FUNCTION SYNTHESIS OF 2-LOOP AND/OR 2-DOF MECHANISMS

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ABSTRACT

FUNCTION SYNTHESIS OF 2-LOOP AND/OR 2-DOF MECHANISMS

Kinematic synthesis problems are important in mechanism and machine science. This thesis focuses on function synthesis. The synthesis of different mechanisms with 1 and 2 degrees-of-freedom (dof) are issued by using least squares and Chebyshev approximations. There are various studies with these approaches in the literature. However, there are not many studies of function synthesis of 2-dof mechanisms in the literature. The aim is to study function synthesis of 2-loop and 2 dof mechanisms. 2-dof planar 5R mechanism, 1-dof Bennet 6R mechanisms, 2-dof 7R double-spherical mechanism and finally 7R planar 2-dof mechanism are worked out for function synthesis. It is shown that the function synthesis problems can be solved analytically and semi analytically. The formulations are applied using MS Excel and the results were verified using Solidworks software. It is seen that the numerical results give reliable results and construction parameters are successfully determined.

ÖZET

İKİ DEVRELİ VE/VEYA İKİ SERBESTLİK DERECELİ MEKANİZMALARIN İŞLEV SENTEZİ

Mekanizma biliminin önemli yapı taşlarından biri kinematik sentez problemleridir. Bu çalışma işlev sentezine odaklanmaktadır. İşlev sentezi çalışmalarında özellikle en küçük kareler ve Çebişev yaklaşım yöntemleri kullanılarak 1 ve 2 serbestlik dereceli (sd) mekanizmaların sentezi çalışılmıştır. Literatürde bu yaklaşım yöntemleri kullanılarak 1-sd mekanizmaların işlev sentezi ile ilgili pek çok çalışma mevcuttur. Ancak literatürde 2-sd mekanizmaların işlev sentezi ile ilgili çok fazla çalışma bulunmamaktadır. Bu çalışmanın amacı 2 devreli ve 2-sd mekanizmaların işlevsel sentezini formüle etmektir. 2-sd düzlemsel 5R mekanizması, 1-sd Bennett mekanizması, 2-sd 7R çift-küresel mekanizma ve son olarak 2-sd düzlemsel 7R mekanizması işlev sentezi problemleri ele alınmıştır. Problemlerin analitik ve yarı analitik çözümleri sunulmuştur. Sayısal çalışmalar MS Excel'e yürütülmüş ve Solidworks çizim programı ile sonuçlar doğrulanmıştır. Sayısal sonuçların güvenilir sonuçlar verdiği ve tasarım parametrelerinin istenen şekilde belirlenebildiği görülmektedir.

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CHAPTER 1

INTRODUCTION

Kinematic synthesis is a well-studied topic in mechanisms machine science. Function synthesis of single loop and single degree-of-freedom (dof) mechanisms are widely studied in the literature. However, there are not much work on function synthesis of multi-loop and multi-dof mechanisms.

This thesis work is on function synthesis of some 2-loop and 2 degree-of-freedom mechanisms. In the following parts, the kinematic synthesis of mechanisms and their types are discussed. Then, the synthesis of function formation is explained. Then the motivation and purpose of this thesis are stated. Finally, the outline of this thesis is presented.

1.1. Kinematic Synthesis of Mechanism

The kinematic and structural analysis and synthesis of mechanisms are fundamental problems in mechanism and machine science.

Two converse problems in the kinematics of mechanisms involve kinematic analysis and synthesis. In kinematic analysis the structure and the link dimensions of the mechanism are known and motions of the links (displacement, velocity & acceleration etc.) are determined for given inputs to the mechanism. Whereas kinematic synthesis deals with the determination of the structure and dimensions of a mechanism for a specified task. Kinematic synthesis is divided to dimensional and structural synthesis. Structural synthesis is also examined in two categories (Figure 1.1).

