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Analysis of the Logistic Growth Model With Taylor Matrix and Collocation Method

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Abstract. Early analysis of infectious diseases is very important in the spread of the disease. The main aim of this study is to make important predictions and inferences for Covid 19, which is the current epidemic disease, with mathematical modeling and numerical solution methods. So we deal with the logistic growth model. We obtain carrying capacity and growth rate with Turkey epidemic data. The obtained growth rate and carrying capacity is used in the Taylor collocation method. With this method, we estimate and making predictions close to reality with Maple. We also show the estimates made with the help of graphics and tables.

1. Introduction

People have made many studies based on science to explain the natural phenomena they encounter. This situation paved the way for the birth of many sciences such as mathematics, physics, engineering, astronomy, chemistry and biology. These sciences converged at one point over time, allowing the emergence of many fields in interaction with each other, generally under the title of applied mathematics. In these areas, mathematical models are used to represent the behavior of a system mathematically. Mathematical models are very important in that they provide information about the problem and allow predictions to be made. Most mathematical models are built with differential equations. Roughly, differential equations are equations that contain derivatives of an unknown function, which we call y(x) and which we want to determine from the equation. Differential equations are of great importance in engineering, because many physical laws and relations appear mathematically in the form of differential equations. In addition, Whenever a physical law involves a rate of change

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of a function, such as velocity, acceleratin, etc., it will be lead to a differential equation. For this reason differential equations occur frequently in physics and engineering. Analytical solutions of these differential equations can be made using various techniques, or many approximate solution methods are presented in cases where analytical solutions cannot be obtained or are difficult to obtain. In this study, the differential equation used in engineering, such as the speed of a falling object, the current in a series-connected electrical circuit and the cooling of the objects, radioactivity, radiocarbondating, Newton laws of cooling, etc., has been used in the modeling of epidemic disease [3]. Mathematical methods used in epidemiology get important contributions to the spread, occurrence, analysis and control of epidemics [5]. Mathematical modeling is appropriate in epidemic diseases and models are generally intented to explain the course of the disease [13]. Mathematical epidemiology is distinct from other epidemiological sciences. Because experiments and observations are not possible for the validity of most models in this field [8]. Verhulst has developed a linear mathematical model to the population problem, assuming the population is a continuous function of time. This mathematical model developed by Verhulst is called logistic differential equation. The logistic growth model is written in the form:

$$\begin{aligned} y^{'}(t) &= ry(t)\left(1-\frac{y(t)}{k}\right) \\ y(0) &= \lambda \end{aligned} \tag{1.1}$$

where $r > 0$ and $k > 0$ are respectively growth rate and carrying capacity.

According to the logistic growth model, growth increases rapidly at the beginning. However growth decreases as it approaches the carrying capacity. The logistic growth model can also be used in the growth of epidemics. As the carrying capacity in the growth of infectious deseases can be taken as the total number of people in the world, it can be defined and modeled as a population growth. Therefore, the logistic growth model can be used to observe the largest outbreak epidemic disease of the present time, Covid 19, and to predict its spread. Covid 19, which emerged in Wuhan, China in December 2019 and continues to spread by affecting the whole world in a very short time, is a virus that shows symptoms of respiratory infection. In order to observe the Covid 19 epidemic and make prediction about the epidemic, some scientists have applied the epidemic data to the logistic growth model. For more details, we refer the reader to [4, 6, 7, 9, 10], The aim of study is to apply the Taylor matrix and collocation method [11, 12], which has been used for differential equations in the literature, to the logistic growth model with Covid 19 datas of Turkey. We develop a method to determine the growth rate carrying capacity and the series truncate limit to give the best approach.

2. Method

In this study, we deal with the logistic growth model for modelling the total number of cases of Covid 19. We use the Taylor matrix and collocation method with Maple to obtain the numerical solutions of this model. Error analysis is performed to show the sensitivity and accuracy of the method presented.

