xi, \\ \lambda_2^\dagger(\xi) &= \frac{-i(\beta\xi-\alpha) - \sqrt{3\beta^2\xi^2 - 2\beta\alpha\xi - \alpha^2 - 4\beta\delta}}{2\beta}, \\ \lambda_3^\dagger(\xi) &= \frac{-i(\beta\xi-\alpha) + \sqrt{3\beta^2\xi^2 - 2\beta\alpha\xi - \alpha^2 - 4\beta\delta}}{2\beta}. \end{aligned} \tag{3.109}$$

(ii) Let $\alpha^2 + 3\beta\delta = 0$. Then (3.104) does not hold for any $k, l \in \mathbb{Z}^+$, which implies the eigenvalue problem (3.103) does not have a nontrivial solution for a pure imaginary eigenvalue. Therefore $\mathrm{Re}(A\varphi, \varphi)_2 < 0$ and real parts of the all eigenvalues of the operator A are strictly negative. On the other hand

$$s^0 = \frac{i\alpha^3}{27\beta^2}$$

is the point where (3.98) assumes triple root (see Appendix B for details). Thus the integrand of (3.102) is continuous for all $r > 0$ and we can shift the contour of integration onto the imaginary axis, provided that we avoid s^0 by a half-circular arc to the right, denoted by C_2^0, with a radius $\rho > 0$ (see Figure 3.6 below). Defining also $C_1^0 := \left(s^0 + i\rho, i\infty\right)$ and $C_3^0 := \left(-i\infty, s^0 - i\rho\right)$,

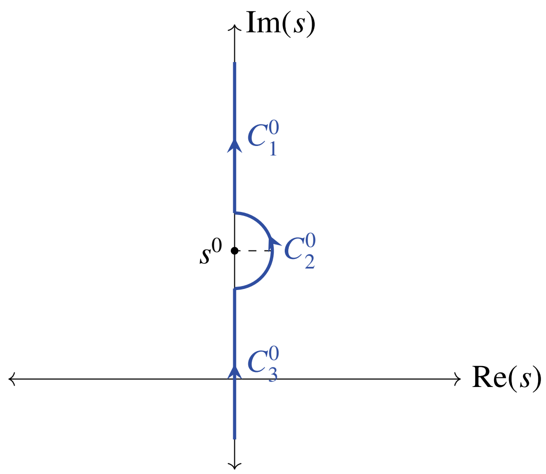

Figure 3.6. Integration path for the case $\alpha^2 + 3\beta\delta = 0$.

we can express (3.102) as

$$
\begin{aligned}
z_m(x,t) =& \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{C_1^0} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{\psi}_m(s)ds \\
&+ \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{C_2^0} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{\psi}_m(s)ds \qquad (3.110)\\
&+ \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{C_3^0} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{\psi}_m(s)ds.
\end{aligned}
$$

Now let us consider changing variables as $s = i\omega(\xi) = i(\beta\xi^3 - \alpha\xi^2 - \delta\xi)$ for the first and the third integrals. For $\alpha^2 + 3\beta\delta = 0$, note that $\omega(\xi)$ is nondecreasing and $\left(\xi^0, s^0\right)$ is the inflection point of $\omega(\xi)$, where recall that $s = s^0$ is also the point for which (3.98) assumes a tripple root. After some calculations, ξ^0 can be obtained as

$$\xi^0 = \frac{\alpha}{3\beta}.$$

See Figure 3.7 for a graphical illustration. Hence for $s \in C_1^0$ and $s \in C_3^0$, $s = i\omega(\xi)$

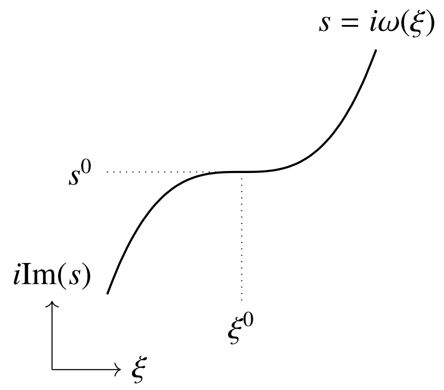

Figure 3.7. Plot of transformation $s = i\omega(\xi)$ when $\alpha^2 + 3\beta\delta = 0$.

is injective. Let us denote their inverse images as $\left(\xi^0 + \eta_1^0, \infty\right)$ and $\left(-\infty, \xi^0 - \eta_2^0\right)$, respectively for some $\eta_1^0, \eta_2^0 > 0$, that depend on ρ. Thus (3.110) becomes

$$\begin{aligned}
z_m(x,t) =& \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{\xi^0+\eta_1^0}^{\infty} e^{i\omega(\xi)t}\frac{\Delta_{j,m}^{\dagger}(\xi)}{\Delta^{\dagger}(\xi)}e^{\lambda_j^{\dagger}(\xi)x}(3\beta\xi^2-2\alpha\xi-\delta)\tilde{\psi}_m^{\dagger}(\xi)d\xi \\
&+\frac{1}{2\pi i}\sum_{j=1}^{3}\int_{C_2^+} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{\psi}_m(s)ds \\
&+\frac{1}{2\pi i}\sum_{j=1}^{3}\int_{-\infty}^{\xi^0-\eta_2^0} e^{i\omega(\xi)t}\frac{\Delta_{j,m}^{\dagger}(\xi)}{\Delta^{\dagger}(\xi)}e^{\lambda_j^{\dagger}(\xi)x}(3\beta\xi^2-2\alpha\xi-\delta)\tilde{\psi}_m^{*}(\xi)d\xi \\
=:& z_{m,1}^0(x,t)+z_{m,2}^0(x,t)+z_{m,3}^0(x,t),
\end{aligned} \tag{3.111}$$

where $\lambda_j^{\dagger}$'s now become

$$\begin{aligned}
\lambda_1^{\dagger}(\xi) &= i\xi, \\
\lambda_2^{\dagger}(\xi) &= \frac{-i(\beta\xi-\alpha)+\sqrt{3}\beta\left|\xi-\xi^0\right|}{2\beta}, \\
\lambda_3^{\dagger}(\xi) &= \frac{-i(\beta\xi-\alpha)-\sqrt{3}\beta\left|\xi-\xi^0\right|}{2\beta}.
\end{aligned} \tag{3.112}$$

(iii) Let $\alpha^2+3\beta\delta<0$. Then (3.104) does not hold for any $k, l \in \mathbb{Z}^+$ and all eigenvalues lie on the left half complex plane. On the other hand, there exits two values of s for which (3.98) assumes double root. These values, say s_1^- and s_2^- with $\mathrm{Re}(s_1^-) > 0 > \mathrm{Re}(s_2^-)$ which are symmetric with respect to the imaginary axis (see Appendix B), are also the branch points of the square root function

$$\sqrt{(s-s_1^-)(s-s_2^-)},$$

where we choose the branch cut as $\left\{s\in\mathbb{C}\,|\,\mathrm{Im}(s)=\mathrm{Im}(s_1^-), \mathrm{Re}(s_2^-)<\mathrm{Re}(s)<\mathrm{Re}(s_1^-)\right\}$. Indeed changing variables as $s = i\mathrm{Im}(s_1^-)+r$ and then performing some calculations, roots of the characteristic equation (3.98) can be expressed as

$$\lambda_j^{\dagger}(r) = \frac{1}{3\beta}\left(i\alpha - \frac{\alpha^2+3\beta\delta}{\Lambda_j(r)} + \Lambda_j(r)\right),$$

where

$$\Lambda_j(r) = -3\beta^{\frac{2}{3}} e^{\frac{2\pi i j}{3}} \left(\frac{1 - \sqrt{r^2 + \frac{4(\alpha^2 + 3\beta\delta)}{729\beta^4}}}{2} \right)^{\frac{1}{3}}, \quad j = 1, 2, 3.$$

Now $\mathrm{Re}(s_1^-)$ and $\mathrm{Re}(s_2^-)$ are the zeros of the square root above.

In conclusion, what distinguishes this case from the previous cases is that, we have now a zero of $\Delta(s)$ that lies on the right half complex plane which is the right end-point of the branch cut. Therefore, to deform the integration path, we first shift the vertical integration line to the left until we meet s_1^-. Then we deform a part of the path near s_1^- by a half-circular arc to the right with a radius $\rho > 0$. Next we deform the rest of the integration path as, first by horizontal line segments to the left starting from the end points of the arc through the imaginary axis and second continuing from the imaginary axis in the vertical direction towards $(+i\infty)$ and $(-i\infty)$, respectively. See Figure 3.8 for the path deformation described here

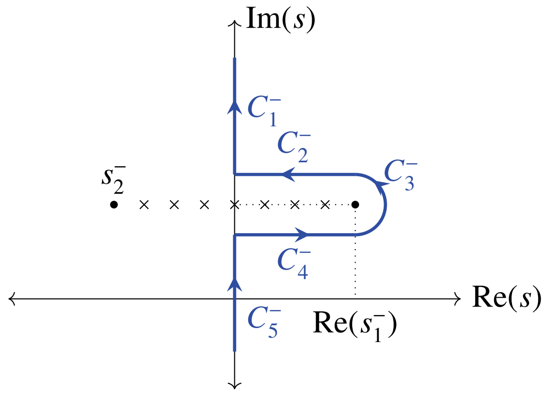

Figure 3.8. Integration path for the case $\alpha^2 + 3\beta\delta < 0$.

Consequently, we can write (3.102) as

$$
\begin{aligned}
z_m(x,t) =& \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{C_1^-} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{\psi}_m(s)ds \\
&+ \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{C_2^-\cup C_3^-\cup C_4^-} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{\psi}_m(s)ds \qquad (3.113)\\
&+ \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{C_5^-} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{\psi}_m(s)ds.
\end{aligned}
$$

Now let us apply change of variable $s = i\omega(\xi) = i(\beta\xi^3 - \alpha\xi^2 - \delta\xi)$ for the first and third integrals. Note that for $\alpha^2 + 3\beta\delta < 0$, this mapping is strictly increasing $s = i\omega(\xi)$ is injective. The inverse image of $\mathrm{Im}(s_1^-)$ under the transformation $\omega(\xi)$ is the point

$$\xi^- = \frac{\alpha}{3\beta}.$$

So the inverse images of C_1^- and C_5^- are $\left(\xi^- + \eta_1^-, \infty\right)$ and $\left(-\infty, \xi^- - \eta_2^-\right)$, respectively, for some $\eta_1^-, \eta_2^- > 0$ that depend on ρ. See Figure 3.9.

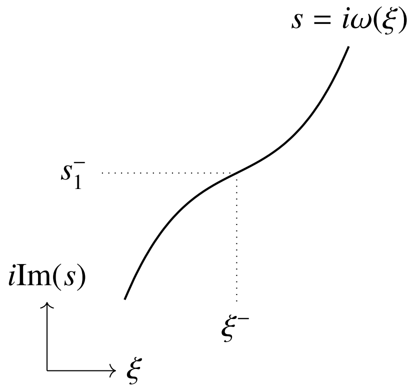

Figure 3.9. Plot of transformation $s = i\omega(\xi)$ when $\alpha^2 + 3\beta\delta < 0$.

Thus (3.113) becomes

$$
\begin{aligned}
z_m(x,t) =& \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{\xi^-+\eta_1^-}^{\infty} e^{i\omega(\xi)t}\frac{\Delta^\dagger_{j,m}(\xi)}{\Delta^\dagger(\xi)}e^{\lambda_j^\dagger(\xi)x}(3\beta^2\xi^2-2\alpha\xi-\delta)\tilde{\psi}^\dagger_m(\xi)d\xi \\
&+\frac{1}{2\pi i}\sum_{j=1}^{3}\int_{C_2^-\cup C_3^-\cup C_4^-} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{\psi}_m(s)ds \\
&+\frac{1}{2\pi i}\sum_{j=1}^{3}\int_{-\infty}^{\xi^--\eta_2^-} e^{i\omega(\xi)t}\frac{\Delta^\dagger_{j,m}(\xi)}{\Delta^\dagger(\xi)}e^{\lambda_j^\dagger(\xi)x}(3\beta^2\xi^2-2\alpha\xi-\delta)\tilde{\psi}^\dagger_m(\xi)d\xi \\
=:& z^-_{m,1}(x,t)+z^-_{m,2}(x,t)+z^-_{m,3}(x,t),
\end{aligned}
\tag{3.114}
$$

where $\lambda_j^\dagger$'s are given by (3.109).

In what follows, we provide estimates for z_m for each m, $m = 1, 2, 3$, respectively. Note that for each solution representation (3.108), (3.111) and (3.114) associated to the different cases of $(\alpha^2+3\beta\delta)$, second integrals are bounded on the corresponding integration paths. Therefore, to obtain desired norm estimates for each z_m, it is enough to study the first and third integrals on their representations (3.108), (3.111) and (3.114), respectively. For the sake of simplicity of the text, we denote those integrals as

$$
I_m(x,t) = \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{\gamma_1}^{\infty} e^{i\omega(\xi)t}\frac{\Delta^\dagger_{j,m}(\xi)}{\Delta^\dagger(\xi)}e^{\lambda_j^\dagger(\xi)x}(3\beta\xi^2-2\alpha\xi-\delta)\tilde{\psi}^\dagger_m(\xi)d\xi \tag{3.115}
$$

and

$$
J_m(x,t) = \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{-\infty}^{\gamma_2} e^{i\omega(\xi)t}\frac{\Delta^\dagger_{j,m}(\xi)}{\Delta^\dagger(\xi)}e^{\lambda_j^\dagger(\xi)x}(3\beta\xi^2-2\alpha\xi-\delta)\tilde{\psi}^\dagger_m(\xi)d\xi, \tag{3.116}
$$

where $\gamma_1 \in \left\{\xi_1^+ + \eta_1^+, \xi^0+\eta_1^0, \xi^- + \eta_1^-\right\}$ and $\gamma_2 \in \left\{\xi_2^+ + \eta_2^+, \xi^0+\eta_2^0, \xi^- + \eta_2^-\right\}$.

Proof Let us start with the case $m = 1$ and let us first obtain large ξ asymptotic behaviours of the ratios $\left|\frac{\Delta^\dagger_{j,1}(\xi)}{\Delta^\dagger(\xi)}\right|$. Using the relation $\lambda_1^\dagger + \lambda_2^\dagger + \lambda_3^\dagger = \frac{i\alpha}{\beta}$, we have

$$\Delta^\dagger(\xi) = e^{\frac{i\alpha L}{\beta}} \Big(e^{-\lambda_1^\dagger(\xi)L}(\lambda_3^\dagger(s) - \lambda_2^\dagger(\xi)) \\ -e^{-\lambda_2^\dagger(\xi)L}(\lambda_3^\dagger(\xi) - \lambda_1^\dagger(\xi)) + e^{-\lambda_3^\dagger(\xi)L}(\lambda_2^\dagger(\xi) - \lambda_1^\dagger(\xi))\Big) \quad (3.117)$$

and

$$\begin{aligned} \Delta^\dagger_{1,1}(\xi) &= e^{\frac{i\alpha L}{\beta}} e^{-\lambda_1^\dagger(\xi)L}\left(\lambda_3^\dagger(\xi) - \lambda_2^\dagger(\xi)\right), \\ \Delta^\dagger_{2,1}(\xi) &= e^{\frac{i\alpha L}{\beta}} e^{-\lambda_2^\dagger(\xi)L}\left(\lambda_1^\dagger(\xi) - \lambda_3^\dagger(\xi)\right), \\ \Delta^\dagger_{3,1}(\xi) &= e^{\frac{i\alpha L}{\beta}} e^{-\lambda_3^\dagger(\xi)L}\left(\lambda_2^\dagger(\xi) - \lambda_1^\dagger(\xi)\right). \end{aligned} \quad (3.118)$$

Using the roots of the characteristic equation (3.109) in the variable ξ, we obtain the following large ξ asymptotics

$$\left|\frac{\Delta^\dagger_{j,1}(\xi)}{\Delta^\dagger(\xi)}\right| \sim \begin{cases} e^{-\dfrac{\sqrt{3}\xi L}{2}}, & j = 1, \\ 1, & j = 2, \\ e^{-\sqrt{3}\xi L}, & j = 3. \end{cases} \quad (3.119)$$

To show that the mapping $x \in [0, L] \to \|\partial_x I_1(x, \cdot)\|_{L^2(0,T)}$ is continuous, let $\{x_n\} \subset [0, L]$ be a sequence such that $x_n \to x$ as $n \to \infty$. Let us differentiate I_1 with respect to its first component and write

$$\partial_x I_1(x,t) - \partial_x I_1(x_n,t) \\ = \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{\gamma_1}^{\infty} e^{i\omega(\xi)t}\lambda_j^\dagger(\xi)\left(e^{\lambda_j^\dagger(\xi)x} - e^{\lambda_j^\dagger(\xi)x_n}\right)\frac{\Delta^\dagger_{j,1}(\xi)}{\Delta^\dagger(\xi)}\tilde{\psi}_1^\dagger(\xi)d\xi. \quad (3.120)$$

Taking L^2-norm in the second component and changing variables as $\mu = \beta\xi^3 - \alpha\xi^2 - \delta\xi$

to get

$$\|\partial_x I_1(x,\cdot) - \partial_x I_1(x_n,\cdot)\|^2_{L^2(0,T)}$$
$$= \left\| \sum_{j=1}^{3} \frac{1}{2\pi} \int_{\gamma_1}^{\infty} e^{i\omega(\xi)t} \lambda_j^\dagger(\xi) \left(e^{\lambda_j^\dagger(\xi)x} - e^{\lambda_j^\dagger(\xi)x_n} \right) \frac{\Delta_{j,1}^\dagger(\xi)}{\Delta^\dagger(\xi)} (3\beta\xi^2 - 2\alpha\xi - \delta) \tilde{\psi}_1^\dagger(\xi) d\xi \right\|^2_{L^2(0,T)}$$
$$\lesssim \sum_{j=1}^{3} \left\| \int_{\omega(\gamma_1)}^{\infty} e^{i\mu t} \lambda_j^\dagger(\theta(\mu)) \left(e^{\lambda_j^\dagger(\theta(\mu))x} - e^{\lambda_j^\dagger(\theta(\mu))x_n} \right) \frac{\Delta_{j,1}^\dagger(\theta(\mu))}{\Delta^\dagger(\theta(\mu))} \tilde{\psi}_1^\dagger(\theta(\mu)) d\mu \right\|^2_{L^2(0,T)}, \tag{3.121}$$

where $\theta(\mu)$ is the real solution of $\mu = \beta\xi^3 - \alpha\xi^2 - \delta\xi$ for $\gamma_1 < \xi < \infty$ (recall that it exists as $\mu = \beta\xi^3 - \alpha\xi^2 - \delta\xi$ is injective on its respective domain for each case of $(\alpha^2 + 3\beta\delta)$. See Figure 3.5, Figure 3.7 and Figure 3.9.) Observe that the function

$$\begin{cases} \lambda_j^\dagger(\theta(\mu)) \left(e^{\lambda_j^\dagger(\theta(\mu))x} - e^{\lambda_j^\dagger(\theta(\mu))x_n} \right) \dfrac{\Delta_{j,1}^\dagger(\theta(\mu))}{\Delta^\dagger(\theta(\mu))} \tilde{\psi}_1^\dagger(\theta(\mu)), & \mu \in (\omega(\gamma_1), \infty), \\ 0, & \text{elsewhere}, \end{cases}$$

is the Fourier transform of the function given by the integral on the last line of (3.121). Therefore, using Plancherel's theorem, we can write

$$\|\partial_x I_1(x,\cdot) - \partial_x I_1(x_n,t)\|^2_{L^2(0,T)} \lesssim$$
$$\sum_{j=1}^{3} \int_{\omega(\gamma_1)}^{\infty} \left| \lambda_j^\dagger(\theta(\mu)) \left(e^{\lambda_j^\dagger(\theta(\mu))x} - e^{\lambda_j^\dagger(\theta(\mu))x_n} \right)^2 \frac{\Delta_{j,1}^\dagger(\theta(\mu))}{\Delta^\dagger(\theta(\mu))} \tilde{\psi}_1^\dagger(\theta(\mu)) \right|^2 d\mu \tag{3.122}$$

for all $x \in [0, L]$. Changing variables back to ξ, we get

$$\|\partial_x I_1(x,\cdot) - \partial_x I_1(x_n,\cdot)\|^2_{L^2(0,T)} \lesssim$$
$$\sum_{j=1}^{3} \int_{\gamma_1}^{\infty} |\lambda_j^\dagger(\xi)|^2 \left(e^{\lambda_j^\dagger(\xi)x} - e^{\lambda_j^\dagger(\xi)x_n} \right)^2 \left| \frac{\Delta_{j,1}^\dagger(\xi)}{\Delta^\dagger(\xi)} \right|^2 (3\beta\xi^2 - 2\alpha\xi - \delta) \left| \tilde{\psi}_1^\dagger(\xi) \right|^2 d\xi. \tag{3.123}$$

Using (3.109) and (3.119), one can obtain the following asymptotic behaviours in ξ

$$
\begin{aligned}
|\lambda_j^\dagger(\xi)|^2 \left(e^{\lambda_j^\dagger(\xi)x} - e^{\lambda_j^\dagger(\xi)x_n}\right)^2 \left|\frac{\Delta_{j,1}^\dagger(\xi)}{\Delta^\dagger(\xi)}\right|^2 &\lesssim \\
|\lambda_j^\dagger(\xi)|^2 \sup_{x\in[0,L]} \left(e^{2\mathrm{Re}(\lambda_j^\dagger(\xi))x}\right) \left|\frac{\Delta_{j,1}^\dagger(\xi)}{\Delta^\dagger(\xi)}\right|^2 &\sim \begin{cases} \xi^2 e^{-\sqrt{3}\xi L}, & j=1, \\ \xi^2, & j=2, \\ \xi^2 e^{-\sqrt{3}\xi L}, & j=3. \end{cases}
\end{aligned}
\tag{3.124}
$$

Then, following from (3.123), we can write

$$
\|\partial_x I_1(x,\cdot) - \partial_x I_1(x_n,\cdot)\|^2_{L^2(0,T)} \lesssim \int_{\gamma_1}^{\infty} \xi^2(3\beta\xi^2 - 2\alpha\xi - \delta)\left|\tilde{\psi}_1^\dagger(\xi)\right|^2 d\xi. \tag{3.125}
$$

Changing variables back to μ via the relation $\mu = \beta\xi^3 - \alpha\xi^2 - \delta\xi$, we conclude

$$
\begin{aligned}
\|\partial_x I_1(x,\cdot) - \partial_x I_1(x_n,\cdot)\|^2_{L^2(0,T)} &\lesssim \int_{\gamma_1}^{\infty} \xi^2(3\beta\xi^2 - 2\alpha\xi - \delta)\left|\tilde{\psi}_1^\dagger(\xi)\right|^2 d\xi \\
&\simeq \int_{\omega(\gamma_1)}^{\infty} (1+\mu^2)^{\frac{1}{3}} \left|\int_0^\infty e^{-i\mu\tau}\psi_1(\tau)d\tau\right|^2 d\mu \\
&\lesssim \|\psi_1\|^2_{H^{1/3}(0,T)} < \infty.
\end{aligned}
\tag{3.126}
$$

Hence, by the dominated convergence theorem

$$
\|\partial_x I_1(x,\cdot) - \partial_x I_1(x_n,\cdot)\|^2_{L^2(0,T)} \to 0 \quad \text{as } n\to\infty,
$$

i.e., the mapping $x \in [0,L] \to \|\partial_x I_1(x,\cdot)\|_{L^2(0,T)}$ is continuous and

$$
\sup_{x\in[0,L]} \|\partial_x I_1(x,\cdot)\|_{L^2(0,T)} \lesssim \|\psi_1\|^2_{H^{1/3}(0,T)}
$$

holds. Applying similar arguments yield the same result for J_1, hence for $z_1 = z[\psi_1, 0, 0]$.

Regarding the solutions $z_2 = z[0, \psi_2, 0]$ and $z_3 = z[0, 0, \psi_3]$, we have

$$\Delta^\dagger_{1,2}(\xi) = \lambda^\dagger_2(\xi)e^{\lambda^\dagger_2(\xi)L} - \lambda^\dagger_3(\xi)e^{\lambda^\dagger_3(\xi)L},$$
$$\Delta^\dagger_{2,2}(\xi) = \lambda^\dagger_3(\xi)e^{\lambda^\dagger_3(\xi)L} - \lambda^\dagger_1(\xi)e^{\lambda^\dagger_1(\xi)L},$$
$$\Delta^\dagger_{3,2}(\xi) = \lambda^\dagger_1(\xi)e^{\lambda^\dagger_1(\xi)L} - \lambda^\dagger_2(\xi)e^{\lambda^\dagger_2(\xi)L},$$

with large ξ asymptotics

$$\left|\frac{\Delta^\dagger_{j,2}(\xi)}{\Delta^\dagger(\xi)}\right| \sim \begin{cases} 1, & j = 1, \\ 1, & j = 2, \\ e^{-\frac{\sqrt{3}\xi L}{2}}, & j = 3, \end{cases}$$

and

$$\Delta^\dagger_{1,3}(\xi) = e^{\lambda^\dagger_3(\xi)L} - e^{\lambda^\dagger_2(\xi)L},$$
$$\Delta^\dagger_{2,3}(\xi) = e^{\lambda^\dagger_1(\xi)L} - e^{\lambda^\dagger_3(\xi)L},$$
$$\Delta^\dagger_{3,3}(\xi) = e^{\lambda^\dagger_2(\xi)L} - e^{\lambda^\dagger_1(\xi)L},$$

with large ξ asymptotics

$$\left|\frac{\Delta^\dagger_{j,3}(\xi)}{\Delta^\dagger(\xi)}\right| \sim \begin{cases} \xi^{-1}, & j = 1, \\ \xi^{-1}, & j = 2, \\ \xi^{-1}e^{-\dfrac{\sqrt{3}\xi L}{2}}, & j = 3, \end{cases}$$

respectively. Considering these changes and using similar arguments to those in (3.120)-(3.125), one can also obtain the continuity of the mappings $x \in [0, L] \to \|\partial_x z_2(x, \cdot)\|_{L^2(0,T)}$ and $x \in [0, L] \to \|\partial_x z_3(x, \cdot)\|_{L^2(0,T)}$ with the time estimates

$$\sup_{x\in[0,L]} \|\partial_x y_2(x, \cdot)\|_{L^2(0,T)} \lesssim \|\psi_2\|^2_{H^{1/3}(0,T)}$$

and

$$\sup_{x\in[0,L]} \|\partial_x y_3(x,\cdot)\|_{L^2(0,T)} \lesssim \|\psi_3\|^2_{L^2(0,T)},$$

respectively. □

Recall that the boundary conditions of the problem (3.72) are given by $\psi_1(t) = -v(0,t) - y(0,t)$, $\psi_2(t) = -v(L,t) - y(L,t)$ and $\psi_3(t) = -v_x(L,t) - y_x(L,t)$. So to fulfill the main purpose of this section, i.e., the validity of the estimates (3.68) and (3.69), we need to show that these trace terms are uniformly bounded by $\|\phi\|_{L^2(0,L)}$ and $\|f\|_{L^1(0,T;L^2(0,L))}$, respectively, on their respective norms. First, note that Lemma 3.5 and Lemma 3.6 directly imply

$$\|\partial_x v(L,t)\|_{L^2(0,T)} \lesssim (1+\sqrt{T})\|\phi\|_{L^2(0,L)} \tag{3.127}$$

and

$$\|\partial_x y(L,t)\|_{L^2(0,T)} \lesssim (1+\sqrt{T})\|f\|_{L^1(0,T;L^2(0,L))}, \tag{3.128}$$

respectively. Regarding the Dirichlet type boundary conditions, applying interpolation argument for (3.78), we can write

$$\|v_1(x,\cdot)\|_{H^{1/3}(0,T)} \lesssim \|\phi^*\|_{L^2(\mathbb{R})}, \tag{3.129}$$

where recall that $v := v_1 + v_2$ is defined in (3.75) and $v_2 := v_{2-} + v_{2+}$ is defined in (3.79). Applying similar argument to those in (3.80)-(3.81), we can obtain that

$$\begin{aligned}\|v_{2-}(x,\cdot)\|^2_{H^{1/3}(0,T)} &\le \|v_{2-}(x,\cdot)\|^2_{H^{1/3}(\mathbb{R})}\\ &= \int_{-\infty}^{\infty} (1+\tau^2)^{\frac{1}{3}} |\hat{v}_{2-}(x,\tau)|^2 d\tau\\ &\le \int_{-\infty}^{\omega(a)} (1+\tau^2)^{\frac{1}{3}} |\frac{|\hat{\phi}^*(\xi_-(\tau))|^2}{|3\beta\xi_-^2(\tau) - 2\alpha\xi_-(\tau) - \delta|^2} d\tau.\end{aligned}$$

Changing variables as $\tau = \omega_-(\xi)$, it follows that

$$\begin{aligned} \|v_{2-}(x,\cdot)\|^2_{H^{1/3}(0,T)} &\leq \int_{-\infty}^{a} \left(1+\omega^2(\xi)\right)^{\frac{1}{3}} \frac{|\hat{\phi}^*(\xi)|^2}{|3\beta\xi^2-2\alpha\xi-\delta|^2}(3\beta\xi^2-2\alpha\xi-\delta)d\xi \\ &\simeq \int_{-\infty}^{a} \left(1+\xi^6\right)^{\frac{1}{3}} \frac{|\hat{\phi}^*(\xi)|^2}{3\beta\xi^2-2\alpha\xi-\delta}d\xi \\ &\simeq \int_{-\infty}^{a} |\hat{\phi}^*(\xi)|^2 d\xi \leq \|\phi^*\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{3.130}$$

$\|v_{2+}(x,\cdot)\|^2_{H^{1/3}(0,T)} \lesssim \|\phi^*\|^2_{L^2(\mathbb{R})}$ can be shown similarly. Combining this together with (3.129) and (3.130), we deduce

$$\|v(x,\cdot)\|_{H^{1/3}(0,T)} \lesssim \|\phi^*\|_{L^2(\mathbb{R})}.$$

Continuity of the mapping $x \to \|v(x,\cdot)\|_{H^{1/3}(0,T)}$ can be shown by the dominated convergence theorem. Hence we get

$$\|v(0,\cdot)\|_{H^{1/3}(0,T)} \lesssim \|\phi^*\|_{L^2(\mathbb{R})} \quad \text{and} \quad \|v(L,\cdot)\|_{H^{1/3}(0,T)} \lesssim \|\phi^*\|_{L^2(\mathbb{R})}. \tag{3.131}$$

Regarding the solution y of (3.71), similar arguments also yield that $x \to \|y(x,\cdot)\|_{H^{1/3}(0,T)}$ is continuous and,

$$\|y(0,\cdot)\|_{H^{1/3}(0,T)} \lesssim \|f^*\|_{L^1(0,T;L^2(\mathbb{R}))} \quad \text{and} \quad \|y(L,\cdot)\|_{H^{1/3}(0,T)} \lesssim \|f^*\|_{L^1(0,T;L^2(\mathbb{R}))}, \tag{3.132}$$

hold.

As a conclusion, using smoothing properties for the Cauchy problem (3.70) obtained in Lemma 3.5, for the inhomogeneous initial-boundary value problem obtained in

Lemma 3.6, together with trace results given by (3.131) and (3.132), we conclude that

$$\begin{aligned}\sup_{x\in[0,L]}\|\partial_x q(x,\cdot)\|_{L^2(0,T)} \le &\sup_{x\in[0,L]}\left\|\partial_x\left[S(\cdot)\phi^*(x)\right]\right\|_{L^2(0,T)} + \sup_{x\in[0,L]}\left\|\partial_x\left[\int_0^{\cdot} S_{\mathbb{R}}(\cdot-\tau)f^*(x,\tau)d\tau\right]\right\|_{L^2(0,T)}\\ &+ \sup_{x\in[0,L]}\|\partial_x z(x,\cdot)\|_{L^2(0,T)}\\ \lesssim&(1+\sqrt{T})\left(\|\phi^*\|_{L^2(\mathbb{R})}+\|f^*\|_{L^1(0,T;L^2(\mathbb{R}))}\right)\\ &+\|\psi_1\|_{H^{1/3}(0,T)}+\|\psi_2\|_{H^{1/3}(0,T)}+\|\psi_3\|_{L^2(0,T)}\\ \lesssim&(1+\sqrt{T})\left(\|\phi^*\|_{L^2(\mathbb{R})}+\|f^*\|_{L^1(0,T;L^2(\mathbb{R}))}\right).\end{aligned}$$

Hence we obtain the following lemma, which summarizes the main results proved in this section.

Lemma 3.8 *Let $T > 0$, $\phi \in L^2(0,L)$ and $f \in L^1(0,T;L^2(0,L))$. Then the semigroup $\{S(t)\}_{t\ge0}$ generated by the operator A given by* (3.49) *with domain* (3.50) *enjoys the following estimates*

$$\sup_{x\in[0,L]}\|\partial_x[S(\cdot)\phi(x)]\|_{L^2(0,T)} \lesssim (1+\sqrt{T})\|\phi\|_{L^2(0,L)} \tag{3.133}$$

and

$$\sup_{x\in[0,L]}\left\|\partial_x\left[\int_0^{\cdot} S(\cdot-\tau)f(x,\tau)d\tau\right]\right\|_{L^2(0,T)} \lesssim (1+\sqrt{T})\|f\|_{L^1(0,T;L^2(0,L))}, \tag{3.134}$$

respectively.

3.3.2. Local well-posedness

With the help of the estimates we derived in Section 3.3.1, now we prove existence of a unique solution w of the target model (3.16) which is local in time. In view of what we presented through (3.51)-(3.53), it suffices to prove existence of a fixed point of the operator

$$\Psi[w_0, \varphi(t)] := S(t)w_0 + \int_0^t S(t-\tau)F(\varphi(\tau))d\tau \tag{3.135}$$

defined on Y_T. This will be shown by applying Banach fixed point theorem below in Proposition 3.2.

Proposition 3.2 *Let $T' > 0$ and $w_0 \in L^2(0, L)$. Then, there exists $T \in (0, T')$ which is independent of size of w_0 such that* (3.16) *possesses a unique local solution $w \in Y_T$.*

Proof Let us first show that Ψ maps Y_T into itself. To see this, first we can write from (3.135) that

$$\|\Psi[w_0, \varphi]\|_{Y_T} \leq \|S(t)w_0\|_{Y_T} + \left\| \int_0^t S(t-\tau)F\varphi(\tau)d\tau \right\|_{Y_T}. \tag{3.136}$$

Using the results in Lemma 3.4 and Lemma 3.8, the first term at the right hand side of (3.136) can be estimated as

$$\|S(t)w_0\|_{Y_T} \lesssim (1 + \sqrt{T})\|w_0\|_{L^2(0,L)}, \tag{3.137}$$

and the second term at the right hand side of (3.136) can be estimated as

$$\begin{aligned} \left\| \int_0^t S(t-s)F\varphi(s)ds \right\|_{Y_T} &\lesssim (1 + \sqrt{T})\| - r\varphi + \beta k_y(\cdot, 0)\varphi_x(0, \cdot)\|_{L^1(0,T;L^2(0,L))} \\ &\leq \sqrt{T}(1 + \sqrt{T})\Big(r\|\varphi\|_{L^2(0,T;L^2(0,L))} \\ &\quad + \beta\|k_y(\cdot, 0)\|_{L^2(0,L)}\|\varphi_x(0, \cdot)\|_{L^2(0,T)}\Big), \end{aligned} \tag{3.138}$$

where we applied Cauchy-Schwarz inequality on the last line. Note that, $\partial_x\varphi(0, \cdot)$ makes sense in $L^2(0, T)$ since $\varphi \in Y_T$. Therefore, it follows from (3.138) that

$$\left\| \int_0^t S(t-s)Fz(s)ds \right\|_{Y_T} \leq c_{\beta,r}\sqrt{T}(1 + \sqrt{T})\left(1 + \|k_y(\cdot, 0)\|_{L^2(0,L)}\right)\|\varphi\|_{Y_T}. \tag{3.139}$$

Combining (3.137) and (3.139), we see that Ψ maps Y_T into itself.