Type of synthesis is defined for a given task to be produced by mechanism as to find the type of mechanism that will best perform the task, such as a linkage, a cam mechanism, a gear train or a combination thereof (Angeles & Bai, 2022). Second phase is the number synthesis, where the number of links and joints are determined for a given type of mechanism in order to achieve the desired motion of the mechanism. These two phases constitute the structural synthesis of the mechanism and can be considered as part of the conceptual design. Lastly, dimensional synthesis

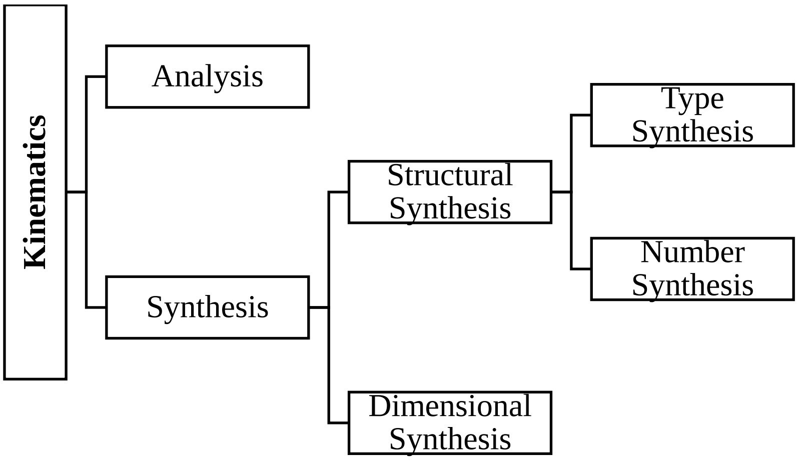


Figure 1.1. Kinematics of Mechanisms

Lastly, dimensional synthesis is synthesis of the metric properties of the mechanisms. In dimensional synthesis, desired motion characteristics of some links (displacement, velocity & acceleration etc.) and the structure of the mechanism are known, whereas the link dimensions of the mechanism are to be determined for given inputs to the mechanism. The kinematic synthesis and analysis are summarized in the Figure 1.2.

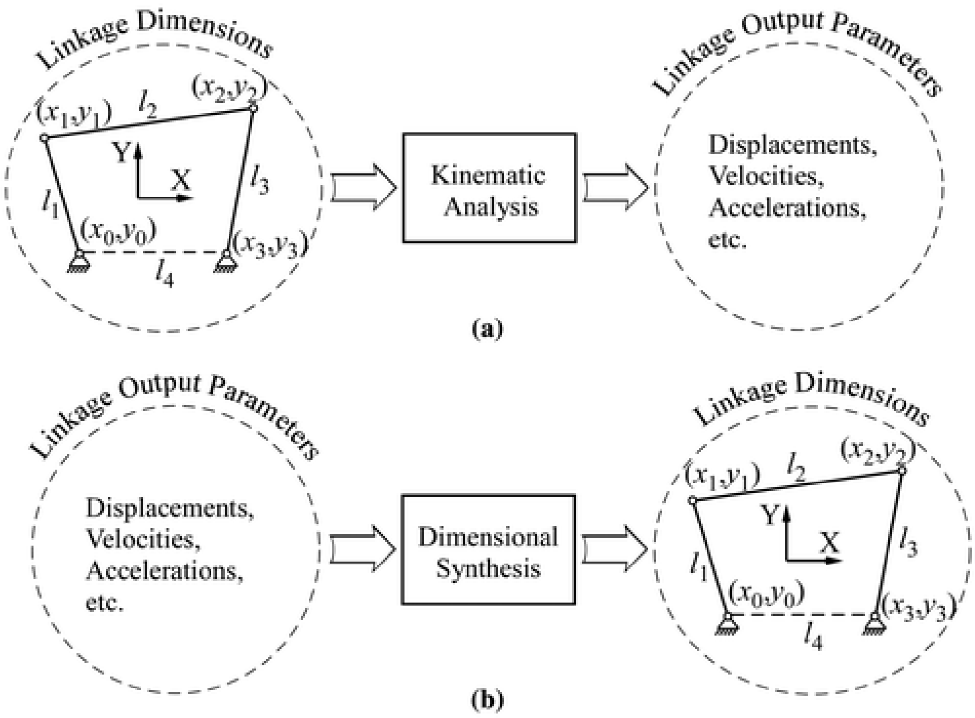


Figure 1.2. a) Kinematic Analysis, b) Kinematic Synthesis

(Source: Lee & Russell, 2018)

1.2. Tasks of Kinematic Synthesis

There are 3 main tasks used in kinematic synthesis of planar mechanisms: path generation, motion generation and function generation (Figure 1.3). The main subject of thesis is function generation.

