FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND

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ABSTRACT

FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND

A unique variation of the inverse problem is the first type of Fredholm integral equation. To address the computing issue, inverse mathematical physics problems have been converted into the first type of Fredholm integral equation. We also use the Landweber iteration as an alternative to the well-known Tikhonov regularization technique , which has been shown to be most effective in solving ill-posed inverse problems. The Landweber iteration is a straightforward and effective technique that exhibits convergence towards the accurate solution given specific conditions. Consequently, it serves as a valuable instrument for resolving inverse problems across diverse domains, including signal processing and geophysics.

Following the examination of the properties of uniqueness and existence pertaining to solutions of integral equations of the first kind, the aforementioned equations are resolved through the utilization of the collocation method. The trapezoidal rule is widely utilized in numerical integration due to its straightforward implementation and computational efficiency. However, it may not be appropriate for integrals with significant oscillatory behavior. In instances of this nature, it may be imperative to employ more sophisticated numerical integration methods, such as Gaussian quadrature or adaptive quadrature, in order to attain precise outcomes. For weakly singular integrals that appear in formulations of integral equations of potential problems in domains with corners and edges, we provide n-points Gaussian quadrature procedures which are particularly useful in numerical integration problems where the integral is difficult to evaluate. The accuracy of the method depends on the number of points used in the procedure, with higher order rules providing more accurate results.

ÖZET

BİRİNCİ TÜR FREDHOLM İNTEGRAL DENKLEMLERİ

Fredholm bütünsel eşitliğinin ilk türü, ters sorunun özel bir türüdür. Matematik fiziğinin tersi sorunları, hesaplama sorunu çözmek için ilk tip Fredholm bütünsel eşitliğine çevrildi. Bütünsel eşitliğin çözümü için tahmin etmeye çalıştığımız proje yöntemini ve kolokasyon yöntemini kullanırız. Ayrıca Tikhonov düzenleme yöntemi iyi bilinir ve alternatif olarak, Landweber iterasyonunu kullanırız.

Trapezoidal kural, sürekli çekirdeklerle bütünleşen bütünsel operatörlerin sayısal entegrasyonu için kullanılırken, zayıf singular çekirgeler başka bir yöntem kullanılarak sayısal entegrasyonda kullanılır. Metodun doğruluğunu kontrol etmek için farklı test durumları dikkate alınır ve yaklaşım ve hata sonuçlarının sırası sayısal örneklerle gösterilir.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

Introduction

1.1 History of Fredholm Integral Equation of First Kind

" Some mathematicians still have a kind of fear whenever they encounter a Fredholm integral equation of the first kind. " ~ $Francesco\,Tricomi$

In the year 1825, the Italian mathematician Abel made a significant contribution to the field by formulating an integral equation related to the well-known tautochrone problem (see references [1], [2] and [3]). Integral equations are a class of equations that involve an unknown function $\varphi(x)$ to be evaluated within the integral sign. The investigation of integral equations is widely recognized as a highly beneficial area of study in both pure and applied mathematics. This approach demonstrates significant utility in addressing a wide range of physical challenges. Numerous initial and boundary value problems pertaining to ordinary differential equations (ODE) and partial differential equations (PDE) have the potential to be reformulated as problems that involve solutions in the form of approximate integral equations.

The most significant and obvious of the three criteria for determining whether a problem is ill-posed is instability under data perturbations. One must therefore take extra precautions because it is very likely that traditional numerical methods will fail when attempting to solve ill-posed problems.

The individual known as **E. Picard** was obligated to furnish substantiation for his essential and complete requirement, now acknowledged as the "**Picard's Criterion**," concerning the existence of solutions to a Fredholm integral equation of the first kind. The individual in question neglected to address the matter of approximate solutions, as their attention was solely directed towards the existence of solutions. The analysis conducted by Picard on Fredholm integral equations of the first kind demonstrates that these equations exhibit characteristics commonly associated with ill-posed problems. The notion of an ill-posed problem, or more precisely, that of a well-posed problem, was first introduced by Hadamard [4] over a century ago. Tikhonov [2] and Phillips [5] each put forth an independent proposition for a regularization technique. The regularization technique involves the substitution of ill-posed problems with well-posed ones. The theoretical contribution made by Tikhonov in 1943 had a notable impact on the stability of solutions pertaining to inverse problems. The author demonstrated that ensuring the stability of the solution to the inverse problem, which involves determining the spatial arrangement of mass beneath a surface that produces a particular gravitational potential on said surface, can be achieved by restricting the permissible mass distributions to a compact subset of a specified function space. This concept is further explained in the scholarly work cited as reference [6].

The text that follows is organized as follows: In Chapter 2, we will provide a general overview of linearly ill-posed problems and the development of appropriate regularization techniques using spectral theory. Additionally, it provides the IFK solution's convergence and accuracy criteria. One such technique is the Landweber iteration, which is introduced and in-depth examined in Chapter 6 using the created framework.

The section on analytical solutions is in Chapter 3. The Tikhonov method and regularization are then discussed in Chapters 4 and 5. Chapter 7 discusses the application of the projection method and the collocation method while also showcasing a few theoretical algorithms. The Gauss-Quadrature for singular kernels was the last point we made. The final section of Chapter 9 provides a succinct summary of our findings. The numerical projection method, Tikhonov regularization method, Landweber iteration method, and Gaussian quadrature approach are included in addition to other numerical solutions to the equation.

CHAPTER 2

Fredholm Integral Equations of the First Kind

Integral equations play a crucial role in diverse fields of science and engineering. Fredholm integral equations are widely recognized as highly valuable in various disciplines, including control systems, economics, electrical engineering, medicine, and more (see [7]). The first kind Fredholm integral equation, denoted as equation (2.1), is expressed as follows:

$$\int_a^b K(x,t)\varphi(t)dt = f(x), \quad a \leq x \leq b. \tag{2.1}$$

In this equation, the functions $f(x)$ and $K(x,t)$ are given and the function $\varphi(x)$ is the unknown quantity that needs to be determined. In general, this kind of integral equation is inverse problem for a given kernel K and driving term f, (see Refs. [8],[3],and [2]).

We consider the integral equations (2.1), as an operator equations of the first kind

$$A\varphi = f \tag{2.2}$$

in appropriate normed function spaces. The notation $A : X \to Y$ denotes a function that maps elements from a set X to a set Y in a one-to-one manner. In other words, for every element $\varphi \in X$, the function A assigns a distinct element $A\varphi \in Y$ [2].

2.1 Ill-posed Problems

According to Hadamard, in order for a mathematical model to be considered "*properly-posed*" or "*well-posed*" in the context of a physical problem, it must satisfy the following three properties [4]:

1. There is a way to solve the issue (Existences).
2. The problem has at most of one solution (Uniqueness).
3. The solution exhibits continuous dependence on the data (Stability).

Definition 2.1 *Consider an operator $A : U \longrightarrow V$ that maps a subset U of a normed space X to a subset V of a normed space Y. The equation*

$$A\varphi = f, \tag{2.3}$$

is called **well-posed** *or* **properly posed** *if $A : U \longrightarrow V$ is bijective and the inverse operator $A^{-1} : V \longrightarrow U$ is continuous. Otherwise, the equation is called* **ill-posed** *or* **improperly posed** *[4].*

By previous definition, there are three types of ill-posedness;

- The equation (2.3) does not have a solution for all $f \in V$ if the mapping A is not surjective, indicating the nonexistence of a solution.
- The equation represented by equation (2.3) may possess multiple solutions in the event that the function A fails to exhibit injectivity (*Non-uniqueness*).
- The solution φ of equation (2.3) exhibits a lack of continuity with respect to the data f in the case where the existence of the operator $A^{-1} : V \longrightarrow U$ is present but its continuity is not guaranteed (*Instability*).

The Fredholm integral equations of the first kind exhibit inherent ill-posedness. The management and condensation of the stability condition pose significant challenges, as a violation of this condition implies that even minor changes in the data can result in highly significant deviations in the solution.

2.2 Compact Self-Adjoint Operators

In this paper, X and Y will always denote Hilbert spaces (see refs. [8], [2],[4], and [9]). In quantum mechanics and functional analysis, Hilbert spaces, a particular kind of mathematical space, are frequently employed. They are distinguished by having an inner product that enables the definition of vector lengths and angles.

Definition 2.2 *[4] A linear mapping A from a Banach space X to a Banach space Y is considered compact if the image of any bounded subset of X under A is a relatively compact set, meaning that its closure is compact.*

Theorem 2.3 *[4] Linear equations $A\varphi = f$ with compact operators $A : X \longrightarrow Y$, where X and Y are normed spaces and $\dim X = \infty$ are always ill-posed .*

Theorem 2.4 *[4] Let us consider two Hilbert spaces, which we will denote as X and Y. We will now introduce a bounded linear operator, denoted as A, that maps from the Hilbert space X to the Hilbert space Y. There exists a linear operator $A^* : Y \longrightarrow X$ that is uniquely determined and possesses the property.*

$$(A\varphi, \psi) = (\varphi, A^*\psi),$$

for all $\varphi \in X$ and $\psi \in Y$, i.e., A and the adjoint of A is defined with respect to the dual systems (X, X) and (Y, Y), which are generated by the scalar products on X and Y respectively. The operator A^* is bounded and

$$\|A\| = \|A^*\|.$$

(Again we use the same symbol $(\cdot, \cdot)$ for the scalar products on X and Y.).

Theorem 2.5 *[4] In the context of a bounded linear operator, it is observed that there exists a certain property*

$$A(X)^\perp = N(A^*) \quad and \quad N(A^*)^\perp = A(X).$$

Note 2.6 *If operator A consists of a finite rank and $f_n \longrightarrow f$, then operator $Kf_n \longrightarrow Kf$. Compact is a characteristic of an operator.*

A linear operator $A : X \longrightarrow X$ that maps a Hilbert space X onto itself is referred to as self-adjoint if it satisfies the condition $A = A^*$, where A^* denotes the adjoint of A, and φ and ψ are arbitrary elements of X. It is crucial to note that, within the framework of a self-adjoint operator, the scalar product $(A\varphi, \varphi)$ is an actual values quantity that is valid for all elements φ belonging to the set X (see refs. [2],[4]), since

$$\overline{(A\varphi,\varphi)} = (\varphi, A\varphi) = (A\varphi,\varphi).$$

In facts, A^* is exists, linear and unique. If A is bounded, then A^* is bounded, and if A is compact, A^* is compact.

Theorem 2.7 *[4] Consider a Hilbert space X and a self-adjoint compact operator $A : X \longrightarrow X$ where A is non-zero. Subsequently, it can be deduced that all eigenvalues of matrix A possess the characteristic of being real numbers. Matrix A possesses at least one eigenvalue that is distinct from zero, and it can have at most a countable collection of eigenvalues that converge solely at zero. According to the cited reference, it is stated that all eigenspaces $N(\lambda I - A)$ corresponding to nonzero eigenvalues λ possess finite dimension. Additionally, the eigenspaces associated with distinct eigenvalues are orthogonal to each other. Assume the sequence (λ_n) of the nonzero eigenvalues to be ordered such that*

$$|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \dots,$$

and denote by $P_n : X \longrightarrow N(\lambda I - A)$ the orthogonal projection operator onto the eigenspace for the eigenvalue (λ_n). Then

$$A = \sum_{n=1}^{\infty} \lambda_n P_n, \tag{2.4}$$

in the sense of norm convergence. The orthogonal projection operator onto the null-space $N(A)$ is denoted as $Q : X \longrightarrow N(A)$ [2]. Then

$$\varphi = \sum_{n=1}^{\infty} P_n\varphi + Q\varphi \tag{2.5}$$

for all $\varphi \in X$.