To see that Ψ is a strict contraction on Y_T, let $\varphi_1, \varphi_2 \in Y_T$ and define $w_1 := \Psi\varphi_1$,

$w_2 := \Psi\varphi_2$. Using the similar arguments as in (3.136)-(3.139), we can write

$$\begin{aligned}\|w_2 - w_1\|_{Y_T} &= \|\Psi\varphi_2 - \Psi\varphi_1\|_{Y_T} \\ &\leq c_{\beta,r}\sqrt{T}(1+\sqrt{T})(1+\|k(\cdot,0)\|_{L^2(0,L)})\|\varphi_2 - \varphi_1\|_{Y_T}.\end{aligned}$$

In order for the map Ψ to be a strict contraction on Y_T, we choose $T \in (0, T')$ such that

$$0 < \sqrt{T}(1+\sqrt{T}) < \left(c_{\beta,r}\left(1+\|k(\cdot,0)\|_{L^2(0,L)}\right)\right)^{-1},$$

which is independent of the size of the initial datum. Now Banach fixed point theorem guarantees that the operator (3.135) has a unique fixed point, i.e., for a given $w_0 \in L^2(0,L)$ and for some $T \in (0,T')$, there exists a unique $w \in Y_T$ such that $w(t;w_0) = \Psi[w_0, w(t;w_0)]$ holds for all $t \in [0,T]$. □

3.3.3. Global well-posedness

Proposition 3.2 given in the previous section provides an existence of a maximal time, say $T_{\max}$ with $T_{\max} < T'$, so that unique solution exists in Y_T for all $T < T_{\max}$. In order to show that w extends globally, i.e. $T_{\max}$ can be taken arbitrarily large, it suffices to prove that $\|w\|_{Y_T} < \infty$ as $T \to T_{\max}^-$. We prove this below in Proposition 3.3.

Proposition 3.3 *Let $w_0 \in L^2(0,L)$. Then solution w of* (3.16) *extends as a global solution in Y_T.*

Proof Multiplying the main equation of (3.16) by $2w$ in $L^2(0,L)$ and following the arguments in (3.35)-(3.39), we can write

$$\begin{aligned}&\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2(0,L)} + \beta|w_x(0,t)|^2 + 2r\,\|w(\cdot,t)\|^2_{L^2(0,L)} \\ &\qquad = 2\beta\mathrm{Re}\int_0^L k_y(x,0)w_x(0,t)\overline{w}(x,t)dx. \quad (3.140)\end{aligned}$$

By using Cauchy's inequality with $\varepsilon > 0$, right hand side of (3.140) can be estimated as

$$2\beta \mathrm{Re} \int_0^L k_y(x,0) w_x(0,t) \overline{w}(x,t) dx \leq \frac{\beta}{2\varepsilon} \int_0^L |k_y(x,0)|^2 |w(x,t)|^2 dx + 2\varepsilon\beta L |w_x(0,t)|^2.$$

Choosing $\varepsilon = \frac{1}{4L}$ and then integrating both sides in the second component over $(0,t)$, (3.140) becomes

$$\begin{aligned} &2\|w(\cdot,t)\|^2_{L^2(0,L)} + \beta \int_0^t |w_x(0,\tau)|^2 d\tau \\ &\qquad \leq 2\|w_0\|^2_{L^2(0,L)} + 4(\beta L \|k_y(\cdot,0)\|^2_{C^\infty([0,L])} - r) \int_0^t \|w(\cdot,\tau)\|^2_{L^2(0,L)} d\tau. \end{aligned} \tag{3.141}$$

Define $E(t) := 2\|w(\cdot,t)\|^2_{L^2(0,L)} + \beta \int_0^t |w_x(0,\tau)|^2 d\tau$. Then, from (3.141),

$$E(t) \leq 2\|w_0\|^2_{L^2(0,L)} + 4 \left| \beta L \|k_y(\cdot,0)\|^2_{C^\infty([0,L])} - r \right| \int_0^t E(\tau) d\tau.$$

Applying Gronwall's inequality, we obtain

$$E(t) = 2\|w(\cdot,t)\|^2_{L^2(0,L)} + \beta \int_0^t |w_x(0,\tau)|^2 d\tau \leq 2\|w_0\|^2_{L^2(0,L)} e^{4\left|\beta L \|k_y(\cdot,0)\|^2_{C^\infty([0,L])} - r\right| t}, \tag{3.142}$$

for all $t \in [0,T]$. Passing to supremum on $[0,T]$ and then letting $T \to T^-_{\max}$, we get

$$\lim_{T \to T_{\max}} \|w\|_{C([0,T];L^2(0,L))} \leq \|w_0\|_{L^2(0,L)} e^{2\left|\beta L \|k_y(\cdot,0)\|^2_{C^\infty([0,L])} - r\right| T_{\max}} < \infty. \tag{3.143}$$

Using

$$\sup_{0 \leq t \leq T} \|w(\cdot,t)\|^2_{L^2(0,L)} \geq \frac{1}{T} \|w\|^2_{L^2(0,T;L^2(0,L))} \tag{3.144}$$

and then letting $T \to T^-_{\max}$, we also get

$$\lim_{T \to T^-_{\max}} \|w\|_{L^2(0,T;L^2(0,L))} \leq \sqrt{T_{\max}} \|w_0\|_{L^2(0,L)} e^{2\left|\beta L \|k_y(\cdot,0)\|^2_{C^\infty([0,L])} - r\right| T_{\max}}. \tag{3.145}$$

Next, we multiply the main equation by $2x\overline{w}$, integrate over $(0,t)\times(0,L)$ and applying the arguments (3.60)-(3.64)

$$\begin{aligned}&\int_0^L x|w(x,t)|^2dx + 3\beta\int_0^t\int_0^L |w_x(x,\tau)|^2dxd\tau + 2r\int_0^L x|w(x,t)|^2dx\\ =&\int_0^L x|w_0(x)|^2dx + \delta\int_0^t\int_0^L |w(x,\tau)|^2dxd\tau\\ &+2\beta\int_0^t\int_0^L xk_y(x,0)w_x(0,\tau)\overline{w}(x,\tau)dxd\tau.\end{aligned}\tag{3.146}$$

Using Cauchy-Schwarz inequality, the last term at the right hand side of (3.146) can be estimated as

$$\begin{aligned}&2\beta\int_0^t\int_0^L xk_y(x,0)(x)w_x(0,\tau)\overline{y}(x,\tau)dxd\tau\\ \le&2\beta L\int_0^t\int_0^L |k_y(x,0)||w_x(0,\tau)||w(x,\tau)|dxd\tau\\ \le&\beta L^2\|k_y(\cdot,0)\|^2_{C^\infty([0,L])}\int_0^t |w_x(0,\tau)|^2d\tau+\beta L\int_0^t\int_0^L |w(x,\tau)|^2dxd\tau.\end{aligned}\tag{3.147}$$

Dropping the first and third terms at the left hand side of (3.146), and using the estimate (3.147), it follows that

$$\begin{aligned}&\|w_x\|^2_{L^2(0,t;L^2(0,L))}\\ \le&\frac{L}{3\beta}\|w_0\|^2_{L^2(0,L)}+\frac{\beta L+\delta}{3\beta}\int_0^t\int_0^L |w(x,\tau)|^2dxd\tau+\frac{L^2\|k_y(\cdot,0)\|^2_{C^\infty([0,L])}}{3}\int_0^t |y_x(0,\tau)|^2d\tau\\ \le&\frac{L}{3\beta}\|w_0\|^2_{L^2(0,L)}+\left(\frac{\beta L+\delta}{6\beta}+\frac{L^2\|k_y(\cdot,0)\|^2_{C^\infty([0,L])}}{3\beta}\right)\int_0^t E(\tau)d\tau.\end{aligned}$$

Passing to limit $T\to T_{\max^-}$ and using (3.142), we get

$$\begin{aligned}&\lim_{T\to T_{\max}}\|w_x\|_{L^2(0,T;L^2(0,L))}\le\sqrt{\frac{L}{3\beta}}\|w_0\|_{L^2(0,L)}\\ &+\left(\sqrt{\frac{\beta L+\delta}{6\beta}}+\frac{L\|k_y(\cdot,0)\|_{C^\infty([0,L])}}{\sqrt{3\beta}}\right)\sqrt{2T_{\max}}\|w_0\|_{L^2(0,L)}e^{2\left|\beta L\|k_y(\cdot,0)\|^2_{C^\infty([0,L])}-r\right|T_{\max}}.\end{aligned}\tag{3.148}$$

Combining the limits (3.145) and (3.148), we deduce that

$$\lim_{T\to T_{\max}^-} \|w\|_{L^2(0,T;H^1(0,L))} \leq \sqrt{\frac{L}{3\beta}}\|w_0\|_{L^2(0,L)} + \left(1 + \sqrt{\frac{\beta L+\delta}{6\beta}} + \frac{L\|k_y(\cdot,0)\|_{C^\infty([0,L])}}{\sqrt{3\beta}}\right) \times \sqrt{2T_{\max}}\|w_0\|_{L^2(0,L)} e^{2\left|\beta L\|k_y(\cdot,0)\|^2_{C^\infty([0,L])} - r\right| T_{\max}} < \infty. \quad (3.149)$$

As a last step, recall from Proposition 3.2 that w is the unique fixed point of the operator (3.135). So it satisfies

$$w(\cdot,t) = S(t)w_0 + \int_0^t S(t-\tau)F(w(\cdot,\tau))d\tau$$

for some $t \in (0,T')$. From Lemma 3.8, we can write

$$\begin{aligned} \sup_{x\in[0,L]} \|\partial_x w(x,\cdot)\|_{L^2(0,T)} &\leq \sup_{x\in[0,L]} \left\| S(\cdot)w_0 + \int_0^\cdot S(\cdot-\tau)[Fw](\tau)d\tau \right\|_{L^2(0,T)} \\ &\lesssim (1+\sqrt{T})\left(\|w_0\|_{L^2(0,L)} + \int_0^T \|[Fw](\cdot,t)\|_{L^2(0,L)}dt\right). \end{aligned} \quad (3.150)$$

Using the definition of Fw, right hand side of (3.150) can be estimated as

$$\begin{aligned} \int_0^T \|[Fw](\cdot,t)\|_{L^2(0,L)}dt &\leq r\int_0^T \|w(\cdot,t)\|_{L^2(0,L)}dt + \beta\|k(\cdot,0)\|_{L^2(0,L)} \int_0^T |w_x(0,t)|dt \\ &\leq \frac{r}{2}\int_0^T \sqrt{E(t)}dt + \sqrt{T\beta}\|k(\cdot,0)\|_{L^2(0,L)}\sqrt{E(t)} \\ &\leq \left(\frac{Tr}{2} + \sqrt{T\beta}\|k(\cdot,0)\|_{L^2(0,L)}\right) \\ &\quad \times \sqrt{2}\|w_0\|_{L^2(0,L)} e^{2\left|\beta L\|k_y(\cdot,0)\|^2_{C^\infty([0,L])} - r\right| T}. \end{aligned} \quad (3.151)$$

Finally using (3.150)-(3.151)

$$\begin{aligned} \lim_{T\to T_{\max}^-} \|w_x\|_{C([0,L];L^2(0,T))} &\lesssim (1+\sqrt{T_{\max}})\|w_0\|_{L^2(0,L)} + (1+\sqrt{T_{\max}}) \\ &\times \left(\frac{T_{\max}r}{2} + \sqrt{T_{\max}\beta}\|k(\cdot,0)\|_{L^2(0,L)}\right)\sqrt{2}\|w_0\|_{L^2(0,L)} e^{2\left|\beta L\|k_y(\cdot,0)\|^2_{C^\infty([0,L])} - r\right| T_{\max}} < \infty. \end{aligned}$$

This completes the proof. □

3.3.4. Higher regularity

In this section, we prove that if $w_0 \in D(A)$, where the operator A and its domain $D(A)$ are defined by (3.49) and (3.50), respectively, then the global solution w of (3.16) enjoys higher regularity.

Proposition 3.4 *Let $w_0 \in D(A)$. Then the global solution $w \in Y_T^0$ of* (3.16) *also belongs to X_T^3.*

Proof Assume v solves the linear model below

$$\begin{cases} iv_t + i\beta v_{xxx} + \alpha v_{xx} + i\delta v_x + irv = i\beta k_y(x,0)v_x(0,t), & (x,t) \in (0,L)\times(0,T), \\ v(0,t) = v(L,t) = v_x(L,t) = 0, & t \in (0,T), \\ v(x,0) = v_0(x), & x \in (0,L), \end{cases} \tag{3.152}$$

where v_0 is set as $v_0 := -\beta w_0''' + i\alpha w_0'' - \delta w_0' - rw_0 + \beta k_y(\cdot,0)w_0'(0)$. Assume also that $v_0 \in L^2(0,L)$. Then from Proposition 3.2 and Proposition 3.3, $v \in Y_T^0$. Set $w(x,t) := w_0(x) + \int_0^t v(x,\tau)d\tau$. Then, w satisfies the initial condition

$$w(x,0) = w_0(x) \tag{3.153}$$

and noting that $w_0 \in D(A)$, w also satisfies the homogeneous boundary conditions

$$\begin{aligned} w(0,t) &= w_0(0) + \int_0^t v(0,\tau)d\tau = w_0(x) + w(0,t) - w(0,0) = 0, \\ w(L,t) &= w_0(L) + \int_0^t v(L,\tau)d\tau = w_0(L) + w(L,t) - w(L,0) = 0, \\ w_x(L,t) &= w_0'(L) + \int_0^t v_x(L,\tau)d\tau = w_0'(L) + w_x(L,t) - w_x(L,0) = 0. \end{aligned} \tag{3.154}$$

Moreover from $w(x,t) = w_0(x) + \int_0^t v(x,\tau)d\tau$, we see that w satisfies the main equation of

(3.16)

$$\begin{aligned}
&(iw_t + i\beta w_{xxx} + \alpha w_{xx} + i\delta w_x + irw)(x,t) - i\beta k_y(x,0)w_x(0,t)) \\
=&iv(x,t) + i\beta w_0'''(x) + \alpha w_0''(x) + i\delta w_0'(x) + irw_0(x) - i\beta k_y(x,0)w_0'(0) \\
&+ \int_0^t \Big((i\beta v_{xxx} + \alpha v_{xx} + i\delta v_x + irv)(x,\tau) - i\beta k_y(x,0)v_x(0,t)\Big)d\tau \\
=&iv(x,t) + i\beta w_0'''(x) + \alpha w_0''(x) + i\delta w_0'(x) + irw_0(x) - i\beta k_y(x,0)w_0'(0) \qquad (3.155)\\
&- \int_0^t iv_t(x,\tau)d\tau \\
=&iv(x,t) + i\beta w_0'''(x) + \alpha w_0''(x) + i\delta w_0'(x) + irw_0(x) - i\beta k_y(x,0)w_0'(0) \\
&- iv(x,t) + iv(x,0) = 0.
\end{aligned}$$

Thus w, defined by $w(x,t) = w_0(x) + \int_0^t v(x,\tau)d\tau$, solves the model (3.16). Now from the main equation, we have

$$i\beta w_{xxx}(x,t) = (-iv - \alpha w_{xx} - i\delta w_x - irw)(x,t) + i\beta k_y(x,0)w_x(0,t). \qquad (3.156)$$

Since $w_x(L,t) = 0$, we can write $w_x(0,t) = -\int_0^L w_{xx}(x,t)dx$. Using this and taking L^2-norms of both sides of (3.156) on $(0,L)$, we get

$$\begin{aligned}
\beta^2\|w_{xxx}(\cdot,t)\|^2_{L^2(0,L)} \le \|v(\cdot,t)\|^2_{L^2(0,L)} &+ \left(\alpha^2 + \beta^2\|k_y(\cdot,0)\|^2_{L^2(0,L)}\right)\|w_{xx}(\cdot,t)\|^2_{L^2(0,L)} \\
&+ \delta^2\|w_x(\cdot,t)\|^2_{L^2(0,L)} + r^2\|w(\cdot,t)\|^2_{L^2(0,L)}. \qquad (3.157)
\end{aligned}$$

Applying Gagliardo-Nirenberg interpolation inequality and then Cauchy's inequality with $\varepsilon > 0$ to the second term at the right hand side of

$$\begin{aligned}
\|w_{xx}(\cdot,t)\|^2_{L^2(0,L)} &\le \|w_{xxx}(\cdot,t)\|^{\frac{4}{3}}_{L^2(0,L)}\|w(\cdot,t)\|^{\frac{2}{3}}_{L^2(0,L)} \\
&\le \varepsilon\|w_{xxx}(\cdot,t)\|^2_{L^2(0,L)} + c_\varepsilon\|w(\cdot,t)\|^2_{L^2(0,L)}.
\end{aligned} \qquad (3.158)$$

Similarly, the third term can be estimated as

$$
\begin{aligned}
\|w_x(\cdot,t)\|^2_{L^2(0,L)} &\le \|y_{xxx}(\cdot,t)\|^{\frac{2}{3}}_{L^2(0,L)}\|y(\cdot,t)\|^{\frac{4}{3}}_{L^2(0,L)} \\
&\le \varepsilon\|w_{xxx}(\cdot,t)\|^2_{L^2(0,L)} + c_\varepsilon\|w(\cdot,t)\|^2_{L^2(0,L)}.
\end{aligned} \tag{3.159}
$$

Employing the estimates (3.158)-(3.159) on (3.157) and choosing $\varepsilon > 0$ sufficiently small, we can write

$$
\|w_{xxx}(\cdot,t)\|^2_{L^2(0,L)} \lesssim \|v(\cdot,t)\|^2_{L^2(0,L)} + \|w(\cdot,t)\|^2_{L^2(0,L)}. \tag{3.160}
$$

Passing to supremum in t over $[0,T]$ and using the fact that the right hand side of (3.160) belongs to $C([0,T],L^2(0,L))$, we get $w \in C([0,T];H^3(0,L))$.

Next, we differentiate the main equation of (3.156) in the first component and taking L^2-norms of both sides on $(0,L)$, we get

$$
\begin{aligned}
\beta^2\|w_{xxxx}(\cdot,t)\|^2_{L^2(0,L)} &\le \|v_x(\cdot,t)\|^2_{L^2(0,L)} + \alpha^2\|w_{xxx}(\cdot,t)\|^2_{L^2(0,L)} \\
&\quad + \left(\delta^2 + \beta^2\|k_{yx}(\cdot,0)\|^2_{L^2(0,L)}\right)\|w_{xx}(\cdot,t)\|^2_{L^2(0,L)} + r^2\|w_x(\cdot,t)\|^2_{L^2(0,L)}.
\end{aligned} \tag{3.161}
$$

Applying Gagliardo–Nirenberg interpolation inequality, Cauchy's inequality with $\varepsilon > 0$ and Poincaré inequality, the second term at the right hand side of (3.161) can be estimated as

$$
\begin{aligned}
\|w_{xxx}(\cdot,t)\|^2_{L^2(0,L)} &\le \|w_{xxxx}(\cdot,t)\|^{\frac{3}{2}}_{L^2(0,L)}\|w(\cdot,t)\|^{\frac{1}{2}}_{L^2(0,L)} \\
&\le \varepsilon\|w_{xxxx}(\cdot,t)\|^2_{L^2(0,L)} + c_\varepsilon\|w(\cdot,t)\|^2_{L^2(0,L)} \\
&\le \varepsilon\|w_{xxxx}(\cdot,t)\|^2_{L^2(0,L)} + c_{\varepsilon,L}\|w_x(\cdot,t)\|^2_{L^2(0,L)}.
\end{aligned} \tag{3.162}
$$

Similarly, the third term of (3.161) can be estimated as

$$
\begin{aligned}
\|w_{xx}(\cdot,t)\|^2_{L^2(0,L)} &\le \|w_{xxxx}(\cdot,t)\|_{L^2(0,L)}\|w(\cdot,t)\|_{L^2(0,L)} \\
&\le \varepsilon\|w_{xxxx}(\cdot,t)\|^2_{L^2(0,L)} + c_\varepsilon\|w(\cdot,t)\|^2_{L^2(0,L)} \\
&\le \varepsilon\|w_{xxxx}(\cdot,t)\|^2_{L^2(0,L)} + c_{\varepsilon,L}\|w_x(\cdot,t)\|^2_{L^2(0,L)}.
\end{aligned} \tag{3.163}
$$

Using (3.162)-(3.163) on (3.161) and choosing $\varepsilon > 0$ sufficiently small, we get

$$\|w_{xxxx}(\cdot,t)\|^2_{L^2(0,L)} \lesssim \|v_x(\cdot,t)\|^2_{L^2(0,L)} + \|w_x(\cdot,t)\|^2_{L^2(0,L)}. \tag{3.164}$$

Right hand side belongs to $L^2(0,T)$, so taking L^2-norms of both sides in the second component yields $w \in L^2(0,T;H^4(0,L))$. Hence $w \in X^3_T$. □

Global well-posedness and higher regularity results of the original plant (3.8) follows from the calculations similar to those in (3.45)-(3.47) which relies on the invertibiliy of the backstepping transformation on $H^l(0,L) \to H^l(0,L)$ and smoothness of the backstepping kernel. Hence we have the following well-posedness and exponential stabilization result.

Theorem 3.1 *Let $T, L, \beta > 0$, $\alpha, \delta \in \mathbb{R}$, $u_0 \in L^2(0,L)$. Assume that the right endpoint feedback controllers are given by*

$$h_0(t) = \int_0^L k(L,y;r)u(y,t)dy, \quad h_1(t) = \int_0^L k_x(L,y;r)u(y,t)dy, \tag{3.165}$$

where k be a smooth backstepping kernel solving (3.14)*. Then, we have the following:*

(i) (Well-posedness) (3.8) *possesses a unique solution $u \in Y^0_T$.*

(ii) (Higher regularity) If $u_0 \in H^3(0,L)$ be such that $w_0 = (I - \Upsilon_k)u_0 \in D(A)$, then the unique solution in (i) also enjoys $u \in X^3_T$.

(iii) (Stabilization) There exists $r > 0$ such that

$$\lambda = \beta\left(\frac{r}{\beta} - \frac{\|k_y(\cdot,0;r)\|^2_{L^2(0,L)}}{2}\right) > 0.$$

For such r, zero equilibrium of the closed-loop system (3.8) *under the influence of the controllers* (3.165) *is exponentially stable. Moreover solution u of the closed-loop system satisfies the following decay estimate*

$$\|u(\cdot,t)\|_{L^2(0,L)} \leq c_k\|u_0\|_{L^2(0,L)}e^{-\lambda t}, \quad \forall t \geq 0,$$

where c_k is a nonnegative constant which depends on k and independent of u_0.

Remark 3.3 *The reason why we in particular mention $L > 0$ in the statement of above theorem is that, stabilization result (as well as the well-posedness) holds for any choice of $L > 0$, i.e., the zero equilibrium to the closed-loop system is exponentially stable either in the case of set of critical length of intervals or in the case of noncritical ones.*

3.4. Numerical simulations

In this part, we present our numerical algorithm and simulations verifying our theoretical stabilization result. Let us first describe our numerical algorithm in three steps. We first obtain an approximation for the backstepping kernel k by solving the integral equation (3.19). Then we solve the modified target equation (3.16) numerically. As a third and final step, we use the invertibility of the backstepping transformation and end up with a numerical approximation to the original plant. Details are given in the below.

Step i. We solve the integral equation

$$G^{j+1}(\bar{x},\bar{y}) = \frac{r}{3\beta}\bar{x}\bar{y} + \int_0^{\bar{y}}\int_0^{\bar{x}}\int_0^{\omega}[DG^j](\xi,\eta)d\xi d\omega d\eta, \quad j = 1,2,\dots,$$

iteratively, where (DG) is given by (3.18) and the iteration is initialized with

$$G^1(\bar{x},\bar{y}) = \frac{r}{3\beta}\bar{x}\bar{y}.$$

Notice that, since the iteration starts with a polynomial, the result of the each step yields again a polynomial. Thus, here, we utilize that summation and multiplication with a scalar of polynomials, their differentiation and integration can be carried out easily by simple algebraic operations. To perform these operations computationally, we express a given $n-$th degree polynomial with complex coefficients, say

$$\begin{aligned} P(\bar{x},\bar{y}) =& \alpha_{0,0} + \alpha_{1,0}\bar{x} + \alpha_{0,1}\bar{y} + \alpha_{2,0}\bar{x}^2 + \alpha_{1,1}\bar{x}\bar{y} + \alpha_{0,2}\bar{y}^2 + \cdots \\ &+ \alpha_{n,0}\bar{x}^n + \alpha_{n-1,1}\bar{x}^{n-1}\bar{y} + \alpha_{n-2,2}\bar{x}^{n-2}\bar{y}^2 + \cdots + \alpha_{0,n}\bar{y}^n, \end{aligned} \tag{3.166}$$

in a more convenient form as

$$[\mathrm{P}] = \begin{bmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} & \alpha_{0,n} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} & \\ \vdots & \vdots & \iddots & & \\ \alpha_{n-1,0} & \alpha_{n-1,1} & & \mathbf{0} & \\ \alpha_{n,0} & & & & \end{bmatrix}. \tag{3.167}$$

Once we introduce this matrix representation $[P]$ of P in our algorithm, then it is easy to perform summation and scalar multiplication. Moreover, using the elementary row and column operations, one can perform the differentiation and integration operations. For instance multiplying the $j-$th row of $[P]$ by $j-1$, writing the result to the $(j-1)-$th row and repeating this process for each j, $j = 2, 3, \ldots, n+1$ yields the matrix representation, $[P_{\bar{x}}]$, of $P_{\bar{x}}(\bar{x},\bar{y})$. Similarly, multiplying the $j-$th row of $[P]$ by $1/j$, writing the result to the $(j+1)-$th row, repeating this process for each j, $j = 1, 2, \ldots, n+1$ and filling the first row by a zero vector yields $\left[\int_0^{\bar{x}} P d\bar{x}\right]$. Differentiation and integration with respect to $\bar{y}$ can be done by performing analogous column operations.

Step ii. Let us consider the uniform discretization of $[0, L]$ with the set of M node points $\{x_m\}_{m=1}^{M}$, $M > 3$, where $x_m = (m-1)h_x$ and $h_x = \frac{L}{M-1}$ is the the uniform spatial grid spacing. Let us introduce the following finite dimensional vector space

$$\mathrm{X}^M \;:=\; \left\{\mathbf{w} = [w_1 \cdots w_M]^T \in \mathbb{C}^M\right\},$$

where each $\mathbf{w} \in \mathrm{X}^M$ satisfies

$$\begin{aligned} w_1(t) = w_M(t) &= 0, & (3.168)\\ \frac{w_{M-2}(t) - 4w_{M-1}(t) + 3w_M(t)}{2h_x} &= 0, & (3.169) \end{aligned}$$

for $t > 0$. Here, $w_m(t)$ is an approximation to $w(x,t)$ at the point $x = x_m$ and, (3.168) and (3.169) correspond to Dirichlet and Neumann type boundary conditions, respectively. Consider the standard forward and backward difference operators $\mathbf{\Delta}_+ : \mathrm{X}^M \to \mathrm{X}^M$ and $\mathbf{\Delta}_- : \mathrm{X}^M \to \mathrm{X}^M$, respectively and let us introduce the following finite difference operators on X^M:

$$\mathbf{\Delta} := \frac{1}{2}\left(\mathbf{\Delta}_+ + \mathbf{\Delta}_-\right), \quad \mathbf{\Delta}^2 := \mathbf{\Delta}_+\mathbf{\Delta}_-, \quad \mathbf{\Delta}^3 := \mathbf{\Delta}_+\mathbf{\Delta}_+\mathbf{\Delta}_-. \tag{3.170}$$

Next assume N be a positive integer, T be the final time and consider the nodal points in time axis $t_n = (n-1)k$, where $n = 1, \ldots, N$ is time index and $h_t = \frac{T}{N-1}$ is the time step size. Let $\mathbf{w^n} = [w_1^n \cdots w_m^n]^T$ be an approximation of the solution at the n-th time step where w_m^n is an approximation to $w(x,t)$ at the point (x_m, t_n). Discretizing (3.16) in space by using the finite difference operators (3.170) and in time by using Crank–Nicolson time stepping, we end up with the discrete problem: Given $\mathbf{w}^n \in \mathrm{X}^M$, find $\mathbf{w}^{n+1} \in \mathrm{X}^M$ such that

$$\left(\mathbf{I}^M + \frac{h_t}{2}\mathbf{A} - \frac{\beta h_t}{2}\mathbf{K}_y^M\mathbf{\Gamma}_0^{1,M}\right)\mathbf{w}^{n+1} = \mathbf{F}\mathbf{w}^n, \quad n = 1, 2, \ldots, N. \tag{3.171}$$

Here $\mathbf{I}^M$ is the identity matrix on X^M, $\mathbf{A}$ is defined as

$$\mathbf{A} := \beta\mathbf{\Delta}^3 - i\alpha\mathbf{\Delta}^2 + \delta\mathbf{\Delta} + r\mathbf{I}^M, \tag{3.172}$$

$\mathbf{K}_y^M$ is an $M \times M$ diagonal matrix, where each element on the diagonal consists of the elements of the form $k_y(x_m, 0)$, $m = 1, \ldots, M$, and $k_y(x, 0)$ is obtained exactly in the previous step, $\mathbf{\Gamma}_0^{1,M}$ is a discrete counterpart of the trace operator $\mathbf{\Gamma}_0^1$ and given by an $M \times M$ matrix

$$\mathbf{\Gamma}_0^{1,M} = \frac{1}{2h_x}\begin{bmatrix} -3 & 4 & -1 & 0 & \cdots & 0 \\ -3 & 4 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -3 & 4 & -1 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.173}$$

and

$$\mathbf{F} := \mathbf{I}^M - \frac{h_t}{2}\mathbf{A} + \frac{\beta h_t}{2}\mathbf{K}_y^M \mathbf{\Gamma}_0^{1,M}.$$

Note that the nonzero elements in the matrix $\mathbf{\Gamma}_0^{1,M}$ given in (3.173) are due to the one–sided second order finite difference approximation to the first order derivative at the point $x = 0$.

Step iii. In this final step, we find the inverse image, u, of w under the backstepping transformation: Given w, we find u by using succession method. More precisely, we set $v := \Upsilon_k u$, therefore we obtain $u = v + w$ and substitute u by $(v + w)$ on the backstepping transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy$$

to get

$$v(x,t) = \int_0^x k(x,y)w(y,t)dy + \int_0^x k(x,y)v(y,t)dy.$$

Now given w obtained numerically in the previous step, we solve this equation successively for v. Using the numerical results for w and v on $u = v + w$, we obtain an approximation for u.

Let us present a numerical simulation that verifies our exponential decay results. We take $M = 1001$ spatial nodes, $N = 5001$ time steps. The iteration for the backstepping kernel is performed $j = 27$ times so that the error is around

$$\max_{(\bar{x},\bar{y})\in\Delta_{\bar{x},\bar{y}}} \left|G^{j+1} - G^j\right| \sim 10^{-14}.$$

We consider the following model

$$
\begin{cases}
iu_t + iu_{xxx} + 2u_{xx} + 8iu_x = 0, & (x,t) \in (0,\pi) \times (0,T), \\
u(0,t) = 0,\ u(\pi,t) = h_0(t),\ u_x(\pi,t) = h_1(t), & t \in (0,T), \\
u(x,0) = 3 - e^{4ix} - 2e^{-2ix}, & x \in (0,L).
\end{cases} \tag{3.174}
$$

Note that the interval length π belongs to set of critical length of intervals and in the absence of controllers, i.e. $h_0(t) \equiv h_1(t) \equiv 0$, the initial state $u(x,0) = 3 - e^{4ix} - 2e^{-2ix}$ does not vary in time, i.e., $u(x,t) = 3 - e^{4ix} - 2e^{-2ix}$ is the time independent solution. Let us choose $r = 0.05$, which yields a positive exponent value λ, defined in Theorem 3.1. Following figures show the behaviour of the solution and the evolution of its $L^2(0,L)-$norm in time.

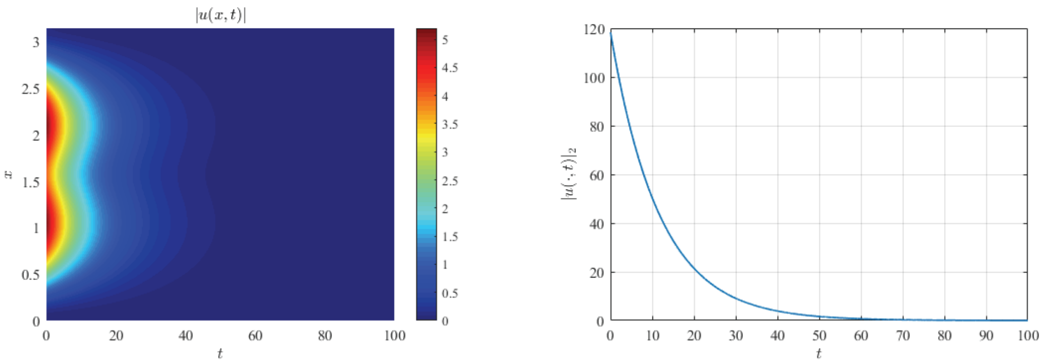

Figure 3.10. Numerical simulations in the presence of controllers. Left: Time evolution of $|u(x,t)|$. Right: Time evolution of $\|u(\cdot,t)\|_{L^2(0,L)}$.

CHAPTER 4

OBSERVER DESIGN FOR HIGHER-ORDER SCHRÖDINGER EQUATION

Designing feedback type controllers requires the measurement of the state at each point of the spatial domain. In this chapter, we consider the case where fully measurement of the state of model

$$\begin{cases} iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = 0, & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = 0,\ u(L,t) = h_0(t),\ u_x(L,t) = h_1(t), & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,L), \end{cases} \tag{4.1}$$

across the domain at any instant, in particular for $t = 0$, is not available. Therefore, in this case, it is not possible to construct a feedback type controller. However, we assume that first and second order boundary traces $y_1(t) = u_x(0,t)$ and $y_2(t) = u_{xx}(0,t)$, are known, say detectable through boundary sensors. Our purpose is to construct an observer model involving y_1, y_2, so that its state is used to construct boundary controllers h_0, h_1. Note that these controllers are imposed at the right endpoint of the spatial domain. So the issue related with right endpoint controllers that we discussed in Section 3.1.2 exists here as well. So in Section 4.1, we apply similar modification on the classical backstepping strategy that we proposed in Section 3.1.3, in order to overcome this issue and introduce our scheme. In Section 4.2 and Section 4.3 we study stabilization of zero equilibrium and well-posedness of (4.1), respectively. Finally, in Section 4.4, we present a numerical simulation that verifies our theoretical exponential decay result.

The results in this chapter were published in a part of our study (Özsarı and Yılmaz, 2022).