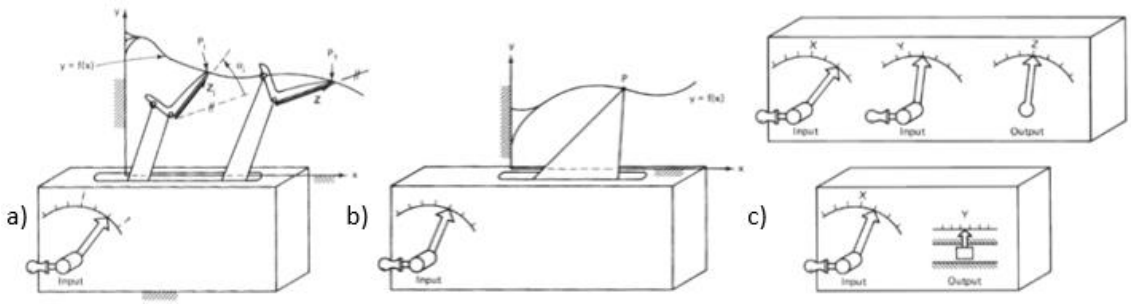


Figure 1.3. a) Motion Generation, b) Path Generation, c) Function Generation
(Source: Sandor and Erdman, 1984)

Path generation occurs when a point on a link is desired to follow a specific path. The problem is to find the dimensions of the mechanism for a certain location on a moving link that is intended to follow a predetermined course. Motion generation synthesis is also called the body guidance where a body is to be moved from one pose to another (Figure 1.4).

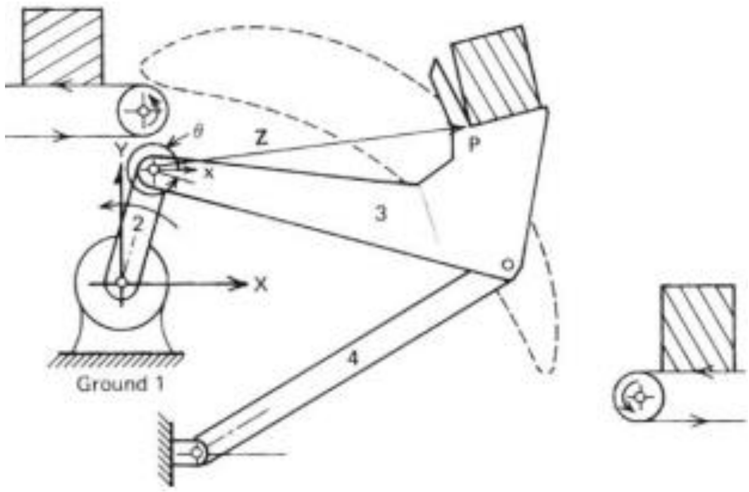


Figure 1.4. Body guidance in a transport machine
(Source: Sandor and Erdman, 1984)

1.2.1. Function Generation Synthesis

In function generation, the function to be generated might have single or multiple inputs or outputs. For a two-input single-output function let the function to be generated be z = f(x, y) for $x_{min} \le x \le x_{max}$, $y_{min} \le y \le y_{max}$ and $x_{min} \le x \le x_{max}$ and $z_{min} \le z \le z_{max}$. The function inputs x and y should be related to the mechanism inputs θ and ϕ and the function output z should be related to the mechanism output ψ. θ, ϕ and ψ are in ranges $\theta_{min} \le \theta \le \theta_{max}$, $\phi_{min} \le \phi \le \phi_{max}$, $\psi_{min} \le \psi \le \psi_{max}$ and typically the limits can be arbitrarily chosen. One can linearly relate x to input θ, y to input ϕ and z to output ψ as

$$\frac{x-x_{min}}{x_{max}-x_{min}}=\frac{\phi-\phi_{min}}{\phi_{max}-\phi_{min}} \ , \ \frac{y-y_{min}}{y_{max}-y_{min}}=\frac{\psi-\psi_{min}}{\psi_{max}-\psi_{min}} \ , \ \frac{z-z_{min}}{z_{max}-z_{min}}=\frac{\theta-\theta_{min}}{\theta_{max}-\theta_{min}} \tag{1.1}$$