CHAPTER 3

Analytical Solution

Analytical solutions to problems are possible. It is possible to change a first-kind equation into a second-kind equation. Then, using the Adomian decomposition method (see [10]), we can apply the currently available techniques of the second kind to the modified problem. Tikhonov and Philips independently developed the regularization technique.

$$f(x) = \int_a^b K(x,t)u(t)dt, \quad x \in [a,b]. \tag{3.1}$$

where $[a,b]$ is a closed and bounded region, a and b are constants, $K(x,t)$ is kernel, $f(x)$ is the data function and $u(x)$ is the unknown function that will be determined. The approximation of Fredholm integral equations of first kind;

$$\alpha u_\alpha(x) = f(x) - \int_a^b K(x,t)u_\alpha(t)dt. \tag{3.2}$$

where α is a small positive parameter which is called the regularization parameter. The equation (3.2) can be written

$$u_\alpha(x) = \frac{1}{\alpha}f(x) - \frac{1}{\alpha}\int_a^b K(x,t)u_\alpha(t)dt. \tag{3.3}$$

Hence, Tikhonov and Philips demonstrated that the solution u_α of Equation (3.2) converges to the solution $u_\alpha(x)$ of Equation (3.1) as the parameter α approaches zero, as referenced in [10]. It was shown that

$$u(x) = \lim_{\alpha \to 0} u_\alpha(x). \tag{3.4}$$

It is of greatest significance to acknowledge that the initial form of the Fredholm integral equation represents a problem that is improperly posed. An ill-posed problem may not have a solution, or if it does, it may not be an unique solution.

Proof We can see on the example 3.1;

Example 3.1

$$\frac{\sin(x)}{2} = \frac{2}{\pi}\int_0^{\pi/2} \sin(x)\sin(t)u(t)dt, \quad 0 < x \leq \pi/2. \tag{3.5}$$

The regularization method is employed to solve the Fredholm integral equation of the first kind

$$\alpha u_\alpha(x) = \frac{\sin(x)}{2} - \frac{2}{\pi}\int_0^{\pi/2} \sin(x)\sin(t)u_\alpha(t)dt. \tag{3.6}$$

Such that,

$$u_\alpha(x) = \frac{1}{2\alpha}\sin(x) - \frac{2}{\pi\alpha}\int_0^{\pi/2} \sin(x)\sin(t)u_\alpha(t)dt. \tag{3.7}$$

By the Adomain decomposition method,

$$u_\alpha(x) = \sum_{n=0}^{\infty} u_{\alpha_n}(x). \tag{3.8}$$

The recurrence relation,

$$u_{\alpha_0}(x) = \frac{1}{2\alpha}\sin(x),$$
$$u_{\alpha_{k+1}}(x) = -\frac{2}{\pi\alpha}\int_0^{\pi/2} \sin(x)\sin(t)u_{\alpha_k}(t)dt, \quad k \geq 0.$$

Consequently, the components are given

$$
\begin{aligned}
u_{\alpha_1}(x) &= -\frac{2}{\pi\alpha}\int_0^{\pi/2} \sin(x)\sin(t)u_{\alpha_0}(t)dt,\\
&= -\frac{2}{\pi\alpha}\sin(x)\int_0^{\pi/2} \sin(t)\frac{1}{2\alpha}\sin(t)dt,\\
&= -\frac{1}{\pi\alpha^2}\sin(x)\underbrace{\int_0^{\pi/2} \sin^2(t)dt}_{=\frac{\pi}{4}},\\
&= -\frac{1}{\pi\alpha^2}.\frac{\pi}{4}.\sin(x) = -\frac{1}{4\alpha^2}\sin(x).
\end{aligned}
$$

$$
\begin{aligned}
u_{\alpha_2}(x) &= -\frac{2}{\pi\alpha}\int_0^{\pi/2} \sin(x)\sin(t)u_{\alpha_1}(t)dt,\\
&= -\frac{2}{\pi\alpha}\sin(x)\int_0^{\pi/2} \sin(t)\left(-\frac{1}{4\alpha^2}\sin(t)\right)dt,\\
&= \frac{1}{2\pi\alpha^3}\sin(x)\underbrace{\int_0^{\pi/2} \sin^2(t)dt}_{=\frac{\pi}{4}},\\
&= \frac{1}{2\pi\alpha^3}\frac{\pi}{4}\sin(x) = \frac{1}{8\alpha^3}\sin(x).
\end{aligned}
$$

$$
\begin{aligned}
u_{\alpha_3}(x) &= -\frac{2}{\pi\alpha}\int_0^{\pi/2} \sin(x)\sin(t)u_{\alpha_2}(t)dt,\\
&= -\frac{2}{\pi\alpha}\sin(x)\int_0^{\pi/2} \sin(t)\left(\frac{1}{8\alpha^3}\sin(t)\right)dt,\\
&= -\frac{1}{4\pi\alpha^4}\sin(x)\underbrace{\int_0^{\pi/2} \sin^2(t)dt}_{=\frac{\pi}{4}},\\
&= -\frac{1}{4\pi\alpha^4}\frac{\pi}{4}\sin(x) = -\frac{1}{16\alpha^4}sin(x).
\end{aligned}
$$

This result gives the approximate solution when it is substituted into (3.8)

$$
\begin{aligned}
u_\alpha(x) &= \frac{1}{2\alpha}\sin(x)\underbrace{\left(1 - \frac{1}{2\alpha} + \frac{1}{4\alpha^2} - \frac{1}{8\alpha^3} + \cdots\right)}_{=\frac{2\alpha}{1}} \\
u_\alpha(x) &= \frac{1}{2\alpha}\sin(x).2\alpha = \sin(x).
\end{aligned}
$$

Hence,

$$
u(x) = \lim_{\alpha \to 0} u_\alpha(x) = \sin(x).
$$

□

It is showed that we get the solution of the equation.

Then, we check the uniqueness ;

$$
\begin{aligned}
\frac{2}{\pi}\int_0^{\pi/2} \sin(x)\sin(t)\varphi(t)dt &= \frac{\sin(x)}{2} \\
\sin(x)\int_0^{\pi/2} \sin(t)\varphi(t)dt &= \frac{\pi\sin(x)}{4} \\
\int_0^{\pi/2} \sin(t)\varphi(t)dt &= \frac{\pi}{4} \\
\int_0^{\pi/2} \sin(t)\frac{\pi}{4}dt = -\frac{\pi}{4}\cos(t)\Big|_0^{\pi/2} &= \frac{\pi}{4} \\
\int_0^{\pi/2} \sin(t)\sin(t)dt &= \frac{\pi}{4}
\end{aligned}
$$

Indeed, it is not a unique solution because the operator is not injective. By definition of ill-posedness (2.1), the operator must be injective. Hence, we cannot solve the integral equations analytically.

CHAPTER 4

Regularization of Ill-posed Problems

" Every restriction corresponds to a law of nature, a regularization of the universe. " ~ Carl Sagan

The utilization of the regularization technique enables the transformation of an ill-posed Fredholm integral equation of the first kind into a problem that is well-posed.

$$\int_a^b K(x,t)\varphi(t)dt = f(x), \quad a \leq x \leq b. \tag{4.1}$$

where $f(x)$ and $K(x,t)$ are known functions and $\varphi(x)$ is the unknown function to be determined. As we did in the section before, this equation will be represented by an abstract equation of the form

$$A\varphi = f, \tag{4.2}$$

where A is a linear compact operator. The comprehension of compact linear operators on Hilbert space is significantly augmented by the theory of singular functions, which was formulated by *E.Schmidt* [11]. Consequently, there is a prevalent endeavor to obtain a particular all-encompassing resolution, commonly acknowledged as the solution with the minimum norm least squares. The application of the Tikhonov regularization method possesses the potential to convert a problem that is initially stated with flaws into one that is approximately well-defined.

The general solution $\varphi = A^{-1}f$ which satisfies the normal equations

$$A^*A\varphi = A^*f, \tag{4.3}$$

where A^* is the adjoint of A. According to ***Schmidt's theory*** [2], the compact self-adjoint operator A^*A is known to have eigenvalues that are non-negative.

4.1 Singular Value Decomposition

The utilization of the mathematical technique known as the *singular value decomposition*(SVD) contributes to the advancement of our comprehension of the process of smoothing and the existence of solutions to Fredholm integral equations of the first kind ([2]).

Definition 4.1 *[2] Let us consider the Hilbert spaces X and Y, and denote by $A : X \to Y$ a compact linear operator. In addition, let $A^* : Y \to X$ denote the adjoint of A. The non-negative square roots of the eigenvalues of the non-negative self-adjoint compact operator $A^*A : X \to X$ are commonly known as the singular values.*

Theorem 4.2 *[2] The sequence (μ_n) represents the nonzero singular values of the compact linear operator A (where A is not equal to zero), and these values are repeated based on their multiplicity. In other words, the repetition is determined by the dimension of the null-spaces $N(\mu_n^2 I - A^*A)$. Then there exist orthonormal sequences (ϕ_n) in X and (g_n) in Y such that*

$$A\varphi_n = \mu_n g_n, \qquad A^* g_n = \mu_n \varphi_n. \tag{4.4}$$

for all $n \in \mathbb{N}$. For each $\varphi \in X$ we have the singular value decomposition

$$\varphi = \sum_{n=1}^{\infty} (\varphi, \varphi_n)\varphi_n + Q\varphi. \tag{4.5}$$

with the orthogonal projection operator $Q : X \to N(A)$ and

$$A\varphi = \sum_{n=1}^{\infty} \mu_n (\varphi, \varphi_n) g_n \tag{4.6}$$

A system denoted as (μ_n, φ_n, g_n), where n belongs to the set of natural numbers, and possessing the aforementioned properties, is referred to as a singular system of matrix A. When the quantity of singular values is limited, the series denoted by equations (4.5) *and (4.6) are transformed into finite sums. It is important to acknowledge that when considering an injective operator A, the orthonormal system* $\{\varphi_n : n \in \mathbb{N}\}$ *derived from the singular system forms a comprehensive system within the space X.*

Note 4.3 *Let* (φ_n) *denote an orthonormal sequence of the eigenelements of* A^*A*, i.e.,* $A^*A\varphi_n = \mu_n^2\varphi_n$.

Theorem 4.4 (Picard) *[2] Let* $A : X \to Y$ *be a compact linear operator with singular system* (μ_n, φ_n, g_n)*. The equation of the first kind*

$$A\varphi = f \tag{4.7}$$

is solvable if and only if f belongs to the orthogonal complement $N(A^*)^\perp$ *and satisfies*

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} |(f, g_n)|^2 < \infty. \tag{4.8}$$

In this case a solution is given by

$$\varphi = \sum_{n=1}^{\infty} \frac{1}{\mu_n} (f, g_n)\varphi_n. \tag{4.9}$$

Picard's theorem demonstrates the ill-posed nature of the equation $A\varphi = f$.

The function f on the right-hand side is required to satisfy Equation (4.8). Picard (2.1) employed the principles of singular system theory to formulate his renowned criterion for the existence of solutions to equations of the aforementioned form (4.1).

Note 4.5 *Equivalent condition:* $f \in range(A)$.

Note 4.6 *According to Picard's theorem, the existence of a solution for equation (4.1) can be guaranteed when a compact linear operator A is involved, provided that the function f belongs to the range of A and satisfies condition (4.8), which requires the singular components of f to decay at a rate that is sufficient. This phenomenon has been identified and labeled as "**Picard's Criterion**"([2]).*

4.2 Regularization Schemes

Definition 4.7 *[2] Let us consider two normed spaces, which we will denote as X and Y. We are given a bounded linear operator $A : X \to Y$ that is injective. Then a family of bounded linear operators $R_\alpha : Y \to X$, $\alpha > 0$, with the property of point-wise convergence*

$$\lim_{\alpha\to 0} R_\alpha A\varphi = \varphi, \quad \varphi \in X$$

is called a regularization scheme for the operator A. The term α is commonly referred to as the regularization parameter.