2.1. **Carrying Capacity in Logistic Differential Equations.**

$$\begin{aligned} y'(t) &= ry(t)\left(1-\frac{1}{k}y(t)\right) \\ y(0) &= y_0 \end{aligned} \tag{2.1}$$

The solution of the logistic initial value problem is

$$y(t)=\frac{ky_0}{y_0+(k-y_0)e^{-rt}} \tag{2.2}$$

if we choose $y_0 = k$ in (2.2), we obtain $y = k$. This is called equilibrium population. One can see easily that $y' > 0$ for $0 < y_0 < k$, we have $y(t) < k$. On the other hand if $y_0 > k$, we have $y' < 0$ and $y(t) > k$. In all cases, the positive or negative numbers in the denominator will be less than y_0 in absolute value and will approach 0 when $t \to \infty$. Hence we obtain

$$\lim_{t\to\infty} y(t) = k\ .$$

It can be concluded that a population that satisfies the logistic differential equation given the initial condition does not increase indefinitely. That is, when $t \to \infty$ converges to the finite limit population. That is called the carrying capacity [1].

2.2. **Method of Solution For Logistic Model.** The logistic differential equation is defined as

$$y'(t)=ry(t)\left(1-\frac{1}{k}y(t)\right) \tag{2.3}$$

where $r > 0$ $k > 0$,

$$y(t)=\sum_{n=0}^{N} y_n t^n, \qquad y_n=\frac{y^n}{n!}, \qquad 0\le t\le m \tag{2.4}$$

with the initial condition Nth order. Our aim in this section is to obtain the numerical solution in terms of truncated Taylor series forms. We consider the approximate solution $y(t)$ and its derivative defined by truncated Taylor series (2.4). If we write (2.4) in the matrix form, we obtain

$$y(t)=S(t)Y \tag{2.5}$$

where

$$S(t)=\begin{bmatrix} 1 & t & t^2 & \cdots & t^N \end{bmatrix}_{1x(N+1)}$$

$$Y=\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix}_{(N+1)x1} .$$

If we write (2.4) derivative in the matrix form, we have

$$y'(t)=S(t)BY \tag{2.6}$$

where

$$B=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)x(N+1)} .$$

Similarly, we can write

$$y^2(t)=S(t)\,S^*(t)\,Y^* \tag{2.7}$$

where

$$S^*(t)=\begin{bmatrix} S(t) & 0 & \cdots & 0 \\ 0 & S(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S(t) \end{bmatrix}_{(N+1)x(N+1)^2}$$

$$Y^*=\begin{bmatrix} y_0Y \\ y_1Y \\ \vdots \\ y_NY \end{bmatrix}_{(N+1)^2x1} .$$

By substituting (2.5), (2.6) and (2.7) into Eq (2.3), we obtain the matrix equation as:

$$-rS(t)\,Y+S(t)\,BY+\frac{r}{k}S(t)\,S^*(t)\,Y^*=0\ . \tag{2.8}$$

Thus, the matrix representation of the logistic differential equation can be written as

$$M(t)\,Y+D(t)\,Y^*=G(t) \tag{2.9}$$

where

$$\begin{aligned} M(t) &= -rS(t)+S(t)\,B \\ D(t) &= \frac{r}{k}S(t)\,S^*(t) \\ G(t) &= 0\ . \end{aligned}$$

If the collocation points which are defined as

$$t_i=\frac{m}{N}i, \qquad i=0,1,...,N$$

substitude into Eq (2.9), we obtain

$$M(t_i)\,Y+D(t_i)\,Y^*=G(t_i)\,, \qquad i=0,1,...,N \tag{2.10}$$

or the matrix equation

$$M\overline{Y}+D\widetilde{Y}=G \tag{2.11}$$

where

$$M\begin{bmatrix} M(t_0) & 0 & \cdots & 0 \\ 0 & M(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M(t_N) \end{bmatrix}_{(N+1)x(N+1)^2}$$

$$\overline{Y} = \begin{bmatrix} Y \\ Y \\ \vdots \\ Y \end{bmatrix}_{(N+1)^2 x1}$$

$$D = \begin{bmatrix} D(t_0) & 0 & \cdots & 0 \\ 0 & D(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D(t_N) \end{bmatrix}_{(N+1)x(N+1)^3}$$

$$\widetilde{Y} = \begin{bmatrix} Y^* \\ Y^* \\ \vdots \\ Y^* \end{bmatrix}_{(N+1)^3 x1} .$$

The matrix form of the condition is written as

$$S(0)Y = \lambda \tag{2.12}$$

and we replaces row matrix (2.12) by any row of the matrix (2.11) and we get the new system depending on conditions. Thus, the logistic equation is transformed into a nonlinear equation under initial conditions. Then, if we calculate the unknown coefficients and replaced in truncated Taylor series in Nth order, a Taylor polynomial solution [2].

3. Numerical Application

In this section, we apply the Covid 19 data to the logistic growth model, which are received from Ministry of Health (Turkey). We obtain necessary results by analyzing the Taylor collocation method using Maple program.