Then desired θ, ϕ and ψ values for given x, y and z are found as follows:

$$\begin{gathered}\theta=\frac{\theta_{max}-\theta_{min}}{x_{max}-x_{min}}\left(x-x_{min}\right)+\theta_{min}, \phi=\frac{\phi_{max}-\phi_{min}}{y_{max}-y_{min}}\left(y-y_{min}\right)+\phi_{min}, \\ \psi=\frac{\psi_{max}-\psi_{min}}{z_{max}-z_{min}}\left(z-z_{min}\right)+\psi_{min}\end{gathered} \tag{1.2}$$

and conversely

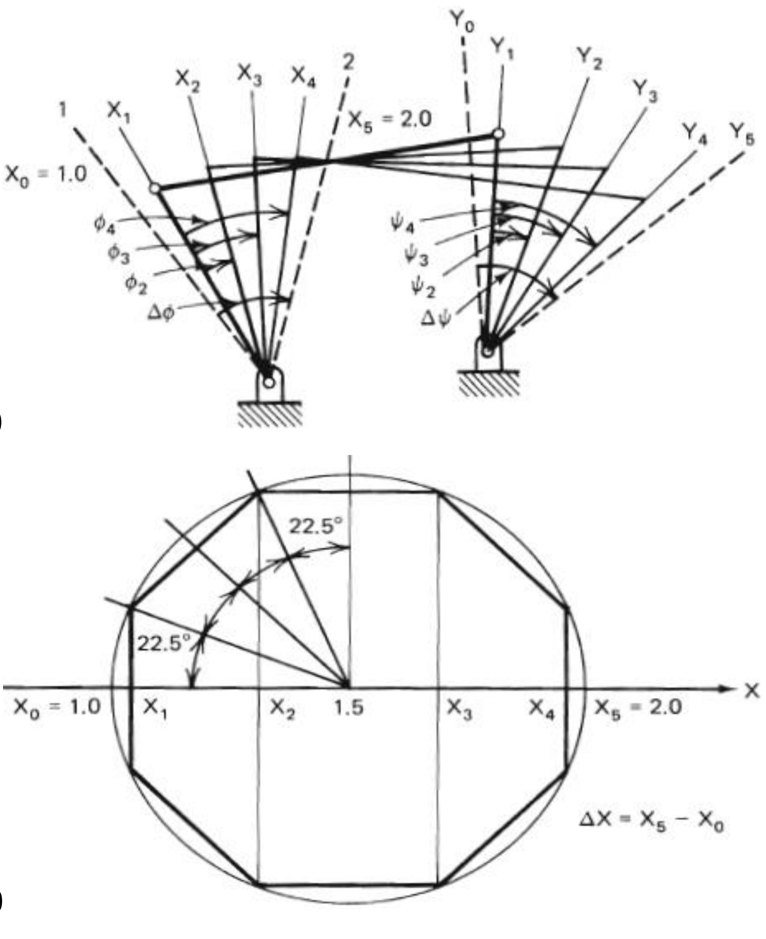
$$\begin{gathered}x=\frac{x_{max}-x_{min}}{\theta_{max}-\theta_{min}}\left(\theta-\theta_{min}\right)+x_{min}, \ y=\frac{y_{max}-y_{min}}{\phi_{max}-\phi_{min}}\left(\phi-\phi_{min}\right)+y_{min}, \\ z=\frac{z_{max}-z_{min}}{\psi_{max}-\psi_{min}}\left(\psi-\psi_{min}\right)+z_{min}\end{gathered} \tag{1.3}$$

Eq. (1.2) is used when determining the design points $\{\theta_i\}_1^N$, $\{\phi_i\}_1^N$ and $\{\psi_i\}_1^N$ in terms of $\{x_i\}_1^N$, $\{y_i\}_1^N$ and $\{z_i\}_1^N=\{f\left(x_i,y_i\right)\}_1^N$. The design points (x_i, y_i) can be selected with equal spacing on the rectangular domain given by $x_{min} \le x \le x_{max}$ and $y_{min} \le y \le y_{max}$, such that $x_i=x_{min}+\frac{i-1}{N-1}\left(x_{max}-x_{min}\right)$ and $y_i=y_{min}+\frac{i-1}{N-1}\left(y_{max}-y_{min}\right)$ for i = 1, 2,…,N.

Eq. (1.3) is used after the synthesis is performed, to check the error in between the desired z = f(x, y) and the generated z with the mechanism. At this step, one shall determine the output values of the mechanism loops for several given input values by solving the I/O relationship.