The efficacy of the Singular Value Decomposition (SVD) in the analysis of first-kind Fredholm integral equations is evident.

According to Picard's Theorem (4.4), the ill-posedness in a first-kind equation with a compact operator can be attributed to the manner in which the eigenvalues behave as the parameters $\mu \to 0$ and $n \to \infty$.

Theorem 4.8 *[2] Consider a compact linear operator $A : X \to Y$ that is injective. Let (μ_n, φ_n, g_n) be the singular system of A, where $n \in \mathbb{N}$. Let $q : (0,\infty) \times (0, \|A\|) \to \mathbb{R}$ be a bounded function. We assume that for each $\alpha > 0$, there exists a positive constant $c(\alpha)$ such that*

$$|q(\alpha,\mu)| \le c(\alpha)\mu, \qquad 0 < \mu \le \|A\|. \tag{4.10}$$

and

$$\lim_{\alpha\to 0} q(\alpha,\mu) = 1, \qquad 0 < \mu \le \|A\|. \tag{4.11}$$

The bounded linear operators $R_\alpha : Y \to X$, where $\alpha > 0$, are defined by

$$R_\alpha f := \sum_{n=1}^{\infty} \frac{1}{\mu_n} q(\alpha,\mu)(f, g_n)\varphi_n, \qquad f \in Y. \tag{4.12}$$

describe a regularization scheme with

$$\|R_\alpha\| \leq c(\alpha). \tag{4.13}$$

We see that "$1/\mu_n$" in that part of theorem 4.8, we have a problem that means $\mu \to 0$, $n \to \infty$. We describe some classical regularization schemes by choosing *the damping(filter function)* **q** appropriately.

Theorem 4.9 *[2] Let $A : X \to Y$ be a compact linear operator. Then for each $\alpha > 0$ the operator $\alpha I + A^*A : X \to X$ has a bounded inverse. Furthermore, if A is injective then*

$$R_\alpha := (\alpha I + A^*A)^{-1}A^* \tag{4.14}$$

describes a regularization scheme with $\|R_\alpha\| \leq 1/2\sqrt{\alpha}$.

Note 4.10 *It is concluded that the operator A is injective. Let (μ_n, φ_n, g_n), where n belongs to the set of natural numbers, be a singular system for the operator A. Then the unique solution φ_α of*

$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f \tag{4.15}$$

can be written in the form

$$\varphi_\alpha = \sum_{n=1}^{\infty} \frac{\mu_n}{\alpha + \mu_n^2}(f, g_n)\varphi_n. \tag{4.16}$$

The equation $A^*A\varphi_n = \mu_n^2\varphi_n$ is indeed utilized and the application of SVD is demonstrated in equation (4.6) to A^*f, we find

$$(\alpha I + A^*A)\varphi_\alpha = \sum_{n=1}^{\infty} \mu_n(f, g_n)\varphi_n = A^*f. \tag{4.17}$$

We try to cut off the "*filter factors*", which is singular value. It turns on the regularization method.

Note 4.11 *In conclusion, the Picard theorem showed that the equation (4.1), with a compact linear operator A, exhibits a solution for a given $f \in N(A^*)^\perp$ if and only if the singular components f decay sufficiently rapidly for the condition (4.8) to hold.*

In the context of discrete mathematics, specifically involving finite-dimensional matrices, it is postulated that the matrix under consideration is either square or possesses a greater number of rows than columns. Then, for any matrix $A \in \mathbb{R}$, mxn with $m \geqslant n$, the SVD takes the form

$$A = U\Sigma V^T = \sum_{i=1}^{n} u_i \sigma_i v_i^T.$$

Here, $\Sigma \in \mathbb{R}^{nxn}$ is a diagonal matrix with the singular values, satisfying

$$\Sigma = diag(\sigma_1, \ldots, \sigma_n), \qquad \sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n \geqslant 0.$$

The matrices $U \in \mathbb{R}^{mxn}$ and $V \in \mathbb{R}^{nxn}$ are composed of the left and right singular vectors

$$U = (u_1, \ldots, u_n), \qquad V = (v_1, \ldots, v_n).$$

and both matrices have orthonormal columns:

$$u_i^T u_j = v_i^T v_j = \Sigma_{ij} \qquad i, j = 1, \ldots, n.$$

The expression of the inverse of matrix A, if it exists, can be demonstrated in a clear and direct manner;

$$A^{-1} = V\Sigma^{-1}U^T$$

Thus we have $\|A^{-1}\| = \Sigma^{-1}$, and it follows immediately

$$condition\ number\ using\ the\ 2-norm : cond_2(A) = \|A\|_2\|A^{-1}\|_2 = \sigma_1/\sigma_n.$$

The aforementioned expression remains valid in the case where matrix A is rectangular and possesses full rank. The computational complexity of all the singular value decomposition algorithms is $O(mn^2)$ floating point operations (flops), under the condition that $m \geqslant n$.

CHAPTER 5

The Tikhonov Regularization

As we did in the section previously, we will represent this equation as an abstract equation of the form

$$A\varphi = f. \tag{5.1}$$

where A be a compact operator that maps from the Hilbert space X to the Hilbert space Y.

We have shown that there is typically more than one possible solution to this equation. To solve this problem, we are looking for a certain general solution, sometimes known as the minimal norm least squares solution. In other words, our program asks that the solution to an imperfectly given issue be exchanged for an approximately sufficient solution.

The $\varphi = A^{-1}f$ is general solution. Hence, it satisfies the normal equations

$$(A^*A)\varphi = A^*f, \tag{5.2}$$

where A^* is the adjoint of A. The eigenvalues of the self-adjoint compact operator represented by A^*A are characterized with non-negativity. Moreover, it can be deduced that the eigenvalues of the operator $A^*A + \alpha I$ on X, with I representing the identity operator [4], are invariably positive for all positive values of α. The issue of determining a solution for the equation is characterized by a bounded inverse in relation to the operator $(A * A + \alpha I)$;

$$(A^*A + \alpha I)\varphi = A^*f. \tag{5.3}$$

turns on approximately well-posed problem. The solution $\varphi(\alpha)$ of the Tikhonov regularization problem, as defined in [2], is referred to as the Tikhonov regularized solution. This solution is obtained according to the theorem.

Theorem 5.1 *[2] Let $A : X \to Y$ be a bounded linear operator and let $\alpha > 0$. For every* *element f in the set Y, there exists a single and distinct element φ_α in the set X such that*

$$\|A\varphi_\alpha - f\|^2 = \inf_{\varphi \in X}\{\|A\varphi_\alpha - f\|^2 + \alpha\|\varphi\|^2\}. \tag{5.4}$$

The minimizer φ_α can be obtained by finding the unique solution of the equation

$$(\alpha I + A^* A)\varphi_\alpha = A^* f, \tag{5.5}$$

and depends continuously on f.

The parameter α, which is greater than zero, in equation (5.4) functions as a regularization parameter. Its purpose is to control the relative impact of the accuracy term and the regularization term in the equation. Let us examine the sequence (μ_n, φ_n, g_n) for $n = 1, 2, \ldots$, which functions as an unique system for the operator A.

Theorem 5.2 *[2] Consider a compact linear operator $A : X \to Y$ that is injective. Then for $f \in A(X)$ the condition $f \in AA^*A(X)(f = AA^*Ag\ for\ some\ g \in X)$ is necessary and sufficient for*

$$\|\varphi_\alpha - A^{-1}f\| = O(\alpha), \quad \alpha \to 0. \tag{5.6}$$

The discrete ill-posed problem is commonly encountered when individual values of matrix A exhibit a gradual convergence towards zero, resulting in an ill-conditioned matrix. This phenomenon is particularly evident in the right-hand side of equation (5.4).

Note 5.3 *The equation (5.4) is equal to*

$$\min_{\varphi}\left\{\|A\varphi - f\|_2^2 + \alpha^2\,\|\varphi\|_2^2\right\}.$$

The condition number of $(\alpha I + A^*A)$ decreases as the size of the matrix A increases. The Tikhonov regularization of the equation corresponds to the regularization operators;

Theorem 5.4 *Let $A : X \to Y$ be an injective bounded linear operator. Then*

$$R_\alpha := (\alpha I + A^* A)^{-1} A^* \tag{5.7}$$

describes a regularization scheme with

$$\|R_\alpha\| \leq \frac{\|A\|}{\alpha}. \tag{5.8}$$

The equation (5.7) yields the Tikhonov approximation to $A^* f$, which corresponds to the solution with the minimum norm of the normal equations.

Hence, it can be inferred that the vectors denoted by R_α are actual approximations to $A^* f$ with respect to the given equation

$$R_\alpha \to A^* f \quad as \qquad\qquad \alpha \to 0.$$

Furthermore, it can be observed that the Tikhonov approximation, represented by the symbol R_α, exhibits continuous dependence on the function f for any fixed positive value of α. This can be attributed to the boundedness of the operator $(\alpha I + A^* A)^{-1} A^*$ for every fixed positive value of α.

Note 5.5 *In summary, the application of the Tikhonov regularization technique allows for the approximation of an ill-posed problem through the formulation of a set of interconnected well-posed problems.*

The initial section of our study demonstrated that the function f specified in equation (4.2) is commonly a quantity that is measured or observed. Consequently, the true f is not readily accessible in practical applications. The optimal outcome that can be expected is the derivation of an approximation f^δ of f that meets the condition:

$$\frac{\|f^\delta - f\|}{\|f\|} = \delta,$$

where δ represents a predetermined limit on the error of measurement. Rather than constructing a regularized approximation using the actual function f, we are constrained to utilize the available data f^δ to construct the regularized approximations.

Numerical solutions are extremely sensitive to noise and perturbations, and certain types of perturbation are caused by rounding error because the problem (5.1) is ill-posed. The regularization method should be used to solve an ill posed problem with noisy data. We include the solutions to linear problems as described by

$$A\varphi^\delta = f^\delta \tag{5.9}$$

where $A \in \mathbb{R}^{m\text{x}n}$ with $m \geq n$ and f^δ is noisy data and known, but represents

$$f^\delta = f + \delta * \eta * \frac{\|f\|}{\|\eta\|}; \tag{5.10}$$

where η is a random variable, δ - noise level.

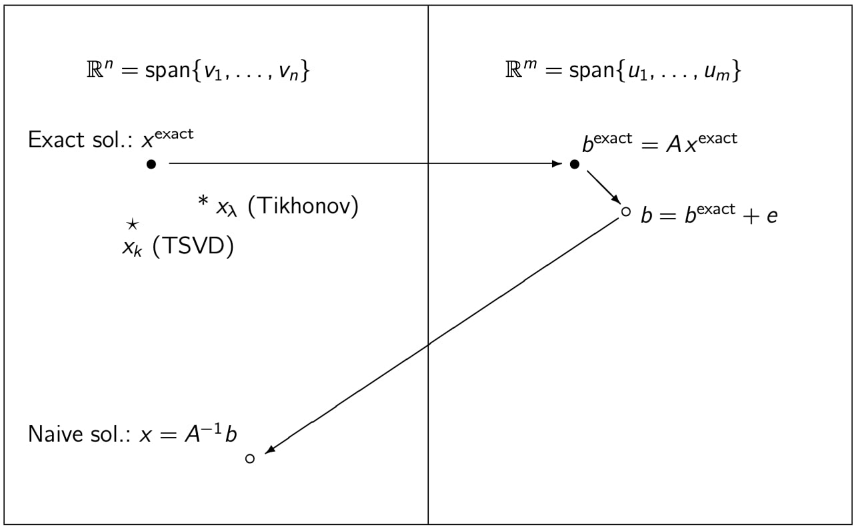

Figure 5.1: The necessity for regularization can be demonstrated through illustration.