4.1. Observer design by the backstepping strategy

Let us recall the observer model

$$\begin{cases} i\hat{u}_t + i\beta\hat{u}_{xxx} + \alpha\hat{u}_{xx} + i\delta\hat{u}_x - p_1(x)(y_1(t) - \hat{u}_x(0,t)) & \\ -p_2(x)(y_2(t) - \hat{u}_{xx}(0,t)) = 0, & (x,t) \in (0,L) \times (0,T), \\ \hat{u}(0,t) = 0, \hat{u}(L,t) = h_0(t), \hat{u}_x(L,t) = h_1(t), & t \in (0,T), \\ \hat{u}(x,0) = \hat{u}_0(x), & x \in (0,L), \end{cases} \tag{4.2}$$

where p_1, p_2 are called observer gains and h_0, h_1 are controllers which are currently unknown. Our first purpose is to obtain observer gains which ensures that $u(\cdot,t) - \hat{u}(\cdot,t) \to 0$ as $t \to \infty$ uniformly. To this end, let us first define the state error $\tilde{u} := u - \hat{u}$, and introduce the error model

$$\begin{cases} i\tilde{u}_t + i\beta\tilde{u}_{xxx} + \alpha\tilde{u}_{xx} + i\delta\tilde{u}_x + p_1(x)\tilde{u}_x(0,t) & \\ +p_2(x)\tilde{u}_{xx}(0,t) = 0, & (x,t) \in (0,L) \times (0,T), \\ \tilde{u}(0,t) = \tilde{u}(L,t) = \tilde{u}_x(L,t) = 0, & t \in (0,T), \\ \tilde{u}(x,0) = \tilde{u}_0(x), & x \in (0,L). \end{cases} \tag{4.3}$$

So, it suffices to find p_1, p_2, which ensures that all solutions of (4.3) decays exponentially to zero. This can be done by treating p_1 and p_2 as control inputs and suitably construct them via the backstepping technique. However, the issue we addressed in Section 3.1.2, which is related with the position of the controllers still exists. More precisely, let us consider the backstepping transformation

$$\tilde{u}(x,t) = \tilde{w}(x,t) - \int_0^x \ell(x,y)\tilde{w}(y,t)dy, \tag{4.4}$$

and the target error model

$$\begin{cases} i\tilde{w}_t + i\beta\tilde{w}_{xxx} + \alpha\tilde{w}_{xx} + i\delta\tilde{w}_x + ir\tilde{w} = 0, & (x,t) \in (0,L) \times (0,T), \\ \tilde{w}(0,t) = \tilde{w}(L,t) = \tilde{w}_x(L,t) = 0, & t \in (0,T), \\ \tilde{w}(x,0) = \tilde{w}_0(x), & x \in (0,L). \end{cases} \tag{4.5}$$

Note that thanks to the weakly damping term $ir\tilde{w}$ in the main equation of (4.5) and homogeneous boundary conditions, it is easy task to show that all solutions decay exponentially to zero. Now if the backstepping kernel ℓ satisfies the following boundary value problem

$$\begin{cases} \beta(\ell_{xxx} + \ell_{yyy}) - i\alpha(\ell_{xx} - \ell_{yy}) + \delta(\ell_x + \ell_y) - r\ell = 0, & (x,y) \in \Delta_{x,y}, \\ \ell(x,x) = \ell(L,y) = \ell_x(L,y) = 0, & x \in [0,L], \\ \ell_x(x,x) = \frac{r}{3\beta}(L-x), & x \in [0,L], \end{cases} \tag{4.6}$$

where $\Delta_{x,y} = \left\{(x,y) \in \mathbb{R}^2 \,|\, y \in (0,x), x \in (0,L)\right\}$ together with the observer gains $p_1(x) = i\beta\ell_y(x,0) - \alpha\ell(x,0)$ and $p_2(x) = -i\beta\ell(x,0)$, then the transformation (4.4) maps the error model (4.3) to the target error model (4.5) (see Appendix A.2 for detailed calculations). Changing variables as $\bar{x} = L - x$, $\bar{y} = x - y$ and defining $H(\bar{x},\bar{y}) \equiv \ell(x,y)$, one can see that the resulting boundary value problem

$$\begin{cases} \beta(H_{\bar{x}\bar{x}\bar{x}} + H_{\bar{y}\bar{y}\bar{y}}) - i\alpha(H_{\bar{x}\bar{x}} - H_{\bar{y}\bar{y}}) + \delta(H_{\bar{x}} + H_{\bar{y}}) - rH = 0, & (x,y) \in \Delta_{x,y}, \\ H(\bar{x},\bar{x}) = H(\bar{x},0) = H_{\bar{x}}(0,\bar{y}) = 0, & x \in [0,L], \\ H_{\bar{x}}(\bar{x},\bar{x}) = \frac{r\bar{x}}{3\beta}, & x \in [0,L], \end{cases} \tag{4.7}$$

where $\Delta_{x,y} = \left\{(\bar{x},\bar{y}) \in \mathbb{R}^2 \,|\, \bar{y} \in (0, L-\bar{x}), \bar{x} \in (0,L)\right\}$, is overdetermined in the sense that there is a mismatch between the boundary conditions:

$$0 = H_{\bar{x}\bar{y}}(0,0) \neq H_{\bar{y}\bar{x}}(0,0) = \frac{r}{3\beta}.$$

This implies that, there does not exist a smooth function that solves (4.7). Following a similar approach as in Section 3.1.3, we shall consider disregarding one boundary condition, here $\ell_x(L,y) = 0$, and take $r > 0$ sufficiently small. Then, the corrected version of the backstepping kernel model (4.10) becomes

$$\begin{cases} \beta(\ell^*_{xxx} + \ell^*_{yyy}) - i\alpha(\ell^*_{xx} - \ell_{yy}) + \delta(\ell^*_x + \ell^*_y) - r\ell^* = 0, & (x,y) \in \Delta_{x,y}, \\ \ell^*(x,x) = \ell^*(L,y) = 0, & x \in [0,L], \\ \ell^*_x(x,x) = \frac{r}{3\beta}(L-x), & x \in [0,L]. \end{cases} \tag{4.8}$$

Now if we use the backstepping transformation, calling it ℓ^*-transformation,

$$\tilde{u}(x,t) = \tilde{w}^*(x,t) - \int_0^x \ell^*(x,y)\tilde{w}^*(y,t)dy, \tag{4.9}$$

where ℓ^* solves the modified model (4.8), then the corresponding target error model for (4.3) becomes

$$\begin{cases} i\tilde{w}^*_t + i\beta\tilde{w}^*_{xxx} + \alpha\tilde{w}^*_{xx} + i\delta\tilde{w}^*_x + ir\tilde{w}^* = 0, & (x,t) \in (0,L)\times(0,T), \\ \tilde{w}^*(0,t) = \tilde{w}^*(L,t) = 0, \tilde{w}^*_x(L,t) = \int_0^L \ell^*_x(L,y)\tilde{w}^*(y,t)dy, & t \in (0,T), \\ \tilde{w}^*(x,0) = \tilde{w}^*_0(x), & x \in (0,L). \end{cases} \tag{4.10}$$

Notice that the inhomogeneous Neumann type boundary condition in (4.10) is occured due to disregarding the condition $\ell_x(L,y) = 0$. Nevertheless, we prove in Proposition 4.1 and Proposition 4.2 that, we still have the exponential decay of solutions of (4.10) provided that r is sufficiently small. Once we show a smooth backstepping kernel ℓ^* and invertibility of backstepping transformation, we deduce that all solutions of error model (4.3) decay to zero exponentially in time, i.e., state of the observer model approaches to the state of the original plant exponentially in time. These are proved in Section 4.2.1.

Next, we also apply backstepping strategy to construct boundary controllers h_0, h_1 to gain exponential stabilization of zero equilibrium of the observer (4.2). Due to the same issues regarding the right endpoint controllers, we use the modified backstepping transformation

$$\hat{w}^*(x,t) = \hat{u}(x,t) - \int_0^x k^*(x,y)\hat{u}(y,t)dy, \tag{4.11}$$

which we call k^*-transformation, where k^* solves the modified backstepping kernel model (3.14) we introduce in Section 3.1.3. Recall that this modification yields a trace term, that shows up in the main equation of associated target model. Therefore the modified target observer model also involves a trace term on its main equation

$$\begin{cases} i\hat{w}_t^* + i\beta\hat{w}_{xxx}^* + \alpha\hat{w}_{xx}^* + i\delta\hat{w}_x^* + ir\hat{w}^* = i\beta k_y^*(x,0)\hat{w}_x^*(0,t) & \\ +[(I-\Upsilon_{k^*})p_1](x)\tilde{w}_x^*(0,t) + [(I-\Upsilon_{k^*})p_2](x)\tilde{w}_{xx}^*(0,t), & (x,t)\in(0,L)\times(0,T), \\ \hat{w}^*(0,t) = \hat{w}^*(L,t) = \hat{w}_x^*(L,t) = 0, & t\in(0,T), \\ \hat{w}^*(x,0) = \hat{w}_0^*(x) := \hat{u}_0(x) - \int_0^x k^*(x,y)\hat{u}(y,t)dy, & x\in(0,L). \end{cases} \tag{4.12}$$

Nevertheless, assuming that $r > 0$ is sufficiently small, we prove that zero equilibrium of (4.12) is exponentially stable. By invertibility of k^*-transformation with a bounded inverse, we deduce exponential stabilization of zero equilibrium of observer model (4.2). These will be proved in Section 4.2.2.

See Figure 4.1 for a graphical illustration of the observer design problem by modified backstepping strategy.

Throughout the following sections, we drop the superscript notation $(\cdot)^*$ and simply write $k, p, \tilde{w}, \hat{w}$ but refering to their modified versions in the above sense.

4.2. Exponential stabilization of zero equilibrium

In this part, we obtain exponential decay of solutions for the plant–observer–error system. This is done by first obtaining decay estimates for solutions of the target error model (4.10) and then for the target observer model (4.12). Bounded invertibility of $\ell-$transformation and $k-$transformation, respectively, yield that the same decay results are also true for error (4.3) and observer models (4.2). Consequently, we obtain exponential stability of zero equilibrium of the original plant (4.1) along with the output feedback

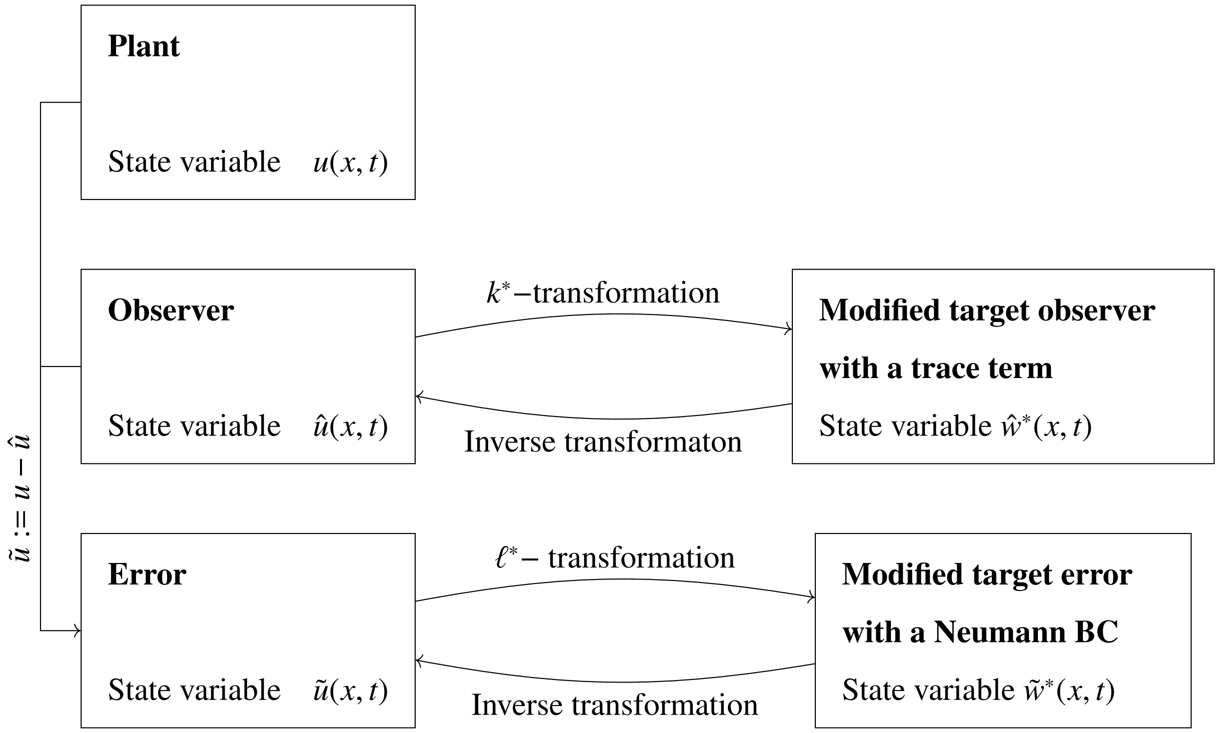

Figure 4.1. Modified backstepping scheme for observer design.

controllers

$$h_0(t) = \int_0^L k(L,y)\hat{u}(y,t)dy, \quad h_1(t) = \int_0^L k_x(L,y)\hat{u}(y,t)dy, \tag{4.13}$$

where k solves (3.14).

4.2.1. Error model

Our first task is to show that backstepping kernel ℓ is a smooth function. To this end, consider changing variables as $\xi = L - y$ and $\eta = L - x$ on (4.8). Then $\Delta_{x,y}$ maps to the same triangular region and the function $P = P(\xi,\eta)$ defined by $P(\xi,\eta) \equiv \ell(x,y)$ solves

$$\begin{cases} \beta(P_{\xi\xi\xi} + P_{\eta\eta\eta}) - i\alpha(P_{\xi\xi} - P_{\eta\eta}) + \delta(P_\xi + P_\eta) - rP = 0, \\ P(\xi,\xi) = P(\xi,0) = 0, \\ P_\xi(\xi,\xi) = -\dfrac{r\xi}{3\beta}. \end{cases}$$

Observe that this model is exactly the same model given we obtained in (3.14), except that r is replaced by $(-r)$. Therefore, we find out that solution k of (3.14) is related with solution ℓ of (4.8) via

$$\ell(x,y) = P(\xi,\eta) = k(\xi,\eta;-r) = k(L-y, L-x;-r). \tag{4.14}$$

Therefore, existence of smooth kernel $\ell = \ell(x,y)$ is guaranteed by smoothness of k which is proved in Lemma 3.1. By Lemma 3.2, this also implies that $\ell-$transform is invertible on $H^l(0,L)$, $l \geq 0$, with a bounded inverse. Hence, exponential decay of the solutions of error model (4.3) directly follows from exponential decay of the solutions of target error model (4.10).

Proposition 4.1 *Let $\beta > 0$, $\alpha, \delta \in \mathbb{R}$ and ℓ be a smooth backstepping kernel that solves* (4.8). *Then for sufficiently small $r > 0$, it is true that*

$$\mu = \beta\left(\frac{r}{\beta} - \frac{\|\ell_x(L,\cdot;r)\|^2_{L^2(0,L)}}{2}\right) > 0. \tag{4.15}$$

Moreover, solution $\tilde{w}$ of (4.10) *satisfies the decay estimate*

$$\|\tilde{w}(\cdot,t)\|_{L^2(0,L)} \leq \|\tilde{w}_0\|_{L^2(0,L)} e^{-\mu t}, \quad \forall t \geq 0. \tag{4.16}$$

Proof Multiplying the main equation of (4.10) by $2\tilde{w}$ in $L^2(0,L)$ and applying similar arguments to those in (3.35)-(3.39), we get

$$\frac{d}{dt}\|\tilde{w}(\cdot,t)\|^2_{L^2(0,L)} + 2r\|\tilde{w}(\cdot,t)\|^2_{L^2(0,L)} + \beta|\tilde{w}_x(0,t)|^2$$
$$= \beta|\tilde{w}_x(L,t)|^2 = \beta\left|\int_0^L \ell_x(L,y)\tilde{w}(y,t)dy\right|^2. \quad (4.17)$$

Dropping the last term on the left hand side and applying Cauchy-Schwarz inequality for the term at the right hand side, it follows that

$$\frac{d}{dt}\|\tilde{w}(\cdot,t)\|^2_{L^2(0,L)} + 2\beta\left(\frac{r}{\beta} - \frac{\|\ell_x(L,\cdot)\|^2_{L^2(0,L)}}{2}\right)\|\tilde{w}(\cdot,t)\|^2_{L^2(0,L)} \le 0. \quad (4.18)$$

Recall from (4.14) that $\ell_x(L,y) = k_x(L-y,0;-r)$, so have

$$\|\ell_x(L,y)\|_{L^2(0,L)} = \|k_x(L-y,0;-r)\|_{L^2(0,L)} = \|k_x(y,0;-r)\|_{L^2(0,L)}.$$

One can also obtain the estimates (3.33) and (3.34) in the proof of Lemma 3.1 by replacing r with $(-r)$. Hence, we can apply the ideas to those given by (3.43)-(3.44) in the proof of Proposition 3.1 and conclude that, sufficiently small $r > 0$ guarantees that μ defined by (4.15) remains positive. □

Following proposition that provides decay estimates for the solutions of target error model in H^3 level as well as for the traces, $\tilde{w}_x(0,t)$ and $\tilde{w}_{xx}(0,t)$. The latter has particular importance later in the proof of the stabilization of observer model.

Proposition 4.2 *Assume the assumption hypothesis of Proposition 4.1 hold. Then solution $\tilde{w}$ of* (4.10) *also satisfies*

$$|\tilde{w}_{xx}(0,t)| + |\tilde{w}_x(0,t)| + \|\tilde{w}(\cdot,t)\|_{H^3(0,L)} \lesssim \|\tilde{w}_0\|_{H^3(0,L)}e^{-\mu t}, \quad \forall t \ge 0. \quad (4.19)$$

Proof We differentiate the main equation of (4.10) with respect to t, multiply by $2\tilde{w}_t$ in

$L^2(0,L)$ and following similar arguments to those in (4.17)-(4.18), we can write

$$\frac{d}{dt}\|\tilde{w}_t(\cdot,t)\|^2_{L^2(0,L)} + 2\beta\left(\frac{r}{\beta} - \frac{\|\ell_x(L,\cdot)\|^2_{L^2(0,L)}}{2}\right)\|\tilde{w}_t(\cdot,t)\|^2_{L^2(0,L)} \leq 0,$$

which implies

$$\|\tilde{w}_t(\cdot,t)\|_{L^2(0,L)} \leq \|\tilde{w}_t(\cdot,0)\|_{L^2(0,L)}e^{-\mu t}. \tag{4.20}$$

Choosing $r > 0$ sufficiently small provides that $\mu > 0$, hence (4.20) becomes a decay estimate. In particular, from the main equation of (4.10) together with (4.20), we can write

$$\begin{aligned}\|\tilde{w}_t(\cdot,t)\|_{L^2(0,L)} &\leq \|\tilde{w}_t(\cdot,0)\|_{L^2(0,L)}e^{-\mu t}\\ &= \|-\beta\tilde{w}_0''' + i\alpha\tilde{w}_0'' - \delta\tilde{w}_0' - r\tilde{w}_0\|_{L^2(0,L)}e^{-\mu t}\\ &\lesssim \|\tilde{w}_0\|_{H^3(0,L)}e^{-\mu t}.\end{aligned} \tag{4.21}$$

On the other hand, again from the main equation of (4.10), we can also write

$$\begin{aligned}\beta\|\tilde{w}_{xxx}(\cdot,t)\|^2_{L^2(0,L)} \leq \alpha\|\tilde{w}_{xx}(\cdot,t)\|^2_{L^2(0,L)} &+ \delta\|\tilde{w}_x(\cdot,t)\|^2_{L^2(0,L)}\\ &+ r\|\tilde{w}(\cdot,t)\|^2_{L^2(0,L)} + \|\tilde{w}_t(\cdot,t)\|^2_{L^2(0,L)}.\end{aligned} \tag{4.22}$$

Applying Gagliardo–Nirenberg interpolation inequality and then Young's inequality with $\varepsilon > 0$, the first term at the right hand side of (4.22) can be estimated as

$$\begin{aligned}\alpha\|\tilde{w}_{xx}(\cdot,t)\|^2_{L^2(0,L)} &\leq c_\alpha\|\tilde{w}_{xxx}(\cdot,t)\|^{\frac{4}{3}}_{L^2(0,L)}\|\tilde{w}(\cdot,t)\|^{\frac{2}{3}}_{L^2(0,L)}\\ &\leq \varepsilon\|\tilde{w}_{xxx}(\cdot,t)\|^2_{L^2(0,L)} + c_{\alpha,\varepsilon}\|\tilde{w}(\cdot,t)\|^2_{L^2(0,L)}.\end{aligned} \tag{4.23}$$

Similarly, the second term at the right hand side of (4.22) can be estimated as

$$\begin{aligned}\delta\|\tilde{w}_x(\cdot,t)\|^2_{L^2(0,L)} &\leq c_\delta\|\tilde{w}_{xxx}(\cdot,t)\|^{\frac{3}{2}}_{L^2(0,L)}\|w(\cdot,t)\|^{\frac{4}{3}}_{L^2(0,L)}\\ &\leq \varepsilon\|\tilde{w}_{xxx}(\cdot,t)\|^2_{L^2(0,L)} + c_{\delta,\varepsilon}\|w(\cdot,t)\|^2_{L^2(0,L)}.\end{aligned} \tag{4.24}$$

Combining (4.23)-(4.24) with (4.22), and then choosing $\varepsilon > 0$ sufficiently small, we can write

$$\|\tilde{w}(\cdot,t)\|^2_{H^3(0,L)} \lesssim \|\tilde{w}(\cdot,t)\|^2_{L^2(0,L)} + \|\tilde{w}_t(\cdot,t)\|^2_{L^2(0,L)}.$$

Using (4.21) and Proposition 4.1, it follows that

$$\|\tilde{w}(\cdot,t)\|_{H^3(0,L)} \lesssim \|\tilde{w}_0\|_{H^3(0,L)} e^{-\mu t}. \tag{4.25}$$

To estimate the trace terms in (4.19), we multiply the main equation of (4.10) by $2(L-x)\tilde{w}_{xx}$ in $L^2(0,L)$ and consider only the imaginary terms to get

$$\begin{aligned}2\mathrm{Re}\int_0^L \tilde{w}_t\overline{\tilde{w}}_{xx}(L-x)dx + 2\beta\mathrm{Re}\int_0^L \tilde{w}_{xxx}\overline{\tilde{w}}_{xx}(L-x)dx \\ + 2\alpha\mathrm{Im}\int_0^L \tilde{w}_{xx}\overline{\tilde{w}}_{xx}(L-x)dx + 2\delta\mathrm{Re}\int_0^L \tilde{w}_x\overline{\tilde{w}}_{xx}(L-x)dx \\ + 2r\mathrm{Re}\int_0^L \tilde{w}\overline{\tilde{w}}_{xx}(L-x)dx = 0.\end{aligned} \tag{4.26}$$

Using integration by parts, the second term can be rewritten as

$$\begin{aligned}2\beta\mathrm{Re}\int_0^L \tilde{w}_{xxx}\overline{\tilde{w}}_{xx}(L-x)dx &= \beta\int_0^L \frac{d}{dx}|\tilde{w}_{xx}|^2(L-x)dx \\ &= \beta\left(-L|\tilde{w}_{xx}(0,t)|^2 + \|\tilde{w}_{xx}(\cdot,t)\|^2_{L^2(0,L)}\right).\end{aligned}$$

The third term vanishes since it is pure real. Again by integration by parts, the fourth term can be expressed as

$$\begin{aligned}2\delta\mathrm{Re}\int_0^L \tilde{w}_x\overline{\tilde{w}}_{xx}(L-x)dx &= \delta\int_0^L \frac{d}{dx}|\tilde{w}_x|^2(L-x)dx \\ &= \delta\left(-L|\tilde{w}_x(0,t)|^2 + \|\tilde{w}_x(\cdot,t)\|^2_{L^2(0,L)}\right).\end{aligned}$$

Therefore, (4.26) becomes

$$L(\beta|\tilde{w}_{xx}(0,t)|^2 + \delta|\tilde{w}_x(0,t)|^2) = 2\mathrm{Re}\int_0^L \tilde{w}_t\overline{\tilde{w}}_{xx}(L-x)dx + \beta\|\tilde{w}_{xx}(\cdot,t)\|^2_{L^2(0,L)}$$
$$+ \delta\|\tilde{w}_x(\cdot,t)\|^2_{L^2(0,L)} + 2r\mathrm{Re}\int_0^L \tilde{w}\overline{\tilde{w}}_{xx}(L-x)dx.$$

Applying Cauchy-Schwarz inequality and then Cauchy's inequality on the first and last terms at the right hand side, using (4.21) and (4.25), we get

$$|\tilde{w}_{xx}(0,t)|^2 + |\tilde{w}_x(0,t)|^2 \lesssim \|\tilde{w}_t(\cdot,t)\|^2_{L^2(0,L)} + \|\tilde{w}(\cdot,t)\|^2_{H^3(0,L)}$$
$$\lesssim e^{-\mu t}\|\tilde{w}_0\|_{H^3(0,L)}.$$

Combining this result with (4.25) finishes the proof. □

Using ℓ−transformation and the fact that kernel $\ell = \ell(x,y)$ is smooth on $\Delta_{x,y}$, we can write

$$\|\tilde{u}(\cdot,t)\|_{H^m(0,L)} \leq \left(1 + \|\ell\|_{H^3(\Delta_{x,y})}\right)\|\tilde{w}(\cdot,t)\|_{H^m(0,L)}, \quad m = 0,3. \tag{4.27}$$

Moreover, due to the invertbility of ℓ−transformation on $H^m(0,L) \to H^m(0,L)$, we can also write

$$\|\tilde{w}_0\|_{H^m(0,L)} \leq \|(I - \Upsilon_\ell)^{-1}\|_{H^m(0,L)\to H^m(0,L)}\|\tilde{u}_0\|_{H^m(0,L)}, \quad m = 0,3. \tag{4.28}$$

Combining (4.27)-(4.28), together with Proposition 4.1 and Proposition 4.2 we conclude

$$\|\tilde{u}(\cdot,t)\| \leq c_{\ell,1}\|\tilde{u}_0\|_{L^2(0,L)}e^{-\mu t}, \quad \forall t \geq 0, \tag{4.29}$$

and

$$\|\tilde{u}(\cdot,t)\|_{H^3(0,L)} \leq c_{\ell,2}\|\tilde{u}_0\|_{H^3(0,L)}e^{-\mu t}, \quad \forall t \geq 0, \tag{4.30}$$

where

$$c_{\ell,1} = \left(1 + \|\ell\|_{L^2(\Delta_{x,y})}\right) \|(I - \Upsilon_\ell)^{-1}\|_{L^2(0,L)\to L^2(0,L)},$$
$$c_{\ell,2} = \left(1 + \|\ell\|_{H^3(\Delta_{x,y})}\right) \|(I - \Upsilon_\ell)^{-1}\|_{H^3(0,L)\to H^3(0,L)}$$

are positive constants which are independent of $\tilde{u}_0$. In other words, $\|u(\cdot,t) - \hat{u}(\cdot,t)\|_{H^m(0,L)} \to 0$, for $m = 0$ and $m = 3$ exponentially in time.

4.2.2. Observer model

In this section, our task is to show that zero equilibrium of observer model (4.2) under the influence of controllers

$$h_0(t) = \int_0^L k(L,y)\hat{u}(y,t)dy, \quad h_1(t) = \int_0^L k_x(L,y)\hat{u}(y,t)dy$$

is exponentially stable. As usual, we obtain this first by studying the associated target model given by (4.12).

Proposition 4.3 *Let $\beta > 0$, $\alpha, \delta \in \mathbb{R}$ and k, ℓ be the smooth backstepping kernels solving* (3.14)*,* (4.8)*, respectively. Then for sufficiently small $r, \varepsilon > 0$, it is true that*

$$\nu := \beta\left(\frac{r}{\beta} - \frac{\|k_y(\cdot,0;r)\|^2_{L^2(0,L)}}{2} - \varepsilon\left(\left\|(I - \Upsilon_k)p_1\right\|^2_{L^2(0,L)} + \left\|(I - \Upsilon_k)p_2\right\|^2_{L^2(0,L)}\right)\right) > 0, \quad (4.31)$$

where $p_1(x) = -i\beta p_y(x,0) + \alpha p(x,0)$, $p_2(x) = i\beta p(x,0)$. Moreover, solution $\hat{w}$ of (4.12) *satisfies the decay estimate*

$$\|\hat{w}(\cdot,t)\|_{L^2(0,L)} \lesssim e^{-\nu t}\left(\|\hat{w}_0\|_{L^2(0,L)} + \|\tilde{w}_0\|_{H^3(0,L)}\right), \quad \forall t \geq 0. \quad (4.32)$$

Proof Multiplying the main equation of (4.12) by $2\hat{w}$ in $L^2(0,L)$ and following the arguments to those given by (3.36)-(3.39), we obtain

$$\begin{aligned} &\frac{d}{dt}\|\hat{w}(\cdot,t)\|^2_{L^2(0,L)} + \beta|\hat{w}_x(0,t)|^2 + 2r\|\hat{w}(\cdot,t)\|^2_{L^2(0,L)} \\ =&2\beta\mathrm{Re}\int_0^L k_y(x,0)\hat{w}_x(0,t)\overline{\hat{w}}(x,t)dx \\ &+ 2\mathrm{Im}\int_0^L (I-\Upsilon_k)p_1(x)\tilde{w}_x(0,t)\overline{\hat{w}}(x,t)dx \\ &+ 2\mathrm{Im}\int_0^L (I-\Upsilon_k)p_2(x)\tilde{w}_{xx}(0,t)\overline{\hat{w}}(x,t)dx. \end{aligned} \tag{4.33}$$

Using Cauchy's inequality and then Cauchy-Schwarz inequality, the first term at the right hand side of (4.33) can be estimated as

$$2\beta\mathrm{Re}\int_0^L k_y(x,0)\hat{w}_x(0,t)\overline{\hat{w}}(x,t)dx \le \beta|\hat{w}_x(0,t)|^2 + \beta\|k_y(\cdot,0)\|^2_{L^2(0,L)}\|\hat{w}(\cdot,t)\|^2_{L^2(0,L)}.$$

Applying Cauchy-Schwarz inequality and Young's inequality with $\varepsilon > 0$ to the second and third terms at the right hand side of (4.33), we get

$$\begin{aligned} &\left|2\mathrm{Im}\int_0^L (I-\Upsilon_k)p_1(x)\tilde{w}_x(0,t)\overline{\hat{w}}(x,t)dx\right| \\ &\qquad\le 2\varepsilon\beta\left\|(I-\Upsilon_k)p_1\right\|^2_{L^2(0,L)}\|\hat{w}(\cdot,t)\|^2_{L^2(0,L)} + \frac{1}{2\varepsilon\beta}|\tilde{w}_x(0,t)|^2 \end{aligned}$$

and

$$\begin{aligned} &\left|2\mathrm{Im}\int_0^L (I-\Upsilon_k)p_2(x)\tilde{w}_x(0,t)\overline{\hat{w}}(x,t)dx\right| \\ &\qquad\le 2\varepsilon\beta\left\|(I-\Upsilon_k)p_2\right\|^2_{L^2(0,L)}\|\hat{w}(\cdot,t)\|^2_{L^2(0,L)} + \frac{1}{2\varepsilon\beta}|\tilde{w}_{xx}(0,t)|^2. \end{aligned}$$

Using these estimates in (4.33), we obtain that

$$\frac{d}{dt}\|\hat{w}(\cdot,t)\|^2_{L^2(0,L)} + 2\beta\nu\|\hat{w}(\cdot,t)\|^2_{L^2(0,L)} \le \frac{1}{2\varepsilon\beta}\left(|\tilde{w}_x(0,t)|^2 + |\tilde{w}_{xx}(0,t)|^2\right), \tag{4.34}$$

where ν is given by (4.31). From Proposition 3.1, we know that $\left(\frac{r}{\beta} - \frac{\|k_y(\cdot,0)\|^2_{L^2(0,L)}}{2}\right)$ remains positive for sufficiently small $r > 0$. Therefore, choosing ε sufficiently small, we are able to guarantee that ν also remains positive. Now applying Proposition 4.2 for the trace terms at the right hand side of (4.34), we get

$$\frac{d}{dt}\|\hat{w}(\cdot,t)\|^2_{L^2(0,L)} + 2\nu\|\hat{w}(\cdot,t)\|^2_{L^2(0,L)} \lesssim \|\tilde{w}_0\|^2_{H^3(0,L)} e^{-2\mu t}.$$

Since $\|\ell_x(L,\cdot)\|_{L^2(0,L)} = \|k_y(\cdot,0)\|_{L^2(0,L)}$ and comparing ν with μ given by (4.15), observe that $\mu > \nu$. Thus, integrating the above inequality in t over $[0,T]$, we finally obtain

$$\begin{aligned} \|\hat{w}(\cdot,t)\|_{L^2(0,L)} &\lesssim e^{-\nu t}\left(\|\hat{w}_0\|_{L^2(0,L)} + \|\tilde{w}_0\|_{H^3(0,L)}\right) \\ &= e^{-\nu t}\left(\|\hat{w}_0\|_{L^2(0,L)} + \|w_0 - \hat{w}_0\|_{H^3(0,L)}\right), \quad \forall t \geq 0, \end{aligned} \tag{4.35}$$

which completes the proof. □

Using the invertibility of the $k-$transformation on $L^2(0,L)$, we can write

$$\|\hat{u}(\cdot,t)\|_2 \leq \|(I + \Upsilon_k)^{-1}\|_{L^2(0,L)\to L^2(0,L)}\|\hat{w}\|_{L^2(0,L)}. \tag{4.36}$$

Since $k = k(x,y)$ is smooth on $\Delta_{x,y}$ and $\ell-$transformation is inverible with a bounded inverse on $H^3(0,L)$, we can also write

$$\|\hat{w}_0\|_{L^2(0,L)} \leq \left(1 + \|\ell\|_{C(\Delta_{x,y})}\right)\|\hat{u}_0\|_{L^2(0,L)} \tag{4.37}$$

and

$$\|w_0 - \hat{w}_0\|_{H^3(0,L)} \leq \|\,(I + \Upsilon_\ell)^{-1}\,\|_{H^3(0,L)\to H^3(0,L)}\|u_0 - \hat{u}_0\|_{H^3(0,L)}. \tag{4.38}$$

Combining (4.36)-(4.38), it follows from (4.35) that

$$\|\hat{u}(\cdot,t)\|_{L^2(0,L)} \lesssim e^{-\nu t}\left(c_k\|\hat{u}_0\|_{L^2(0,L)} + c_{k,\ell}\|u_0 - \hat{u}_0\|_{H^3(0,L)}\right) \tag{4.39}$$

where

$$c_k = \left(1 + \|\ell\|_{C(\Delta_{x,y})}\right) \|(I + \Upsilon_k)^{-1}\|_{L^2(0,L)\to L^2(0,L)}$$
$$c_{k,\ell} = \|(I + \Upsilon_\ell)^{-1}\|_{H^3(0,L)\to H^3(0,L)} \|(I + \Upsilon_k)^{-1}\|_{L^2(0,L)\to L^2(0,L)}$$

are nonnegative constants that are independent of the initial u_0, $\hat{u}_0$.

To derive exponential stabilization of zero equilibrium to original plant (4.1), recall that u is given via the relation $u := \hat{u} + \tilde{u}$. Thus, combining the decay estimates (4.29) and (4.39), we conclude that

$$\begin{aligned} \|u(\cdot,t)\|_{L^2(0,L)} &= \|(\hat{u} + \tilde{u})(\cdot,t)\|_{L^2(0,L)} \\ &\le \|\hat{u}(\cdot,t)\|_{L^2(0,L)} + \|\tilde{u}(\cdot,t)\|_{L^2(0,L)} \\ &\lesssim \left(c_k\|\hat{u}_0\|_{L^2(0,L)} + c_{k,\ell}\|u_0 - \hat{u}_0\|_{H^3(0,L)}\right) e^{-\nu t} + c_\ell \|u_0 - \hat{u}_0\|_{L^2(0,L)} e^{-\mu t} \end{aligned}$$

for all $t \ge 0$.

4.3. Well-posedness

In this section, our task is to show that plant-observer-error system is well-posed. Analysis is carried out through the associated target models. Notice that, in order to guarantee that the boundary traces $\hat{w}_x(0,t), \hat{w}_{xx}(0,t)$ make sense, one requires that $\hat{w}$ is continuously differentiable up to order two. Since $C^2([0,L])$ is continuously embedded in $H^3(0,L)$, then it suffices to prove existence existence of a solution $\hat{w}$ of the target observer model (4.12) in X_T^3. But then, this requires that the source term

$$f(x,t) := [(I - \Upsilon_k)p_1](x)\tilde{w}_x(0,t) + [(I - \Upsilon_k)p_2](x)\tilde{w}_{xx}(0,t)$$

involving in the main equation of (4.12) belongs to $W^{1,1}(0,T;L^2(0,L))$, i.e., $\tilde{w}_{xt}(0,t)$ and $\tilde{w}_{xxt}(0,t)$ make sense in $L^1(0,T)$, which also requires that solution $\tilde{w}$ of the target error model (4.5) exists in $H^6(0,L)$ level. In view of these observations, in Section 4.3.1 we prove existence of a unique solution of the target error model in X_T^6 space and in Section

4.3.2 we prove existence of a unique solution of the target observer model in X_T^3 space. Then, using the invertibility of the associated backstepping transformations, we conclude well-posedness of the error and observer models, respectively, hence the original plant.