1.3. Spacing of Design Point

The main spacing type for design points is the equal spacing in which the design points are equally spaced in their domain. Equal spacing is one of the most commonly used type of spacing. Another well-known type of spacing for a single variable is the Chebyshev spacing.



a)

b)

Figure 1.5. a) Schematic of the function generation for 4 design points with equal spacing b) Schematic of the function generation for 4 design points with Chebyshev spacing (Source: Sandor and Erdman, 1984)

Computationally the design points of equal spacing in Eq. (1.4) and Chebyshev spacing in Eq. (1.5) can be found as follows:

$$x_i = x_{n-0} + \frac{x_{n+1} - x_0}{n} \quad \text{for } i = 1 \ldots n \tag{1.4}$$

$$x_i = \frac{x_{n+1} + x_0}{2} - \frac{x_{n+1} - x_0}{2} \cos\frac{(2i-1)\pi}{2n} \quad \text{for } i = 1 \ldots n \tag{1.5}$$

1.4. Theory of Function Approximation

Three main approximation methods are interpolation approximation, least square approximation and best (Chebyshev) approximation.

During the approximation process, if exists, nonlinear system of equations may be linearized using Lagrange parameters.

1.4.1. Interpolation Approximation

In interpolation approximation, an approximation function $F(\overline{C},\overline{\mathbf{x}})$ is used to approximate a function $F(\overline{\mathbf{x}})$ such that the difference $F(\overline{C},\overline{\mathbf{x}})$ – $F(\overline{\mathbf{x}})$ in certain points, called the precision points, is set to zero. Here, $\overline{\mathbf{x}}$ is the set of function inputs and $\overline{C}$ are the construction parameters of the approximator. The aim is to determine the $\overline{C}$ parameters.

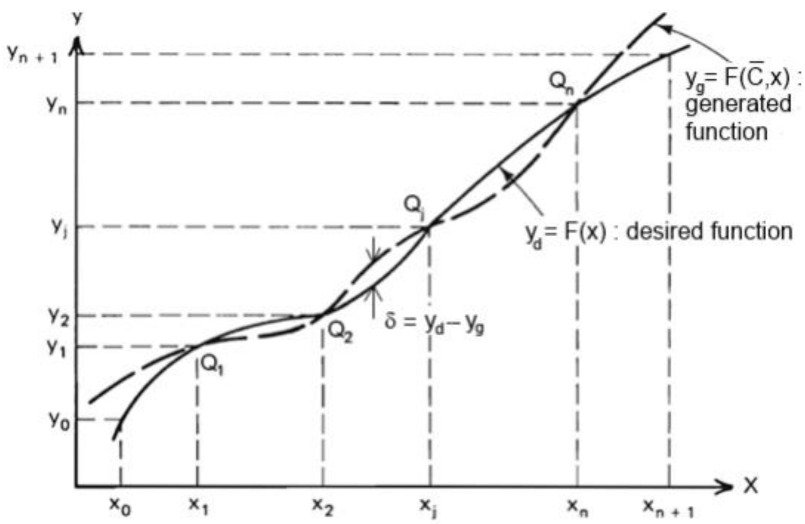


Figure 1.6. Interpolation approximation

(Source: Sandor and Erdman, 1984)

In Figure 1.6,

$y_d = F(\bar{\mathbf{x}})$: given function

$y_m = F(\bar{C},\bar{\mathbf{x}})$: generated function

$\delta(\bar{\mathbf{x}}) = F(\bar{C},\bar{\mathbf{x}}) - F(\bar{\mathbf{x}})$: structural error at $\bar{\mathbf{x}}$

Assume that $F(\bar{C},\bar{\mathbf{x}})$ can be expressed in the following polynomial form:

$$y_{gen} = F(\bar{C},\bar{\mathbf{x}}) = \sum_{j=1}^{n} P_j f_j(\bar{\mathbf{x}}) \tag{1.6}$$

where P_j are functions of the construction parameters $\bar{C}$ and f_i are linearly independent continuous functions. At the precision points x, i = 1, …, n have $y_g = y_d$;

$$y_d = F_i(\bar{\mathbf{x}}) = \sum_{j=1}^{n} P_j f_j(\bar{x}_j) \text{ for } j = 1,\ldots,n \tag{1.7}$$