The coefficient matrices of discrete ill-posed problems typically have a very high condition number.This implies that the simplistic approach is highly susceptible to any alteration in the right-hand side, which serves as a representation of the errors in the data.

Then classical perturbation theory leads to the bound

$$\frac{\|\varphi_{exact} - \varphi\|_2}{\|\varphi_{exact}\|_2} \leq cond(A) \frac{\|f^\delta\|_2}{\|f_{exact}\|_2} \tag{5.11}$$

Given the large condition number of matrix A, it can be inferred that the difference between the estimated value φ and the exact value φ_{exact} can be significant. Rather than constructing a regularized approximation using the actual function f, we are constrained to utilize the provided data f^δ and construct regularized approximations as follows:

$$\varphi_\alpha^\delta = (\alpha I + A^* A)^{-1} A^* f^\delta$$

It has been established that the estimations f_α derived from clean data f converge towards the solution of minimum norm least squares $A^* f$. Hence, it is justifiable to draw a comparison between φ_α^δ and φ_α.

$$\varphi_\alpha^\delta - \varphi_\alpha = (\alpha I + A^* A)^{-1} A^* (f^\delta - f).$$

The results of this method are demonstrated through the emphasis on the following outcomes:

$$\|AA^*(\alpha I + A^* A)^{-1}\|_2 \leq 1, \qquad \|(\alpha I + A^* A)^{-1}\|_2 \leq 1/\alpha.$$

and hence

$$\|\varphi_\alpha^\delta - \varphi_\alpha\|_2 \leq \delta / 2\sqrt{\alpha} \tag{5.12}$$

The previously mentioned inequality denotes a stability threshold for the approximation φ_α^δ. Given these considerations, it can be concluded that as the regularization parameter α approaches zero, the process becomes unstable, while keeping the error level δ constant. Optimal regularization parameter selection involves determining the appropriate value based on the data error. In accordance with Tikhonov's proposition, it can be stated that the selection of a choice $\alpha = \alpha(\delta)$ results in the implementation of a regular algorithm for the ill-posed problem (5.5). Assuming that

$$\alpha(\delta) \to 0 \quad and \quad \varphi_\alpha^\delta(\delta) \to A^* f \quad as \quad \delta \to 0.$$

The investigation of the regularization technique can be best carried out in the context of a Hilbert space, where the integral operator is represented by a compact operator denoted as $A : X \to Y$. In this context, X and Y represent Hilbert spaces. The norm of the variable X should exhibit adequate robustness in order to impose favorable structural characteristics on the solution, whereas the norm of the variable Y should be appropriately flexible to accommodate realistic data functions. The regularization method involves seeking a solution to the equation

$$A\varphi = f. \tag{5.13}$$

is sought by minimizing the functional

$$\|A\varphi_\alpha - f\|_2 = \|A\varphi - f\|_2 + \alpha\|\varphi\|_2, \tag{5.14}$$

where $\alpha > 0$ is a regularization parameter. The minimizer φ_α of this functional is the solution of the well-posed second-kind equation

$$(A^* A)\varphi_\alpha + \alpha\varphi_\alpha = A^* f. \tag{5.15}$$

Therefore, the function h exhibits both uniqueness and stability when subjected to perturbations of the data function f. The discrepancy principle, introduced by V.A. Morozov [9], is an a posteriori parameter choice strategy that was informally utilized by D.L. Phillips [9]. According to this principle, there is a unique $\alpha(\delta)$ satisfying

$$\|A\varphi_{\alpha(\delta)}^\delta - f^\delta\| = \delta \quad and \quad \|\varphi_{\alpha(\delta)}^\delta - \varphi\| \to 0. \quad as \quad \delta \to 0.$$

Moreover, if the function φ lies within the range of the adjoint operator A^* then

$$\|\varphi^{\delta}_{\alpha(\delta)} - \varphi\| = O(\sqrt{\delta})$$

but this order is best possible [9]. The authors T. Raus, H. Gfrerer, and H. Engl have developed discrepancy principles that achieve the optimal order $O(\delta^{2/3})$ (refer to [9]). One can employ an iterative approach to scale the brick wall of ordinary Tikhonov regularization, which has a complexity of $O(\delta^{2/3})$. In iterated Tikhonov regularization, the functionals are successively minimized,

$$\|A\varphi_{\alpha} - f^{\delta}\|_2 = \min_{\varphi}\{\|A\varphi_{\alpha} - f^{\delta}\|_2 + \alpha\|\varphi_{\alpha}\|_2\}, \tag{5.16}$$

A suitable set of regularization parameters is α. With this approach, any p in the range of $[0, 1)$ is approximated to an order of $O(\delta^p)$.

CHAPTER 6

The Landweber Method

The Landweber iteration, also known as the Landweber algorithm, was developed to tackle non-linear problems with constraints and has now been expanded to solve ill-posed linear inverse problems. Louis Landweber[12] first suggested the method in the 1950s, and it can today be seen as a specific example of many other, more generic methods. The Landweber method iteratively minimizes the residual error between the observed data and the estimated solution.

We examine the integral equation, employing iterative solvers based on Landweber type iterative methods.

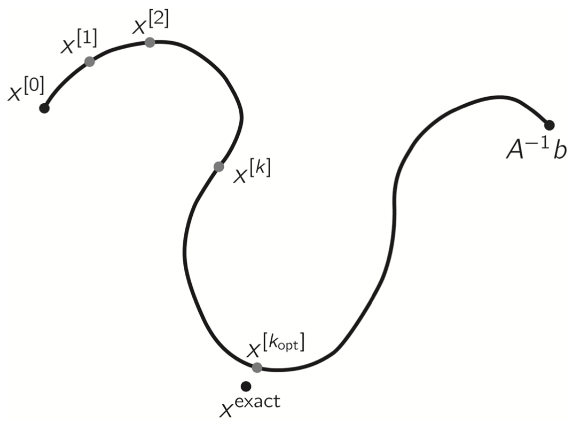

Figure 6.1: The fundamental notion of semiconvergence. In the initial iterations, the iterates x_k exhibit a tendency to progressively improve as approximations to the precise solution x_{exact}.

The first requirement is to enhance existing iterative methods because they only require matrix-vector products. This makes them computationally efficient and suitable for large-scale problems. However, these methods may converge slowly or even fail to converge for ill-conditioned matrices. Therefore, there is a need to develop new preconditioning techniques to improve the convergence rate of iterative methods.

We must use iterative techniques that do not use a fixed regularization parameter but instead use the number of iterations as a regularization parameter to meet the second requirement. Remember that iterative methods always produce a series of iterations $x_1, x_2, \ldots, x_k$ that converge to some solution, starting with a user-specified starting vector x_0(often the zero vector). In the case of $k = 1, 2, \ldots$ and for some iterative methods, the initial iterations of x_k resemble regularized(filtered) solutions. They progressively become closer and closer to the exact solution x_{exact}. We may view the Landweber algorithm as solving:

$$\min_x \|Ax - f\|_2 \tag{6.1}$$

using an iterative method. We examine a Landweber method

$$x_{n+1} = x_n + \omega A^T(f - Ax_n), \quad n = 0, 1, 2, \ldots, k. \tag{6.2}$$

where ω is a relaxation parameter and real number that must satisfy $0 < \omega < 2\|A^T A\|_2^{-1} = 2/\sigma_1^2$.

Theorem 6.1 *[13] Let ω be a positive real number. The iterates of equation (6.1) will converge to a solution of equation (6.2) if and only if the value of ω satisfies the condition $0 < \omega < 2/\sigma_1^2$, where σ_1 represents the greatest singular value of matrix A. If x_0 belongs to the range of the transpose of matrix A over the real numbers, then x_k is the only solution that has the smallest Euclidean norm.*

In each iteration, the computation involves determining the residual vector $r_k = b - Ax_k$ and subsequently multiplying it with A^T and ω. This correction is then added to the current iterate x_k in order to obtain the subsequent iterate. Specifically, we can write the kth iterate as

$$x_k = V\Phi_k\Sigma^{-1}U^T f,$$

where the elements of the diagonal matrix $\Phi_k = diag(\phi_1^k, \ldots, \phi_k^k)$ are the filter factors for x_k, which are given by

$$\phi_i^k = 1 - (1 - \omega\sigma_i^2)^k, \quad i = 1, 2, \ldots, n.$$

Additionally, we have that for small singular values σ_i, $\phi_i^k \approx k\omega\sigma_i^2$, meaning that they degrade at the same rate as the Tikhonov filter factors.

Note 6.2 *It is important to acknowledge that we consider the problem, which encompasses a broader scope than solving the linear equation $A\varphi = f$. It should be noted that a linear system of equations has the potential to exhibit consistency when considering noise-free data, but may demonstrate inconsistency when considering the presence of noise in the data of interest. As a result, the scope of the comparison between the minimization problem and the linear system is more extensive. If the singular values of matrix A exhibit a gradual decrease towards zero and the matrix A is characterized by ill-conditioning, the reduction of equation (6.1) generally gives rise to a discrete ill-posed problem.*

CHAPTER 7

Projection Method

Integral equations are discretized with the intention of converting them into systems of linear algebraic equations that can be solved numerically to obtain approximations of solutions. The process of finding a solution to an equation of the first kind

$$A\varphi = f. \tag{7.1}$$

The projection method is frequently employed for numerical solutions of an injective compact operator $A : X \to Y$ from a Banach space X to a Banach space Y, without the inclusion of regularization techniques.

Definition 7.1 *[2], Let X denote a normed space and U be a nontrivial subspace contained within X. A linear operator $P : X \to U$ that is bounded and satisfies the condition $P\varphi = \varphi$ for all $\varphi \in U$ is referred to as a projection operator from X onto U.*

Definition 7.2 *[2], Consider two Banach spaces, denoted by X and Y, and let A be a bounded linear operator mapping from X to Y. It is given that A is an injective operator. Consider two sequences of sub-spaces, denoted as $X_n \subset X$ and $Y_n \subset Y$. It is given that the dimension of each subspace in both sequences is equal to n. Additionally, we have projection operators $P_n : Y \to Y_n$. For given $f \in A(X)$ the projection method approximates the solution $\varphi \in X$ of*

$$A\varphi = f \tag{7.2}$$

for $\varphi \in X$ by projected equation

$$P_n A\varphi_n = P_n f \quad \varphi_n \in X_n. \tag{7.3}$$

We describe projection method as a general tool for approximately solving linear operator equations by projection them onto sub-spaces which is finite dimensional.

Note 7.3 *In a general, it can be stated that projection methods are classified as semi-discrete methods. Therefore, numerical computations will be employed to obtain an approximate version of form*

$$P_n A_n \tilde{\varphi}_n = P_n f_n. \tag{7.4}$$

where A_n is some approximation to A and f_n approximates f.

A theorem can be stated as follows:

Theorem 7.4 *[2], Let us consider a bijective bounded linear operator $A : X \longrightarrow Y$, where X and Y are Banach spaces. We assume that the projection operators $P_n : Y \longrightarrow Y_n$ and the approximating bounded linear operators $A_n : X \longrightarrow Y$ point-wise convergence*

$$P_n A_n - P_n A \longrightarrow 0, \quad n \longrightarrow \infty,$$

and

$$\sup_{\varphi \in X_n, \|\varphi\|=1} \|P_n A_n - P_n A\| \longrightarrow 0, \quad n \longrightarrow \infty,$$

is satisfied. For sufficiently large values of n, the approximate equation (7.4) exhibits a unique solution, and an error estimate can be obtained

$$\|\tilde{\varphi}_n - \varphi\| \leq C\{\inf_{\psi \in X_n} \|\psi - \varphi\| + \|(P_n A_n - P_n A)\varphi\| + \|P_n(f_n - f)\|\} \tag{7.5}$$

for the solution φ of (7.1) and some constant C.