4.3.1. Error model

For a moment, let us consider the model

$$\begin{cases} i\tilde{w}_t + i\beta\tilde{w}_{xxx} + \alpha\tilde{w}_{xx} + i\delta\tilde{w}_x + ir\tilde{w} = 0, & (x,t) \in (0,L) \times (0,T), \\ \tilde{w}(0,t) = \tilde{w}(L,t) = 0,\ \tilde{w}_x(L,t) = \psi(t), & t \in (0,T), \\ \tilde{w}(x,0) = \tilde{w}_0(x), & x \in (0,L). \end{cases} \tag{4.40}$$

Note that the original target error model (4.10) is the closed-loop version of (4.40) with the inhomogeneous Neumann type boundary condition

$$\psi(\tilde{w}(t)) = \int_0^L \ell_x(L,y)\tilde{w}(y,t)dy.$$

Our task here is first to obtain a-priori estimates for the model (4.40), then apply fixed point argument to show that same results also hold for (4.10).

To this end, let us first define $\zeta(\cdot,t) := e^{rt}\tilde{w}(\cdot,t)$ with initial and boundary data

$$\zeta(\cdot,0) := \phi(\cdot) = \tilde{w}_0(\cdot), \quad \zeta_x(L,t) := g(t) = \psi(t)e^{rt}.$$

Then, ζ solves the following model

$$\begin{cases} i\zeta_t + i\beta\zeta_{xxx} + \alpha\zeta_{xx} + i\delta\zeta_x = 0, & (x,t) \in (0,L) \times (0,T), \\ \zeta(0,t) = \zeta(L,t) = 0,\ \zeta_x(L,t) = g(t), & t \in (0,T), \\ \zeta(x,0) = \phi(x), & x \in (0,L). \end{cases} \tag{4.41}$$

So the results we obtain below for $\zeta-$model also be hold for (4.40). Let us denote solution of (4.41) as $\zeta = \zeta[\phi, g]$ in terms of initial and boundary data. Regarding the solution $\zeta = \zeta[\phi, 0]$ of (4.41), we already derived in Section 3.3.1 that, an application of operator semigroup theory together with Lemma 3.4 yields that if $\phi_0 \in L^2(0, L)$, then there exists a unique mild solution $\zeta = \zeta[\phi, 0] \in X_T^0$. Therefore, it is enough to show that $\zeta = \zeta[0, g] \in X_T^0$, which is now our task. Observe that the model (4.41) is a particular case of the $z-$model given by (3.72) with $\psi_1 = \psi_2 = 0$ and $\psi_3 = g$, that we analyzed in Section 3.3.1.2. Let us consider zero extension of g from $(0, T)$ to $\mathbb{R}$, which we still denote by the same notation. Then, in view of our work we performed in Section 3.3.1.2, we express solution $\zeta = \zeta[0, g]$ as follows

$$\zeta(x,t) = \frac{1}{2\pi i}\sum_{j=1}^{3}\int_C e^{st}\frac{\Delta_j(s)}{\Delta(s)}e^{\lambda_j(s)x}\tilde{g}(s)ds, \tag{4.42}$$

where C is chosen as C^+, C^0 or C^- by considering the sign of the quantity $\alpha^2 + 3\beta\delta$, namely $\alpha^2 + 3\beta\delta > 0$, $\alpha^2 + 3\beta\delta = 0$ or $\alpha^2 + 3\beta\delta < 0$, respectively, and Δ and Δ_j's are the determinants that arise due to the application of Cramer's rule for the following linear system

$$\begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1(s)L} & e^{\lambda_2(s)L} & e^{\lambda_3(s)L} \\ \lambda_1(s)e^{\lambda_1(s)L} & \lambda_2(s)e^{\lambda_2(s)L} & \lambda_3(s)e^{\lambda_3(s)L} \end{pmatrix}\begin{pmatrix} c_1(s) \\ c_2(s) \\ c_3(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \tilde{g}(s) \end{pmatrix}. \tag{4.43}$$

Proceeding similarly to those in (3.108)-(3.114), we conclude that it is enough to study the improper parts of (4.42), which are, after changing variables as $s = i\omega(\xi) = i(\beta\xi^3 - \alpha\xi^2 - \delta\xi)$, given by

$$I(x,t) = \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{\gamma_1}^{\infty} e^{i\omega(\xi)t}\frac{\Delta_j^*(\xi)}{\Delta^*(\xi)}e^{\lambda_j^*(\xi)x}(3\beta\xi^2 - 2\alpha\xi - \delta)\tilde{g}^*(\xi)d\xi \tag{4.44}$$

and

$$J(x,t) = \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{-\infty}^{\gamma_2} e^{i\omega(\xi)t}\frac{\Delta_j^*(\xi)}{\Delta^*(\xi)}e^{\lambda_j^*(\xi)x}(3\beta\xi^2 - 2\alpha\xi - \delta)\tilde{g}^*(\xi)d\xi. \tag{4.45}$$

Lemma 4.1 *Let $T > 0$ and $g \in L^2(0, T)$. Then, solution $\zeta = \zeta[0, g]$ of* (4.41) *belongs to* *the space X_T^0 and satisfies*

$$\|\zeta\|_{C([0,T];L^2(0,L))} + \|\zeta\|_{L^2(0,T;H^1(0,L))} \lesssim \|g\|_{L^2(0,T)}.$$

Proof Recall from the proof of Lemma 3.7 that, we have

$$\Delta^\dagger(\xi) = e^{\frac{i\alpha L}{\beta}} \Big(e^{-\lambda_1^\dagger(\xi)L}(\lambda_3^\dagger(s) - \lambda_2^\dagger(\xi)) \\ - e^{-\lambda_2^\dagger(\xi)L}(\lambda_3^\dagger(\xi) - \lambda_1^\dagger(\xi)) + e^{-\lambda_3^\dagger(\xi)L}(\lambda_2^\dagger(\xi) - \lambda_1^\dagger(\xi)) \Big) \quad (4.46)$$

and

$$\begin{aligned} \Delta_1^\dagger(\xi) &= e^{\lambda_3^\dagger(\xi)L} - e^{\lambda_2^\dagger(\xi)L}, \\ \Delta_2^\dagger(\xi) &= e^{\lambda_1^\dagger(\xi)L} - e^{\lambda_3^\dagger(\xi)L}, \\ \Delta_3^\dagger(\xi) &= e^{\lambda_2^\dagger(\xi)L} - e^{\lambda_1^\dagger(\xi)L}, \end{aligned}$$

with large ξ asymptotics

$$\left| \frac{\Delta_j^\dagger(\xi)}{\Delta^\dagger(\xi)} \right| \sim \begin{cases} \xi^{-1}, & j = 1, \\ \xi^{-1}, & j = 2, \\ \xi^{-1} e^{-\frac{\sqrt{3}\xi L}{2}}, & j = 3. \end{cases} \quad (4.47)$$

We take L^2-norm of I with respect to its first component and apply Lemma 2.5 in (Bona et. al., 2003) to get

$$\|I(\cdot,t)\|^2_{L^2(0,L)} \lesssim \sum_{j=1}^{3} \int_{\gamma_1}^{\infty} \left(e^{L\mathrm{Re}(\lambda_j^\dagger(\xi))} + 1 \right)^2 \left| \frac{\Delta_j^\dagger(\xi)}{\Delta^\dagger(\xi)} \right|^2 \left| 3\beta\xi^2 - 2\alpha\xi - \delta \right|^2 \left| \tilde{g}^\dagger(\xi) \right|^2 d\xi. \quad (4.48)$$

Using the asymptotic behaviors (4.47) as $\xi \to \infty$, we can write

$$\left(e^{L\mathrm{Re}(\lambda_j^\dagger(\xi))} + 1 \right)^2 \left| \frac{\Delta_j^\dagger(\xi)}{\Delta^\dagger(\xi)} \right|^2 \sim \xi^{-2}, \quad j = 1, 2, 3. \quad (4.49)$$

Consequently, the integral on (4.48) is equivalent to

$$\int_{\gamma_1}^{\infty} \frac{1}{\xi^2} \left|3\beta\xi^2 - 2\alpha\xi - \delta\right|^2 \left|\tilde{g}^\dagger(\xi)\right|^2 d\xi$$

for each $j = 1, 2, 3$. Changing variables as $\mu = \beta\xi^3 - \alpha\xi^2 - \delta\xi$

$$\begin{aligned} \|I(\cdot, t)\|^2_{L^2(0,L)} &\lesssim \int_{\omega(\gamma_1)}^{\infty} \frac{\left|3\beta\xi^2(\mu) - 2\alpha\xi(\mu) - \delta\right|}{\xi^2(\mu)} \left|\int_0^\infty e^{-i\mu\tau} g(\tau) d\tau\right|^2 d\mu \\ &\simeq \int_{\omega(\gamma_1)}^{\infty} |\hat{g}(\mu)|^2 \, d\mu \\ &\leq \|g\|^2_{L^2(\mathbb{R})} = \|g\|^2_{L^2(0,T)}, \end{aligned}$$

where $\xi(\mu)$ is the invese of the change of variable (indeed $\mu = \beta\xi^3 - \alpha\xi^2 - \delta\xi$ is injective on $\xi \in (\omega(\gamma_1), \infty)$ for each case of $(\alpha^2 + 3\beta\delta)$, thanks to our work throughout (3.105)-(3.114)). Passing to supremum in t over $[0, T]$, we obtain

$$\|I\|_{C([0,T];L^2(0,L))} \lesssim \|g\|_{L^2(0,T)}. \tag{4.50}$$

Next, we differentiate I with respect to its first component, take L^2-norm on $(0, T)$ and change variables as $\mu = \beta\xi^3 - \alpha\xi^2 - \delta\xi$ to get

$$\begin{aligned} &\|\partial_x I(x, \cdot)\|^2_{L^2(0,T)} \\ &= \left\| \sum_{j=1}^{3} \frac{1}{2\pi} \int_{\gamma_1}^{\infty} e^{i\omega(\xi)t} \lambda_j^\dagger(\xi) e^{\lambda_j^\dagger(\xi)x} \frac{\Delta_j^\dagger(\xi)}{\Delta^\dagger(\xi)} (3\beta\xi^2 - 2\alpha\xi - \delta)\tilde{g}^\dagger(\xi) d\xi \right\|^2_{L^2(0,T)} \\ &\lesssim \sum_{j=1}^{3} \left\| \int_{\omega(\gamma_1)}^{\infty} e^{i\mu t} \lambda_j^\dagger(\theta(\mu)) e^{\lambda_j^\dagger(\theta(\mu))x} \frac{\Delta_j^\dagger(\theta(\mu))}{\Delta^\dagger(\theta(\mu))} \tilde{g}^\dagger(\theta(\mu)) d\mu \right\|^2_{L^2(0,T)}, \end{aligned} \tag{4.51}$$

where $\theta(\mu)$ is the real solution of $\mu = \beta\xi^3 - \alpha\xi^2 - \delta\xi$ for $\gamma_1 < \xi < \infty$. Observe that the function

$$\begin{cases} \lambda_j^\dagger(\theta(\mu)) e^{\lambda_j^\dagger(\theta(\mu))x} \dfrac{\Delta_j^\dagger(\theta(\mu))}{\Delta^\dagger(\theta(\mu))} \tilde{g}^\dagger(\theta(\mu)), & \mu \in (\omega(\gamma_1), \infty), \\ 0, & \text{elsewhere}, \end{cases}$$

is the Fourier transform of the function given by the integral. So, using the Plancherel's theorem, we can write

$$\|\partial_x I_1(x,\cdot)\|^2_{L^2(0,T)} \lesssim \sum_{j=1}^{3} \int_{\omega(\gamma_1)}^{\infty} \left| \lambda_j^\dagger(\theta(\mu)) e^{\lambda_j^\dagger(\theta(\mu))x} \frac{\Delta_j^\dagger(\theta(\mu))}{\Delta^\dagger(\theta(\mu))} \tilde{g}^\dagger(\theta(\mu)) \right|^2 d\mu \tag{4.52}$$

for all $x \in [0, L]$. It follows that

$$\begin{aligned} \|\partial_x I_1\|^2_{L^2(0,L;L^2(0,T))} &\leq \sup_{x\in[0,L]} \|\partial_x I_1(x,\cdot)\|^2_{L^2(0,T)} \\ &\lesssim \sum_{j=1}^{3} \int_{\gamma_1}^{\infty} |\lambda_j^\dagger(\xi)|^2 \sup_{x\in[0,L]} \left(e^{2\mathrm{Re}(\lambda_j^\dagger(\xi))x} \right) \left| \frac{\Delta_j^\dagger(\xi)}{\Delta^\dagger(\xi)} \right|^2 \\ &\quad \times (3\beta\xi^2 - 2\alpha\xi - \delta) \left| \tilde{g}^\dagger(\xi) \right|^2 d\xi. \end{aligned} \tag{4.53}$$

Using representations for $\lambda_j^\dagger$'s given by (3.109) and (4.47), one can obtain the following asymptotic behaviors in ξ

$$|\lambda_j^\dagger(\xi)|^2 \sup_{x\in[0,L]} \left(e^{2\mathrm{Re}(\lambda_j^\dagger(\xi))x} \right) \left| \frac{\Delta_j^\dagger(\xi)}{\Delta^\dagger(\xi)} \right|^2 \sim \begin{cases} 1, & j = 1, \\ 1, & j = 2, \\ e^{-\sqrt{3}\xi L}, & j = 3. \end{cases} \tag{4.54}$$

Then, by (4.54) in (4.53), and changing variables back as $\mu = \beta\xi^3 - \alpha\xi^2 - \delta\xi$, we get

$$\begin{aligned} \|\partial_x I_1\|^2_{L^2(0,L;L^2(0,T))} &\leq \sup_{x\in[0,L]} \|\partial_x I_1(x,\cdot)\|^2_{L^2(0,T)} \\ &\lesssim \int_{\gamma_1}^{\infty} (3\beta\xi^2 - 2\alpha\xi - \delta) |\tilde{g}^\dagger(\xi)|^2 d\xi \\ &\lesssim \int_{\omega(\gamma_1)}^{\infty} \left| \int_0^\infty e^{-i\mu\tau} g(\tau) d\tau \right|^2 d\mu \\ &\lesssim \|g\|^2_{L^2(\mathbb{R}^+)} = \|g\|^2_{L^2(0,T)}. \end{aligned} \tag{4.55}$$

Changing the integration order on $\|\partial_x I_1\|^2_{L^2(0,L;L^2(0,T))}$ and using Poincaré inequality, we conclude

$$\|I\|_{L^2(0,T;H^1(0,L))} \lesssim \|g\|_{L^2(0,T)}.$$

Applyying similar procedure yields the same result for J. This completes the proof. □

Solutions of the models (4.40) and (4.41), and the associated initial and boundary conditions are related via the relations

$$\tilde{w}(\cdot,t) = \zeta(\cdot,t)e^{-rt}, \quad \psi(t) = g(t)e^{-rt}, \quad \tilde{w}_0 = \phi.$$

Consequently, estimations we obtained for $\zeta = \zeta[\phi, g]$ also holds true for the solution $\tilde{w}$ of the model (4.40). Hence, we have the following lemma.

Lemma 4.2 *Let $\tilde{w}_0 \in L^2(0,L)$ and $g \in L^2(0,T)$. Then solution $\tilde{w}$ of* (4.40) *belongs to the space X_T^0.*

Recall that the trace terms $\tilde{w}_x(0,t)$, $\tilde{w}_{xx}(0,t)$, are involved in the main equation of target observer model. To guarantee that these terms make sense and the fact that $C^2([0,L])$ is continuously embedded in $H^3(0,L)$, if suffices the prove well-posedness of the model (4.40) in $H^3(0,L)$ level.

Lemma 4.3 *Let $\tilde{w}_0 \in H^3(0,L)$, $\psi \in H^1(0,T)$ and suppose that (w_0, ψ) satisfy the compatibility conditions* (2.12). *Then $\tilde{w} \in X_T^3$ and $\tilde{w}_t \in X_T^0$.*

Proof Let $\tilde{v}$ solves the linear model below

$$\begin{cases} i\tilde{v}_t + i\beta\tilde{v}_{xxx} + \alpha\tilde{v}_{xx} + i\delta\tilde{v}_x + ir\tilde{v} = 0, & (x,t) \in (0,L)\times(0,T), \\ \tilde{v}(0,t) = \tilde{v}(L,t) = 0, \tilde{v}_x(L,t) = \psi'(t), & t \in (0,T), \\ \tilde{v}(x,0) = \tilde{v}_0(x), & x \in (0,L), \end{cases} \tag{4.56}$$

where we set $\tilde{v}_0 := -\beta\tilde{w}_0''' + i\alpha\tilde{w}_0'' - \delta\tilde{w}_0' - r\tilde{w}_0 \in L^2(0,L)$. According to Lemma 4.2, following estimate is true

$$\|\tilde{v}\|_{X_T^0} \lesssim \|\tilde{v}_0\|_{L^2(0,L)} + \|\psi'\|_{L^2(0,T)}.$$

Set $\tilde{w}(x,t) := \tilde{w}_0(x) + \int_0^t \tilde{v}(x,\tau)d\tau$. Then, $\tilde{w}$ satisfies the initial condition

$$\tilde{w}(x,0) = \tilde{w}_0(x).$$

Considering the compatibility conditions (2.12) and the relation $\tilde{w}_t(\cdot,t) = \tilde{v}(\cdot,t)$, $\tilde{w}$ also satisfies the boundary conditions

$$\begin{aligned}
\tilde{w}(0,t) &= \tilde{w}_0(0) + \int_0^t \tilde{v}(0,\tau)d\tau = \tilde{w}_0(0) + \tilde{w}(0,t) - \tilde{w}(0,0) = 0,\\
\tilde{w}(L,t) &= \tilde{w}_0(L) + \int_0^t \tilde{v}(L,\tau)d\tau = \tilde{w}_0(L) + \tilde{w}(L,t) - \tilde{w}(L,0) = 0,\\
\tilde{w}_x(L,t) &= \tilde{w}_0'(L) + \int_0^t \tilde{v}_x(L,\tau)d\tau = \tilde{w}_0'(L) + \tilde{w}_x(L,t) - \tilde{w}_x(L,0) = \psi(t).
\end{aligned}$$

Moreover, from the main equation of (4.56), we see that $\tilde{w}$ satisfies the main equation of

$$\begin{aligned}
&(i\tilde{w}_t + i\beta\tilde{w}_{xxx} + \alpha\tilde{w}_{xx} + i\delta\tilde{w}_x + ir\tilde{w})(x,t)\\
=&i\tilde{v}(x,t) + \int_0^t (i\beta\tilde{v}_{xxx} + \alpha\tilde{v}_{xx} + i\delta\tilde{v}_x + ir\tilde{v})(x,\tau)d\tau\\
&+ i\beta\tilde{w}_0'''(x) + \alpha\tilde{w}_0''(x) + i\delta\tilde{w}_0'(x) + ir\tilde{w}_0(x)\\
=&i\tilde{v}(x,t) - \int_0^t i\tilde{v}_t(x,\tau)d\tau + i\beta\tilde{w}_0'''(x) + \alpha\tilde{w}_0''(x) + i\delta\tilde{w}_0'(x) + ir\tilde{w}_0(x)\\
=&i\tilde{v}(x,0) + i\beta\tilde{w}_0'''(x) + \alpha\tilde{w}_0''(x) + i\delta\tilde{w}_0'(x) + ir\tilde{w}_0(x) = 0.
\end{aligned}$$

Thus w defined by $w(x,t) = w_0(x) + \int_0^t \tilde{v}(x,\tau)d\tau$, solves the model (4.40). Applying same arguments to those given by (3.156)-(3.164), one can show that $\tilde{w} \in C([0,T];H^3(0,L))$ and $\tilde{w} \in L^2(0,T;H^4(0,L))$, hence $\tilde{w} \in X_T^3$. □

The above analysis, regarding the open-loop version of (4.10), implicitly defines a continuous data-to-solution map

$$\Gamma : (\tilde{w}_0,\psi) \in H^3(0,L) \times H^1(0,T) \to \tilde{w} \in Q_T, \tag{4.57}$$

where $Q_T := \left\{\varphi \in X_T^3 \mid \varphi_t \in X_T^0\right\}$ equipped with the norm $\left(\|\varphi\|_{X_T^3} + \|\varphi_t\|_{X_T^0}\right)$. However, notice that the original boundary condition of the target error model is of feedback type, of the form

$$\psi(\tilde{w}) := \tilde{w}_x(L,t) = \int_0^L \ell_x(L,y)\tilde{w}(y,t)dy.$$

Therefore, one needs to show that the mapping Γ has a fixed point, that is, for a given $\tilde{w}_0$, one needs to show the existence of a $\tilde{w}$ so that $\Gamma[\tilde{w}_0, \psi(\tilde{w})] = \tilde{w}$ holds. To this end, first notice that for any $\tilde{w}^* \in Q_T$ and since ℓ is smooth, we have

$$\begin{aligned} \|\psi(\tilde{w}^*)\|_{H^1(0,T)} &= \left\| \int_0^L \ell_x(L,y)\tilde{w}^*(y,\cdot)dy \right\|_{H^1(0,T)} \\ &\leq \sqrt{T}\|\ell_x(L,\cdot)\|_{L^2(0,L)} \left(\|\tilde{w}^*\|_{X_T^0} + \|\tilde{w}_t^*\|_{X_T^0} \right) < \infty, \end{aligned} \tag{4.58}$$

so $\psi(\tilde{w}^*) \in H^1(0,T)$. Let $\tilde{w}_1, \tilde{w}_2 \in Q_T$. Using the result in Lemma 4.3 and the estimate in (4.58), we get

$$\begin{aligned} \|\Gamma[\tilde{w}_0, \psi(\tilde{w}_1)] - \Gamma[\tilde{w}_0, \psi(\tilde{w}_2)]\|_{\tilde{Q}_T} &\leq C\|\psi(\tilde{w}_1) - \psi(\tilde{w}_2)\|_{H^1(0,T)} \\ &\leq C\sqrt{T}\|\tilde{w}_1 - \tilde{w}_2\|_{Q_T}. \end{aligned} \tag{4.59}$$

If we choose $T > 0$ sufficiently small, then the constant at the right hand side becomes lesser than one. Consequently, for such T, we can guarantee that the mapping Γ becomes contraction. Hence, thanks to Banach fixed point theorem, this yields the unique local solution, $\tilde{w}$, of (4.10). Note that exponential decay results we proved in Proposition 4.1 and Proposition 4.2 implies that local solution remains uniformly bounded in time. This implies that the local solution is global in time. Finally, using the $\ell-$transformation given by (4.9) and its invertibility on $H^3(0,L)$ which is proved in Lemma 3.2, we conclude the well-posedness of the error model (4.3).

Proposition 4.4 *Let $T, L > 0$, $\tilde{u}_0 \in H^3(0,L)$, $p_1(x) = i\beta\ell_y(x,0) - \alpha\ell(x,0)$ and $p_2(x) = -i\beta\ell(x,0)$. Assume that the couple $(\tilde{w}_0, \psi(\tilde{w}))$ satisfies the compatibility conditions* (2.12), *where $\tilde{w}_0$ is defined as $\tilde{w}_0 := (I - \Upsilon_\ell)^{-1}\tilde{u}_0$, $\psi(\tilde{w})(t) = \int_0^L \ell_x(L,y)\tilde{w}(y,t)dy$ and $\tilde{w}$ solves* (4.10). *Then the global solution $\tilde{u}$ belongs to the space X_T^3.*

4.3.2. Observer model

Let us consider the target observer model

$$\begin{cases} i\hat{w}_t + i\beta\hat{w}_{xxx} + \alpha\hat{w}_{xx} + i\delta\hat{w}_x + ir\hat{w} = i\beta k_y(x,0)\hat{w}_x(0,t) \\ +f(x,t), & (x,t) \in (0,L)\times(0,T), \\ \hat{w}(0,t) = \hat{w}(L,t) = \hat{w}_x(L,t) = 0, & t \in (0,T), \\ \hat{w}(x,0) = \hat{w}_0(x) := \hat{u}_0(x) - \int_0^x k(x,y)\hat{u}(y,t)dy, & x \in (0,L), \end{cases} \tag{4.60}$$

where $f(x,t) := [(I-\Upsilon_k)p_1](x)\tilde{w}_x(0,t) + [(I-\Upsilon_k)p_2](x)\tilde{w}_{xx}(0,t)$. Let us denote the solution as $\hat{w} = \hat{w}[\hat{w}_0, f]$. Notice that this model (4.60) with $f \equiv 0$ is exactly the same model as (3.16), that we studied its well-posedness in Chapter 3. So from Proposition 3.3, we know that $\hat{w}_0 \in L^2(0,L)$ guarantees the existence of a unique solution $\hat{w}[\hat{w}_0, 0]$ in the space X_T^0. This leads to a one parameter family of bounded evolution operators $W(\cdot)$, defined on $L^2(0,L)$ and maps into X_T^0 such that, $\hat{w}$ can be expressed as $\hat{w}[\hat{w}_0, 0] = W(t)\hat{w}_0$ for all $t \geq 0$.

Now let us consider $\hat{w}[0, f]$, i.e., solution of the model (4.60) in the presence of an interior source f and with a zero initial state. Application of Duhamel's principle yields following representation

$$\hat{w}[0, f(\cdot,t)] = \int_0^t W(t-\tau)f(\cdot,\tau)d\tau. \tag{4.61}$$

Recall that $k = k(x,y)$, $p_1(x) = -i\beta p_y(x,0) + \alpha p(x,0)$ and $p_2(x) = i\beta p(x,0)$ are smooth functions on their respective domains. Moreover, if $\tilde{w}_0 \in H^3(0,L)$, then by Proposition 4.2, we have $w_x(0,t), w_{xx}(0,t) \in L^1(0,T)$. Consequently $f \in L^1(0,T; H^\infty(0,L))$. Now taking L^2-norm of both sides of (4.61) in the first component, we get

$$\begin{aligned} \|\hat{w}[0, f(\cdot,t)]\|_{L^2(0,L)} &\leq \int_0^t \|W(t-\tau)f(\cdot,\tau)\|_{L^2(0,L)}\, d\tau \\ &\lesssim \int_0^t \|f(\cdot,\tau)\|_{L^2(0,L)} d\tau \leq \|f\|_{L^1(0,T;L^2(0,L))}. \end{aligned} \tag{4.62}$$

Passing to supremum on both sides in t over $[0,T]$ implies

$$\|\hat{w}[0,f(\cdot,t)]\|_{C([0,T];L^2(0,L))} \leq \|f\|_{L^1(0,T;L^2(0,L))} < \infty.$$

Since f is smooth in its first component and using same arguments in (4.62), we also get

$$\begin{aligned}\|\partial_x\hat{w}[0,f(\cdot,t)]\|_{L^2(0,L)} &\leq \int_0^t \left\|\partial_x\left[W(t-\tau)f(\cdot,\tau)\right]\right\|_{L^2(0,L)} d\tau \\ &\leq \int_0^t \|f(\cdot,\tau)\|_{L^2(0,L)} d\tau \leq \|f\|_{L^1(0,T;L^2(0,L))}.\end{aligned} \tag{4.63}$$

Combining (4.62)-(4.63) and integrating both sides in t over $[0,T]$ yields

$$\|\hat{w}[0,f]\|_{L^2(0,T;H^1(0,L))} \leq \sqrt{T}\|f\|_{L^1(0,T;L^2(0,L))},$$

i.e., $\hat{w}[0,f] \in X_T^0$. Together with $\hat{w}[\hat{w}_0,0] \in X_T^0$, we have $\hat{w} = \hat{w}[\hat{w}_0,f] \in X_T^0$.

Finally, using the $k-$transformation given by (4.11) and its invertibility on $L^2(0,L)$ which we stated in Lemma 3.2, observer model (4.2) along with the controllers $h_0(\hat{u}), h_1(\hat{u})$, is also well-posed in X_T^0. Under the assumption of hypothesis of Proposition 4.4, we have the following proposition.

Proposition 4.5 *Let $T, L > 0$, $\hat{u}_0 \in L^2(0,L)$. Suppose that the observer gains are given by $p_1(x) = i\beta\ell_y(x,0) - \alpha\ell(x,0)$, $p_2(x) = -i\beta\ell(x,0)$, where ℓ is a smooth solution of* (4.8)*. Suppose also that the right endpoint boundary controllers are given by*

$$h_0(t) = \int_0^L k(L,y)\hat{u}(y,t)dy, \quad h_1(t) = \int_0^L k_x(L,y)\hat{u}(y,t)dy,$$

where k is a smooth solution of (3.14)*. Then the global solution $\hat{u}$ belongs to X_T^0.*

Notice that well-posedness result we stated above in Proposition 4.5 is not sufficient to guarantee that $\hat{w}_x(0,t), \hat{w}_{xx}(0,t)$ make sense and one needs to prove a regularity result in a higher level. As we explained at the beginning of this section, it suffices to prove a regularity result in $H^3(0,L)$ level. But then, this requires that $f(x,t) := [(I -$

$\Upsilon_k)p_1](x)\tilde{w}_x(0,t) + [(I-\Upsilon_k)p_2](x)\tilde{w}_{xx}(0,t) \in W^{1,1}(0,T)$, i.e., $\tilde{w}_{xt}(0,t), \tilde{w}_{xxt}(0,t) \in L^1(0,T)$. However, hypotheses of Proposition 4.4 does not guarantee that this remains true. Instead, one needs to prove a regularity result for the target error model (4.10) in $H^6(0,L)$ level. This requires that initial and boundary data are compatible in a higher sense.

Proposition 4.6 *Assume $\tilde{w}_0 \in H^6(0,L)$, $\psi \in H^2(0,L)$, where assume that the couple $(\tilde{w}_0, \psi)$ satisfies the higher compatibility conditions* (2.13)-(2.14). *Then solution $\tilde{w}$ of* (4.40) *belongs to the space X_T^6.*

Remark 4.1 *Proposition 4.6 provides the higher regularity result for the open loop model* (4.40)*, i.e., the Neumann boundary condition is defined as $\tilde{w}_x(L,t) = \psi(t)$. However, recall that the original boundary condition of target error model* (4.10) *is of the form*

$$\psi(\tilde{w})(t) = \int_0^L \ell_x(L,y)\tilde{w}(y,t)dy, \tag{4.64}$$

which is feedback type. So applying a similar fixed point argument to those that we performed through made in (4.57)-(4.59)*, one can also show that results for the Proposition 4.6 also holds for the target error model* (4.10).

Proof Let $\tilde{y}$ solves the below linear model

$$\begin{cases} i\tilde{y}_t + i\beta\tilde{y}_{xxx} + \alpha\tilde{y}_{xx} + i\delta\tilde{y}_x + ir\tilde{y} = 0, & (x,t) \in (0,L)\times(0,T), \\ \tilde{y}(0,t) = \tilde{y}(L,t) = 0,\ \tilde{y}_x(L,t) = \psi''(t), & t \in (0,T), \\ \tilde{y}(x,0) = \tilde{y}_0(x) := -\beta\tilde{v}_0''' + i\alpha\tilde{\tilde{v}}_0'' - \delta\tilde{v}_0' - r\tilde{v}_0, & x \in (0,L), \end{cases} \tag{4.65}$$

where $\tilde{v}_0$ is the initial datum of the model (4.56) given by $\tilde{v}_0 := -\beta\tilde{w}_0''' + i\alpha\tilde{w}_0'' - \delta\tilde{w}_0' - r\tilde{w}_0$. Suppose that $\tilde{y}_0 \in L^2(0,L)$, then $\tilde{v}_0 \in H^3(0,L)$ which also implies $\tilde{w}_0 \in H^6(0,L)$. Assume that $\psi'' \in L^2(0,T)$. Then, by Lemma 4.2, $\tilde{y} \in X_T^0$ and the estimate

$$\|\tilde{y}\|_{X_T^0} \lesssim \|\tilde{y}_0\|_{L^2(0,L)} + \|\psi''\|_{L^2(0,T)}$$

holds true. Define $\tilde{v}(x,t) = \tilde{v}_0(x) + \int_0^t \tilde{y}(x,\tau)d\tau$. Recall also the relation $\tilde{w}(x,t) = \tilde{w}_0(x) +$

$\int_0^t \tilde{v}(x,\tau)d\tau$ we set in the proof of Lemma 4.3. Therefore, $\tilde{w}$ and $\tilde{y}$ are related via the relation

$$\tilde{w}(x,t) := \tilde{w}_0(x) + t\tilde{v}_0(x) + \int_0^t \int_0^\tau \tilde{y}(x,s)dsd\tau. \tag{4.66}$$

Taking $t = 0$ on (4.66), we see that $\tilde{w}$ satisfies the initial condition $\tilde{w}(x,0) = \tilde{w}_0(x)$. Moreover for $x = 0$ and due to the boundary condition $\tilde{y}(0,t) = 0$

$$\begin{aligned}\tilde{w}(0,t) &= \tilde{w}_0(0) + t\tilde{v}_0(0) + \int_0^t \int_0^\tau \tilde{y}(0,s)dsd\tau \\ &= \tilde{w}_0(0) + t\tilde{v}_0(0) = 0,\end{aligned}$$

if and only if $\tilde{w}_0(0) = 0$ and $v_0(0) = (-\beta\tilde{w}_0''' + i\alpha\tilde{w}_0'' - \delta\tilde{w}_0' - r\tilde{w}_0)(0) = 0$. Similarly for $x = L$

$$\begin{aligned}\tilde{w}(L,t) &= \tilde{w}_0(L) + t\tilde{v}_0(L) + \int_0^t \int_0^\tau \tilde{y}(L,s)dsd\tau \\ &= \tilde{w}_0(L) + t\tilde{v}_0(L) = 0\end{aligned}$$

if and only if $\tilde{w}_0(L) = 0$ and $\tilde{v}_0(L) = (-\beta\tilde{w}_0''' + i\alpha\tilde{w}_0'' - \delta\tilde{w}_0' - r\tilde{w}_0)(L) = 0$. Differentiating both sides of (4.66) with respect to x, taking $x = L$ and using the boundary condition $\tilde{y}_x(L,t) = \psi''(t)$, we write

$$\begin{aligned}\tilde{w}_x(L,t) &= \tilde{w}_0'(L) + t\tilde{v}_0'(L) + \int_0^t \int_0^\tau \tilde{y}_x(L,s)dsd\tau \\ &= \tilde{w}_0'(L) + t\tilde{v}_0'(L) + \int_0^t \left(\psi'(\tau) - \psi'(0)\right)d\tau \\ &= \tilde{w}_0'(L) + t\tilde{v}_0'(L) + \psi(t) - \psi(0) - t\psi'(0),\end{aligned}$$

where $(\cdot)'$ denotes ordinary derivative with respect to the associated independent variable. $\tilde{w}_x(L,t) = \psi(t)$ holds if and only if $\tilde{w}_0'(L) = \psi(0)$ and $\tilde{v}_0'(L) = (-\beta\tilde{w}_0''' + i\alpha\tilde{w}_0'' - \delta\tilde{w}_0' - r\tilde{w}_0)'(L) = \psi'(0)$. Consequently, $\tilde{w}$ defined by (4.66) satisfies the boundary conditions of (4.10), provided that the couple $(\tilde{w}_0, \psi)$ satisfies higher compatibility conditions (2.13)-

(2.14). Moreover, $\tilde{w}$ also satisfies the main equation of (4.10). Indeed, we have

$$\begin{aligned}
&(i\tilde{w}_t + i\beta\tilde{w}_{xxx} + \alpha\tilde{w}_{xx} + i\delta\tilde{w}_x + ir\tilde{w})(x,t) - i\beta k_y(x,0)\tilde{w}_x(0,t)) \\
=& i\tilde{v}_0(x) + \int_0^t i\tilde{y}(x,s)ds + \left(i\beta\tilde{w}_0'''(x) + \alpha\tilde{w}_0''(x) + i\delta\tilde{w}_0'(x) + ir\tilde{w}_0(x)\right) \\
&+ t\left(i\beta\tilde{v}_0'''(x) + \alpha\tilde{v}_0''(x) + i\delta\tilde{v}_0'(x) + ir\tilde{v}_0(x)\right) \\
&+ \int_0^t \int_0^\tau (i\beta\tilde{y}_{xxx}(x,s) + \alpha\tilde{y}_{xx}(x,s) + i\delta\tilde{y}_x(x,s) + ir\tilde{y}(x,s))\, dsd\tau \\
=& i\tilde{v}_0(x) + \int_0^t i\tilde{y}(x,s)ds - i\tilde{v}_0(x) - it\tilde{y}_0(x) + \int_0^t \int_0^\tau (-i\tilde{y}_s(x,s))dsd\tau \\
=& \int_0^t i\tilde{y}(x,s)ds - it\tilde{y}_0(x) + \int_0^t (-i\tilde{y}(x,\tau) + i\tilde{y}(x,0))\, d\tau = 0.
\end{aligned}$$

Hence, we conclude that $\tilde{w}$ defined by (4.66) solves the model (4.10).