3.1. **Turkey and Covid 19.**

3.1.1. *The Numbers of Death.* If we solve the logistic differential equation (2.2), we obtain

$$37 = \frac{k}{1 + (k-1)\, e^{-6r}} \tag{3.1}$$

for $t = 6$ and

$$356 = \frac{k}{1 + (k-1)\, e^{-16r}} \tag{3.2}$$

for $t = 16$. We calculate the growth rate and carrying capacity for the numbers of death as 0.6193081195 and 362.3979687 using Eqs (3.1) and (3.2) with maple program, respectively. We plot the Figure 1 using these values with maple program. Figure 1 gives for 480 days comparison of the approximate solution with the numbers of actual death in the following. We can see the rate of the numbers of death gradually increases.

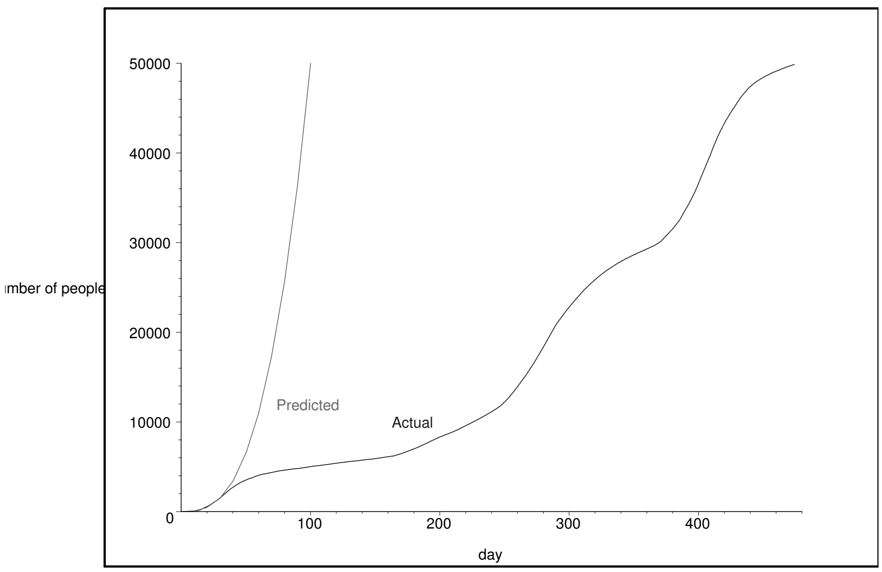

Figure 1. Time evolution of the numbers of death predicted by the logistic growth model with Taylor collocation method and the numbers of actual daeath, which number of healthy people living in Turkey from the date of the first case March 11, 2020 through July 3, 2021, obtained by Eq. (2.1). The parameters are r=0.6193081195 and k=362.3979687.

3.1.2. *The Numbers of Case.* If we solve the logistic differential equation (2.2), we obtain

$$98 = \frac{k}{1 + (k-1)\, e^{-6r}} \tag{3.3}$$

for $t = 6$ and

$$460916 = \frac{k}{1 + (k-1)\, e^{-258r}} \tag{3.4}$$

for $t = 258$. We calculate the growth rate and carrying capacity for the numbers of case as -3.548090934 and 1.979327005 using Eqs (3.3) and (3.4) with maple program, respectively. We plot the Figure 2 using these values with maple program. Figure 2 gives for 480 days comparison of the approximate solution with the numbers of actual case in the following. We can see the rate of the numbers of case gradually increases.