In other words,the projection operator is a linear operator that operates between two finite-dimensional spaces. The projection method is a technique that reduces the solution of a linear system to finite dimensions.

The projection method is a flexible tool in mathematical modeling because it can also be used to approximate partial differential equation solutions. As a result, the projection method is effective in terms of computation and can be used to solve a variety of issues. It is crucial to remember that the basis functions chosen to represent the solution have an impact on the method's accuracy.

7.1 The Collocation Method

The first kind of linear integral equations can be expressed as follows:

$$\int_a^b K(x,t)\varphi(t)dt = f(x), \quad a \leq x \leq b. \tag{7.6}$$

In this equation, the kernel K and the function f are known, while φ represents the unknown function that needs to be determined. The Fredholm integral equations of first kind were discretized using a discretization scheme based on the projection method.

Definition 7.5 *[2] Consider two sequences of sub-spaces, denoted as $X_n \subset X$ and $Y_n \subset Y$, where the dimension of each subspace is n. We define the equation (7.1) to be satisfied only at a finite number of points, which we refer to as "collocation points". Let Y be the set defined as the image of the function C on the graph G. We proceed by selecting n points, denoted as $x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}$, from the graph G.*

Then the collocation method approximates the solution of (7.1) by an element $\varphi_n \in X_n$ satisfying

$$(A\varphi_n)(x_j) = f(x_j), \quad j = 1, \dots, n. \tag{7.7}$$

Let $X_n = span\{u_1, \dots, u_n\}$. Then, we represent φ_n as a linear combination

$$\varphi_n = \sum_{k=1}^{n} \gamma_k u_k, \tag{7.8}$$

and immediately see that (7.7) is equivalent to the linear system [2]

$$\sum_{k=1}^{n} \gamma_k (Au_k)(x_j) = f(x_j), \quad j = 1, \dots, n. \tag{7.9}$$

for the coefficients $\gamma_1, \ldots, \gamma_n$.

The method can be understood as a projection method that utilizes the interpolation operator $P_n : Y \to Y_n$. The establishment of the equivalence of equation (7.7) can be achieved by taking into account the fact that the interpolating function is uniquely determined by its values at the interpolation points [2].

$$P_n A \varphi_n = P_n f. \tag{7.10}$$

Moreover, let us now assume that we have chosen a basis for a finite dimensional sub-spaces which is spanned by the columns of the matrix $\varphi = span\{u_1, \ldots, u_n\} \in \mathbb{R}^{n \times n}$. Within this particular context, our primary aim is to identify a resolution, articulated within the selected framework, that effectively fulfills the given right-hand side denoted as f. We can formulate this as a constrained least squares problem:

$$\min_{\varphi} \|A\varphi - f\|_2 \quad s.t \quad \varphi = span\{u_1, \ldots, u_n\}. \tag{7.11}$$

It is worth mentioning that the collocation method can be considered as a projection method, wherein a solution to equation (7.1) is sought within the subspace formed by the basis functions $u_1, \ldots, u_n$.

The constraint in equation (7.11) can be restated as the condition that φ is equal to the product of u_k and γ, where γ is an unknown vector in $\mathbb{R}^k$. This leads to a regularized solution expressed in the more computation formulation

$$\varphi^k = u_k \gamma_k \quad , \gamma^k = argmin_{\gamma} \|(Au_k)\gamma - f\|_2 \tag{7.12}$$

The problem denoted as $\|(AX_k)\gamma - f\|_2$ in equation (7.12) is commonly referred to as the projected problem because it is obtained by projecting the original problem onto the subspace spanned by n dimensions, namely $span\{u_1, \ldots, u_n\}$. If the value of k is not excessively large, it is possible to calculate the matrix $(A\varphi_k) \in \mathbb{R}^{n\text{x}k}$ in an explicit manner. Subsequently, the projected problem, which refers to the least squares problem for γ, can be solved.

As a special case of the projection problem, if $\varphi_k = (u_1, \ldots, u_k)$, meaning that the basis vectors consist of the first k right singular vectors of matrix A, then the projected problem takes the form

$$\|U\Sigma V^T\varphi_k\gamma - f\|_2 = \left\| U\Sigma \begin{pmatrix} I_k \\ 0 \end{pmatrix} \gamma - f \right\|_2$$
$$= \left\| \begin{pmatrix} | & & | \\ u_1 & \dots & u_k \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{pmatrix} \gamma - f \right\|_2 .$$

It follows immediately that the elements of $\gamma^{(k)}$ are $\gamma^{(k)} = u_i^T f/\sigma_i$, and there for the projected solution $\varphi^{(k)} = (u_1, \dots, u_k)\gamma^{(k)}$ is the solution.

A finite-dimensional family of functions can be used to numerically approximate the solution to the equation $A\varphi = f$. The method is referred to as the projection method because the exact solution of the equation is projection into a space of finite dimensions. The collocation method is most well-known.

Examining the matrix's condition number is one method that can be used to make a prediction about how well or ill the matrix A behaves in response to perturbations. This index is defined in $\|\varphi - \varphi_\alpha\|_2 = \|O\|_2$ as

$$cond(A) = \|A\|_2\|A^{-1}\|_2 = \sigma_1/\sigma_n. \tag{7.13}$$

where σ_1 and σ_n are the greatest and the smallest singular values of A, respectively. Matrix ill-conditioning is respected as a high condition number. A matrix is considered singular if it has an uncountable number of elements. When the matrix is obtained through the discretization of an integral equation, it becomes apparent that the singular values of matrix A exhibit a gradual decline until they eventually converge to a stable value close to zero, approximately σ_1 times the precision of the computational machine. As a result, the condition number $cond(A)$, which serves as a numerical indicator of a matrix's sensitivity, exhibits an unbounded nature and is roughly inversely proportional to the precision of the computing machine. Ill-conditioning is a well-established phenomenon that can lead to numerical instability during computations involving matrix A. To mitigate this concern, various regularization techniques can be employed to improve the computational stability and enhance the accuracy of the results.

7.1.1 The Collocation Method Application

Using the quadrature rule to approximate $\int_a^b K(x,t)\varphi(t)dt$ say

$$A_n\varphi(x_i) = \int_a^b K(x,t)\varphi(t)dt \cong \sum_{j=1}^{n} \gamma_j K(x_i,t_j)u(t_j).$$

So the equation (7.6) can be replaced by

$$A_n\varphi(x_i) = \sum_{j=1}^{n} \gamma_j K(x_i,t_j)u(t_j) = f(x_i), \quad a \le x \le b \tag{7.14}$$

In the collocation method the values of $\varphi(t_j)$, $j = 1,2,\ldots,n$ are found so that the equation (7.14) is verified for all points $x_1, x_2, \ldots, x_n$, in $[a,b]$. Although it's not necessary to assume $m = n$, m and n are frequently chosen to be equal, and $x_i = \frac{(b-a)i}{n}$ is chosen as $x_i = t_i$, $i = 1,2,\ldots,n$.

$$\sum_{j=1}^{n} \gamma_j K(x_i,t_j)u(t_j) = f(x_i), \quad i = 1,2,\ldots,n. \tag{7.15}$$

Taking $A = (a_{ij})$ matrix such that $a_{ij} = \gamma_j k(x_i,t_j)$ for $a \le i,j \le n$, the unknown vector $\vec{\gamma} = (\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_n))^T = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T$ and the right hand side $\vec{F} = (f_1, f_2, \ldots, f_n)^T$, the equation (7.7) can be approximated by the matrix equation

$$A\vec{\gamma_n} = \vec{F}. \tag{7.16}$$

Now applying Trapezoidal rule on equation (7.14) then we obtain

$$h\left[\frac{1}{2}A_{i0}\varphi_0 + A_{i1}\varphi_1 + \cdots + \frac{1}{2}A_{in}\varphi_n\right] = f(x_i), \quad i = 0,1,2,\ldots,n.$$

where $h = (b - a)/n$ and its general form is written as

$$A\varphi = F \tag{7.17}$$

Hence, we have the linear system then we can solve the finite linear system.

CHAPTER 8

Numerical Quadrature for Singular Kernel

In this paper, we propose a methodology for the development of high-order quadrature methods that can be used to numerically compute singular integrals. Over the past few years, researchers have developed generalized Gaussian quadratures as a solution for situations where direct application of Gaussian quadratures is not feasible. In order to ascertain the optimal number of quadrature points, our proposed approach employs an adaptive algorithm and generalized Gaussian quadratures [14]. The efficacy and accuracy of numerical integration for singular integrals can be significantly improved through the utilization of high-order quadrature rules, as evidenced by previous studies.

Then, for any point $y \in (-1, 1)$, the quadrature rules

$$\int_{-1}^{1} \frac{1}{2} \log((y - x)^2)\phi(x)dx \approx \sum_{n=1}^{N} W_n(y)\phi(x_n) \tag{8.1}$$

have the degree $N - 1$. Now, suppose that $x_1, x_2, \ldots, x_N$ denotes the $N-$Legendre nodes on $[-1, 1]$.

Furthermore, we have provided a description of the quadrature formula for integrals of the form (8.1), in which the point of evaluation y lies within the interval of integration. It can be confidently asserted that the quadrature formula will yield exact integration for all functions f conforming to the structure described in equation (7.17).

Theorem 8.1 *[14] Let $x_1, x_2, \ldots, x_N$ and $w_1, w_2, \ldots, w_N$ represent the N nodes and weights of the Gaussian quadrature on the interval $[-1, 1]$. Let us assume that φ : $[-1, 1] \to \mathbb{R}$ is a function that is sufficiently smooth and $P_j(x)$ and $Q_j(x)$ represent the j-th Legendre polynomial and Legendre function of the second kind, respectively. Finally,* *suppose that the coefficients $W_{1,1}, W_{1,2}, \ldots, W_{1,N}$ is defined by the formulae*

$$W = w_n \Bigg((P_0(x_n) - P_1(x_n))R_0(y) + \sum_{j=1}^{N-2} (P_{j-1}(x_n) - P_{j+1}(x_n))R_j(y) \\ + P_{N-2}(x_n)R_{N-1}(y) + P_{N-1}(x_n)R_N(y) \Bigg). \tag{8.2}$$

for all $n = 1, 2, \ldots, N$, with $[(N + j - 3)/2]$ denoting the integer part of $(N + j - 3)/2$, and the mappings $R_j : (-1, 1) \to \mathbb{R}$ defined by the formula

$$R_j(y) = Q_j(y) + \frac{1}{4} log \left((y-1)^2 \right). \tag{8.3}$$

Then, for any point $y \in (-1, 1)$, the quadrature rules

$$\int_{-1}^{1} \frac{1}{2} \log((y-x)^2)\phi(x)dx \approx \sum_{n=1}^{N} W_n(y)\phi(x_n) \tag{8.4}$$

have the degree $N - 1$, respectively.