Now leaving $\tilde{w}_{xxx}$ on the main equation of the $\tilde{w}-$model (4.40), using $\tilde{w}_t = \tilde{v}$ and applying similar arguments in (3.156)-(3.159) and in (3.162)-(3.163), we can write

$$\|\tilde{w}_{xxx}(\cdot,t)\|_{L^2(0,L)} \lesssim \|\tilde{v}(\cdot,t)\|_{L^2(0,L)} + \|\tilde{w}(\cdot,t)\|_{L^2(0,L)} \tag{4.67}$$

and

$$\|\tilde{w}_{xxxx}(\cdot,t)\|_{L^2(0,L)} \lesssim \|\tilde{v}_x(\cdot,t)\|_{L^2(0,L)} + \|\tilde{w}_x(\cdot,t)\|_{L^2(0,L)}. \tag{4.68}$$

Notice that $\tilde{y}_0 \in L^2(0,L)$ implies $\tilde{v}_0 \in H^3(0,L)$. Moreover, assume that $\psi' \in H^1(0,T)$. If $(\tilde{v}_0, \psi') \in H^3(0,L) \times H^1(0,T)$ satisfying compatibility conditions (2.12), then by Lemma 4.3, the function $\tilde{v}$ that solves (4.56) with initial and boundary data $(\tilde{v}_0, \psi') \in H^3(0,L) \times H^1(0,T)$ belongs to the space X_T^3. Similarly, again by the Lemma 4.3, we also have $\tilde{w} \in X_T^3$ if $\tilde{w}_0 \in H^3(0,L)$ (which follows from $\tilde{v}_0 \in H^3(0,L)$) and $\psi \in H^1(0,T)$. Consequently, from the estimates (4.67) and (4.68), we obtain $\tilde{w}_{xxx} \in C([0,T]; H^3(0,L))$ and $\tilde{w}_{xxx} \in L^2(0,T; H^4(0,L))$. Moreover, by Lemma 4.3, assumptions of Proposition 4.6 also imply that $\tilde{w} \in X_T^3$. Combining these results, we deduce that $\tilde{w} \in X_T^6$. Note that the assumptions $\psi' \in H^1(0,T)$ and $\psi \in H^1(0,T)$ imply $\psi \in H^2(0,T)$. □

Assumptions of Proposition 4.6 make sure that $\|f(\cdot,t)\|_{L^2(0,L)} \in W^{1,1}(0,T)$. So under those assumptions, now we can prove the following proposition.

Proposition 4.7 *Let $\hat{w}_0 \in D(A)$ and assumption hypothesis of Proposition 4.6 hold. Then $\hat{w} \in X_T^3$.*

Proof Proof can be done by following similar arguments to those in the proof of Proposition 3.4 (or Lemma 4.3). Let us introduce a new variable $\hat{v} = \hat{w}_t$ that solves

$$\begin{cases} i\hat{v}_t + i\beta\hat{v}_{xxx} + \alpha\hat{v}_{xx} + i\delta\hat{v}_x + ir\hat{v} = i\beta k_y(x,0)\hat{v}_x(0,t) & \\ +f_t(x,t), & (x,t) \in (0,L)\times(0,T), \\ \hat{v}(0,t) = \hat{v}(L,t) = \hat{v}_x(L,t) = 0, & t \in (0,T), \\ \hat{v}(x,0) = \hat{v}_0(x), & x \in (0,L), \end{cases} \tag{4.69}$$

where $\hat{v}_0 := -\beta\hat{w}_0''' + i\alpha\hat{w}_0'' - \delta\hat{w}_0' - ir\hat{w}_0 + \beta k_y(\cdot,0)\hat{w}_0(0,0) - if(\cdot,0) \in L^2(0,L)$ and $f(x,t) := [(I-\Upsilon_k)p_1](x)\tilde{w}_x(0,t) + [(I-\Upsilon_k)p_2](x)\tilde{w}_{xx}(0,t)$. Well-posedness of this model is studied in Section 4.3.2 and if $\tilde{\hat{v}}_0 \in L^2(0,L)$ and $f_t \in L^1(0,T;L^2(0,L))$ then $\hat{v} \in X_T^0$. Set $\hat{w}(x,t) := \hat{w}_0(x) + \int_0^t \hat{\hat{v}}(x,\tau)d\tau$. Applying the arguments (3.153)-(3.155), one can show that if $\hat{w}_0 \in D(A)$, then $\hat{w}$ solves the target observer model (4.12) with the initial condition $\hat{w}_0$. Now from its main equation, we write

$$i\beta\hat{w}_{xxx}(x,t) = (-i\hat{v} - \alpha\hat{w}_{xx} - i\delta\hat{w}_x - ir\hat{w})(x,t) + i\beta k_y(x,0)\hat{w}_x(0,t) + f(x,t). \tag{4.70}$$

Using $\hat{w}_x(0,t) = -\int_0^L \hat{w}_{xx}(x,t)dx$ and taking L^2-norms of both side in x, we get

$$\begin{aligned}\beta^2\|\hat{w}_{xxx}(\cdot,t)\|^2_{L^2(0,L)} \le \|\hat{v}(\cdot,t)\|^2_{L^2(0,L)} + \left(\alpha^2 + \beta^2\|k_y(\cdot,0)\|^2_{L^2(0,L)}\right)\|\hat{w}_{xx}(\cdot,t)\|^2_{L^2(0,L)} \\ + \delta^2\|\hat{w}_x(\cdot,t)\|_2^2 + r^2\|\hat{w}(\cdot,t)\|^2_{L^2(0,L)} + \|f(\cdot,t)\|^2_{L^2(0,L)}.\end{aligned}$$

Applying similar arguments to those in (3.158)-(3.160), we can write

$$\|\hat{w}_{xxx}(\cdot,t)\|_2^2 \lesssim \|\hat{v}(\cdot,t)\|_2^2 + \|\hat{w}(\cdot,t)\|_2^2 + \|f(\cdot,t)\|_2^2.$$

Notice that $\hat{v}, \hat{w} \in C([0,T];L^2(0,L))$ and from Proposition 4.2, supremum of $\tilde{w}_x(0,t)$, $\tilde{w}_{xx}(0,t)$

exist. So passing to supremum in t over $[0,T]$ on both sides, we get $\hat{w} \in C([0,T];H^3(0,L))$.

Next, we differentiate both sides of (4.70) with respect to x and applying the arguments (3.161)-(3.164), we obtain

$$\|\hat{w}_{xxxx}(\cdot,t)\|_2^2 \lesssim \|\hat{v}_x(\cdot,t)\|_2^2 + \|\hat{w}(\cdot,t)\|_2^2 + \|f_x(\cdot,t)\|_2^2. \tag{4.71}$$

Notice that $\hat{v},\hat{w} \in L^2(0,T;H^1(0,L))$, so the first two terms on the right hand side of (4.71) make sense in $L^2(0,T)$. From the estimate in Proposition 4.2, we can also deduce that $\tilde{w}_x(0,t), \tilde{w}_{xx}(0,t) \in L^2(0,T)$. Hence, right hand side of (4.71) belongs to $L^2(0,T)$, which implies $\hat{w} \in L^2(0,T;H^4(0,L))$. Combining this with previous result, we conclude $\hat{w} \in X_T^3$.

□

Well-posedness results for the observer model (4.2) and the error model (4.3) follow from the invertibility of the $k-$transformation on $H^3(0,L)$ and $\ell-$transformation on $H^6(0,L)$, respectively, which is given by Lemma 3.2. Now states of the plant-observer-error models are related via the relation $u := \hat{u} + \tilde{u}$. So combining the well-posedness results regarding the observer and error models, we obtain well-posedness of the original plant in X_T^3 level.

The theorem below summarizes the well-posedness and stabilization results of the plant-observer-error system.

Theorem 4.1 *Let $T, L, \beta > 0$, $\alpha, \delta \in \mathbb{R}$. Suppose that the right endpoint controllers are given by*

$$h_0(t) = \int_0^L k(L,y;r)\hat{u}(y,t)dy, \quad h_1(t) = \int_0^L k_x(L,y;r)\hat{u}(y,t)dy, \tag{4.72}$$

and observer gains are given by $p_1(x) = i\beta\ell_y(x,0) - \alpha\ell(x,0)$ and $p_2(x) = -i\beta\ell(x,0)$ where k and ℓ are smooth solutions of (3.14) *and* (4.8)*, respectively.*

(i) (Well-posedness) Suppose $\hat{u}_0 \in H^3(0,L)$ be such that $\hat{w}_0 := (I - \Upsilon_k)\hat{u}_0 \in D(A)$. Suppose also $\tilde{u}_0 \in H^6(0,L)$ be such that $\tilde{w}_0$, given implicitly by relation $\tilde{u}_0 = (I-\Upsilon_\ell)\tilde{w}_0$ and $\psi(\tilde{w}(\cdot,t)) = \int_0^L \ell_x(L,y)\tilde{w}(y,t)dy$ satisfy the higher compatibility conditions. Then plant-observer-error system has a unique solution $(u,\hat{u},\tilde{u}) \in X_T^3 \times X_T^3 \times X_T^6$.

(ii) (Stabilization) There exists $r > 0$ sufficiently small so that the zero equilibrium to the plant (3.8) *under the influence of the controllers* (4.72) *is exponentially stable. Moreover, for such r, there exists μ and ν given by* (4.15) *and* (4.31) *and satisfying $\mu > \nu > 0$ such that the decay estimates*

$$\begin{aligned}
\|u(\cdot,t)\|_{L^2(0,L)} &\lesssim \left(c_k\|\hat{u}_0\|_{L^2(0,L)} + c_{k,\ell}\|u_0-\hat{u}_0\|_{H^3(0,L)}\right)e^{-\nu t} + c_{\ell,1}\|u_0-\hat{u}_0\|_{L^2(0,L)}e^{-\mu t},\\
\|\hat{u}(\cdot,t)\|_{L^2(0,L)} &\lesssim e^{-\nu t}\left(c_k\|\hat{u}_0\|_{L^2(0,L)} + c_{k,\ell}\|u_0-\hat{u}_0\|_{L^2(0,L)}\right),\\
\|(u-\hat{u})(\cdot,t)\|_{L^2(0,L)} &\lesssim c_{\ell,1}\|u_0-\hat{u}_0\|_{L^2(0,L)}e^{-\mu t},\\
\|(u-\hat{u})(\cdot,t)\|_{H^3(0,L)} &\lesssim c_{\ell,2}\|u_0-\hat{u}_0\|_{H^3(0,L)}e^{-\mu t},
\end{aligned}$$

hold true for all $t \geq 0$, where $u_0 := \hat{u}_0 + \tilde{u}_0$ and $c_k, c_{k,\ell}, c_{\ell,1}, c_{\ell,2}$ are nonnegative constants that are independent of $u_0, \hat{u}_0, \tilde{u}_0$.

4.4. Numerical simulations

In this part, we present our numerical algorithm and simulations verifying our theoretical stabilization result.

We introduce our algorithm in five steps. First we obtain an approximation for the backstepping kernel ℓ. This allows us to get an approximation for observer gains p_1 and p_2, which are included in the main equation of error and observer models. In the second and third steps, we obtain a numerical solution for the error model (4.3) and target observer model (4.12), respectively. Next, as the fourth step, we follow the same arguments to those we presented in Step (iii) of Section 3.4 to use the invertibility of the backstepping transformation (4.9) to get a numerical approximation for the solution of the observer model. At the fifth and the last step, we deduce numerical solution of the original plant u, using the relation $u = \hat{u} + \tilde{u}$. Let us now give the steps in details.

Step i. Recall that ℓ solving (4.8) and k solving (3.14) are related via the relation $\ell(x,y) = k(L-y, L-x; -r)$. Changing variables as $\bar{x} = x - y$, $\bar{y} = y$, we transform $k-$model to $G-$model (3.13). Therefore we have

$$G(\xi,\eta;-r) = G(L-\bar{y}, L-\bar{x}-\bar{y}; -r) = k(L-y, L-x; -r) = \ell(x,y),$$

where $\xi = L - \bar{y}$, $\eta = L - \bar{x} - \bar{y}$. So, to get an approximation for $\ell = \ell(x, y)$, it is enough to get an approximation for $G = G(\xi, \eta)$. But this was already done in Step (i) of Section 3.4. Thus, applying the same procedure by replacing r with $-r$, we obtain $G(\xi, \eta; -r)$, therefore $\ell(x, y)$. Using the approximation for $\ell(x, y)$, we also obtain an approximation for the observer gains $p_1(x) = -i\beta\ell_y(x, 0) + \alpha\ell(x, 0)$ and $p_2(x) = i\beta\ell(x, 0)$.

Step ii. To solve (4.3), we apply the same discretization procedure as we introduced in the second step of Section 3.4. Note that the additional trace terms, $\tilde{u}_x(0, t)$, $\tilde{u}_{xx}(0, t)$, included in the main equation of (4.3) are approximated by the following one sided second order finite differences

$$\begin{aligned} \tilde{u}_x(0, t) &\approx \frac{-3\tilde{u}_0(t) + 4\tilde{u}_1(t) - \tilde{u}_2(t)}{2h_x}, \\ \tilde{u}_{xx}(0, t) &\approx \frac{2\tilde{u}_0(t) - 5\tilde{u}_1(t) + 4\tilde{u}_2(t) - \tilde{u}_3(t)}{h_x^2}. \end{aligned} \tag{4.73}$$

Step iii. To solve (4.12) numerically, we apply the same discretization procedure in Step (ii) of Section 3.4. Note that the previous step, in particular, allows us to obtain an approximation for the trace terms $\tilde{u}_x(0, t)$, $\tilde{u}_{xx}(0, t)$. Notice also that, using $\ell-$transformation we can write $\tilde{w}_x(0, t) = \tilde{u}_x(0, t)$ and $\tilde{w}_{xx}(0, t) = \tilde{u}_{xx}(0, t)$. So these approximations can be implemented as known inputs in the discretization scheme. Discrete counterpart Υ_k^M, of Υ_k included in the main equation of (4.12), can be obtained by applying a suitable numerical integration technique, for instance composite trapezoidal rule, as follows

$$\Upsilon_k^M = h_x \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2}k(x_2, x_1) & \frac{1}{2}k(x_2, x_2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2}k(x_{M-1}, x_1) & k(x_{M-1}, x_2) & \cdots & \frac{1}{2}k(x_{M-1}, x_{M-1}) & 0 \\ \frac{1}{2}k(x_M, x_1) & k(x_M, x_2) & \cdots & k(x_M, x_{M-1}) & \frac{1}{2}k(x_M, x_M) \end{bmatrix}.$$

Step iv. Next we use same procedure in Step (iii) of Section 3.4 to get an approximation for inverse image $\hat{u}$, of $\hat{w}$.

Step v. Finally we get a numerical approximation for u via the relation $u = \hat{u} + \tilde{u}$.

Now we present a numerical experiment that verifies exponential decay of solutions of plant-observer-error system. We used $M = 1001$ spatial nodes, $N = 5001$ time steps. The iteration for G, so to obtain an approximation for k and ℓ, is performed up several times so that the error is $\max_{(\bar{x},\bar{y})\in\Delta_{\bar{x},\bar{y}}} |G^{j+1} - G^j| \sim 10^{-14}$ level. Consider the model

$$\begin{cases} iu_t + iu_{xxx} + 2u_{xx} + 8iu_x = 0, & (x,t) \in (0,\pi) \times (0,T), \\ u(0,t) = 0, u(\pi,t) = h_0(t), u_x(\pi,t) = h_1(t), & t \in (0,T). \end{cases}$$

Note that the interval length π belongs to set of critical lengths of intervals. Therefore, in the absence of controllers, the model may assume a time independent solution. Let us initialize the error model as $\tilde{u}(x,0) = 3 - e^{4ix} - 2e^{-2ix}$ and observer model as $\hat{u}(x,0) \equiv 0$. We take $r = 0.05$. This choice yields positive vales $\mu > \nu > 0$ where μ and ν are defined in (4.15) and (4.31), respectively.

The figure below on the left present the behaviour of the solution of the original plant in time. On the right we present time evolution of L^2-norms of each of the states of plant-observer-error system.

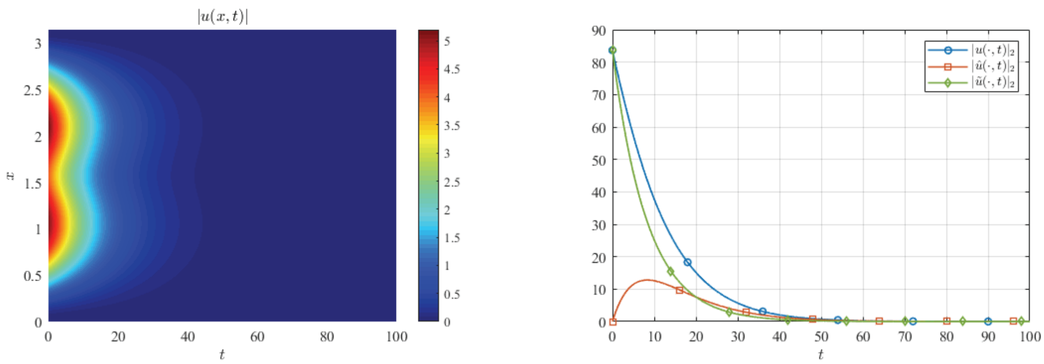

Figure 4.2. Numerical simulations in the presence of controllers. Left: Time evolution of $|u(x,t)|$. Right: Time evolution of $||u(\cdot,t)||_{L^2(0,L)}$, $||\hat{u}(\cdot,t)||_{L^2(0,L)}$ and $||\tilde{u}(\cdot,t)||_{L^2(0,L)}$.

CHAPTER 5

FINITE DIMENSIONAL BACKSTEPPING CONTROLLER DESIGN FOR INFINITE DIMENSIONAL DISSIPATIVE SYSTEMS

In this chapter, we introduce finite dimensional version of classical backstepping strategy for stabilizing zero equilibrium to dissipative systems. More precisely, the boundary controller that we construct by the backstepping method now involves a projection of the state u, onto a finite dimensional space. As a canonical example, we study stabilization of zero equilibrium to the nonlinear reaction diffusion equation. However, our strategy is applicable also to other evolutionary equations and system of equations which has finite dimensional long time behaviour such as complex Ginzburg-Landau equation, various kind of dissipative wave equations, or system of dissipative PDEs such as Fitzhugh-Nagumo system, phase-field systems.

To this end, let us consider the following system:

$$\begin{cases} u_t - \nu u_{xx} - \alpha u + u^3 = 0, & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = 0,\ u(L,t) = h(t), & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,L). \end{cases} \tag{5.1}$$

Here $\nu, \alpha > 0$ are given constants and $h(t) = h(u(\cdot,t))$ is a soughtafter feedback control that acts through Dirichlet actuation at $x = L$ and involves only finitely many Fourier sine modes of u. In the absence of control input and for certain values of ν, α and L, zero equilibrium solution may be either asymptotically stable or unstable. To be more precise, if $\nu\lambda_1 - \alpha > 0$, (5.1) has a unique equilibrium solution $u \equiv 0$. For this case, it is exponentially stable. Conversely, if $\nu\lambda_1 - \alpha < 0$, then there exist at least two nontrivial stationary solutions, exactly two of which are asymptotically stable, and all solutions bifurcate from the zero equilibrium. So for this case, $u \equiv 0$ is no more stable. Our aim is to construct a feedback law of the form

$$h(t) = h(P_N u(\cdot,t)) = \int_0^L \xi(y)\Gamma[P_N u](y,t)dy,$$

such that all solutions are steered asymptotically to zero. Here, Γ is a linear bounded operator on a certain L^2-based functional space, ξ is a suitable smooth function to be constructed, and P_N is the projection operator

$$P_N\varphi(x) = \sum_{j=1}^{N} e_j(x)\left(e_j(\cdot),\varphi(\cdot)\right)_2, \quad e_j(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{j\pi x}{L}\right).$$

Remark 5.1 *The above model considers Dirichlet actuation at the right endpoint of the spatial interval. We can also consider a controller which acts to the model from the left endpoint, or a Neumann type controller which acts either from left or from right. No matter the choice is, our strategy we introduce below still works. Note that if one considers a Neumann actuation, then one should consider a change in the boundary condition of the associated target model from Dirichlet type to Neumann type.*

Let us now describe our strategy. We first consider the following linearized model of (5.1) around zero solution:

$$\begin{cases} u_t - \nu u_{xx} - \alpha u = 0, & (x,t) \in (0,L)\times(0,T),\\ u(0,t) = 0,\ u(L,t) = h(t), & t \in (0,T),\\ u(x,0) = u_0(x), & x \in (0,L), \end{cases} \tag{5.2}$$

and apply the backstepping method in order to stabilize zero equilibrium of (5.2). Once we construct finite dimensional feedback control for our purposes, we use it for stabilizing the zero equilibrium of the nonlinear model (5.1). To design a finite dimensional controller via the backstepping method, we use the same three step strategy we explained in Section 3.1. However, there is a novel change in the choice of associated target model, where its choice relies on finite dimensionality of the asymptotic in time behaviour of dissipative models. Indeed, aiming to convert the boundary feedback actuation to an interior damping, motivates us to suggest the following nontrivial choice target model

$$\begin{cases} w_t - \nu w_{xx} - \alpha w + \mu P_N w = 0, & (x,t) \in (0,L) \times (0,T), \\ w(0,t) = w(L,t) = 0, & t \in (0,T), \\ w(x,0) = w_0(x), & x \in (0,L), \end{cases} \tag{5.3}$$

which is a suitable one for our purposes. Recall that dissipative dynamical systems possess finite dimensional long time dynamics, since such systems possess a finite number of determining modes. Therefore, (5.3) is indeed a suitable choice, since the additional damping term $\mu P_N w$ involving in its main equation is capable of preventing large fluctuations or uncontrolled growth of the solution for large N due to the term $-\alpha w$. In fact, as we show in Proposition 5.1 that for a given $\mu > \alpha - \nu\lambda_1$, if N is sufficiently large, then all solutions can be steered to zero with respect to L^2 and H^1 metric.

Now let us define the backstepping transformation as

$$u(x,t) = w(x,t) + \int_0^x k(x,y) P_N w(y,t) dy. \tag{5.4}$$

After some calculations, which we detail in Section 5.1.1, we see that the linear plant (5.2) maps to linear target (5.3), if k satisfies the following boundary value problem

$$\begin{cases} \nu(k_{xx} - k_{yy}) + \mu k = 0, & (x,y) \in \Delta_{x,y}, \\ k(x,0) = 0, \quad k(x,x) = -\dfrac{\mu x}{2\nu}, & x \in [0,L], \end{cases} \tag{5.5}$$

where $\Delta_{x,y} := \left\{(x,y) \in \mathbb{R}^2 \,|\, x \in (0,L), y \in (0,x)\right\}$. Notice that compared to the previous applications of backstepping method in Chapter 3 and Chapter 4, now the backstepping transformation (5.4) involves a projection operator inside the integral term. This is because, once we show the existence of a backstepping kernel k and then by taking $x = L$ on (5.4), we are allowed to express the control input as

$$h(t) = \int_0^L k(L,y) P_N w(y,t) dy \tag{5.6}$$

which involves finitely many sine modes of w.

Finally, proving invertibility of the backstepping with a bounded inverse on a suitable space, allows us to conclude exponential decay of solutions of original linear plant (5.2). Moreover derivation of the inverse backstepping operator also allows us to replace w by u via the inverse relation, which makes the control input (5.6) of feedback type. A crucial part in this step is that, one needs to make a delicate proof of invertibility and obtain an explicit inverse relation so that the control input needs not only to be feedback type but also needs to involve finitely many Fourier sine modes of u. More precisely, one needs to find an inverse relation, which relates not only w to u but actually $P_N w$ to $P_N u$, so that the controller (5.6) depends on the projection $P_N u$. We experienced that usual methods for proving bounded invertibility of the standard backstepping transformation fails in the current finite dimensional case and one needs to give a new proof of such result. This is the second novelity of our proposed strategy and is proved in Lemma 5.2.

Remark 5.2 *Unlike the way we defined in Chapter 3, now state of the target model, w, is given in the backstepping transformation* (5.4) *implicitly. This is due to overcome some mathematical difficulties that occurs during calculations, when we map the linear plant* (5.2) *to the linear target* (5.3).

Once we gain the exponential stabilization of zero equilibrium to the linear closed-loop system (5.2), then we turn our attention to nonlinear plant (5.1) and study exponential stabilization of the associated zero equilibrium. This is done by using the same backstepping transformation (5.4) along with the controller (5.6). Hence, it is enough to study the associated nonlinear target model to the nonlinear plant.

Remark 5.3 *In the nonlinear case, our stabilization is only in local manner, that is, we have a smallness assumption on the initial datum for stabilizing the zero equilibrium. This is because the backstepping transformation turns the original nonlinear plant into the target model in which the monotone structure of the nonlinear term, u^3, is disrupted. So applying multipliers to derive L^2-based norm estimates yields Bernoulli type differential inequalities which forces us to put a smallness assumption on the initial datum. Note that smallness condition for stabilization of nonlinear evolutionary equations via backstepping controllers is common, see for instance (Cerpa and Coron, 2013). On the other hand, it is not a necessary assumption for local existence of solutions.*

5.1. Exponential stabilization of zero equilibrium

In this section, we construct a boundary feedback controller which involves finitely many Fourier sine modes of the state that exponentially stabilizes zero equilibrium of the linear plant and nonlinear plant. We do this in the following order: First, in Section 5.1.1, we state existence of a smooth kernel and prove that the backstepping transform is invertible. Next, in Section 5.1.2, we prove exponential decay of solutions to the linear and nonlinear target model in L^2 and in H^1 level. In Section 5.1.3, we also derive the minimal number of Fourier modes that provides exponential decay of solutions. Invertibility of the backstepping transformation provides that the same result also hold for linear and nonlinear original plants along with the boundary controller that involves only finitely many Fourier modes of u.

5.1.1. Smooth backstepping kernel and invertibility of the backstepping transformation

We want to find sufficient conditions for the backstepping kernel so that the backstepping transformation becomes a mapping between the target model and the linearized plant. To this end, we differentiate (5.4) with respect to t, use the main equation in (5.2) and linearity of P_N to get

$$\begin{aligned} u_t(x,t) &= w_t(x,t) + \int_0^x k(x,y)(P_N u)_t(y,t)dy \\ &= w_t(x,t) + \int_0^x k(x,y)(P_N u_t)(y,t)dy \\ &= w_t(x,t) + \int_0^x k(x,y)\left(\nu P_N \partial_y^2 w + \alpha P_N w - \mu P_N w\right)(y,t)dy. \end{aligned} \tag{5.7}$$

P_N projects onto the span of the first N eigenfunctions of the operator $-\frac{d^2}{dy^2}$ subject to homogeneous Dirichlet boundary conditions. Therefore, P_N and $\frac{d^2}{dy^2}$ commute and we can write

$$u_t(x,t) = w_t(x,t) + \int_0^x k(x,y)\left(\nu\partial_y^2(P_N w) + \alpha P_N w - \mu P_N w\right)(y,t)dy.$$

Integrating bu parts, we get

$$\begin{aligned}u_t(x,t) =& w_t(x,t) + \nu\Big(k(x,x)(P_N w(x,t))_x - k(x,0)(P_N w(y,t))_y\Big|_{y=0} \\ & -k_y(x,x)P_N w(x,t)\Big) \\ & + \int_0^x \left(\nu k_{yy} + (\alpha-\mu)k\right)(x,y)P_N w(y,t)dy.\end{aligned} \tag{5.8}$$

Next, we differentiate (5.4) with respect to x twice to get

$$\begin{aligned}-\nu u_{xx}(x,t) =& -\nu w_{xx}(x,t) - \nu\int_0^x k_{xx}(x,y)P_N w(y,t)dy - \nu k_x(x,x)P_N w(x,t) \\ & -\nu\frac{d}{dx}k(x,x)P_N w(x,t) - \nu k(x,x)(P_N w(x,t))_x.\end{aligned} \tag{5.9}$$

Adding (5.8) and (5.9) side by side, together with

$$-\alpha u(x,t) = -\alpha w(x,t) - \alpha\int_0^x k(x,y)P_N w(y,t)dy, \tag{5.10}$$

we obtain

$$\begin{aligned}0 =& -\mu P_N w(x,t) - \int_0^x \left(\nu(k_{xx}-k_{yy}) + \mu k\right)(x,y)P_N w(y,t)dy \\ & -\nu\left(k_x(x,x) + k_y(x,x) + \frac{d}{dx}k(x,x)\right)P_N w(x,t) - \nu k(x,0)(P_N w(y,t))_y\Big|_{y=0}.\end{aligned} \tag{5.11}$$

Forcing $k(x,0) = 0$, we see that the last term in (5.11) vanishes. Using the relation $\frac{d}{dx}k(x,x) = k_x(x,x) + k_y(x,x)$, we get $\left(2\nu\frac{d}{dx}k(x,x) + \mu\right)P_N w(x,t) = 0$, which implies

$$\frac{d}{dx}k(x,x) = -\frac{\mu}{2\nu}. \tag{5.12}$$

Integrating both sides of this equality over $(0,x)$ and using $k(x,0) = 0$, we obtain

$$k(x,x) = -\frac{\mu x}{2\nu}. \tag{5.13}$$

Hence, considering the term inside the integral on (5.11) together with the conditions (5.12) and (5.13), we see that if k satisfies the following boundary value problem

$$\begin{cases} \nu(k_{xx} - k_{yy}) + \mu k = 0, & (x,y) \in \Delta_{x,y}, \\ k(x,0) = 0, \quad k(x,x) = -\frac{\mu x}{2\nu}, & x \in [0,L], \end{cases} \tag{5.14}$$

on the triangular region $\Delta_{x,y} = \left\{(x,y) \in \mathbb{R}^2 : x \in (0,L), y \in (0,x)\right\}$, then the backstepping transformation (5.4) successfully maps the linear plant (5.2) to the associated target model (5.3).

Existence of a solution to a model similar to (5.14) is studied at Section 4 in (Krstic and Smyshlyaev, 2008), which is based on method of successive approximations. So applying similar arguments to those, one can show that the series

$$-\frac{\mu y}{2\nu} \sum_{m=0}^{\infty} \left(-\frac{\mu}{4\nu}\right)^m \frac{(x^2 - y^2)^m}{m!(m+1)!} \tag{5.15}$$

and its partial derivatives of arbitrary order are uniformly and absolutely convergent on the closure of $\Delta_{x,y}$, and (5.15) solves (5.14). Thus, we can consider k as in the form (5.15).

Lemma 5.1 *There exists a smooth function k that solves* (5.14) *and given by* (5.15).

Remark 5.4 *Thanks to existence of a backstepping kernel, we take $x = L$ on the backstepping transformation* (5.4) *and using $w(L,t) = 0$ to obtain that control input h has the form*

$$h(t) = \int_0^L k(L,y) P_N w(y,t) dy. \tag{5.16}$$

Below we obtain a representation for the inverse of the backstepping transformation. So replacing this representation by w, we will see that (5.16) *can be written as a feedback.*

Our next task is to prove the invertibility of the backstepping transformation with a bounded inverse on $H^l(0,L)$, for $l = 0$ and $l = 1$. Let $\Upsilon_k : H^l(0,L) \to H^l(0,L)$ be the integral operator defined by $(\Upsilon_k \psi)(x) := \int_0^x k(x,y)\psi(y)dy$. We have the following invertibility result.

Lemma 5.2 *Let $N \geq 1$, then $I + \Upsilon_k P_N : H^l(0,L) \to H^l(0,L)$, $l = 0,1$ is invertible with a bounded inverse. Moreover $(I+\Upsilon_k P_N)^{-1}$ can be written as $I - \Phi_N$, where $\Phi_N : L^2(0,L) \to H^l(0,L)$ is a linear bounded operator and iteratively given by the relation*

$$\begin{aligned}
\Phi_1\varphi &\equiv \frac{1}{1+\beta_1}\Upsilon_k P_1\varphi, \quad \beta_1 = (e_1, \Upsilon_k e_1)_2,\\
\Phi_N\varphi &\equiv (I-\Phi_{N-1})[\Upsilon_k P_N\varphi] - \frac{((I-\Phi_{N-1})[\Upsilon_k P_N\varphi], e_N)_2}{1+((I-\Phi_{N-1})[\Upsilon_k e_N], e_N)_2}(I-\Phi_{N-1})[\Upsilon_k e_N],
\end{aligned} \tag{5.17}$$

for $N \geq 2$.

Proof We write the backstepping transformation in operator form as $\varphi = (I + \Upsilon_k P_N)\psi$. Set $v := \Upsilon_k P_N \psi$. Then, we have $\psi = \varphi - v$ and we get $v = \Upsilon_k P_N(\varphi - v)$. Given $\psi \in H^l(0,L)$, we have $(I+\Upsilon_k P_N)\psi \in H^l(0,L)$. Therefore, we have the inclusion $R(I+\Upsilon_k P_N) \subset H^l(0,L)$. We will prove the invertibility with induction on N.

(i) First we consider the case $N = 1$. We can write

$$v = \Upsilon_k P_1(\varphi - v) = (\hat{\varphi}_1 - \hat{v}_1)\Upsilon_k e_1,$$

where $\hat{\varphi}_1$ and $\hat{v}_1$ are the first Fourier sine modes of φ and of v, respectively. Now, we look for a solution in the form $v = \alpha_1 \Upsilon_k e_1$. Such solution can be constructed by finding α_1 that satisfies

$$\alpha_1 \Upsilon_k e_1 = (\hat{\varphi}_1 - \alpha_1\beta_1)\Upsilon_k e_1,$$

where $\beta_1 = \int_0^L e_1(s)[\Upsilon_k e_1](s)ds$. This is the case if we choose

$$\alpha_1(1+\beta_1) = \hat{\varphi}_1.$$

Therefore, if $\beta_1 \neq -1$, then we can choose $\alpha_1 = \frac{\hat{\varphi}_1}{1+\beta_1}$. Then we can write v as

$$v = \frac{\hat{\varphi}_1}{1+\beta_1}\Upsilon_k e_1 = \frac{1}{1+\beta_1}\Upsilon_1 P_N \varphi,$$

and therefore ψ as

$$\psi = \varphi - v = \varphi - \frac{1}{1+\beta_1}\Upsilon_1 P_N \varphi.$$

We remark that the condition $\beta_1 \neq -1$ can always be guaranteed by slightly modifying μ if necessary. This follows from the explicit solution representation of $k = k(x, y; \mu)$ given in (5.15) and the definition of β_1. So this shows surjectivity of $I + \Upsilon_k P_1$. Note that we in particular have $v \in H^l(0, L)$ (indeed better than this) so that $\psi = \varphi - v \in H^l(0, L)$.

Now we set

$$\Phi_1 \varphi \equiv \frac{1}{1+\beta_1}\Upsilon_k P_1 \varphi.$$

It is easy to verify that $(I - \Phi_1)$ is both a right inverse and a left inverse for $(I + \Upsilon_k P_1)$ and moreover $\Phi_1 : L^2(0, L) \to H^l(0, L)$ is a linear bounded operator. Therefore, $(I + \Upsilon_k P_1)^{-1}$ exists and is given by $(I + \Upsilon_k P_1)^{-1}\varphi = (I - \Phi_1)\varphi$.

(ii) We assume that there exists some $K \geq 1$ such that the statement of the lemma holds true for $N = K$.