Eq. (1.7) is a linear set of equations in P_i as shown in Eq. (1.8) and P_i can be computed as in Eq. (1.9);

$$\begin{bmatrix} f_1(\bar{x}_1) & f_2(\bar{x}_1) & \ldots & f_n(\bar{x}_1) \\ f_1(\bar{x}_2) & f_2(\bar{x}_2) & \ldots & f_n(\bar{x}_2) \\ \ldots & \ldots & \ldots & \ldots \\ f_1(\bar{x}_n) & f_2(\bar{x}_n) & \ldots & f_n(\bar{x}_n) \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \ldots \\ P_n \end{bmatrix} = \begin{bmatrix} F_1(\bar{x}_1) \\ F_2(\bar{x}_2) \\ \ldots \\ F_n(\bar{x}_n) \end{bmatrix} \tag{1.8}$$

$$\begin{bmatrix} P_1 \\ P_2 \\ \ldots \\ P_n \end{bmatrix} = \begin{bmatrix} f_1(\bar{x}_1) & f_2(\bar{x}_1) & \ldots & f_n(\bar{x}_1) \\ f_1(\bar{x}_2) & f_2(\bar{x}_2) & \ldots & f_n(\bar{x}_2) \\ \ldots & \ldots & \ldots & \ldots \\ f_1(\bar{x}_n) & f_2(\bar{x}_n) & \ldots & f_n(\bar{x}_n) \end{bmatrix}^{-1} \begin{bmatrix} F_1(\bar{x}_1) \\ F_2(\bar{x}_2) \\ \ldots \\ F_n(\bar{x}_n) \end{bmatrix} \tag{1.9}$$

1.4.2. Least Square Approximation

In least square approximation, the number of design points, n, have to be strictly larger than the number of construction parameters, m. In this type of approximation, the aim is to minimize the sum of the squares of the structural errors, at the design points given $\mathbf{x}_i$ for i = 1, …, n. For

$$\delta = \sum_{j=1}^{m} P_j f_j(\mathbf{x}_i) - F(\mathbf{x}_i) = 0 \text{ for } j = 1,...,n \tag{1.10}$$

The sum of the squares of the structural errors is given by

$$S = \sum_{j=1}^{m} \delta_i^2 = \sum_{i=1}^{m} \left[\sum_{j=1}^{m} P_j f_j(\bar{\mathbf{x}}_i) - F(\bar{\mathbf{x}}_i) \right]^2 \tag{1.11}$$

We equate the derivatives of S with respect to P_j to zero to find its minimum:

$$\frac{1}{2}\frac{dS}{dP_j} = \sum_{i=1}^{n} f_j(\bar{\mathbf{x}}_i) \left[\sum_{j=1}^{m} P_j f_j(\bar{\mathbf{x}}_i) - F(\bar{\mathbf{x}}_i) \right] = 0 \tag{1.12}$$

Eqs. (1.12) are linear in P_j's:

$$\begin{bmatrix} \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i) f_1(\bar{\mathbf{x}}_i) & \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i) f_2(\bar{\mathbf{x}}_i) & \cdots\cdots & \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i) f_m(\bar{\mathbf{x}}_i) \\ \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i) f_1(\bar{\mathbf{x}}_i) & \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i) f_2(\bar{\mathbf{x}}_i) & \cdots\cdots & \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i) f_m(\bar{\mathbf{x}}_i) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i) f_1(\bar{\mathbf{x}}_i) & \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i) f_2(\bar{\mathbf{x}}_i) & \cdots\cdots & \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i) f_m(\bar{\mathbf{x}}_i) \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \cdots\cdots \\ P_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i) F(\bar{\mathbf{x}}_i) \\ \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i) F(\bar{\mathbf{x}}_i) \\ \\ \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i) F(\bar{\mathbf{x}}_i) \end{bmatrix} \tag{1.13}$$

Eqs. (1.13) are linear in P_j, hence P_j can be determined uniquely. However, there are some restrictions on P_j in order to obtain a mechanism. If P_j's are greater than the number of the construction parameters, then P_j can be solved with Lagrange parameters.