Note 8.2 *For any natural number n and $x \in [a, b]$, the Legendre differential equation [14] is*

$$(1 - x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + n(n+1)u = 0. \tag{8.5}$$

A potential resolution to the Legendre differential equation (8.5) is the Legendre polynomial $P_n(x) : [-1, 1] \to \mathbb{R}$, defined by the three-term recursion formula

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x). \tag{8.6}$$

with

$$P_0(x) = 1,$$
$$P_1(x) = x.$$

The Legendre function of the second kind [14] $Q_n : \mathbb{C}\ [-1, 1] \to \mathbb{C}$*, defined by the three-term recursion formula*

$$Q_{n+1}(z) = \frac{2n+1}{n+1} z Q_n(z) - \frac{n}{n+1} Q_{n-1}(z). \tag{8.7}$$

with

$$Q_0(z) = \frac{1}{2} \log(\frac{z+1}{z-1}),$$
$$Q_1(z) = \frac{z}{2} \log(\frac{z+1}{z-1}) - 1,$$

It is evident that the function $Q_n(z)$ *exhibits a branch cut in the complex z-plane along the real axis spanning from* -1 *to* 1*. In accordance with established convention, the function* Q_n *is defined on the branch cut. The real numbers can be mapped to the set of real numbers using the given formula*

$$Q_n(z) = \frac{1}{2} \lim_{h \to 0} (Q_n(x + ih) + Q_n(x - ih)). \tag{8.8}$$

CHAPTER 9

Numerical Experiments

The objective of this section is to demonstrate the precision and effectiveness of the proposed methodologies by showcasing multiple examples of Fredholm integral equations of the first kind (see Refs. [4] and [15]). The discretized manifestation of the problem can be represented as the discrete ill-posed problem [2]. The minimization problem is replaced by a linear system of equations, as the discrete version under consideration is a consistent linear system

$$A\varphi = f.$$

We consider noise-free and noisy data with noise levels, $\left(\frac{\|f-f^{\delta}\|_2}{\|f\|_2}\right)$. To conduct error analysis, the relative error is calculated. The numerical examples are compared to each other by computing relative error, $\left(\frac{\|\varphi-\widetilde{\varphi}\|_2}{\|\varphi\|_2}\right)$. A natural number N represents the number of points.

The objective of this concluding section is to provide numerical illustrations that support the theoretical findings of this study. The methods employed in this study encompass the Collocation method, Tikhonov regularization method, Landweber iteration, and the Generalized Gaussian Quadrature method. These methods will be implemented using appropriate algorithms and the Matlab software. Subsequently, a comparison will be made between the exact solution and the approximate solution using a suitable number of n points.

Example 9.1 *The integral equation, as referenced in [16], can be solved using the collocation method. The equation is defined as follows:*

$$\int_0^1 e^{xt}\varphi(t)dt = \frac{e^{x+1}-1}{x+1}, \quad 0 \le x \le 1.$$

The operator is injective. The function $\varphi(x) = e^x$ represents a unique solution. Before solving the example, we will proof that the operator is injective.

Proof *First of all, $\varphi(x)$ be integrable on $[0,1]$ and e^{xt} be monotonic for x is fixed.*

1. *If $A\varphi = 0 \Rightarrow \varphi = 0$, we are done.*
2. *Assume $\varphi \neq 0$ on Ω^+,*

$$\varphi > 0 \Rightarrow \exists \Omega^- \quad \varphi < 0, \quad x \in \Omega^-$$

Case $\quad x = 0$,

$$\int_0^1 \varphi(t)dt = 0 \Rightarrow \int_{\Omega^+} \varphi(t)dt = -\int_{\Omega^-} |\varphi(t)|dt,$$
$$\int_{\Omega^+} \varphi(t)dt + \int_{\Omega^-} |\varphi(t)|dt = 0.$$

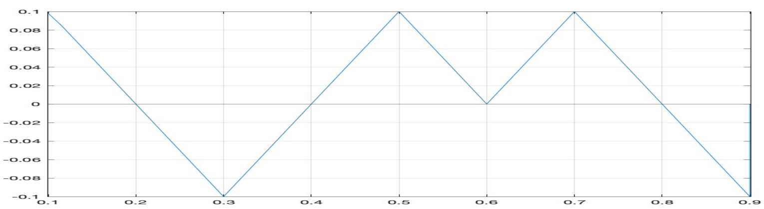

Figure 9.1: The graph shows Ω^+ and Ω^-

We are done.

Case $\quad x \neq 0$, *By extreme value theorem, $\exists t_0, t_1$ such that for each $x \in [0, 1]$.*

- *Since φ is non-negative,*

$$\int_0^1 e^{xt_0}\varphi(t)dt \leq \int_0^1 e^{xt}\varphi(t)dt \leq \int_0^1 e^{xt_1}\varphi(t)dt.$$

By the intermediate value theorem, φ attains every value of the interval $[t_0, t_1]$, so for some t in $[0, 1]$.

- *If φ is negative on* $[0, 1]$

$$\int_0^1 e^{xt_1}\varphi(t)dt \leq \int_0^1 e^{xt}\varphi(t)dt \leq \int_0^1 e^{xt_0}\varphi(t)dt.$$

and we still get the same result as above. We are done. $\square$

In order to estimate the value of an integral, the trapezoidal rule can be employed in the following manner:

$$\int_0^1 e^{xt}\varphi(t)dt \approx h\left[\frac{1}{2}e^0\varphi(0) + \sum_{i=1}^{n-1} e^{ihx}\varphi(jh) + \frac{1}{2}e^x\varphi(1)\right] = f(x_i).$$

where $b = 1$, $a = 0$, $x_n = \sum_{i=1}^{n} a + (i-1)h$, $h = \frac{b-a}{n}$ *and* $0 \leq t \leq 1$.

We use the exact solution φ and its approximate one φ_n as well as the relative error $\frac{\|\varphi-\varphi_n\|_2}{\|\varphi\|_2}$ of the example 9.1 in a natural number N where N is the number of collocation points, for $N = 8, 16$ *and* 32.

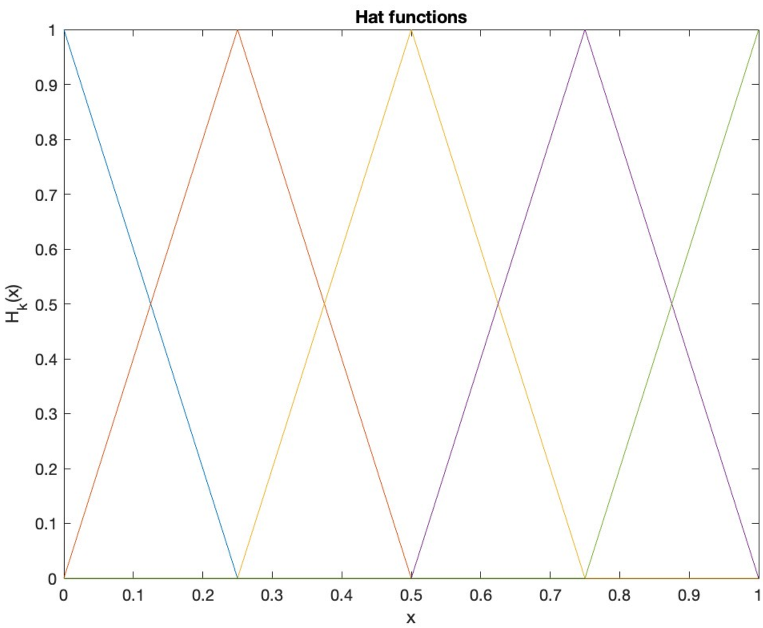

Figure 9.2: Basis functions the set of triangular functions(Hat function)

Table 9.1: Relative errors of the collocation method of $N = 8, 16$ and 32 based on piece-wise linear function.

N	Condition (A)	Error
8	1.18182e+18	0.009032
16	1.32322e+18	0.019635
32	1.44785e+18	0.021673

The table 9.1 shows the solution of integral equation with basis of Hat function in figure 9.2. Furthermore, it is observed from the data presented in Table 9.1 that as the value of N increases, the accuracy of the exact solution decreases. Hence, it is possible to modify the basis by employing a spline basis, as illustrated in Figure 9.3.

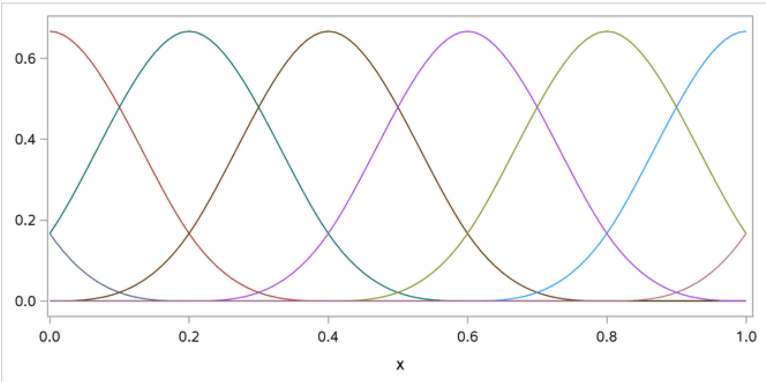

Figure 9.3: The basis of B-Spline

The integral equations are solved using the spline basis depicted in Figure 9.3.

Table 9.2: Relative errors of the collocation method of $N = 8, 16$ and 32 based on Spline basis.

N	Condition (A)	Error
8	1.01955e+17	0.009031
16	1.03098e+18	0.011764
32	1.24485e+18	0.017624

The table 9.2 shows the solution of integral equation with spline basis in figure 9.3. Increasing N will result in a closer representation of the matrix(A), but it will become more ill-conditioned.

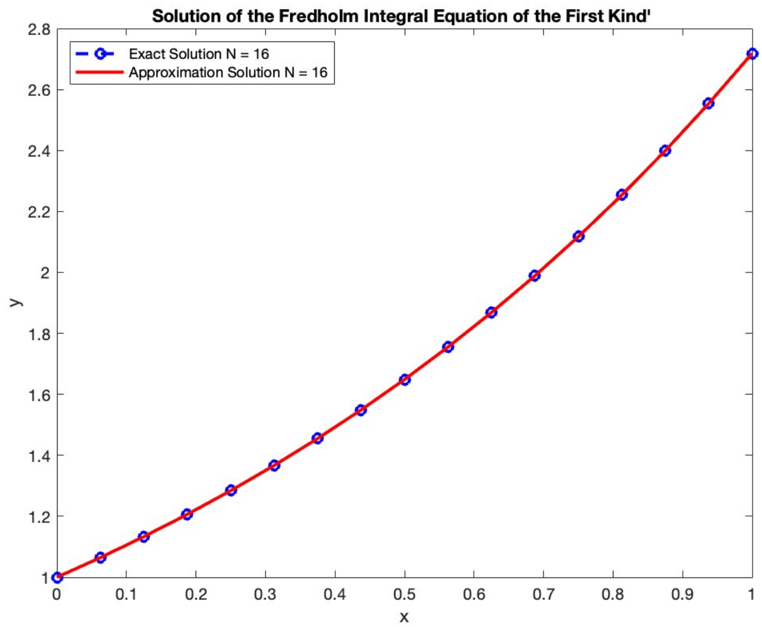

Figure 9.4: Graph of approximation solution of integral equation with exact solution based on hat function

We can therefore see that by using a collocation method with a different basis, we can approach the exact solution very closely. Furthermore, the results are improved as seen in the table 9.2, as the spline basis in figure 9.3 is a smoother function than the hat function in figure 9.2. Thus, it is clear that by applying a collocation method with a different basis, we can come close to finding the precise solution.

As the dimensions of the matrix increase, the solvability of the equation becomes progressively more challenging due to the escalating condition number. This phenomenon leads to a deviation between the approximate solution and the exact solution. In tjis case, it is imperative to employ the regularization approach.

Example 9.2 *To solve the Fredholm integral equation of the first kind, which is same as previous example 9.1, but now, we will now employ the Tikhonov method.*

$$\int_0^1 e^{xt}\varphi(t)dt = \frac{e^{x+1}-1}{x+1}, \quad 0 \le x \le 1.$$

Let $x_i = 0 + i/N, i = 1, 2, \ldots, N+1 \subset [0,1]$ *is the collocation points. Firstly, we apply the Tikhonov regularization method with* $\alpha = 0.0001(1e-04)$ *and* I *is identity matrix without perturbation data .*

Table 9.3: Relative errors of the Tikhonov regularization of $N = 32, 64$ and 128.