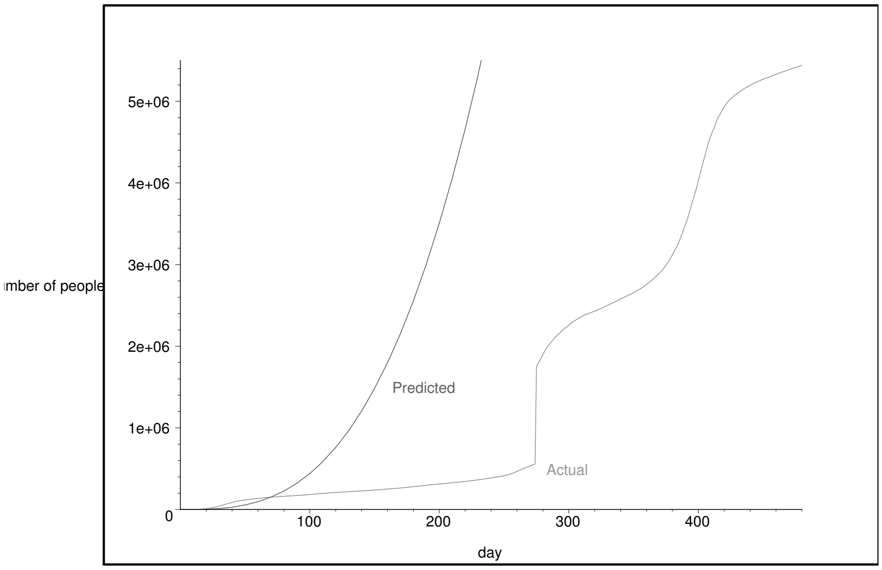

Figure 2. Time evolution of the numbers of case predicted by the logistic growth model with Taylor collocation method and the numbers of actual case, which number of healthy people living in Turkey from the date of the first case March 11, 2020 through July 3, 2021, obtained by Eq. (2.1). The parameters are r=−3.548090934 and k=1.979327005.

3.1.3. *The Numbers of Total Healing.* If we solve the logistic differential equation (2.2), we obtain

$$10453 = \frac{42k}{42 + (k - 42)\, e^{-22r}} \tag{3.5}$$

for $t = 22$ and

$$11976 = \frac{42k}{42 + (k - 42)\, e^{-23r}} \tag{3.6}$$

for $t = 23$. We calculate the growth rate and carrying capacity for the numbers of total healing as 0.4698047529 and 664420.9304 using Eqs (3.5) and (3.6) with maple program, respectively. We plot the Figure 3 using these values with maple program. Figure 3 gives for 480 days comparison of the approximate solution with the numbers of actual total healing in the following. We can see the rate of the numbers of total healing gradually increases.

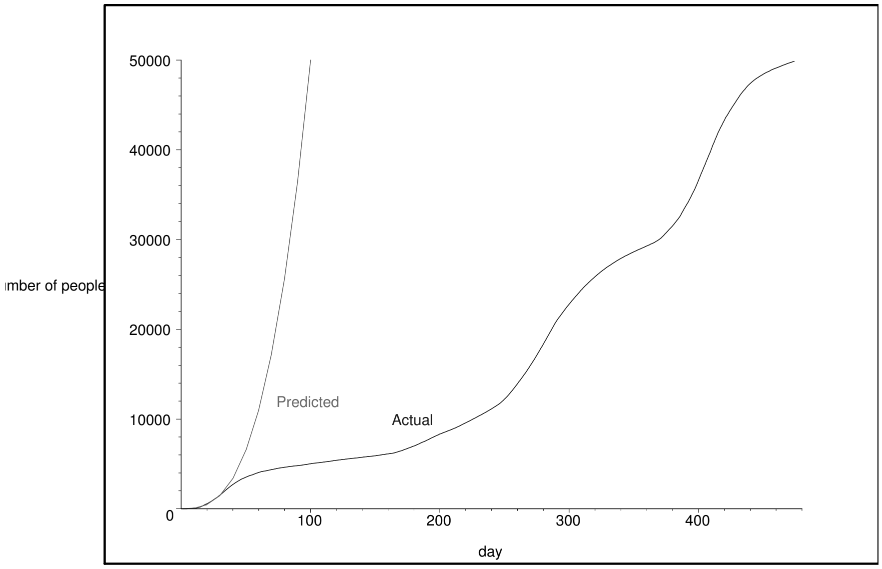

Figure 3. Time evolution of the numbers of total healing predicted by the logistic growth model with Taylor collocation method and the numbers of actual total healing, which number of healthy people living in Turkey from the date of the first case March 11, 2020 through July 3, 2021, obtained by Eq. (2.1). The parameters are r=0.4698047529 and k=664420.9304.

4. Conclusion

In this study, the numbers of death, the numbers of total healing and the numbers of case, associated with Covid 19 are handled in Turkey. Applying datas to the logistic growth model, we have tried to estimates that can be used to quantify the temporal evolution of Covid 19 in the country using Taylor matrix and collocation method. The numbers of death, the numbers of case and the numbers of total healing are solved for $N = 3$ in Figure 1 - Figure 3, respectively. It is possible to make predictions for the occurences, progression and analysis of Covid 19 with this method. Also, this process is very useful as the results can be obtained easily with a computer program.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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