(iii) Now, we claim that the statement of the lemma must also be true for $N = K + 1$. Replacing N by $(K + 1)$ we have

$$\begin{aligned} v &= \Upsilon_k P_{K+1}(\varphi - v) \\ &= \Upsilon_k P_{K+1}\varphi - \Upsilon_k P_K v - \Upsilon_k E_{K+1} v, \end{aligned} \tag{5.18}$$

where we used $P_{K+1} = P_K + E_{K+1}$ with E_{K+1} being the projection onto the $(K+1)-$th Fourier sine mode. Rearranging the terms, we get

$$(I + \Upsilon_k P_K)v = \Upsilon_k P_{K+1}\varphi - \hat{v}_{K+1}\Upsilon_k e_{K+1},$$

where $\hat{v}_{K+1}$ is the $(K + 1)-$th Fourier sine mode of v. By using the induction assumption in Step (ii), we obtain

$$v = (I - \Phi_K)[\Upsilon_k P_{K+1}\varphi] - \hat{v}_{K+1}(I - \Phi_K)[\Upsilon_k e_{K+1}].$$

Multiplying the both sides of the above equation with e_{K+1} in $L^2(0, L)$, we get

$$\hat{v}_{K+1} = \frac{((I - \Phi_K)[\Upsilon_k P_{K+1}\varphi], e_{K+1})_2}{1 + ((I - \Phi_K)[\Upsilon_k e_{K+1}], e_{K+1})_2}.$$

Note that $\hat{v}_{K+1}$ is well-defined provided the denominator does not vanish above, but this can always be guaranteed by slightly modifying the parameter μ in the backstepping kernel model if necessary. Therefore, we find

$$v = (I - \Phi_K)[\Upsilon_k P_{K+1}\varphi] - \frac{((I - \Phi_K)[\Upsilon_k P_{K+1}\varphi], e_{K+1})_2}{1 + ((I - \Phi_K)[\Upsilon_k e_{K+1}], e_{K+1})_2}(I - \Phi_K)[\Upsilon_k e_{K+1}]. \quad (5.19)$$

Denote the right hand side of (5.19) by $\Phi_{K+1}\varphi$. Then, using the relation $\psi = \varphi - v$, we can explicitly write ψ as

$$\psi = \varphi - \Phi_{K+1}\varphi.$$

This shows the surjectivity of $(I + \Upsilon_k P_{K+1})$ and it has a right inverse given by $(I - \Phi_{K+1})$. Moreover, for given φ, v is uniquely determined, and therefore the right inverse is unique. This implies the right inverse is also a left inverse, $(I + \Upsilon_k P_{K+1})$ is invertible and we have

$$\psi = (I + \Upsilon_k P_{K+1})^{-1}\varphi = \varphi - v = (I - \Phi_{K+1})\varphi.$$

It is easy to check that $\Phi_{K+1} : L^2(0, L) \to H^l(0, L)$ is a linear bounded operator. This follows from the definition of Φ_{K+1} together with the induction assumption in Step (ii).

□

Remark 5.5 *Observe that the operator Φ_N defined by the iteration* (5.17) *is a projection operator, i.e., $\Phi_N\varphi = \Phi_N P_N\varphi$. So using this property and replacing w in the control input $h(t) = \int_0^L k(x, L)P_N w(y, t)dy$ with the inverse relation $w = (I - \Phi_N)u$, we see that h can be* *expressed as*

$$\begin{aligned} h(t) = \int_0^L k(L,y)P_N w(y,t)dy &= \int_0^L k(L,y)P_N (I - \Phi_N)\, u(y,t)dy \\ &= \int_0^L k(L,y)P_N (I - \Phi_N P_N)\, u(y,t)dy \\ &= \int_0^L k(L,y)\Gamma_N[P_N u](y,t)dy, \end{aligned}$$

where $\Gamma_N := I - P_N\Phi_N : H^l(0,L) \to H^l(0,L)$ *is a bounded operator. Hence, the control input is not only of feedback type but also is calculated by using only finitely many Fourier sine modes of the state. Consequently we comment that, the way we construct the inverse backstepping operator is crucial for our purposes, since it allows us to write the control input involving finitely many Fourier modes of* u*.*

Remark 5.6 *From computational point of view, we will see in Section 5.3 that defining* Φ_N*'s iteratively as in* (5.17) *also allows us to construct their numerical approximations easily.*

5.1.2. Stabilization of zero equilibrium

In this part, we prove stabilization of zero equilibrium to the linear and nonlinear plants, both in L^2 and in H^1 level. We carry out the proofs through the associated target models, and then conclude the same results for original plants, by using the invertibility of the backstepping transformation with a bounded inverse.

5.1.2.1. Linear target model

Regarding the linear target model, we have following exponential decay result.

Proposition 5.1 *Let* $\nu, \alpha > 0$*,* $w_0 \in H^l(0,L)$*, where* $l = 0$ *or* $l = 1$*,* $\alpha - \nu\lambda_1 \geq 0$*. For a* *given $\mu > \alpha - \nu\lambda_1$, suppose that N is chosen such that*

$$N > \max\left\{\frac{\mu}{2\nu\lambda_1} - 1, \frac{\mu}{\mu + \nu\lambda_1 - \alpha} - 1\right\}. \tag{5.20}$$

Then solution w of (5.3) *satisfies the decay estimate*

$$\|w(\cdot, t)\|_{H^l(0,L)} \leq e^{-\gamma t}\|w_0\|_{H^l(0,L)}, \quad \forall t \geq 0,$$

where γ is given by

$$\gamma = \nu\lambda_1 - \alpha + \mu\left(1 - \frac{1}{N+1}\right) > 0. \tag{5.21}$$

Remark 5.7 *If one uses all Fourier sine modes ($N = \infty$), then the main equation of the target model takes the form*

$$w_t - \nu w_{xx} - \alpha w + \mu w = 0. \tag{5.22}$$

Using standard multipliers, one can see that $\|w(\cdot, t)\|_{H^l(0,L)} = O(e^{-(\nu\lambda_1-\alpha+\mu)t})$, $t \geq 0$. Hence, the condition $\mu > \alpha - \nu\lambda_1$ is necessary for solutions to decay. Therefore, this condition, which also appears in the above proposition is a natural assumption.

Proof Let us multiply the main equation of (5.3) by $2w$ in $L^2(0, L)$, apply integration by parts for the second term and get

$$\frac{d}{dt}\|w(\cdot, t)\|^2_{L^2(0,L)} + 2\nu\|w_x(\cdot, t)\|^2_{L^2(0,L)} - 2\alpha\|w(\cdot, t)\|^2_{L^2(0,L)} + 2\mu\int_0^L w(x, t)P_N w(x, t)dx = 0. \tag{5.23}$$

Using the Cauchy-Schwarz inequality and Cauchy's inequality with $\varepsilon_1 > 0$, we see that the last term on the left hand side of (5.23) is bounded from below as

$$
\begin{aligned}
&2\mu \int_0^L w(x,t) P_N w(x,t) dx \\
=& -2\mu \int_0^L w(x,t)\,(w - P_N w)\,(x,t) dx + 2\mu \|w(\cdot,t)\|^2_{L^2(0,L)} \\
\geq& -2\mu \|w(\cdot,t)\|_{L^2(0,L)} \|\,(w - P_N w)\,(\cdot,t)\|_{L^2(0,L)} + 2\mu \|w(\cdot,t)\|^2_{L^2(0,L)} \\
\geq& (2\mu - \varepsilon_1 \mu) \|w(\cdot,t)\|^2_{L^2(0,L)} - \frac{\mu}{\varepsilon_1} \|\,(w - P_N w)\,(\cdot,t)\|^2_{L^2(0,L)}.
\end{aligned}
\tag{5.24}
$$

Using this bound in (5.23), we get

$$
\begin{aligned}
\frac{d}{dt} \|w(\cdot,t)\|^2_{L^2(0,L)} &+ 2\nu \|w_x(\cdot,t)\|^2_{L^2(0,L)} - 2\alpha \|w(\cdot,t)\|^2_{L^2(0,L)} \\
&+ (2\mu - \mu\varepsilon_1) \|w(\cdot,t)\|^2_{L^2(0,L)} - \frac{\mu}{\varepsilon_1} \|\,(w - P_N w)\,(\cdot,t)\|^2_{L^2(0,L)} \leq 0.
\end{aligned}
$$

Employing the Poincaré type inequality for the last term on the left hand side, we obtain

$$
\begin{aligned}
\frac{d}{dt} \|w(\cdot,t)\|^2_{L^2(0,L)} &+ 2\left(\nu - \frac{\mu}{2\varepsilon_1 \lambda_1 (N+1)^2}\right) \|w_x(\cdot,t)\|^2_{L^2(0,L)} \\
&+ (2\mu - \mu\varepsilon_1 - 2\alpha) \|w(\cdot,t)\|^2_{L^2(0,L)} \leq 0.
\end{aligned}
\tag{5.25}
$$

Given μ and then chosen N specified below, there exists $\varepsilon_1 > 0$ such that

$$
\nu - \frac{\mu}{2\varepsilon_1 \lambda_1 (N+1)^2} > 0 \tag{5.26}
$$

holds. Therefore, for such choice of ε_1, we can apply the Poincaré inequality for the second term on the left hand side of (5.25) to get

$$
\frac{d}{dt} \|w(\cdot,t)\|^2_{L^2(0,L)} + 2\left[\nu\lambda_1 - \alpha + \mu\left(1 - \frac{\varepsilon_1}{2} - \frac{1}{2\varepsilon_1 (N+1)^2}\right)\right] \|w(\cdot,t)\|^2_{L^2(0,L)} \leq 0. \tag{5.27}
$$

To maximize the damping effect, therefore the exponential decay rate, we maximize the inner parenthesis in the second term with respect to ε_1. For this purpose, we consider this term as a function of ε_1 and see that maximum occurs if $\varepsilon_1 = \frac{1}{N+1}$. Note that (5.26) with the specific choice $\varepsilon_1 = \frac{1}{N+1}$ holds, provided that

$$N > \frac{\mu}{2\nu\lambda_1} - 1. \tag{5.28}$$

Therefore, under condition (5.28) on N, we get

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2(0,L)} + 2\gamma\|w(\cdot,t)\|^2_{L^2(0,L)} \leq 0, \tag{5.29}$$

where γ is given by (5.21). In order to obtain a decay result, we must have $\gamma > 0$ which imposes the condition

$$N > \frac{\mu}{\mu + \nu\lambda_1 - \alpha} - 1. \tag{5.30}$$

Finally, integrating (5.29), we deduce the following exponential decay result

$$\|w(\cdot,t)\|_{L^2(0,L)} \leq e^{-\gamma t}\|w_0\|_{L^2(0,L)}, \quad \forall t \geq 0, \tag{5.31}$$

under the assumption (5.20).

Next, we show exponential decay of solutions of (5.3) in $H^1(0,L)$. We multiply the main equation of (5.3) by $-2w_{xx}$ in $L^2(0,L)$, integrate the third term by parts and obtain

$$\frac{d}{dt}\|w_x(\cdot,t)\|^2_{L^2(0,L)} + 2\nu\|w_{xx}(\cdot,t)\|^2_{L^2(0,L)} \\ - 2\alpha\|w_x(\cdot,t)\|^2_{L^2(0,L)} - 2\mu\int_0^L w_{xx}(x,t)P_N w(x,t)dx = 0. \tag{5.32}$$

Applying the Cauchy-Schwarz inequality and then Cauchy's inequality with $\varepsilon_2 > 0$, the last term at the left hand side of (5.32) can be bounded from below as

$$
\begin{aligned}
& -2\mu \int_0^L w_{xx}(x,t) P_N w(x,t)dx \\
=& 2\mu \int_0^L w_{xx}(x,t)(w - P_N w)(x,t)dx - 2\mu \int_0^L w_{xx}(x,t) w(x,t)dx \\
\geq & -2\mu \|w_{xx}(\cdot,t)\|_{L^2(0,L)} \|(w - P_N w)(\cdot,t)\|_{L^2(0,L)} + 2\mu \|w_x(\cdot,t)\|^2_{L^2(0,L)} \\
\geq & -\mu\varepsilon_2 \|w_{xx}(\cdot,t)\|^2_{L^2(0,L)} - \frac{\mu}{\varepsilon_2} \|(w - P_N w)(\cdot,t)\|^2_{L^2(0,L)} + 2\mu \|w_x(\cdot,t)\|^2_{L^2(0,L)}.
\end{aligned}
\tag{5.33}
$$

Next we use the Poincaré type inequality to get

$$
\begin{aligned}
-2\mu \int_0^L w_{xx}(x,t) P_N w(x,t)dx \geq & -\mu\varepsilon_2 \|w_{xx}(\cdot,t)\|^2_{L^2(0,L)} \\
& + \left(2\mu - \frac{\mu}{\varepsilon_2 \lambda_1 (N+1)^2}\right) \|w_x(\cdot,t)\|^2_{L^2(0,L)}.
\end{aligned}
\tag{5.34}
$$

Combining the above estimate with (5.32), we obtain

$$
\begin{aligned}
\frac{d}{dt} \|w_x(\cdot,t)\|^2_{L^2(0,L)} & + 2\left(\nu - \frac{\mu\varepsilon_2}{2}\right) \|w_{xx}(\cdot,t)\|^2_{L^2(0,L)} \\
& + 2\left(\mu - \alpha - \frac{\mu}{2\varepsilon_2 \lambda_1 (N+1)^2}\right) \|w_x(\cdot,t)\|^2_{L^2(0,L)} \leq 0.
\end{aligned}
\tag{5.35}
$$

Now for a given μ, we can find ε_2 such that

$$
\nu - \frac{\mu\varepsilon_2}{2} > 0. \tag{5.36}
$$

For this choice of ε_2, we apply the Poincaré inequality to the second term at the left hand side of (5.35) and get

$$
\frac{d}{dt} \|w_x(\cdot,t)\|^2_{L^2(0,L)} + 2\left[\nu\lambda_1 - \alpha + \mu\left(1 - \frac{\varepsilon_2 \lambda_1}{2} - \frac{1}{2\varepsilon_2 \lambda_1 (N+1)^2}\right)\right] \|w_x(\cdot,t)\|^2_{L^2(0,L)} \leq 0. \tag{5.37}
$$

Observe that the inner parenthesis, therefore the exponential decay rate, is maximized for the choice $\varepsilon_2 = \frac{1}{\lambda_1(N+1)}$. Using this choice we rewrite (5.37) and get

$$\frac{d}{dt}\|w_x(\cdot,t)\|^2_{L^2(0,L)} + 2\gamma\|w_x(\cdot,t)\|^2_{L^2(0,L)} \le 0, \tag{5.38}$$

where γ is given by (5.21). Note that letting $\varepsilon_2 = \frac{1}{\lambda_1(N+1)}$ in (5.36) yields the same condition (5.28) on N as we obtained in the L^2-decay case. Finally, combining (5.38) with (5.29) and using definition of H^1-space, we obtain the following decay result

$$\|w(\cdot,t)\|_{H^1(0,L)} \le e^{-\gamma t}\|w_0\|_{H^1(0,L)}, \quad \forall t \ge 0,$$

under condition (5.20) on N. □

Now we show that exponential decay results in $H^l(0,L)$ for linear target model is also true for linear plant. To this end, we first use smoothness of the backstepping kernel k, definition of the projection operator P_N and Proposition 5.1 to get

$$\begin{aligned}\|u(\cdot,t)\|_{H^l(0,L)} &\le \left(1 + \|k\|_{L^2(\Delta_{x,y})}\right)\|w(\cdot,t)\|_{H^l(0,L)} \\ &\le \left(1 + \|k\|_{L^2(\Delta_{x,y})}\right)e^{-\gamma t}\|w_0\|_{H^l(0,L)}.\end{aligned} \tag{5.39}$$

Next, by the Lemma 5.2, backstepping transformation is invertible with a bounded inverse and we can also write

$$\|w_0\|_{H^l(0,L)} \le \|I + \Phi_N\|_{H^l(0,L)\to H^l(0,L)}\|u_0\|_{H^l(0,L)}. \tag{5.40}$$

Combining (5.39)-(5.40)

$$\|u(\cdot,t)\|_{H^l(0,L)} \le c_k e^{-\gamma t}\|u_0\|_{H^l(0,L)},$$

where

$$c_k = \left(1 + \|k\|_{L^2(\Delta_{x,y})}\right)\|I + \Phi_N\|_{H^l(0,L)\to H^l(0,L)} \tag{5.41}$$

is a nonnegative constant independent of the initial datum.

Hence, we proved the following exponential stabilization of zero equilibrium to the linear plant (5.2).

Proposition 5.2 *Let $\nu, \alpha > 0$, $\alpha - \nu\lambda_1 \geq 0$, and $u_0 \in H^l(0, L)$, where $l = 0$ or $l = 1$. Assume that the feedback controller is given by*

$$h(t) = \int_0^L k(L, y; \mu)\Gamma[P_N u](y, t)dy, \tag{5.42}$$

where k is a smooth solution of (5.14), $\Gamma[P_N u] := (I - P_N \Phi_N)[P_N u]$ and $\Phi_N : H^l(0, L) \to L^2(0, L)$ is a linear bounded operator. For $\mu > \alpha - \nu\lambda_1$, if N satisfies

$$N > \max\left\{\frac{\mu}{2\nu\lambda_1} - 1, \frac{\mu}{\mu + \nu\lambda_1 - \alpha} - 1\right\},$$

then zero equilibrium to the closed-loop system (5.2) is exponentially stable. Moreover all solutions satisfies the exponential decay estimate

$$\|u(\cdot, t)\|_{H^l(0,L)} \lesssim c_k e^{-\gamma t}\|u_0\|_{H^l(0,L)}, \quad t \geq 0,$$

where c_k is a nonnegative constant independent of initial datum and γ is given by (5.21).

Remark 5.8 *In the above proposition, the decay rate γ can be made as large as desired by choosing μ large enough. So this result is actually rapid stabilization of the zero equilibrium. Ofcourse, to gain a rapid decay by increasing the value of μ, one requires a stronger control effort. This will be illustrated as a numerical example in Section 5.3.*

5.1.2.2. Nonlinear target model

If we apply backstepping transformation to the nonlinear plant (5.1), there exists an extra term in the main equation of the associated nonlinear target model, due to the term u^3 involving in the main equation of (5.1). Let us denote this extra term as Fw

derive a representation for this term. Applying the arguments in (5.7)-(5.10), we get

$$\begin{aligned} -u^3 = & -\mu P_N w(x,t) - Fw - \int_0^x \Big(\nu(k_{xx} - k_{yy}) + \mu k\Big)(x,y) P_N w(y,t) dy \\ & - \int_0^x k(x,y) P_N(Fw)(y,t) dy - \nu \left(k_x(x,x) + k_y(x,x) + \frac{d}{dx} k(x,x)\right) P_N w(x,t) \\ & - \nu k(x,0)(P_N w(y,t))_y \Big|_{y=0}. \end{aligned} \tag{5.43}$$

In order to guarantee that (5.43) holds where k satisfies (5.14), we must have

$$u^3 = (I + \Upsilon_k P_N) Fw.$$

Using the invertibility of the backstepping transformation, we see that Fw can be explicitly expressed as

$$Fw = (I - \Phi_N)\left[((I + \Upsilon_k P_N) w)^3\right], \tag{5.44}$$

where Φ_N is defined by (5.17) in Lemma 5.2. Hence the target model corresponding to the nonlinear plant is given by

$$\begin{cases} w_t - \nu w_{xx} - \alpha w + \mu P_N w + Fw = 0, & (x,t) \in (0,L) \times (0,T), \\ w(0,t) = 0, \ w(L,t) = 0, & t \in (0,T), \\ w(x,0) = w_0(x), & x \in (0,L), \end{cases} \tag{5.45}$$

where Fw is given by (5.44).

Proposition 5.3 *Let $\nu, \alpha > 0$, $\alpha - \nu\lambda_1 \geq 0$, $w_0 \in H^l(0,L)$, $l = 0$ or $l = 1$, where $\|w_0\|_{H^l(0,L)}$ is sufficiently small. For a given $\mu > \alpha - \nu\lambda_1$, if N satisfies* (5.20)*, then solution w of* (5.45) *has the decay estimate*

$$\|w(\cdot,t)\|_{H^l(0,L)} \lesssim e^{-\left(\gamma - \frac{\varepsilon_3 \lambda_1}{2}\right)t} \|w_0\|_{H^l(0,L)}, \quad \forall t \geq 0, \tag{5.46}$$

where γ is given by (5.21) *and $\varepsilon_3 > 0$ satisfies $2\nu - \varepsilon_3 - \frac{\mu}{\lambda_1(N+1)} > 0$.*

Proof Multiplying the main equation of (5.45) by $2w$ in $L^2(0,L)$ and applying the arguments (5.23)-(5.25) by chooisng $\varepsilon_1 = \frac{1}{N+1}$ we get

$$\frac{d}{dt}\|w(\cdot,t)\|^2 + 2\left(\nu - \frac{\mu}{2\lambda_1(N+1)}\right)\|w_x(\cdot,t)\|^2 + (2\mu - \frac{\mu}{N+1} - 2\alpha)\|w(\cdot,t)\|^2$$
$$= -2\int_0^L (I-\Phi_N)\left[((I+\Upsilon_k P_N)w(x,t))^3\right]w(x,t)dx. \quad (5.47)$$

Applying the Cauchy-Schwarz inequality and using that inverse backstepping transformation, $I-\Phi_N$, is bounded on $L^2(0,L)$, right hand side of (5.47) can be estimated as

$$\begin{aligned}&-2\int_0^L (I-\Phi_N)\left[((I+\Upsilon_k P_N)w(x,t))^3\right]w(x,t)dx\\ \le&2\left\|(I-\Phi_N)\left[((I+\Upsilon_k P_N)w)^3\right](\cdot,t)\right\|_{L^2(0,L)}\|w(\cdot,t)\|_{L^2(0,L)}\\ \le&2\left\|((I+\Upsilon_k P_N)w)^3(\cdot,t)\right\|_{L^2(0,L)}\|w(\cdot,t)\|_{L^2(0,L)}.\end{aligned} \quad (5.48)$$

Now we apply the Gagliardo-Nirenberg inequality for the first multiplicant in the last line of (5.48) and use the boundedness of the backstepping transformation on $L^2(0,L)$, to get

$$\begin{aligned}\left\|((I+\Upsilon_k P_N)w)^3(\cdot,t)\right\|_{L^2(0,L)} =&\|(I+\Upsilon_k P_N)w\|^3_{L^6(0,L)}\\ \le&c_L\|\partial_x((I+\Upsilon_k P_N)w)(\cdot,t)\|_{L^2(0,L)}\\ &\times\|(I+\Upsilon_k P_N)w(\cdot,t)\|^2_{L^2(0,L)}\\ \le&c_{k,L}\|w_x(\cdot,t)\|_{L^2(0,L)}\|w(\cdot,t)\|^2_{L^2(0,L)}.\end{aligned} \quad (5.49)$$

Combining this estimate with (5.48) then applying Cauchy's inequality with $\varepsilon_3 > 0$, we get

$$\begin{aligned}-2\int_0^L (I-\Phi_N)\left[((I+\Upsilon_k P_N)w(x,t))^3\right]w(x,t)dx \le& c_{k,L}\|w_x(\cdot,t)\|_{L^2(0,L)}\|w(\cdot,t)\|^3_{L^2(0,L)}\\ \le& \varepsilon_3\|w_x(\cdot,t)\|^2 + c_{k,L,\varepsilon_3}\|w(\cdot,t)\|^6.\end{aligned}$$

Therefore, it follows from (5.47) that

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2(0,L)} + 2\left(\nu - \frac{\varepsilon_3}{2} - \frac{\mu}{2\lambda_1(N+1)}\right)\|w_x(\cdot,t)\|^2_{L^2(0,L)} + \left(2\mu - \frac{\mu}{N+1} - 2\alpha\right)\|w(\cdot,t)\|^2_{L^2(0,L)} \leq c_{k,L,\varepsilon_3}\|w(\cdot,t)\|^6_{L^2(0,L)}. \quad (5.50)$$

Observe that given μ and choosing N in a suitable manner specified below, one can find $\varepsilon_3 > 0$ such that the second term at the left hand side can be made nonnegative, i.e., the following inequality holds

$$\nu - \frac{\mu}{2\lambda_1(N+1)} \geq \frac{\varepsilon_3}{2} > 0.$$

This yields a condition $N > \frac{\mu}{2\nu\lambda_1} - 1$, which is the same one as (5.28) in the linear case. Assuming that this condition is being satisfied and therefore applying the Poincaré inequality, we end up with

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2(0,L)} + 2\left(\gamma - \frac{\varepsilon_3\lambda_1}{2}\right)\|w(\cdot,t)\|^2_{L^2(0,L)} \leq c_{k,L,\varepsilon_3}\|w(\cdot,t)\|^6_{L^2(0,L)}, \quad (5.51)$$

where $\gamma > 0$ is given by (5.21). Now if μ is sufficiently large that satisfies $\mu > \nu\lambda_1 - \alpha \geq 0$ and N satisfies (5.30), then (5.51) implies L^2-decay of the solutions exponentially provided that $\|w_0\|_{L^2(0,L)}$ is small enough. This follows from Lemma 5.3 below.

Next, we prove exponential decay of solutions of (5.45) in $H^1(0,L)$ level. To this end, we take the L^2-inner product of the main equation of (5.45) by $-2w_{xx}$ and proceed as in (5.33)-(5.35) by choosing $\varepsilon_2 = \frac{1}{\lambda_1(N+1)}$ and get

$$\frac{d}{dt}\|w_x(\cdot,t)\|^2 + 2\left(\nu - \frac{\mu}{2\lambda_1(N+1)}\right)\|w_{xx}(\cdot,t)\|^2_{L^2(0,L)} + 2\left(\mu - \alpha - \frac{\mu}{2(N+1)}\right)\|w_x(\cdot,t)\|^2_{L^2(0,L)} \leq 2\int_0^L (I-\Phi_N)\left[((I+\Upsilon_k P_N)w(x,t))^3\right]w_{xx}(x,t)dx. \quad (5.52)$$

We apply Cauchy-Schwarz inequality and using that $I - \Phi_N$ is a bounded operator on

$L^2(0, L)$, right hand side of (5.52) can be estimated as

$$2\int_0^L (I - \Phi_N)\left[((I + \Upsilon_k P_N)w(x,t))^3\right] w_{xx}(x,t)dx \\ \leq 2c_k \left\|((I + \Upsilon_k P_N)w)^3(\cdot,t)\right\|_{L^2(0,L)} \|w_{xx}(\cdot,t)\|_{L^2(0,L)}. \quad (5.53)$$

Applying the arguments in (5.49) for the first multiplicant on the right hand side of (5.53), then Poincaré inequality and Cauchy's inequality with $\varepsilon_3 > 0$, we can bound the right hand side of (5.53) by

$$\begin{aligned} &2\left\|((I + \Upsilon_k P_N)w)^3(\cdot,t)\right\|_{L^2(0,L)} \|w_{xx}(\cdot,t)\|_{L^2(0,L)} \\ &\leq \left(c_{k,L}\|w_x(\cdot,t)\|_{L^2(0,L)}\|w(\cdot,t)\|^2_{L^2(0,L)}\right) \|w_{xx}(\cdot,t)\|_{L^2(0,L)} \\ &\leq c_{k,L}\|w_x(\cdot,t)\|^3_{L^2(0,L)}\|w_{xx}(\cdot,t)\|_{L^2(0,L)} \\ &\leq c_{k,L,\varepsilon_3}\|w_x(\cdot,t)\|^6_{L^2(0,L)} + \varepsilon_3\|w_{xx}(\cdot,t)\|^2_{L^2(0,L)}. \end{aligned} \quad (5.54)$$

Combining (5.53)-(5.54), it follows from (5.52) that

$$\frac{d}{dt}\|w_x(\cdot,t)\|^2_{L^2(0,L)} + 2\left(\nu - \frac{\varepsilon_3}{2} - \frac{\mu}{2\lambda_1(N+1)}\right)\|w_{xx}(\cdot,t)\|^2_{L^2(0,L)} \\ + 2\left(\mu - \alpha - \frac{\mu}{2(N+1)}\right)\|w_x(\cdot,t)\|^2_{L^2(0,L)} \leq c_{k,L,\varepsilon_3}\|w_x(\cdot,t)\|^6_{L^2(0,L)}. \quad (5.55)$$

Given μ and choosing N in a suitable way, one can find $\varepsilon_3 > 0$ such that

$$\nu - \frac{\mu}{2\lambda_1(N+1)} \geq \frac{\varepsilon_3}{2} > 0,$$

which yields the same condition (5.28) on N. So assuming that (5.28) holds, we can apply the Poincaré inequality and get

$$\frac{d}{dt}\|w_x(\cdot,t)\|^2_{L^2(0,L)} + 2\left(\gamma - \frac{\varepsilon_3\lambda_1}{2}\right)\|w_x(\cdot,t)\|^2_{L^2(0,L)} \leq c_{k,L,\varepsilon_3}\|w_x(\cdot,t)\|^6_{L^2(0,L)}, \quad (5.56)$$

where $\gamma > 0$ is given by (5.21). If $\mu > \nu\lambda_1 - \alpha$, then (5.56) combined with (5.51) yields H^1-decay of solutions. This follows from Lemma 5.3 below. □

Lemma 5.3 *Let $y = y(t) > 0$ satisfy*

$$\frac{d}{dt}y(t) + (2\gamma - \varepsilon_3\lambda_1)\,y(t) - c_3y^3(t) \leq 0, \quad \forall t \geq 0, \tag{5.57}$$

where γ is given by (5.21)*, $c_3 > 0$ and $\varepsilon_3 > 0$ be such that $2\nu - \varepsilon_3 - \frac{\mu}{\lambda_1(N+1)} > 0$. If $y(0)$ is sufficiently small and μ is sufficiently large, so that $\gamma > \frac{\varepsilon_3\lambda_1}{2}$, then $y(t) \sim O\left(e^{-(2\gamma-\varepsilon_3\lambda_1)t}\right)$.*

Proof (5.57) is a Bernoulli type differential inequality. Solving this inequality directly yields

$$y^2(t) \leq \left(\left(\frac{1}{y^2(0)} - \frac{c_3}{2\gamma - \varepsilon_3\lambda_1}\right)e^{(4\gamma-2\varepsilon_4\lambda_1)t} + \frac{c_3}{2\gamma - \varepsilon_4\lambda_1}\right)^{-1}. \tag{5.58}$$

Under the assumption $y(0) \leq \sqrt{\frac{2\gamma-\varepsilon_3\lambda_1}{c_3}}$, it follows from the above inequality that

$$y(t) \lesssim y(0)e^{-(2\gamma-\varepsilon_3\lambda_1)t}, \quad \forall t \geq 0,$$

which completes the proof. □

To turn back to the original nonlinear plant, we apply the same arguments to those in (5.39)-(5.40) and obtain

$$||u(\cdot,t)||_{H^l(0,L)} \lesssim c_k e^{-\gamma' t}||u_0||_{H^l(0,L)},$$

where $c_k > 0$ is given by (5.41), $\gamma' = \gamma - \frac{\varepsilon_3\lambda_1}{2}$, provided that $||u_0||_{H^l(0,L)}$ is sufficiently small. As the stabilization of linear problem, here the decay rate γ' can be made arbitrarily large by choosing μ as large enough. Consequently, we have the following rapid stabilization result for the nonlinear closed-loop model.

Proposition 5.4 *Let $\nu, \alpha > 0$, $\alpha - \nu\lambda_1 \geq 0$. Let $u_0 \in H^l(0,L)$, where $l = 0$ or $l = 1$ and suppose that $||u_0||_{H^l(0,L)}$ is sufficiently small. Assume that the feedback controller is given by*

$$h(t) = \int_0^L k(L,y;\mu)\Gamma[P_N u](y,t)dy,$$

where k is a smooth solution of (5.14)*,* $\Gamma[P_N u] := (I - P_N\Phi_N)[P_N u]$ *and* $\Phi_N : H^l(0, L) \to L^2(0, L)$ *is a linear bounded operator. For prescibed* $\mu > \alpha - \nu\lambda_1$*, if N satisfies*

$$N > \max\left\{\frac{\mu}{2\nu\lambda_1} - 1, \frac{\mu}{\mu + \nu\lambda_1 - \alpha} - 1\right\},$$

then zero equilibrium to the closed-loop system (5.2) *is exponentially stable. Moreover, for any* γ' *with* $0 < \gamma' < \gamma$*, following exponential decay estimate holds*

$$\|u(\cdot, t)\|_{H^l(0,L)} \lesssim c_k e^{-\gamma' t}\|u_0\|_{H^l(0,L)}, \quad \forall t \geq 0,$$

where c_k *is a nonnegative constant independent of initial datum and* γ*, given by* (5.21) *can be made arbitrarily large by choosing* μ *large enough.*

5.1.3. Minimal number of Fourier modes for exponential decay

In the previous section, we focused on the rapid stabilization problem with finite dimensional boundary feedback controllers. That is, for given problem parameters ν, α, L and prescribed exponential decay rate, we sought to answer whether we are able to stabilize the zero equilibrium for some $N > 0$. One can also be interested in determining the minimal number of Fourier sine modes of u to gain exponential stabilization of zero equilibrium for some unprescribed and not necessarily arbitrarily large decay rate. To answer the latter question, suppose that the instability level of the model is N. That is, we suppose that the problem parameters ν, α, L be such that

$$\lambda_N \leq \frac{\alpha}{\nu} < \lambda_{N+1}. \tag{5.59}$$

Suppose also that the control input involves M Fourier sine modes of u:

$$h(t) = \int_0^L k(L, y)\Gamma[P_M u](y, t)dy. \tag{5.60}$$

So our aim is to find minimal value of M. Recall that the norm estimates we performed on the previous section to gain L^2-decay is true under the assumptions (5.26) and (5.27), that is we require that

$$\mu < 2\varepsilon_1 \nu \lambda_1 (M+1)^2$$

and

$$\mu > \frac{\alpha - \nu\lambda_1}{1 - \frac{\varepsilon_1}{2} - \frac{1}{2\varepsilon_1 (M+1)^2}}.$$

To guarantee that such μ exists, we must have

$$\frac{\alpha - \nu\lambda_1}{1 - \frac{\varepsilon_1}{2} - \frac{1}{2\varepsilon_1 (M+1)^2}} < 2\varepsilon_1 \nu \lambda_1 (M+1)^2. \tag{5.61}$$

From the inequality (5.61), we can write

$$\frac{\alpha}{\nu} < \lambda_1 (M+1)^2 (2\varepsilon_1 - \varepsilon_1^2) = \lambda_{M+1}(2\varepsilon_1 - \varepsilon_1^2). \tag{5.62}$$

Recall that our previous choice on ε_1 is to maximize the damping effect. However, now our choice is to ensure the inequality (5.61), therefore (5.62) to be hold. Therefore, we choose $\varepsilon_1 = 1$, which maximizes the right hand side of (5.62), and (5.62) becomes

$$\frac{\alpha}{\nu} < \lambda_{M+1}(2\varepsilon_1 - \varepsilon_1^2)\Big|_{\varepsilon_1 = 1} = \lambda_{M+1}.$$

Similarly, throughout the calculations that we gain H^1-decay of solutions, we assumed (5.36) and (5.37), that is we assumed

$$\mu < \frac{2\nu}{\varepsilon_2}$$

and

$$\mu > \frac{\alpha - \nu\lambda_1}{1 - \frac{\varepsilon_2 \lambda_1}{2} - \frac{1}{2\varepsilon_2 \lambda_1 (M+1)^2}}.$$

Such μ exists if

$$\frac{\alpha - \nu\lambda_1}{1 - \dfrac{\varepsilon_2\lambda_1}{2} - \dfrac{1}{2\varepsilon_2\lambda_1(M+1)^2}} < \frac{2\nu}{\varepsilon_2} \tag{5.63}$$

or equivalently

$$\frac{\alpha}{\nu} < \frac{2}{\varepsilon_2} - \frac{1}{\varepsilon_2^2\lambda_1(M+1)^2} = \frac{2}{\varepsilon_2} - \frac{1}{\varepsilon_2^2\lambda_{M+1}} \tag{5.64}$$

holds. Right hand side of (5.64) is maximized for $\varepsilon_2 = \frac{1}{\lambda_{M+1}}$. Therefore, for such value of ε_2, we get

$$\frac{\alpha}{\nu} < \frac{2}{\varepsilon_2} - \frac{1}{\varepsilon_2^2\lambda_{M+1}}\Big|_{\varepsilon_2=\lambda_{M+1}^{-1}} = \lambda_{M+1}.$$

Based on the above discussion, we see that M must be chosen such that

$$\frac{\alpha}{\nu} < \lambda_{M+1} \tag{5.65}$$

in both cases. However, we assumed that the instability level of the model is N, that is we assumed that (5.59) holds. Hence the minimum value for M that (5.65) holds true is N. Note that taking $M = N$ and, choosing $\varepsilon_1 = 1$ in the L^2-decay case and $\varepsilon_2 = \lambda_{M+1}^{-1}$ in H^1-decay case, exponential decay rate of solutions to the linear target model becomes

$$\rho = \nu\lambda_1 - \alpha + \frac{\mu}{2}\left(1 - \frac{1}{(N+1)^2}\right) > 0 \tag{5.66}$$

where μ obeys

$$2(\alpha - \nu\lambda_1)\left(1 - \frac{1}{(N+1)^2}\right)^{-1} < \mu < 2\nu\lambda_{N+1}. \tag{5.67}$$

Assumptions on the stabilization of zero equilibrium to the nonlinear model are same as the linear case. Therefore, the above work is also valid for the nonlinear target model, provided that the initial datum $\|w_0\|_{H^l(0,L)}$ is sufficiently small and exponential decay rate is given by

$$\rho' := \rho - \frac{\varepsilon_3\lambda_1}{2}$$

where $\varepsilon_3 > 0$ satisfying $2\nu - \varepsilon_3 - \frac{\mu}{\lambda_1(N+1)} > 0$.