N	Condition $(\alpha I + A^* A)$	Error
32	1.9627e+04	0.018481
64	1.8960e+04	0.013629
128	1.8632e+04	0.010490

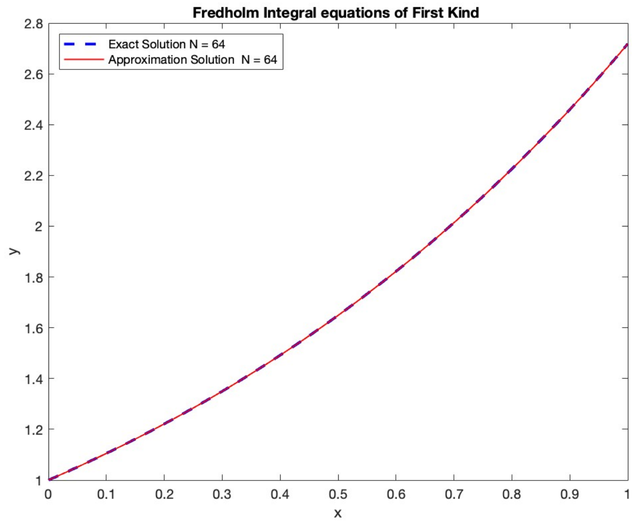

Figure 9.5: Graph of approximation solution of integral equation by Tikhonov regularization without noisy data.

In conclusion, the Tikhonov regularization technique is employed in the presence of noisy data. Consequently, a random distributed perturbation, acquired through the utilization of the Matlab command "randn," is incorporated into the right hand side. We obtain the vector f^δ *:*

$$f^\delta = f + \delta * \eta * \frac{\|f\|}{\|\eta\|},$$

where η *is a random variable,* δ *- noise level,*

$$\frac{\|(f^\delta - f)\|}{\|f\|} = \delta,$$

and the function "randn(.)" generates arrays of normally distributed random numbers with mean 0*, "randn(size(f))" returns an array of random entries of the same size as* f*. We get this equation,*

$$A\varphi^\delta = f^\delta.$$

As a result, we use collocation points and Tikhonov regularization to solve the integral equation.

We use the exact solution φ *and its approximate one* φ_n *as well as the relative error* $(\|\varphi - \varphi_n\|/\|\varphi\|)$ *of the example 9.2 in some arbitrary points for* $N = 32, 64,$ *and* 128*. We choose* $\delta = 0.1$ *and* $\eta = randn(N)$*.*

Table 9.4: Relative errors of the Tikhonov regularization of $N = 32, 64$, and 128 .

N	Condition $(\alpha I + A^* A)$	Error
32	1.9627e+04	0.027268
64	1.8960e+04	0.017671
128	1.8632e+04	0.010430

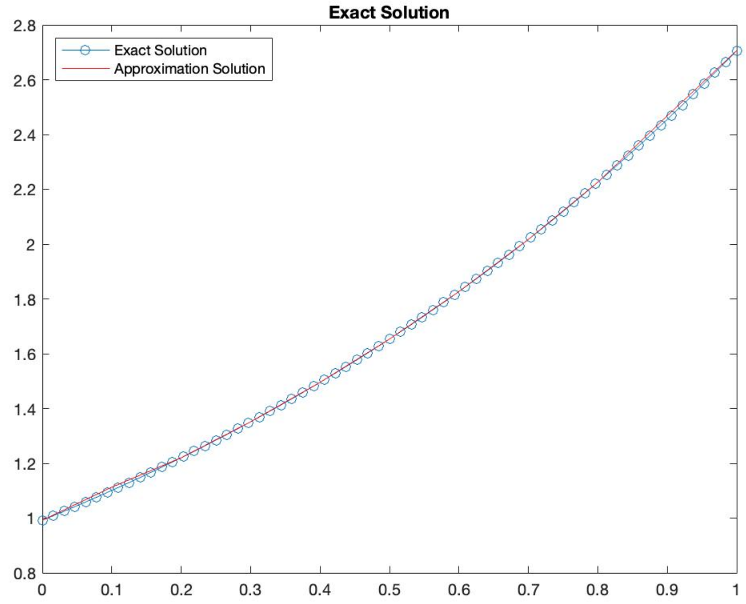

Figure 9.6: Graph of solution of integral equation with approximation solution by Tikhonov regularization with noisy data

Tikhonov regularization can be applied to integral equations with or without noisy data, but it is typically applied with noisy data. The table 9.4 demonstrates that while the number of conditions is rising, the number of errors is falling. The exact result is observable. Additionally, Tikhonov regularization is a helpful method for resolving ambiguous problems where the solution might not be obvious or unique. Additionally, it can aid in stabilizing the solution and avoiding over-fitting.

In conclusion, if we carefully examine the figure 9.6, we can see that the integral equation has noisy data. By using Tikhonov regularization, we are able to find the approximation solution even with noisy data. Tikhonov regularization, which involves including a regularization term in the objective function, is a frequently employed technique for resolving imprecise problems. This technique aids in solution stabilization and lessens the impact of data noise.

Example 9.3 *The Fredholm integral equation of the first kind, previously discussed, is being addressed using the Landweber iteration method with collocation points.*

$$\int_0^1 e^{xt}\varphi(t)dt = \frac{e^{x+1}-1}{x+1}, \quad 0 \le x \le 1.$$

We know that the operator is injective. The unique solution is $\varphi(x) = e^x$. Firstly, we apply the Landweber iteration without noisy data. The algorithm is given by the update

$$x_{k+1} = x_k + \omega A^*(f - Ax_k).$$

where the relaxation factor ω satisfies

$$0 < \omega < 2\|A^T A\|^{-1} = 2/\sigma_1^2.$$

where σ_1 is the largest singular value of A. We choose $\omega = \frac{(f-Ax_0)(f-Ax_0)}{(f-Ax_0)^T A^T (f-Ax_0)}$, where $x_0 = 0$ is initial vector.

Table 9.5: Residual and error of the Landweber iteration of $N = 32, 64$, and 128.

N	Residual: $\|(f - Ax_k)\|_2$	Error
32	0.00174	0.01827
64	0.00036	0.01708
128	0.00010	0.01625

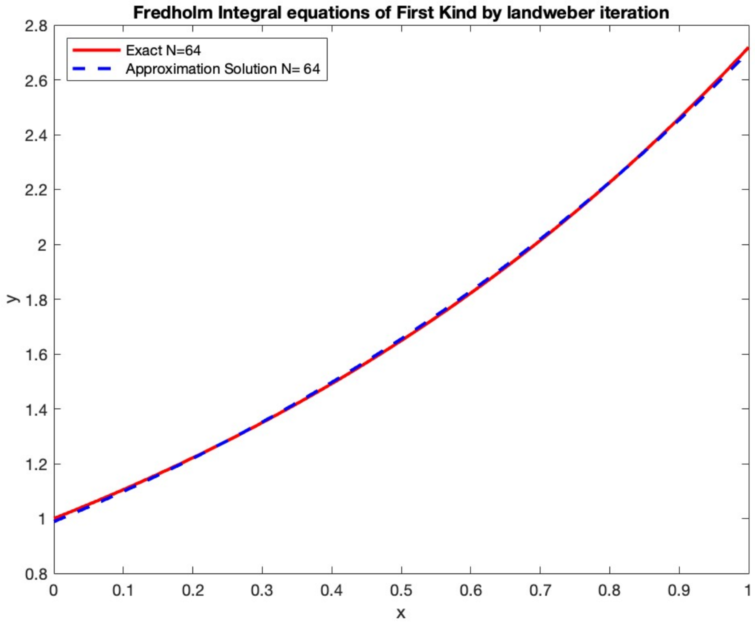

Figure 9.7: Graph of approximation solution of integral equation by Landweber iteration without noisy data

Then, we apply the Landweber iteration with noisy data. Consequently, the introduction of a randomly distributed perturbation to each data function, we obtain the vector f^{δ} *:*

$$f^{\delta} = f + \delta * \eta * \frac{\|f\|}{\|\eta\|},$$

where η *is a random variable,* δ *- noise level and the function "randn(.)" generates arrays of normally distributed random numbers with mean* 0*, "randn(size(f))" returns an array of random entries of the same size as* f*. We get this equation,*

$$A\varphi = f^{\delta}.$$

Moreover, We may view the Landweber method with noisy data as solving:

$$\min_{x \in \mathbb{R}} \|A\varphi - f^{\delta}\|_2.$$

We choose $\delta = 0.1$ and $\eta = randn(N, 1)$. The results of our approach are displayed :

Table 9.6: Residual and errors of the Landweber iteration of $N = 32, 64$, and 128.

N	$\|(f^{\delta} - Ax_k)\|_2$	Error
32	0.00206	0.02533
64	0.00046	0.01512
128	0.00020	0.01121

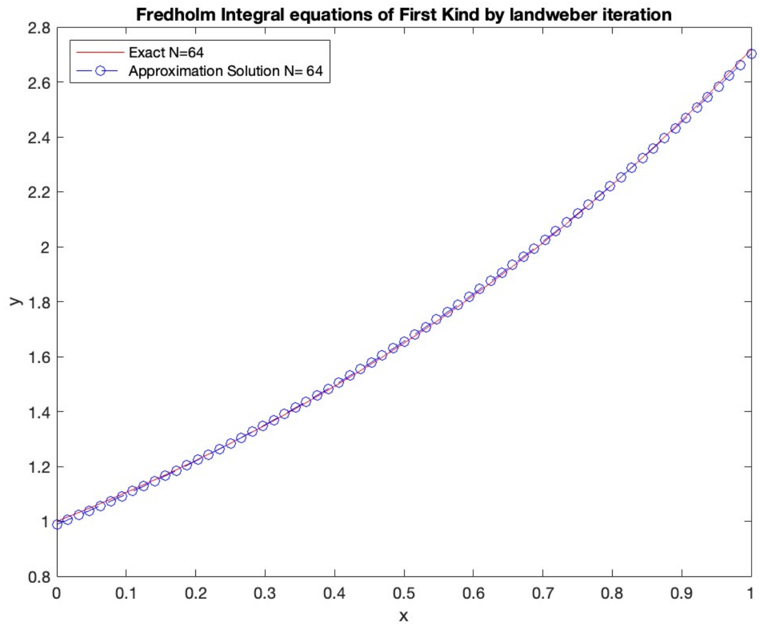

Figure 9.8: Graph of approximation solution of integral equation by Landweber iteration with noisy data

The Landweber iteration method has the potential to be used as an alternative for Tikhonov regularization. The state of relaxation is imperative in this particular case. Consequently, we ascertain the precise solution to the problem.

The Landweber iteration method is considered to be simpler in comparison to Tikhonov regularization due to its avoidance of the need to compute the singular value decomposition. If the relaxation factor is not selected with caution, the solution may exhibit *slow convergence and exhibit noise. Hence, determining the exact solution may be used as a helpful indicator for evaluating the efficacy of the Landweber iteration. Additionally, a variety of inverse problems, such as signal processing and image reconstruction, can be solved using Landweber iteration. Due to its slow convergence rate, it might not be the most effective approach for large-scale issues.*

Example 9.4 *In order to solve the Fredholm integral equation of the first kind, which is presented without an exact solution.*

$$\int_0^1 e^{xt}\varphi(t)dt = x\cos(x), \quad 0 \le x \le 1.$$

We apply collocation method with collocation points and piece-wise liner functions(Hat functions).

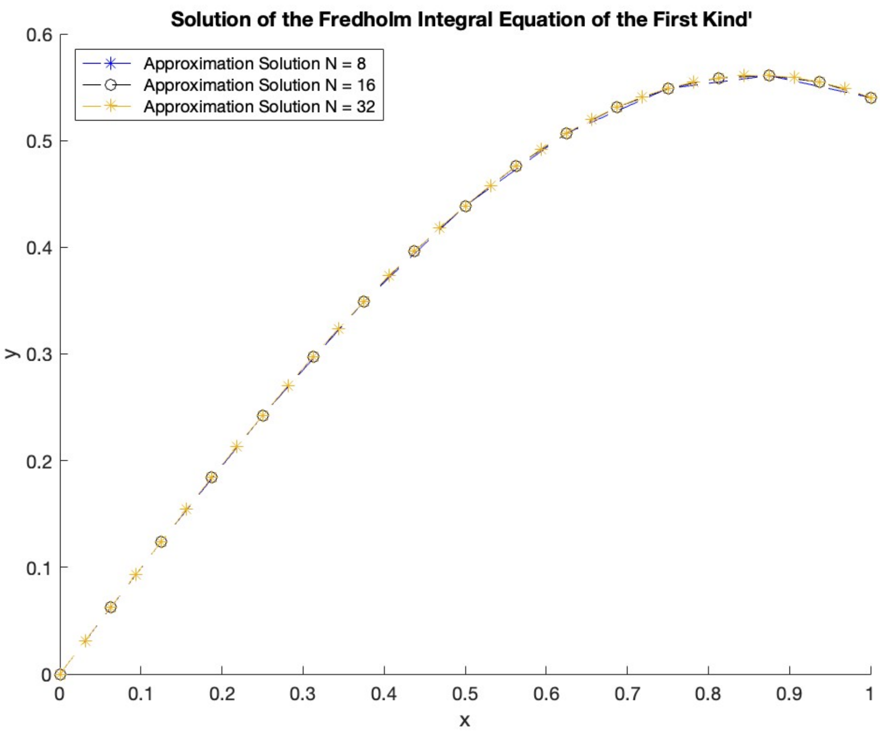

Figure 9.9: Solution of Integral Equation without exact solution

Also, condition number of matrix A: $1.02485e+04$*. Actually, we find the approximation solution which looks like* $\sin(x)$ *function.*

Example 9.5 *The Fredholm integral equation of the first kind [[14]] which is given*

$$\int_{-1}^{1} \frac{1}{2} \log((y-x)^2)\phi(x)dx \approx \sum_{n=1}^{N} W_n(y)\phi(x_n), \tag{9.1}$$

where

$$\phi = \sin(2x) + \cos(3x).$$

We use the quadrature to evaluate integral of (9.1) for the singularities with function $\phi : [-1, 1] \to \mathbb{R}$,

Table 9.7: 14-node quadrature formula for $y = -0.98628$

x_n	w_n
-0.9863	-.1749507
-0.9285	-.243983
-0.8273	-.203561
-0.6873	-.215990
-0.5152	-.107525
-0.3191	-.119635
-0.1080	0.108820
0.1080	-.1913486
0.3191	0.903821
0.5152	0.448256
0.6873	0.104767
0.8273	0.561625
0.9285	0.607434
0.9863	0.213079

The table 9.7 shows that the integral equation is satisfied by the Gaussian quadrature rule for singular kernels method (8.1) [14] . This rule is widely used in numerical analysis for approximating integrals with singularities. The table provides the weights *and nodes necessary for accurate approximation using this method.*

Example 9.6 *We consider the following integral in article "Quadrature rules for weakly singular, strongly singular, and hyper-singular integrals in boundary integral equation methods [15]":*

$$\int_{-1}^{1} \frac{(1-\tau^2)^{1/2}}{\tau^2+\lambda^2} \ln|t-\tau| d\tau = \pi \ln 2 + \frac{\pi}{\lambda}(1+\lambda^2)^{1/2} \ln \frac{\sqrt{\lambda^2+t^2}}{\lambda+\sqrt{1+\lambda^2}}. \tag{9.2}$$

The integral in (9.2) contains logarithmic singularities at $\tau = t$. For integrals of the type (9.2) in the context of related singular integral equations, a number of special-purpose quadrature rules have been proposed (see Refs. [15] and [14]).

First, We choose the $\lambda = 5$, Then, we can solve by Gaussian quadrature method for singular kernel.

Table 9.8: Residual and errors of the Gaussian quadrature method of $N = 8, 16$, and 32.

N	$\|f - A\phi_k\|_2$	Error
8	0.00260	0.06287
16	0.00326	0.06300
32	0.00430	0.56051

The table 9.8 illustrates that the integral equation is compact and ill-posed after $N = 16$, which prevents convergence to the exact solution.

In addition, we apply the Tikhonov regularization method with Gaussian quadrature nodes, $\lambda = 5$, identity matrix I and $\alpha = 0.001(1e-03)$ without perturbing the data. It has been demonstrated that this approach works well for resolving ill-posed issues, particularly those involving singular kernels. It is crucial to remember that the regularization parameters that are chosen can significantly affect how accurate the solution is, so care should be taken to choose the correct values.

Table 9.9: Relative errors of the Tikhonov regularization of $N = 32, 64$ and 128

N	Condition $(\alpha I + A^* A)$	Error
32	1.08446e+03	0.025501176
64	1.09924e+03	0.010415111
128	1.11911e+03	0.010101289

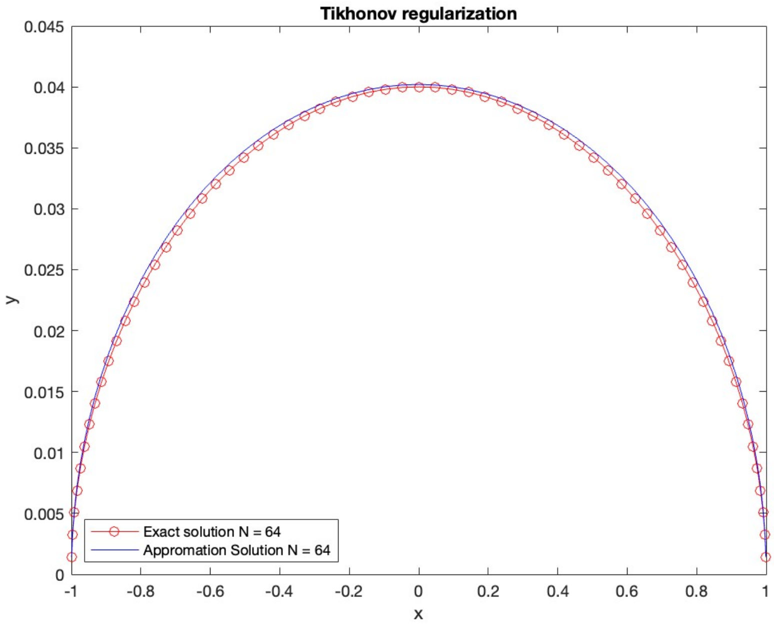

Figure 9.10: The graph presented illustrates the approximate solution of an integral equation without noisy data.

Therefore, the Tikhonov regularization technique is employed in the presence of noisy data. We obtain the vector f^δ :

$$f^\delta = f + \delta * \eta * \frac{\|f\|}{\|\eta\|},$$

where η is a random variable, δ - noise level. We have this equation,

$$A\varphi = f^{\delta}.$$

We choose $\delta = 0.1$ and $\eta = randn(N, 1)(Matlab - command)$. Moreover, we apply the Tikhonov regularization method without perturbation data with $\alpha = 0.001(1e - 03)$ and I is identity matrix.

Table 9.10: Relative errors of the Tikhonov regularization method of $N = 32, 64$, and 128.

N	Condition $(\alpha I + A^*A)$	Error
32	1.08446e+03	0.03545011
64	1.09924e+03	0.01141511
128	1.11911e+03	0.01210128

,

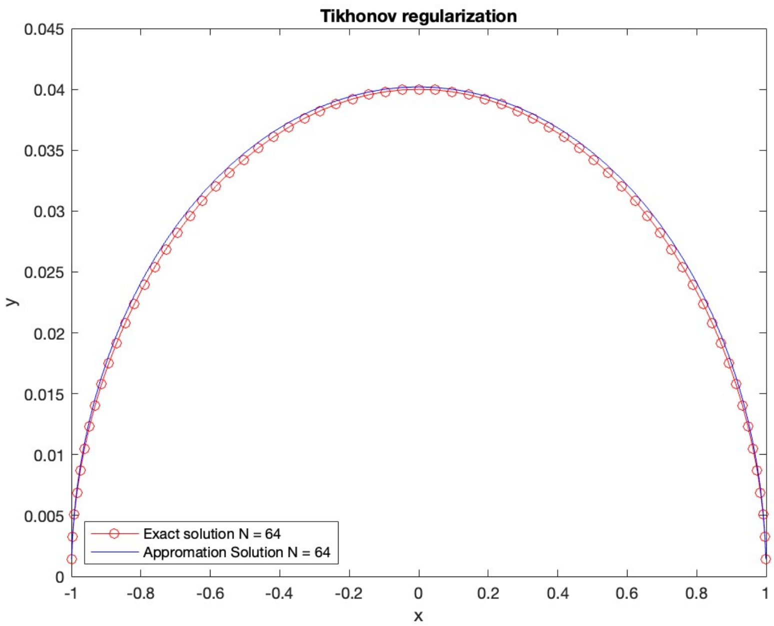

Figure 9.11: The graph presented illustrates the approximation solution of an integral equation using data that contains noise.

The Gaussian Quadrature rule is initially employed in the context of solving the Fredholm integral equation of the first kind, which involves a singular kernel. Hence, it is evident that the solution fails to converge at the given amount of $N = 16$. The observed phenomenon can be explained by the ill-conditioned characteristics of the matrix, which arises from the existence of a singular kernel. As a result, numerical instability occurs. Regularization techniques are frequently utilized to address the issue of ill-conditioned matrices that possess singular kernels. The process of truncated singular value decomposition entails the truncation of singular values that fall below a specific threshold, thereby diminishing the influence of the singular kernel. In contrast, Tikhonov regularization includes a regularization component to enhance the stability of the solution by establishing a balance between accurately fitting the data and minimizing the norm of the solution. Both methodologies have been demonstrated to be effective in augmenting numerical stability and achieving precise solutions for equations of the first kind featuring singular kernels.

The application of Tikhonov regularization is examined both in the presence and absence of noisy data. The inclusion of a regularization parameter in the Tikhonov regularization technique works to enhance the stability of the solution. This parameter provides for finding a balance between the accuracy and smoothness of the solution. The graph 9.11 depicted in the table 9.10 demonstrates how closely the result matches the correct response.

CHAPTER 10

Conclusion

In this article, we examine the resolution of the Fredholm integral equation and subsequently discuss the necessary conditions to attain a possible, unique, and correctly available solution. To effectively examine the resolution of the Fredholm integral equation, it is imperative to possess a comprehensive comprehension of the fundamental mathematical structure and methodologies employed. This encompasses the investigation of the characteristics of the kernel function, analysis of conditions for convergence, and exploration of diverse numerical techniques for solving equations of this nature. Moreover, a comprehensive comprehension of linear algebra and functional analysis is imperative in order to guarantee a rigorous examination of the existence, uniqueness, and stability of the solution.

The application of the collocation method has been employed for the resolution of Fredholm integral equations of the first order, utilizing hat functions and spline bases. When a spline basis is used, the results are improved. After $N = 16$, our solution gets worse as the number of conditions increases. To try and improve the accuracy of our solution, we can try increasing the quantity of collocation points or using a different numerical approach. It is crucial to remember that the basis functions selected can have a significant impact on the method's effectiveness. We can use the generalized Gaussian quadrature rule. These methods have been found to be effective in dealing with singular kernels and large values of N. Furthermore, the Tikhonov method possesses the benefit of exhibiting computational efficiency and straightforward implementation.

It is important to emphasize that regularization techniques, like Tikhonov regularization, can help produce more stable solutions for problems, whether or not they involve noisy data. To prevent over fitting or under fitting the data, it is crucial to carefully choose the regularization parameter. Additionally, incorporating prior knowledge or constraints into the regularization term can further improve the accuracy of the solution. For problems of first kind, a specific type of iterative regularization known as the Landweber-type iterative method can also produce reliable and stable results. The self-regularization property of this method can also help with the analysis of the iterative process's convergence.

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