Now we use the same invertibility arguments (5.39)-(5.40) to turn back to the original plant and obtain the following result.

Proposition 5.5 *Let $\nu, \alpha, L > 0$ be such that*

$$\begin{aligned} \lambda_j^* &= \nu\lambda_j - \alpha \le 0, \quad j = 1, \dots, N, \\ \lambda_{N+1}^* &= \nu\lambda_{N+1} - \alpha > 0, \end{aligned}$$

i.e., the instability level is N. Suppose that $u_0 \in H^l(0, L)$, $l = 0, 1$ and in addition, if the model is nonlinear suppose that $\|u_0\|_{H^l(0,L)}$ is sufficiently small. Then, it suffices to employ a controller, $h(t) = h(P_N u(\cdot, t))$ of the form (5.42)*, that involves only first N Fourier sine modes to achieve exponential stabilization of zero equilibrium.*

5.2. Well-posedness

In this part, we prove well-posedness of the linearized model (5.2) in Section 5.2.1 and then well-posedness of the nonlinear model (5.1) in Section 5.2.2.

5.2.1. Linear model

Well-posedness of the linear target model (5.3) can be proved rigorously by the Galerkin-Faedo method. At first one constructs an approximate weak solution in terms of the finitely many basis functions of $H_0^1(0, L)$, which yields a system of ODEs. Existence of a solution to that system follows from theory of ODEs. Next, using multiplier method, one obtains uniform bounds for solutions to the approximate finite dimensional system. With the help of these uniform bounds and compactness arguments, one can extract a subsequence of the sequence of approximate solutions to the finite dimensional system so that, it converges to the solution of the original model. Using the arguments (5.39)-(5.40) this result is also true for the linear plant (5.2) This procedure is rather standard and considering the volume of the thesis, we skip the proof and only state our result.

Proposition 5.6 *Let $T, \nu, \alpha > 0$. Suppose that $u_0 \in H^1(0, L)$. Then* (5.2) *admits a unique* *weak solution such that*

$$u \in L^\infty(0,T;H^1(0,L)) \cap L^2(0,T;H^2(0,L) \cap H^1(0,L))$$

and

$$u_t \in L^2(0,T;L^2(0,L)).$$

5.2.2. Nonlinear model

Now we consider the nonlinear target model (5.45). We perform the well-posedness analysis by using fixed point arguments combined with operator semigroup theory.

Remark 5.9 *Well-posedness analysis of the nonlinear target model can be also carried out by the Galerkin-Faedo procedure combined with compactness arguments. However, due to the nonlinear term*

$$Fw = (I - \Phi_N)\left[((I + \Upsilon_k P_N)w)^3\right]$$

in the main equation, the uniform bounds that we need can be achieved under a smallness assumption on the initial datum $\|w_0\|_{H^1(0,L)}$. However, if we apply fixed point argument, we do not need to suppose such smallness assumption at least for the existence and uniqueness of local solution. Therefore, we prefer to perform the well-posedness analysis via the fixed point argument.

In order to apply fixed point argument, we need a priori estimates for the linear nonhomogeneous model

$$\begin{cases} q_t - \nu q_{xx} - \alpha q + \mu P_N q = f(x,t), & (x,t) \in (0,L) \times (0,T), \\ q(0,t) = q(L,t) = 0, & t \in (0,T), \\ q(x,0) = q_0(x), & x \in (0,L). \end{cases} \tag{5.68}$$

To this end, let us multiply the main equation of (5.68) by $2q$ in $L^2(0,L)$ and integrate by parts to get

$$\frac{d}{dt}\|q(\cdot,t)\|^2 + 2\nu\|q_x(\cdot,t)\|^2 - 2\alpha\|q(\cdot,t)\|^2 + 2\mu\int_0^L q(x,t)P_N q(x,t)dx \leq 2\int_0^L |f(x,t)||q(x,t)|dx. \quad (5.69)$$

Integrating over $[0,t]$, we infer existence of nonnegative constants C_i, $i = 1,2,3$ that depend on ν, α, μ such that

$$\begin{aligned} \|q(\cdot,t)\|^2 + \int_0^t \|q_x(\cdot,s)\|^2 ds \leq& C_1\int_0^t \|q(\cdot,s)\|^2 ds + C_2\|q_0\|^2 \\ &+ C_3\int_0^t \|q(\cdot,s)\|\|f(\cdot,s)\|ds \\ \leq& C_1 T \sup_{t\in[0,T]} \|q(\cdot,t)\|^2 + C_2\|q_0\|^2 \\ &+ \varepsilon \sup_{t\in[0,T]} \|q(\cdot,t)\|^2 + C_\varepsilon\|f\|^2_{L^1(0,T;L^2(0,T))}, \end{aligned} \quad (5.70)$$

where the last inequality follows from Cauchy's inequality with $\varepsilon > 0$. Then, taking supremum with respect to t on $[0,T]$, and for sufficiently small and fixed T and ε, we get

$$\|q\|_{X_T^0} \leq c_T(\|q_0\| + \|f\|_{L^1(0,T;L^2(0,L))}), \quad (5.71)$$

where c_T depends on $\nu, \alpha, \mu, \varepsilon, T$ and is uniformly bounded in T.

Next, we prove a linear nonhomogeneous estimate in X_T^1. We multiply the main equation by $-2q_{xx}$ in $L^2(0,L)$ and get

$$\frac{d}{dt}\|q_x(\cdot,t)\|^2 + 2\nu\|q_{xx}(\cdot,t)\|^2 - 2\alpha\|q_x(\cdot,t)\|^2 - 2\mu\int_0^L q_{xx}(x,t)P_N q(x,t)dx \leq 2\int_0^L |f(x,t)||q_{xx}(x,t)|dx.$$

Using similar arguments, we obtain

$$\|q\|_{X_T^1} \le c_T(\|q_0\|_{H^1(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))}) \tag{5.72}$$

for sufficiently small T. We proved the following lemma.

Lemma 5.4 *Let $q_0 \in H^l(0,L)$ where $l = 0$ or $l = 1$ and $f \in L^1(0,T;L^2(0,L))$. Then, there exists a $T > 0$ such that solution q of* (5.68) *belongs to space X_T^l and following regularity estimate holds*

$$\|q\|_{X_T^l} \le c_T(\|q_0\|_{H^l(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))}),$$

where $C_T > 0$ depends on $\nu, \alpha, \mu, \varepsilon, T$.

Now let us go on with the nonlinear target model (5.45) and express it in the abstract operator theoretic form as

$$\begin{cases} \dfrac{dw}{dt} = Aw + F(w), \quad t \in (0,T), \\ w(0) = w_0, \end{cases} \tag{5.73}$$

where $F(w)$ is given by (5.44) and the operator A is defined as $A\varphi := \nu\varphi'' + \alpha\varphi - \mu P_N\varphi$ with domain $D(A) := \left\{\varphi \in H^2(0,L) \,|\, \varphi(0) = \varphi(L) = 0\right\}$. An application of operator semigroup theory as we performed in details in Section 3.3.1 implies that A is the infinitesimal generator of a C_0−semigroup of contractions, $\{S(t)\}_{t\ge 0}$, on $L^2(0,L)$. Now application of Duhamel's principle yields that solution w of (5.73), if exists, satisfies the following implicit relation

$$w(t;w_0) = S(t)w_0 + \int_0^t S(t-\tau)F(w(\tau;w_0))d\tau. \tag{5.74}$$

In order to prove existence of a function w that satisfies the relation (5.74), we need to show that the mapping

$$\Psi[w_0, z(t)] := S(t)w_0 + \int_0^t S(t-\tau)F(z(\tau))d\tau, \tag{5.75}$$

has a fixed point in X_T^l. This will be proved by Banach fixed point theorem.

To this end, we use the estimates obtained in Lemma 5.4 and then the boundedness of the operator $(I + \Phi_N)$ on $L^2(0, L)$ which we obtained in Lemma 5.2, to get

$$\begin{aligned}
\|\Psi z\|_{X_T^l} &\leq \left\| S(t)w_0 + \int_0^t S(t-\tau)Fz(\tau)d\tau \right\|_{X_T^l} \\
&\leq c_T\|w_0\|_{H^l(0,L)} + c_T \int_0^T \left\|(I - \Phi_N)[((I + \Upsilon_k P_N)z)^3(\cdot, t)]\right\|_{L^2(0,L)} dt \\
&\leq c_T\|w_0\|_{H^l(0,L)} + c_{T,k} \int_0^T \left\|((I + \Upsilon_k P_N)z)^3(\cdot, t)\right\|_{L^2(0,L)} dt.
\end{aligned} \tag{5.76}$$

Using the arguments in (5.49), we can bound the second term on the right hand side of (5.76) as

$$\begin{aligned}
\int_0^T \left\|((I + \Upsilon_k P_N)z)^3(\cdot, t)\right\| dt &\leq \int_0^T \left(c_{k,L}\|z_x(\cdot, t)\|\|z(\cdot, t)\|^2\right) dt \\
&\leq c_{k,L} \sqrt{T}\|z\|^2_{C([0,T];L^2(0,L))}\|z_x\|_{L^2(0,T;L^2(0,L))} \\
&\leq c_{k,L} \sqrt{T}\|z\|^3_{X_T^l}.
\end{aligned} \tag{5.77}$$

Combining (5.76)-(5.77), we get

$$\|\Psi z\|_{X_T^l} \leq c_T \left(\|w_0\|_{H^l(0,L)} + \sqrt{T}\|z\|^3_{X_T^l}\right), \tag{5.78}$$

where c_T is uniformly bounded in T, and it also depends on parameters ν, α, μ, k, L, which are fixed throughout.

To prove local existence, it is enough to show that the mapping (5.74) has a fixed point for some $T > 0$. Let $B_R^T := \left\{z \in X_T^l \mid \|z\|_{X_T^l} \leq R\right\}$, where $R = 2c_T\|w_0\|_{H^l(0,L)}$. Then,

$$\|\Psi z\|_{X_T^l} \leq \frac{R}{2} + c_T \sqrt{T}R^3.$$

Let us choose $T > 0$ small enough that $2c_T \sqrt{T}R^2 < 1$ holds. This yields $\|\Psi z\|_{X_T^l} < R$, i.e., Ψ maps B_R^T onto itself for sufficiently small $T > 0$.

Next, we show that Ψ is a contraction on B_R^T for sufficiently small T. Let $z_1, z_2 \in$

B_R^T. Then

$$\begin{aligned}
\|\Psi z_1 - \Psi z_2\|_{X_T^l} =& \left\| \int_0^t S(\cdot - s)\,(Fz_1 - Fz_2)\,ds \right\|_{X_T^l} \\
\leq & c_T \int_0^T \|Fz_1 - Fz_2\|_{L^2(0,L)}\,dt \\
=& c_T \int_0^T \left\| (I - \Phi_N) \left[((I + \Upsilon_k P_N) z_1)^3 - ((I + \Upsilon_k P_N) z_2)^3 \right] \right\|_{L^2(0,L)} dt \\
\leq & c_T \int_0^T \left\| ((I + \Upsilon_k P_N) z_1)^3 - ((I + \Upsilon_k P_N) z_2)^3 \right\|_{L^2(0,L)} dt \\
\leq & c_T \int_0^T \Big\| ((I + \Upsilon_k P_N)(z_1 - z_2)) \Big(((I + \Upsilon_k P_N) z_1)^2 \\
& + ((I + \Upsilon_k P_N) z_2)^2 \Big) \Big\|_{L^2(0,L)} dt \\
\leq & c_T \int_0^T \left(\|(I + \Upsilon_k P_N) z_1\|_\infty^2 + \|(I + \Upsilon_k P_N) z_2\|_\infty^2 \right) \\
& \times \|(I + \Upsilon_k P_N)(z_1 - z_2)\|_{L^2(0,L)} dt \\
\leq & c_T \int_0^T \left(\|\partial_x z_1\|_{L^2(0,L)} \|z_1\|_{L^2(0,L)} + \|\partial_x z_2\|_{L^2(0,L)} \|z_2\|_{L^2(0,L)} \right) \|z_1 - z_2\|_{L^2(0,L)} dt \\
\leq & c_T \sqrt{T} (\|z_1\|_{X_T^l}^2 + \|z_2\|_{X_T^l}^2) \|z_1 - z_2\|_{X_T^l},
\end{aligned} \tag{5.79}$$

where we used Gagliardo-Nirenberg's inequality, boundedness of the backstepping transformation on $L^2(0, L)$ as well as boundedness of its inverse. Now for sufficiently small $T > 0$ which also guarantees that $c_T T^{\frac{1}{2}} R^2 < \frac{1}{2}$, Ψ becomes a contraction on a closed ball B_R^T. This yields a unique local solution in B_R^T.

To show that the local solution is unique in X_T^l, let us suppose there are two solutions $z_1 = \Psi z_1$ and $z_2 = \Psi z_2$. Then, applying the above arguments and choosing T smaller if necessary, we obtain

$$\|z_1 - z_2\|_{X_T^l} = \|\Psi z_1 - \Psi z_2\|_{X_T^l} \leq 2 c_T T^{\frac{1}{2}} R \|z_1 - z_2\|_{X_T^l} \leq \frac{1}{2} \|z_2 - z_1\|_{X_T^l},$$

which is possible only if $z_1 \equiv z_2$.

Finally, using the exponential decay results we stated in Proposition 5.3, the local solution remains uniformly bounded in X_T^l which implies that the local solution extends globally.

We finish this part by summarizing our well-posedness and stabilization results regarding the linear and nonlinear closed-loop systems.

Theorem 5.1 (Linear model) *Let $T, \nu, \alpha > 0$, $u_0 \in H^1(0, L)$. Assume that the feedback controller is given by*

$$h(t) = \int_0^L k(L, y; \mu)\Gamma[P_N u](y, t), \tag{5.80}$$

where k is a smooth solution of (5.14), $\Gamma[P_N u] := (I - P_N\Phi_N)[P_N u]$ *and* $\Phi_N : H^l(0, L) \to L^2(0, L)$ *is a linear bounded operator defined by the relation* (5.17). *Then we have the following:*

(i) (Well-posedness) The problem (5.2) *has a unique weak solution u such that*

$$u \in L^\infty(0, T; H^1(0, L)) \cap L^2(0, T; H^2(0, L) \cap H^1(0, L))$$

and

$$u_t \in L^2(0, T; L^2(0, L)).$$

(ii) (Stabilization) Let $u_0 \in H^l(0, L)$, $l = 0$ or $l = 1$. For prescribed $\mu > \alpha - \nu\lambda_1$, if N satisfies

$$N > \max\left\{\frac{\mu}{2\nu\lambda_1} - 1, \frac{\mu}{\mu + \nu\lambda_1 - \alpha} - 1\right\},$$

then zero equilibrium to the closed-loop system (5.2) *becomes exponentially stable in $H^l(0, L)$, respectively. Moreover, all solutions satisfies the exponential decay estimate*

$$\|u(\cdot, t)\|_{H^l(0,L)} \lesssim c_k e^{-\gamma t}\|u_0\|_{H^l(0,L)}, \quad \forall t \geq 0,$$

where $c_k > 0$ depends on the kernel k, independent of initial datum and γ, given by (5.21)*, can be made as large as desired by choosing μ large enough.*

Theorem 5.2 (Nonlinear model) *Let $\nu, \alpha > 0$, $u_0 \in H^l(0, L)$ where $l = 0$ or $l = 1$. Assume that the feedback controller is given by*

$$h(t) = \int_0^L k(L, y; \mu)\Gamma[P_N u](y, t), \tag{5.81}$$

where k is a smooth solution of (5.14), $\Gamma[P_N u] := (I - P_N\Phi_N)[P_N u]$ *and* $\Phi_N : H^l(0,L) \to L^2(0,L)$ *is a linear bounded operator defined by the relation* (5.17). *Then we have the following:*

(i) (Local well-posedness) The problem (5.2) *has a unique local solution* $u \in X_T^l$.

(ii) (Global well-posedness and stabilization) Suppose that $\|u_0\|_{H^l(0,L)}$ *is sufficiently small. For prescribed* $\mu > \alpha - \nu\lambda_1$, *if N satisfies*

$$N > \max\left\{\frac{\mu}{2\nu\lambda_1} - 1, \frac{\mu}{\mu + \nu\lambda_1 - \alpha} - 1\right\},$$

then the local solution in X_T^l *extends globally and zero equilibrium to the closed-loop system* (5.2) *becomes exponentially stable. Moreover, for any* γ' *with* $0 < \gamma' < \gamma$, *global solution has the following exponential decay estimate*

$$\|u(\cdot,t)\|_{H^l(0,L)} \lesssim c_k e^{-\gamma' t}\|u_0\|_{H^l(0,L)}, \quad \forall t \geq 0,$$

where $c_k > 0$ *depends on k, independent of initial datum and* γ, *given by* (5.21), *can be made as large as desired by choosing* μ *large enough.*

5.3. Numerical simulations

In this part, we present our numerical algorithm and and some experiments verifying our theoretical exponential decay results.

Although our algorithm is based on the same three steps, which we followed in Section 3.4, there are some differences in details, especially on derivation of the inverse backstepping transformation. Also, unlike to the cases in Section 3.4 and Section 4.4, now one of the models we consider is nonlinear. So, we would like to present our algorithm in details.

Step i. Initial datum of the target model is given implicitly by $u_0 = (I + \Upsilon_k P_N)w_0$. To find an explicit approximation for w_0, and therefore to initialize the target model, we need to find an approximation for the inverse of the backstepping transformation. Recall that w_0 can be expressed explicitly as $w_0 = u_0 - \Phi_N u_0$, where $\Phi_N u_0$ is given via the iteration

$$\Phi_N u_0 \equiv (I - \Phi_{N-1})[\Upsilon_k P_N u_0] - \frac{((I - \Phi_{N-1})[\Upsilon_k P_N u_0], e_N)_2}{1 + ((I - \Phi_{N-1})[\Upsilon_k e_N], e_N)_2}(I - \Phi_{N-1})[\Upsilon_k e_N] \quad (5.82)$$

for $N = 2, 3, \ldots$ with

$$\Phi_1 u_0 \equiv \frac{1}{1+\beta_1}\Upsilon_k P_1 u_0, \quad \beta_1 = (e_1, \Upsilon_k e_1)_2. \quad (5.83)$$

(5.82)-(5.83) includes the operators Υ_k and P_N. So at first, we need to express their discrete counterparts. To this end, let $N_x > 1$ be a positive integer, $1 \le j \le i \le N_x$, and (x_i, y_j) be distinct points of the triangular region $\Delta_{x,y}$, where

$$\delta_x = \frac{L}{N_x - 1}, \quad x_i = (i-1)\delta_x, \quad y_j = (j-1)\delta_x.$$

We derive k approximately by setting

$$k^M(x_i, y_j) = \frac{-\mu y_i}{2\nu} \sum_{m=0}^{M} \left(-\frac{\mu}{4\nu}\right)^m \frac{(x_i^2 - y_j^2)^m}{m!(m+1)!},$$

where M is an appropriate value in the sense that $\max\limits_{1 \le j \le i \le N_x} \left|k^{M+1}(x_i, y_j) - k^M(x_i, y_j)\right|$ is lesser than a tolerance value. Using k^M and applying composite trapezoidal rule for the integration, discrete counterpart $\boldsymbol{\Upsilon}_k^h$, of Υ_k can be expressed by an N_x dimensional square matrix as

$$
\mathbf{\Upsilon}_k^h = \delta_x \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{2}k^M(x_2,x_1) & \frac{1}{2}k^M(x_2,x_2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2}k^M(x_{N_x-1},x_1) & k^M(x_{N_x-1},x_2) & \cdots & \frac{1}{2}k^M(x_{N_x-1},x_{N_x-1}) & 0 \\ \frac{1}{2}k^M(x_{N_x},x_1) & k^M(x_{N_x},x_2) & \cdots & k^M(x_{N_x},x_{N_x-1}) & \frac{1}{2}k^M(x_{N_x},x_{N_x}) \end{bmatrix}.
$$

Discrete counterpart $\mathbf{P}_N^h$, of the operator P_N can also be expressed by an N_x dimensional matrix $\mathbf{P}_N^h = \delta_x \mathbf{W}\mathbf{W}^T$, where

$$
\mathbf{W} = \sqrt{\frac{2}{L}} \begin{bmatrix} \sin\left(\frac{\pi x_1}{L}\right) & \sin\left(\frac{2\pi x_1}{L}\right) & \cdots & \sin\left(\frac{N\pi x_1}{L}\right) \\ \sin\left(\frac{\pi x_2}{L}\right) & \sin\left(\frac{2\pi x_2}{L}\right) & \cdots & \sin\left(\frac{N\pi x_2}{L}\right) \\ \vdots & \vdots & & \vdots \\ \sin\left(\frac{\pi x_{N_x}}{L}\right) & \sin\left(\frac{2\pi x_{N_x}}{L}\right) & \cdots & \sin\left(\frac{N\pi x_{N_x}}{L}\right) \end{bmatrix}.
$$

Next, we obtain $\Phi_N u_0$ via the iteration (5.82)-(5.83). To perform this, observe that one requires information related with Φ_{N-1}. More precisely, one explicitly needs to know $\Phi_{N-1}[\Upsilon_k P_N u_0]$ and $\Phi_{N-1}[\Upsilon_k e_N]$ to derive $\Phi_N u_0$. Let us show this situation schematically as follows:

$$
\left.\begin{array}{l} \Phi_{N-1}[\Upsilon_k P_N u_0] \\ \Phi_{N-1}[\Upsilon_k e_N] \end{array}\right\} \rightarrow \Phi_N u_0. \tag{5.84}
$$

It describes that the left end part of the arrow is required as an input to derive the right end part of the arrow. However, the inputs on the left end part of the above scheme, $\Phi_{N-1}[\Upsilon_k P_N u_0]$ and $\Phi_{N-1}[\Upsilon_k e_N]$, are unknowns as well, since they depend on Φ_{N-2} due to iteration (5.82). In other words, considering (5.82) for $N-1$ and replacing u_0 with $\Upsilon_k P_N u_0$, we can see that we explicitly need to know $\Phi_{N-2}[(\Upsilon_k P_{N-1})(\Upsilon_k P_N)u_0]$ and $\Phi_{N-2}[\Upsilon_k e_{N-1}]$. Similarly, replacing u_0 with $\Upsilon_k e_N$, on explicitly needs to have $\Phi_{N-2}[(\Upsilon_k P_{N-1})(\Upsilon_k e_N)]$ and $\Phi_{N-2}[\Upsilon_k e_{N-1}]$. In view of these new unknowns, we enlarge the scheme (5.84) as follows:

$$\left.\begin{array}{l}
\left.\begin{array}{l}\Phi_{N-2}[(\Upsilon_k P_{N-1})(\Upsilon_k P_N)u_0] \\ \Phi_{N-2}[\Upsilon_k e_{N-1}]\end{array}\right\} \to \Phi_{N-1}[\Upsilon_k P_N u_0] \\
\left.\begin{array}{l}\Phi_{N-2}[(\Upsilon_k P_{N-1})(\Upsilon e_N)] \\ \Phi_{N-2}[\Upsilon_k e_{N-1}]\end{array}\right\} \to \Phi_{N-1}[\Upsilon_k e_N]
\end{array}\right\} \to \Phi_N u_0. \tag{5.85}$$

The functions lying at the most left end of scheme (5.85) are still unknown since they depend on Φ_{N-3}. In order to see what we need to determine them, we need to go back one more step. After some calculations and considering the required *ingredients*, the scheme (5.85) is enlarged as the following one:

$$\left.\begin{array}{l}
\left.\begin{array}{l}
\left.\begin{array}{l}\Phi_{N-3}[(\Upsilon_k P_{N-2})(\Upsilon_k P_{N-1})(\Upsilon_k P_N)u_0] \\ \Phi_{N-3}[\Upsilon_k e_{N-2}]\end{array}\right\} \to \Phi_{N-2}[(\Upsilon_k P_{N-1})(\Upsilon_k P_N)u_0] \\
\left.\begin{array}{l}\Phi_{N-3}[(\Upsilon_k P_{N-2})[\Upsilon_k e_{N-1}]] \\ \Phi_{N-3}[\Upsilon_k e_{N-2}]\end{array}\right\} \to \Phi_{N-2}[\Upsilon_k e_{N-1}]
\end{array}\right\} \to \Phi_{N-1}[\Upsilon_k P_N u_0] \\
\left.\begin{array}{l}
\left.\begin{array}{l}\Phi_{N-3}[(\Upsilon_k P_{N-2})(\Upsilon_k P_{N-1})[(\Upsilon_k e_N)]] \\ \Phi_{N-3}[\Upsilon_k e_{N-2}]\end{array}\right\} \to \Phi_{N-2}[(\Upsilon_k P_{N-1})(\Upsilon e_N)] \\
\left.\begin{array}{l}\Phi_{N-3}[(\Upsilon_k P_{N-2})[\Upsilon_k e_{N-1}]] \\ \Phi_{N-3}[\Upsilon_k e_{N-2}]\end{array}\right\} \to \Phi_{N-2}[\Upsilon_k e_{N-1}]
\end{array}\right\} \to \Phi_{N-1}[\Upsilon_k e_N]
\end{array}\right\} \to \Phi_N u_0.$$

Each backward step to determine Φ_j gives rise to a new unknown including Φ_{j-1}, $j = 3, \ldots, N$. Therefore, in order to understand what total information is required to derive $\Phi_N u_0$, we need to go backward step by step until we hit Φ_2, since it depends on Φ_1, which is already known and defined as (5.83). Stepping back in this manner, one can conclude that to derive $\Phi_N u_0$ explicitly, we need

$$\begin{array}{l}
(\Upsilon_k P_1)[\Upsilon_k e_2] \\
(\Upsilon_k P_1)(\Upsilon_k P_2)[\Upsilon_k e_3] \\
\quad\vdots \\
(\Upsilon_k P_1)(\Upsilon_k P_2)\cdots(\Upsilon_k P_{N-1})[\Upsilon_k e_N]
\end{array}$$

and

$$(\Upsilon_k P_1)(\Upsilon_k P_2)\cdots(\Upsilon_k P_{N-1})(\Upsilon_k P_N)u_0$$

as inputs. Once we obtain $\mathbf{\Phi}_N^h \mathbf{u_0}$, where $\mathbf{\Phi}_N^h$ represents the discrete counterpart of Φ_N obtained by using discrete operators $\mathbf{\Upsilon}_k^h$, $\mathbf{P}_j^h$, $j = 1, \ldots, N$ defined above and $\mathbf{u}_0 = [u_0(x_1)\cdots u_0(N_x)]^T$, we write

$$\mathbf{w}_0 = \mathbf{u}_0 + \mathbf{\Phi}_N^h \mathbf{u_0},$$

and obtain an approximation, $\mathbf{w}_0$, for a numerical approximation $\mathbf{w}_0$, to the initial datum w_0 of the target model.

Step ii. Let us first consider the linear target model (5.3). Define the finite dimensional space $\mathrm{X}^h := \left\{\varphi = [\varphi_1 \ \cdots \ \varphi_{N_x}]^T \in \mathbb{R}^{N_x} \,|\, \varphi_1 = \varphi_{N_x} = 0\right\}$ and the operator $\mathbf{A} : \mathrm{X}^h \to \mathrm{X}^h$,

$$\mathbf{A} := -\nu\mathbf{\Delta} - \alpha\mathbf{I}_{N_x} + \mu\mathbf{P}_N^h,$$

where $\mathbf{\Delta}$ is the N_x dimensional tridiagonal matrix obtained by applying central difference approximation to the second order derivative and $\mathbf{I}_{N_x}$ is the N_x dimensional identity matrix.

Let N_t denotes the number of time steps, T is the final time and $\delta_t = \frac{T}{N_t - 1}$. Let $\mathbf{w}^n = [w_1^n \ \cdots \ w_{N_x}^n]^T \in \mathrm{X}^h$ be an approximation of the solution w at $n-$th time level. Using Crank-Nicolson time stepping, which yields unconditionally numerical stable results in time, we end up with the following fully discrete problem for the linear target model (5.3)

$$\begin{cases} \text{Given } \mathbf{w}^n \in \mathrm{X}^h, \text{ find } \mathbf{w}^{n+1} \in \mathrm{X}^h \text{ such that} \\ \left(\mathbf{I}_{N_x} + \frac{\delta_t}{2}\mathbf{A}\right)\mathbf{w}^{n+1} = \mathbf{B}_l^n, \quad n = 0, 1, \ldots, N_t, \end{cases} \tag{5.86}$$

where

$$\mathbf{B}_l^n := \left(\mathbf{I}_{N_x} - \frac{\delta_t}{2}\mathbf{A}\right)\mathbf{w}^n. \tag{5.87}$$

(5.86)-(5.87) is linear in $\mathbf{w}^{n+1}$, hence can be solved directly.

Nonlinear case. Recall that the nonlinear target model (5.45) involves an additional term

$$Fw = (I - \Phi_N)\left[((I + \Upsilon_k P_N)w)^3\right]$$

on the left hand side of the main equation. Applying the same discretization procedure as in the linear case, we obtain the following fully discrete system of equations for the nonlinear target model (5.45)

$$\left(\mathbf{I}_{N_x} + \frac{\delta_t}{2}\mathbf{A}\right)\mathbf{w}^{n+1} + \frac{\delta_t}{2}\left(\mathbf{I}_{N_x} + \boldsymbol{\Upsilon}_k^h\mathbf{P}_N^h\right)^{-1}\left[\left(\left(\mathbf{I}_{N_x} + \boldsymbol{\Upsilon}_k^h\mathbf{P}_N^h\right)\mathbf{w}^{n+1}\right)^3\right] = \mathbf{B}_l^n + \mathbf{B}_{nl}^n, \tag{5.88}$$

where $\mathbf{B}_l^n$ is given by (5.87) and

$$\mathbf{B}_{nl}^n = -\frac{\delta_t}{2}\left(\mathbf{I}_{N_x} + \boldsymbol{\Upsilon}_k^h\mathbf{P}_N^h\right)^{-1}\left[\left(\left(\mathbf{I}_{N_x} + \boldsymbol{\Upsilon}_k^h\mathbf{P}_N^h\right)\mathbf{w}^n\right)^3\right].$$

Observe that (5.88) is nonlinear in the unknown $\mathbf{w}^{n+1}$. To linearize this, let $\mathbf{w}^{n,p}$ be an approximation to the unknown $\mathbf{w}^{n+1}$ and consider performing the iteration

$$\begin{cases} \mathbf{w}^{n,p+1} = \mathbf{w}^{n,p} + \mathbf{dw}, & p = 0, 1, 2, \ldots, \\ \mathbf{w}^{n,0} = \mathbf{w}^n, \end{cases} \tag{5.89}$$

to obtain a better approximation $\mathbf{w}^{n,p+1}$, until $\max\limits_{1\le i\le N_x} |\mathbf{dw}|$ is small enough. To determine $\mathbf{dw}$ in each iteration, we replace $\mathbf{w}^{n+1}$ by $(\mathbf{w}^{n,p} + \mathbf{dw})$ for the linear term and by $\mathbf{w}^{n,p}$ for the nonlinear term on (5.88), and then write

$$\left(\mathbf{I}_{N_x} + \frac{\delta_t}{2}\mathbf{A}\right)(\mathbf{w}^{n,p} + \mathbf{dw}) + \frac{\delta_t}{2}\left(\mathbf{I}_{N_x} + \boldsymbol{\Upsilon}_k^h\mathbf{P}_N^h\right)^{-1}\left[\left(\left(\mathbf{I}_{N_x} + \boldsymbol{\Upsilon}_k^h\mathbf{P}_N^h\right)(\mathbf{w}^{n,p})\right)^3\right]$$
$$= \mathbf{B}_l^n + \mathbf{B}_{nl}^n. \tag{5.90}$$

Rearranging (5.90) with respect to **dw**, we get

$$\left(\mathbf{I}_{N_x} + \frac{\delta_t}{2}\mathbf{A}\right)\mathbf{dw} = \mathbf{B}_l^n + \mathbf{B}_{nl}^n - \left(\mathbf{I}_{N_x} + \frac{\delta_t}{2}\mathbf{A}\right)\mathbf{w}^{n,p} - \frac{\delta_t}{2}\left(\mathbf{I}_{N_x} + \Upsilon_k^h\mathbf{P}_N^h\right)^{-1}\left[\left(\left(\mathbf{I}_{N_x} + \Upsilon_k^h\mathbf{P}_N^h\right)\mathbf{w}^{n,p}\right)^3\right], \quad (5.91)$$

which is now linear in **dw**. Hence for given $\mathbf{w}^{n,p}$, (5.91) can be solved directly to obtain **dw**. Once $\max_{1\le i\le N_x}|\mathbf{dw}|$ is small enough, we stop the iteration (5.89) and set $\mathbf{w}^{n+1} := \mathbf{w}^{n,p} + \mathbf{dw}$. Note that inverse backstepping operators involving in the second and forth terms on the right hand side of (5.91) are evaluated via the procedure we introduced in Step (i) in each time iteration.

Step iii. Once we obtain approximate solutions for the linear and nonlinear target models, we substitute them into the discrete backstepping transformation

$$\mathbf{u}^n = \left(\mathbf{I}_{N_x} + \Upsilon_k^h\mathbf{P}_N^h\right)\mathbf{w}^n, \quad 1 \le n \le N_t, \quad (5.92)$$

to get the corresponding approximate solutions for the linear and nonlinear plant, respectively.

Below we give several numerical simulations that verify our theoretical stabilization results for both linear and nonlinear models. Results are obtained by taking $N_x = 1000$ spatial nodes and $N_t = 1000$ time steps. The value M for the partial sum of the series representation of k is chosen such that $\max_{1\le j\le i\le N_x}|k^{M+1}(x_i, y_j) - k^M(x_i, y_j)| \sim 10^{-12}$.

Experiment 1 - Stabilization with a single Fourier mode. Consider the linear problem

$$\begin{cases} u_t - u_{xx} - 2u = 0, & (x,t) \in (0,\pi)\times(0,T), \\ u(0,t) = 0,\ u(\pi,t) = h(t), & t \in (0,T), \\ u(x,0) = x(\pi - x), & x \in (0,\pi). \end{cases} \quad (5.93)$$

Observe that $\nu = 1, \alpha = 2$, and $\lambda_1 < 2 < \lambda_2$ where $\lambda_j = j^2$, $j = 1, 2$. So, in the absence of control, the instability level of (5.93) is one and solution grows up in time (see Figure 5.1).

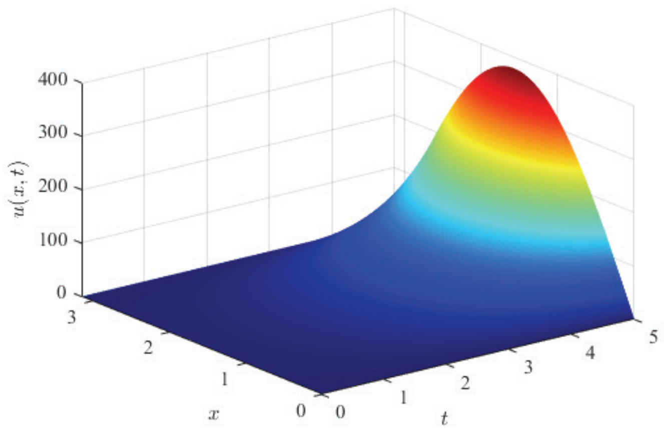

Figure 5.1. Uncontrolled approximate solution of the linear model (5.93).

As the instablity level is one, we should be able to stabilize the zero solution by a controller involving only a single Fourier mode. So let us take $N = 1$. Then, in order to fulfill the condition (5.67), μ belongs to $\left(\frac{8}{3}, 8\right)$. Let us choose $\mu = 3$. See Figure 5.2 for the numerical simulation of the stabilized model and for time evolution of L^2-norm of the corresponding solution.

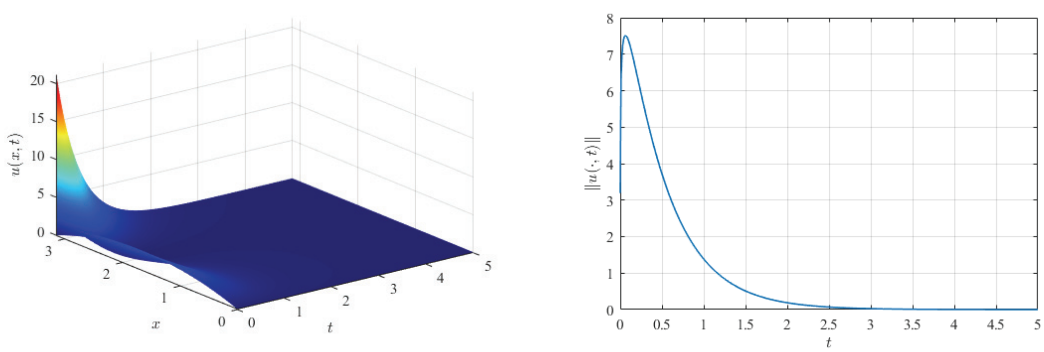

Figure 5.2. Simulations for the stabilized linear model (5.93). Left: 3d plot. Right: Time evolution of L^2-norm.

Experiment 2 - Stabilization of a linear model that assumes a stationary solution. Let us consider another linear model

$$\begin{cases} u_t - u_{xx} - u = 0, & x \in (0, 2\pi) \times (0, T), \\ u(0, t) = 0, \ u(2\pi, t) = h(t), & t \in (0, T), \\ u(x, 0) = \sin x, & x \in (0, 2\pi). \end{cases} \tag{5.94}$$

In this model, note that $\alpha = 1$ is the second eigenvalue of $-\frac{d^2}{dx^2}$ on $(0, 2\pi)$ subject to homogeneous Dirichlet type boundary condition. Therefore, if $h(t) = 0$, then the corresponding eigenfunction, $\sin x$, is a nontrivial stationary solution of (5.94). This shows that zero equilibrium is not asymptotically stable.

To stabilize it, let us choose $\mu = 1.3$ (indeed we can choose higher) which satisfies $\mu > \alpha - \nu\lambda_1 = \frac{3}{4}$. Then, we see that choosing $N = 2$ fulfills the criteria

$$N = 2 > \max\left\{\frac{\mu}{2\nu\lambda_1} - 1, \frac{\mu}{\mu + \nu\lambda_1 - \alpha} - 1\right\} = \max\left\{\frac{16}{10}, \frac{15}{11}\right\}.$$

See Figure 5.3 for the numerical simulation of the stabilized model and for time evolution of L^2-norm of the corresponding solution.

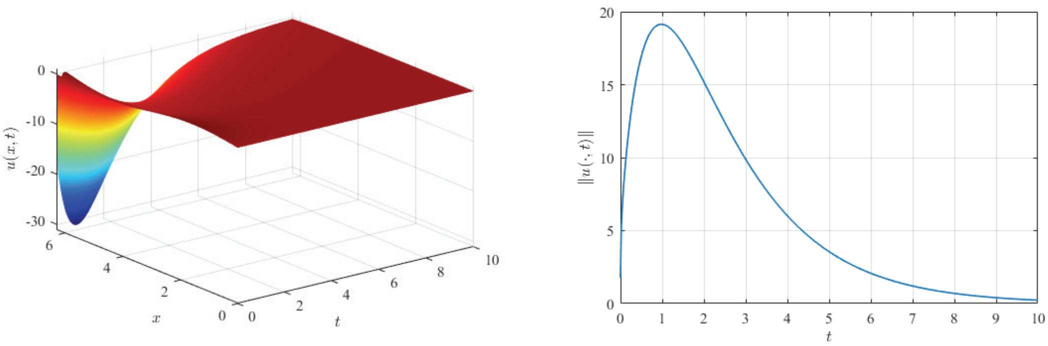

Figure 5.3. Simulations for the stabilized linear model (5.94). Left: 3d plot. Right: Time evolution of L^2-norm.

Experiment 3 - Rapid stabilization of a nonlinear model. As a final simulation, let us consider the following nonlinear model

$$\begin{cases} u_t - u_{xx} - 15u + u^3 = 0, & x \in (0,1) \times (0,T), \\ u(0,t) = 0, u(1,t) = h(t), & t \in (0,T), \\ u(x,0) = \sin(2\pi x) - 2\sin(3\pi x), & x \in (0,1). \end{cases} \tag{5.95}$$

Note that $\alpha - \nu\lambda_1 = 15 - \pi^2 > 0$. Therefore, if there is no control input acting on the model, then the given initial state evolves into a nontrivial equilibrium in time (see Figure 5.4).

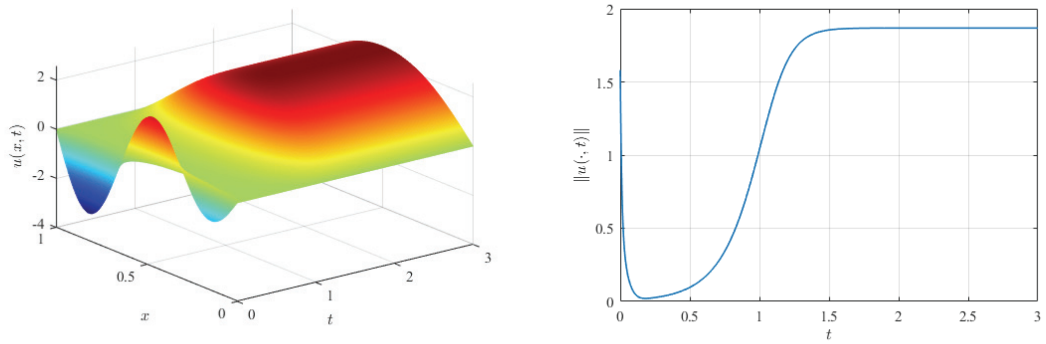

Figure 5.4. Simulations for uncontrolled model (5.95). Left: 3d plot. Right: Time evolution of L^2-norm.

Let us take $\mu = 8 > \alpha - \nu\lambda_1 = 15 - \pi^2$. Then choosing $N = 2$ fulfills the condition

$$N = 2 > \max\left\{\frac{\mu}{2\nu\lambda_1} - 1, \frac{\mu}{\mu + \nu\lambda_1 - \alpha} - 1\right\} = \max\left\{\frac{4}{\pi^2} - 1, \frac{8}{\pi^2 - 7} - 1\right\},$$

and ε_3, stated in Proposition 5.3, be such that $0 < \varepsilon_3 < 2 - \frac{8}{3\pi^2}$. See Figure 5.5 for the numerical simulations of the stabilized model.

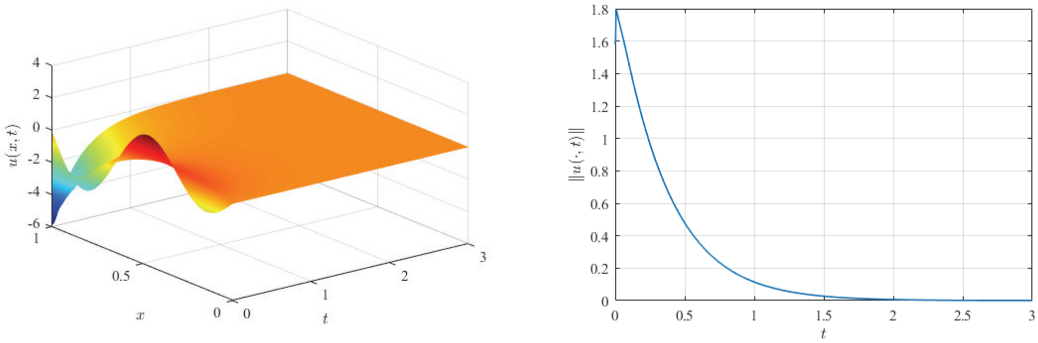

Figure 5.5. Simulations for the stabilized nonlinear model (5.95). Left: 3d plot. Right: Time evolution of L^2-norm.

To observe the effect of μ on the decay rate of L^2-norm of u, let us now increase the value of μ by $\mu = 10$ and by $\mu = 15$, and compare the associated simulations with the results we obtained by the previous choice $\mu = 8$. Note that, $N = 2$ still fulfills the criteria (5.20) for both choices of μ so that, the associated control inputs involve same amount of Fourier sine mode.

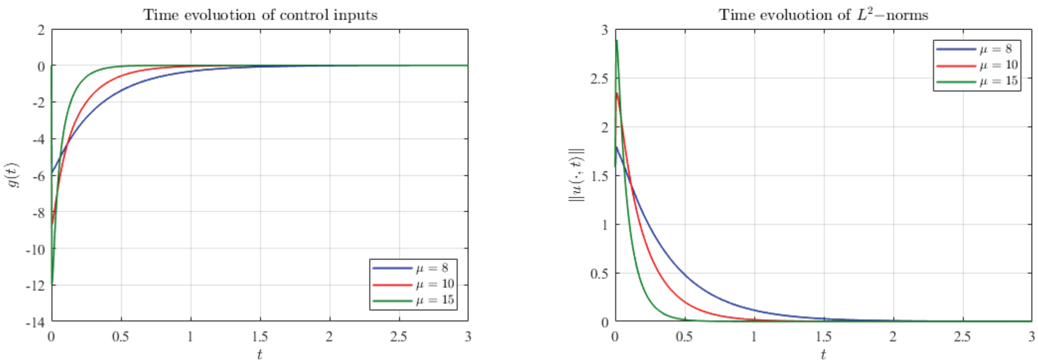

Figure 5.6. Time evolution of control inputs obtained by various values of μ that involve $N = 2$ Fourier modes (left) and L^2-norms of the associated approximate solutions (right).

On the left of Figure 5.6, we observe that higher values of μ result in greater control effort in the early times of the evolution. In response to this, decay rates of the corresponding L^2-norms of u decay more rapidly in time.

CHAPTER 6

CONCLUSION

In this thesis, we studied various kind of stabilization problems of some evolutionary PDEs by controlling the models from the boundary of the region of the evolution. Control inputs were constructed by employing an analog of backstepping method. However, the standard way is not directly applicable to the models we considered so that, we modified the method for the sake of our purposes. However, from the well-posedness point of view, these modifications come along with some difficulties and mathematical challanges.

In Chapter 3, we considered linear higher-order Schrödinger equation posed on a finite interval, where there is a single boundary condition imposed at the left endpoint and control inputs acting from the opposite end. We faced that applying standard backstepping strategy yields an overdetermined backstepping kernel model, which cannot possess a smooth solution. Due to the fact that smoothness of the backstepping kernel is crucial, we modify the method by correcting the backstepping kernel model so that, the corrected model possesses a smooth solution. On the other hand, this is not the case if one considers a single controller acting from the left endpoint and one can obtain rapid stabilization of zero equilibrium. So the drawback in the case of right endpoint controllers is, one loses rapid stabilization of the zero equilibrium. Nevertheless, we still achieved exponential decay of the solutions in L^2-norm with a certain decay rate either the length of the interval belongs to the set of critical length of intervals or not. This is in particular important because, in the critical length of interval case, the problem is not exact controllable from right endpoint neither in the case of Dirichlet actuation nor in the case of Neumann actuation. From the well-posedness point of view, due to the modification of the backstepping method, one needs to obtain time-space regularity estimates which also requires to reveal smoothing properties for associated Cauchy problems and an initial-boundary value problem with inhomogeneous boundary conditions.

In Chapter 4, we considered the case where the state of the model is not fully measurable at any moment and assumed partial measurements, first order and second order boundary measurements were available. This allowed us to design an observer model so that its state were used in the boundary stabilizers. The control input were placed at the right endpoint of the domain. Therefore, the issues related with right endpoint controllers that occurs in Chapter 3 still remains. To overcome this situation, we used a similar modification on the backstepping strategy. In particular, this process yielded a target error model which has inhomogeneous Neumann type boundary condition of feedback type. Well-posedness of this model was carried out by similar techniques we applied in Chapter 3 combined with fixed point arguments. From the stabilization point of view, we still success on gaining exponential stabilization of zero equilibrium either in the case of critical length of interval or noncritical one. The drawback is that, we lose rapid stabilization of the zero equilibrium compared to the left endpoint controller case.

In the above problems, the controllers involve full state of the solution. Contrary in Chapter 5, we introduced a machinery for constructing controllers that involve finitely many Fourier modes of the solution. That is, the controller contains only the projection of the state onto a finite-dimensional space. This corresponds the influence the asymptotic behaviour of the trajectories in the inifite dimenisonal dynamical system by finite dimensional effects through the boundary of the region of the evolution. Our motivation rely on the fact that infinite dimensional dynamical systems generated by dissipative models has finite dimensional asymptotic behaviour in time and such systems possesses finitely many determining parameters. We combined this idea with backstepping strategy and considered nonlinear reaction-diffusion equation as a canonical example. We precisely calculate the number of Fourier modes in order the gain rapid stabilization of zero equilibrium. In addition, we find the minimal number of Fourier modes that guarantees exponential stabilization of zero equilibrium with a certain exponential decay rate. Our result is meaningful: We find that the sufficient number of Fourier modes is exactly same as the instability level of the problem. We want to underline that, this is the first result in the literature that stabilizes zero equilibrium to infinite dimensional dissipative systems by finite dimensional backstepping type controllers. The drawback here is, regarding the nonlinear model, we have a smallness assumption on the initial condition, that corresponds to local rapid stabilization of zero equilibrium. Note that this is common for stabilization of nonlinear PDEs via backstepping controllers since backstepping transformation turns the original nonlinear plant into a target model in which the monotonic structure of the nonlinear term is disrupted.

Apart from the theoretical findings, we derived numerical simulations verifying our theoretical stabilization results.

Since we succeeded in proving exponential stabilization of zero equilibrium to nonlinear reaction-diffusion model by finite dimensional feedback controllers, our further plan is to apply our strategy to various kind of dissipative models such as Ginzburg-Landau equation, dissipative wave equations and various kind of dissipative systems such as Fitzhugh-Nagumo system, original Burger's equations, phase-field system. In particular, regarding the nonlinear models, our aim is also to gain global stabilization of zero equilibrium by finding a way to remove smallness assumption on the initial data.

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APPENDIX A

DEDUCTION OF BACKSTEPPING KERNEL MODELS

A.1. Kernel model (3.10)

In this section, we present the details of the calculations that gives the backstepping kernel model (3.10). Differentiating both sides of (3.9) with respect to t we get

$$
\begin{aligned}
iw_t(x,t) =& iu_t(x,t) - \int_0^x ik(x,y)u_t(y,t)dy \\
=& iu_t(x,t) + \int_0^x k(x,y)(i\beta u_{yyy} + \alpha u_{yy} + i\delta u_y)(y,t)dy \\
=& iu_t(x,t) \\
&+ i\beta \left(k(x,y)u_{yy}(y,t) - k_y(x,y)u_y(y,t) + k_{yy}(x,y)u(y,t) \Big|_0^x - \int_0^x k_{yyy}(x,y)u(y,t)dy \right) \\
&+ \alpha \left(k(x,y)u_y(y,t) - k_y(x,y)u(y,t) \Big|_0^x + \int_0^x k_{yy}(x,y)u(y,t)dy \right) \\
&+ i\delta \left(k(x,y)u(y,t) \Big|_0^x - \int_0^x k_y(x,y)u(y,t)dy \right).
\end{aligned}
$$

Using the boundary condition $u(0,t) = 0$, rearrenging the last expression in terms of $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, $u_x(0,t)$ and $u_{xx}(0,t)$, we obtain

$$
\begin{aligned}
iw_t(x,t) =& iu_t(x,t) + \int_0^x \left(-i\beta k_{yyy} + \alpha k_{yy} - i\delta k_y \right)(x,y)u(y,t)dy \\
&+ \left(i\beta k_{yy}(x,x) - \alpha k_y(x,x) + i\delta k(x,x) \right) u(x,t) \\
&+ \left(-i\beta k_y(x,x) + \alpha k(x,x) \right) u_x(x,t) + i\beta k(x,x)u_{xx}(x,t) \\
&+ \left(i\beta k_y(x,0) - \alpha k(x,0) \right) u_x(0,t) - i\beta k(x,0)u_{xx}(0,t).
\end{aligned}
\tag{A.1}
$$

Next we differentiate both sides of (3.9) with respect to x up to the order three and multiply the results by $i\delta$, α and $i\beta$ respectively to obtain

$$\begin{aligned} i\delta w_x(x,t) &= i\delta u_x(x,t) - i\delta\frac{\partial}{\partial x}\int_0^x k(x,y)u(y,t)dy \\ &= i\delta u_x(x,t) - \int_0^x i\delta k_x(x,y)u(y,t)dy - i\delta k(x,x)u(x,t), \end{aligned} \tag{A.2}$$

$$\begin{aligned} \alpha w_{xx}(x,t) =& \alpha u_{xx}(x,t) - \alpha\frac{\partial}{\partial x}\int_0^x k_x(x,y)u(y,t)dy - \alpha\frac{\partial}{\partial x}\left(k(x,x)u(x,t)\right) \\ =& \alpha u_{xx}(x,t) - \int_0^x \alpha k_{xx}(x,y)u(y,t)dy + \alpha\left(-k_x(x,x) - \frac{d}{dx}k(x,x)\right)u(x,t) \\ &- \alpha k(x,x)u_x(x,t) \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} i\beta w_{xxx}(x,t) =& i\beta u_{xxx}(x,t) - i\beta\frac{\partial}{\partial x}\int_0^x k_{xx}(x,y)u(y,t)dy \\ &- i\beta\frac{\partial}{\partial x}\left(\left(k_x(x,x) + \frac{d}{dx}k(x,x)\right)u(x,t) + k(x,x)u_x(x,t)\right) \\ =& i\beta u_{xxx}(x,t) - \int_0^x i\beta k_{xxx}(x,y)u(y,t)dy \\ &+ i\beta\left(-k_{xx}(x,x) - \frac{d}{dx}k_x(x,x) - \frac{d^2}{dx^2}k(x,x)\right)u(x,t) \\ &+ i\beta\left(-k_x(x,x) - 2\frac{d}{dx}k(x,x)\right)u_x(x,t) - i\beta k(x,x)u_{xx}(x,t). \end{aligned} \tag{A.4}$$

Adding (A.1)-(A.4) side by side together with

$$irw(x,t) = iru(x,t) - ir\int_0^x k(x,y)u(y,t)dy,$$

and then using the main equation of the linear plant, we obtain

$$
\begin{aligned}
& iw_t + i\beta w_{xxx} + \alpha w_{xx} + i\delta w_x + irw \\
&= \int_0^x \Big(-i\beta(k_{xxx} + k_{yyy}) - \alpha(k_{xx} - k_{yy}) - i\delta(k_x + k_y) - irk\Big)(x,y)u(y,t)dy \\
&\quad + \Bigg(i\beta\bigg(k_{yy}(x,x) - k_{xx}(x,x) - \frac{d}{dx}k_x(x,x) - \frac{d^2}{dx^2}k(x,x)\bigg) \\
&\quad + \alpha\bigg(-k_y(x,x) - k_x(x,x) - \frac{d}{dx}k(x,x)\bigg) + ir\Bigg)u(x,t) \\
&\quad - i\beta\bigg(k_y(x,x) + k_x(x,x) + 2\frac{d}{dx}k(x,x)\bigg)u_x(x,t) \\
&\quad + \Big(i\beta k_y(x,0) - \alpha k(x,0)\Big)u_x(0,t) - i\beta k(x,0)u_{xx}(0,t).
\end{aligned} \tag{A.5}
$$

We set

$$k(x,0) = 0 \quad \text{and} \quad k_y(x,0) = 0 \tag{A.6}$$

so the last line of (A.5) vanishes. Using the relation $\frac{d}{dx}k(x,x) = k_x(x,x) + k_y(x,x)$, we see that the fifth line of (A.5) vanishes if $\frac{d}{dx}k(x,x) = 0$. Thanks to $k(x,0) = 0$, this implies that

$$k(x,x) = 0. \tag{A.7}$$

Next, we differentiate $\frac{d}{dx}k(x,x) = k_x(x,x) + k_y(x,x)$ with respect to x and use $\frac{d}{dx}k(x,x) = 0$ to obtain $k_{yy}(x,x) = -2k_{xy}(x,x) - k_{xx}(x,x)$. Using this result, we deduce that third and forth lines of (A.5) vanishes if $\frac{d}{dx}k_x(x,x) = \frac{r}{3\beta}$ holds. Note that $k_y(x,0) = 0 \Rightarrow k_x(x,0) = 0 \Rightarrow k_x(0,0) = 0$, thus integrating both sides of $\frac{d}{dx}k_x(x,x) = \frac{r}{3\beta}$ over $(0,x)$, we get

$$k_x(x,x) = \frac{rx}{3\beta}. \tag{A.8}$$

Consequently, under the assumptions (A.6)-(A.8) and if

$$\beta(k_{xxx} + k_{yyy}) - i\alpha(k_{xx} - k_{yy}) + \delta(k_x + k_y) + rk = 0, \quad (x,y) \in \Delta_{x,y},$$

right hand side of (A.5) vanishes and we obtain that

$$iw_t + i\beta w_{xxx} + \alpha w_{xx} + i\delta w_x + irw = 0.$$

Note also that taking $x = 0$ in the backstepping transformation implies $w(0,t) = u(0,t) = 0$ and, taking $x = L$ implies

$$w(L,t) = u(L,t) - \int_0^L k(L,y)u(y,t)dy = 0$$

and

$$w_x(L,t) = u_x(L,t) - \int_0^L k_x(L,y)u(y,t)dy - k(0,0)u(0,t) = 0.$$

So the boundary conditions of (3.3) are being satisfied without any extra conditions on k.

As a conclusion, linear plant (3.8) is mapped to target model (3.3), if k solves the boundary value problem (3.10).

A.2. Kernel model (4.6)

In this section, we present the details of the calculations that gives the backstepping kernel model (4.6). Differentiating (4.4) with respect to t, we get

$$\begin{aligned}
i\tilde{u}_t(x,t) =& i\tilde{w}_t(x,t) - \int_0^x i\ell(x,y)\tilde{w}_t(y,t)dy \\
=& i\tilde{w}_t(x,t) + \int_0^x \ell(x,y)(i\beta\tilde{w}_{yyy} + \alpha\tilde{w}_{yy} + i\delta\tilde{w}_y + ir\tilde{w})(y,t)dy \\
=& i\tilde{w}_t(x,t) \\
&+ i\beta\left(\ell(x,y)\tilde{w}_{yy}(y,t) - \ell_y(x,y)\tilde{w}_y(y,t) + \ell_{yy}(x,y)\tilde{w}(y,t)\Big|_0^x \right.\\
&\left. - \int_0^x \ell_{yyy}(x,y)\tilde{w}(y,t)dy\right) \\
&+ \alpha\left(\ell(x,y)\tilde{w}_y(y,t) - \ell_y(x,y)\tilde{w}(y,t)\Big|_0^x + \int_0^x \ell_{yy}(x,y)\tilde{w}(y,t)dy\right) \\
&+ i\delta\left(\ell(x,y)\tilde{w}(y,t)\Big|_0^x - \int_0^x \ell_y(x,y)\tilde{w}(y,t)dy\right) \\
&+ ir\int_0^L \ell(x,y)\tilde{w}(y,t)dy.
\end{aligned}$$

Using the boundary condition $w(0,t) = 0$ and rearranging the last expression in terms of $\tilde{w}(x,t)$,$\tilde{w}_x(x,t)$, $\tilde{w}_{xx}(x,t)$, $\tilde{w}_x(0,t)$ and $\tilde{w}_{xx}(0,t)$, we obtain

$$\begin{aligned}
i\tilde{u}_t(x,t) =& i\tilde{w}_t(x,t) + \int_0^x \left(-i\beta\ell_{yyy} + \alpha\ell_{yy} - i\delta\ell_y + ir\ell\right)(x,y)\tilde{w}(y,t)dy \\
&+ \left(i\beta\ell_{yy}(x,x) - \alpha\ell_y(x,x) + i\delta\ell(x,x)\right)\tilde{w}(x,t) \\
&+ \left(-i\beta\ell_y(x,x) + \alpha\ell(x,x)\right)\tilde{w}_x(x,t) + i\beta\ell(x,x)\tilde{w}_{xx}(x,t) \\
&+ \left(i\beta\ell_y(x,0) - \alpha\ell(x,0)\right)\tilde{w}_x(0,t) - i\beta\ell(x,0)\tilde{w}_{xx}(0,t).
\end{aligned} \tag{A.9}$$

Next we differentiate (4.4) up to order three and multiply the results by $i\delta$, α and $i\beta$ respectively to obtain

$$\begin{aligned} i\delta\tilde{u}_x(x,t) &= i\delta\tilde{w}_x(x,t) - i\delta\frac{\partial}{\partial x}\int_0^x \ell(x,y)\tilde{w}(y,t)dy \\ &= i\delta\tilde{w}_x(x,t) - \int_0^x i\delta\ell_x(x,y)\tilde{w}(y,t)dy - i\delta\ell(x,x)\tilde{w}(x,t), \end{aligned} \tag{A.10}$$

$$\begin{aligned} \alpha\tilde{u}_{xx}(x,t) =& \alpha\tilde{w}_{xx}(x,t) - \alpha\frac{\partial}{\partial x}\int_0^x \ell_x(x,y)\tilde{w}(y,t)dy - \alpha\frac{\partial}{\partial x}\left(\ell(x,x)\tilde{w}(x,t)\right) \\ =& \alpha\tilde{w}_{xx}(x,t) - \int_0^x \alpha\ell_{xx}(x,y)\tilde{w}(y,t)dy \\ &+ \alpha\left(-\ell_x(x,x) - \frac{d}{dx}\ell(x,x)\right)\tilde{w}(x,t) - \alpha\ell(x,x)\tilde{w}_x(x,t) \end{aligned} \tag{A.11}$$

and

$$\begin{aligned} i\beta\tilde{u}_{xxx}(x,t) =& i\beta\tilde{w}_{xxx}(x,t) - i\beta\frac{\partial}{\partial x}\int_0^x \ell_{xx}(x,y)\tilde{w}(y,t)dy \\ &- i\beta\frac{\partial}{\partial x}\left(\left(\ell_x(x,x) + \frac{d}{dx}\ell(x,x)\right)\tilde{w}(x,t) + \ell(x,x)\tilde{w}_x(x,t)\right) \\ =& i\beta\tilde{w}_{xxx}(x,t) - \int_0^x i\beta\ell_{xxx}(x,y)\tilde{w}(y,t)dy \\ &+ i\beta\left(-\ell_{xx}(x,x) - \frac{d}{dx}\ell_x(x,x) - \frac{d^2}{dx^2}\ell(x,x)\right)\tilde{w}(x,t) \\ &+ i\beta\left(-\ell_x(x,x) - 2\frac{d}{dx}\ell(x,x)\right)\tilde{w}_x(x,t) - i\beta\ell(x,x)\tilde{w}_{xx}(x,t). \end{aligned} \tag{A.12}$$

From (A.10) and (A.11) we also have

$$p_1(x)\tilde{u}_x(0,t) = p_1(x)\tilde{w}_x(0,t) \tag{A.13}$$

and

$$p_2(x)\tilde{u}_{xx}(0,t) = p_2(x)(\tilde{w}_{xx}(0,t) - \ell(0,0)\tilde{w}_x(0,t)). \tag{A.14}$$

Adding (A.9)-(A.14) side by side we obtain

$$
\begin{aligned}
& i\tilde{u}_t + i\beta\tilde{u}_{xxx} + \alpha\tilde{u}_{xx} + i\delta\tilde{u}_x + p_1(x)\tilde{u}_x(0,t) + p_2(x)\tilde{u}_{xx}(0,t) \\
=& i\tilde{w}_t + i\beta\tilde{w}_{xxx} + \alpha\tilde{w}_{xx} + i\delta\tilde{w}_x && \text{(A.15)} \\
& + \int_0^x \left(-i\beta(\ell_{xxx} + \ell_{yyy}) - \alpha(\ell_{xx} - \ell_{yy}) - i\delta(\ell_x + \ell_y) + ir\ell\right)(x,y)\tilde{w}(y,t)dy && \text{(A.16)} \\
=& \left(i\beta\left(\ell_{yy}(x,x) - \ell_{xx}(x,x) - \frac{d}{dx}\ell_x(x,x) - \frac{d^2}{dx^2}\ell(x,x)\right)\right. \\
& \left. +\alpha\left(-\ell_y(x,x) - \ell_x(x,x) - \frac{d}{dx}\ell(x,x)\right)\right)\tilde{w}(x,t) && \text{(A.17)} \\
& - i\beta\left(\ell_y(x,x) + \ell_x(x,x) + 2\frac{d}{dx}\ell(x,x)\right)\tilde{w}_x(x,t) && \text{(A.18)} \\
& + \left(i\beta\ell_y(x,0) - \alpha\ell(x,0) + p_1(x) - \ell(0,0)p_2(x)\right)\tilde{w}_x(0,t) && \text{(A.19)} \\
& + (-i\beta\ell(x,0) + p_2(x))\tilde{w}_{xx}(0,t). && \text{(A.20)}
\end{aligned}
$$

Note that, taking $x = L$ on (4.4) and using the boundary condition $\tilde{u}(L,t) = 0$, we must have $\ell(L,y) = 0$ in order to ensure $\tilde{w}(L,t) = 0$. On the other hand, using the relation $\frac{d}{dx}\ell(x,x) = \ell_x(x,x) + \ell_y(x,x)$, we get from (A.18) that

$$\frac{d}{dx}\ell(x,x) = 0, \tag{A.21}$$

which, thanks to $\ell(L,y) = 0$ implies

$$\ell(x,x) = 0.$$

Next, we differentiate $\frac{d}{dx}\ell(x,x) = \ell_x(x,x) + \ell_y(x,x)$ with respect to x and use $\frac{d}{dx}\ell(x,x) = 0$ to obtain $\ell_{yy}(x,x) = -2\ell_{xy}(x,x) - \ell_{xx}(x,x)$. Using this result in (A.17), we deduce that

$$\frac{d}{dx}\ell_x(x,x) = -\frac{r}{3\beta}$$

which, due to the implications $\ell(L, y) = 0 \Rightarrow \ell_y(L, y) = 0 \Rightarrow \ell_y(L, L) = 0 \Rightarrow \ell_x(L, L) = 0$, is equivalent to

$$\ell_x(x, x) = \frac{r}{3\beta}(L - x).$$

On the other hand we obtain from (A.19)-(A.20) that

$$\begin{aligned} p_1(x) &= -i\beta\ell_y(x, 0) + \alpha\ell(x, 0), \\ p_2(x) &= i\beta\ell(x, 0). \end{aligned} \tag{A.22}$$

Note that for $x = 0$ in (4.4), we have $\tilde{u}(0, t) = \tilde{w}(0, t) = 0$. For $x = L$ and thanks to $\ell(L, y) = 0$, we have $\tilde{u}(L, t) = \tilde{w}(L, t) = 0$. Also for $x = L$ on (A.10), we see that $\tilde{w}_x(L, t) = 0$ holds if $\ell_x(L, y) = 0$.

As a conclusion, the error model (4.3) is mapped to the target error model (4.5), if ℓ solves the boundary value problem (4.6) together with the observer gains defined as (A.22).

APPENDIX B

INVESTIGATION OF ZEROS OF THE CHARACTERISTIC EQUATION (3.98)

In this part, we show that the characteristic equation

$$s + \beta\lambda^3 + i\alpha\lambda^2 + \delta\lambda = 0 \tag{B.1}$$

has double or possibly triple roots only for finitely many values of s in the complex plane. Locations of those values of s are directly related with the sign of the quantity $(\alpha^2 + 3\beta\delta)$.

To this end, let us denote the roots of (B.1) by $\lambda_j = \lambda_j(s)$, $j = 1, 2, 3$, and assume that two roots, say $\lambda_2(s)$ and $\lambda_3(s)$ are equal for some $s \in \mathbb{C}$. Then, for such possible values of s, $\lambda_1(s)$ and $\lambda_2(s)$ satisfy

$$\lambda_1(s) + 2\lambda_2(s) = \frac{i\alpha}{\beta}, \tag{B.2}$$

$$2\lambda_1(s)\lambda_2(s) + \lambda_2^2(s) = \frac{\delta}{\beta}, \tag{B.3}$$

$$\lambda_1(s)\lambda_2^2(s) = -\frac{s}{\beta}. \tag{B.4}$$

Let us express the roots as $\lambda_1 = a + ib$ and $\lambda_2 = \lambda_3 = c + id$, where a, b, c, d are real valued functions of s. For simplicity, in the rest of this part, we drop s dependence of the functions. Now from the real and imaginary parts of (B.2)-(B.3), we have

$$a + 2c = 0, \tag{B.5}$$

$$b + 2d = \frac{\alpha}{\beta}, \tag{B.6}$$

$$2ac - 2bd + c^2 - d^2 = \frac{\delta}{\beta}, \tag{B.7}$$

$$ad + bc + cd = 0. \tag{B.8}$$

Using (B.5) in (B.8), we get $c(b-d) = 0$. So we have two cases.

(i) Let $b = d$. By (B.6), $b = d = \frac{\alpha}{3\beta}$. Substituting these into (B.7) yields

$$c(2a + c) = \frac{\alpha^2 + 3\beta\delta}{3\beta^2}.$$

Using (B.5), we get

$$a^2 = -\frac{4(\alpha^2 + 3\beta\delta)}{9\beta^2}, \quad c^2 = -\frac{\alpha^2 + 3\beta\delta}{9\beta^2}.$$

Now $\alpha^2 + 3\beta\delta > 0$ yields a contradiction. $\alpha^2 + 3\beta\delta = 0$ implies $\lambda_1 = \lambda_2 = \lambda_3 = \frac{i\alpha}{3\beta}$. For this case, we see from (B.4) that the only possible value for s is

$$s = s^0 := \frac{i\alpha^3}{27\beta^2}.$$

Now let $\alpha^2 + 3\beta\delta < 0$. Then we have

$$a_{1,2} = \mp\frac{2\sqrt{-(\alpha^2 + 3\beta\delta)}}{3\beta}, \quad c_{1,2} = \pm\frac{\sqrt{-(\alpha^2 + 3\beta\delta)}}{3\beta}.$$

Note that using (B.5) and $b = d$, we obtain from (B.4) there are two possible values of s, denoted by s_1^- and s_2^-, which are given by

$$\mathrm{Re}\,(s_1^-) = \frac{2}{27\beta^2}(-\alpha^2 - 3\beta\delta)^{3/2}, \quad \mathrm{Re}\,(s_2^-) = -\frac{2}{27\beta^2}(-\alpha^2 - 3\beta\delta)^{3/2}$$

and

$$\mathrm{Im}\,(s_1^-) = \mathrm{Im}\,(s_2^-) = -\frac{\alpha}{27\beta^2}(2\alpha^2 + 9\beta\delta).$$

(ii) Let $c = 0$. Then, by (B.5), $a = 0$. Using this, direct calculation from (B.6)-(B.7) yields

$$b_{1,2} = \frac{\alpha \mp 2\sqrt{\alpha^2 + 3\beta\delta}}{3\beta}, \quad d_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 + 3\beta\delta}}{3\beta}.$$

Observe that $\alpha^2 + 3\beta\delta < 0$ yields a contradiction and $\alpha^2 + 3\beta\delta = 0$ ends up with the triple root case investigated in (i). Now let $\alpha^2 + 3\beta\delta > 0$. From (B.4), we see that there are two possible values of s, denoted by s_1^+ and s_2^+, given by

$$s = s_1^+ := \frac{i}{27\beta^2}\left(\alpha^3 - 3\alpha(\alpha^2 + 3\beta\delta) + 2(\alpha^2 + 3\beta\delta)^{3/2}\right)$$

and

$$s = s_2^+ := \frac{i}{27\beta^2}\left(\alpha^3 - 3\alpha(\alpha^2 + 3\beta\delta) - 2(\alpha^2 + 3\beta\delta)^{3/2}\right).$$

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