For example, consider the case where there are four construction parameters, but six P_j's. Then, P_{m+1} and P_{m+2} can be defined in terms of the other P_j's and two Lagrange parameters $\lambda_1 = P_{m+1}$ and $\lambda_2 = P_{m+2}$ are introduced as two more construction parameters. In order to linearize the system, let $P_j = \ell_j + m_j\lambda_1 + n_j\lambda_2$ for j = 1, 2,…, n. Eqs. (1.10) become;

$$\sum_{j=1}^{m}\left(\ell_j + m_j\lambda_1 + n_j\lambda_2\right) f_j(\mathbf{x}) + \lambda_1 f_{m+1}(\mathbf{x}) + \lambda_2 f_{m+2}(\mathbf{x}) - F(\mathbf{x}) = 0 \tag{1.14}$$

Eqs. (1.14) should be satisfied for all design points, hence the coefficients of λ_1, λ_2 and the rest can be dissected as follows:

$$\sum_{j=1}^{m} \ell_j f_j(\mathbf{x}) - F(\mathbf{x}) = 0 \tag{1.15}$$

$$\sum_{j=1}^{m} m_j f_j(\mathbf{x}) + f_{m+1}(\mathbf{x}) = 0 \tag{1.16}$$

$$\sum_{j=1}^{m} n_j f_j(\mathbf{x}) + f_{m+2}(\mathbf{x}) = 0 \tag{1.17}$$

In least squares approximation the number of design points, n, is necessarily greater than the number of construction parameters and the aim is to minimize the square-sum of the errors at these design points. At each design point i, the square sum of the errors corresponding to Eqs. (1.15)-(1.17) are defined as

$$S_\ell = \sum_{i=1}^{n}\left[\sum_{j=1}^{m} \ell_j f_{ji} - F(\mathbf{x}_i)\right]^2 \tag{1.18}$$

$$S_m = \sum_{i=1}^{n}\left[\sum_{j=1}^{4} m_j f_{ji} + f_{m+1}\right]^2 \tag{1.19}$$

$$S_n = \sum_{i=1}^{n}\left[\sum_{j=1}^{m} n_j f_{ji} + f_{m+2}\right]^2 \tag{1.20}$$

where $f_{ji} = f_j(\mathbf{x}_i), f_{5i} = f_5(\mathbf{x}_i), f_{6i} = f_6(\mathbf{x}_i)$ and $F_i = F(\mathbf{x}_i)$. In order to find the minimum of the square sums, the derivatives of Eqs. (1.18)-(1.20) with respect to ℓ_j, m_j, n_j are set to zero to obtain

$$\frac{1}{2}\frac{dS_\ell}{d\ell_j} = \sum_{i=1}^{n}\left[f_{1i}\ell_1 + f_{2i}\ell_2 + \ldots + f_{mi}\ell_m - F_i\right]f_{ji} = 0 \qquad \text{for } j = 1, 2\ldots m \tag{1.21}$$

$$\frac{1}{2}\frac{dS_\ell}{d\ell_j} = \sum_{i=1}^{n}\left[f_{1i}\ell_1 + f_{2i}\ell_2 + \ldots + f_{mi}\ell_m - f_{(m+1)i}\right]f_{ji} = 0 \qquad \text{for } j = 1, 2\ldots m \tag{1.22}$$

$$\frac{1}{2}\frac{dS_\ell}{d\ell_j} = \sum_{i=1}^{n}\left[f_{1i}\ell_1 + f_{2i}\ell_2 + \ldots + f_{mi}\ell_m - f_{(m+2)i}\right]f_{ji} = 0 \qquad \text{for } j = 1, 2\ldots m \tag{1.23}$$

ℓ_i, m_i and n_i are linearly fund as

$$\begin{bmatrix} \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i)f_1(\bar{\mathbf{x}}_i) & \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i)f_2(\bar{\mathbf{x}}_i) & \ldots\ldots\ldots & \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i)f_m(\bar{\mathbf{x}}_i) \\ \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i)f_1(\bar{\mathbf{x}}_i) & \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i)f_2(\bar{\mathbf{x}}_i) & \ldots\ldots\ldots & \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i)f_m(\bar{\mathbf{x}}_i) \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i)f_1(\bar{\mathbf{x}}_i) & \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i)f_2(\bar{\mathbf{x}}_i) & \ldots\ldots\ldots & \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i)f_m(\bar{\mathbf{x}}_i) \end{bmatrix} \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ldots\ldots \\ \ell_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i)F(\bar{\mathbf{x}}_i) \\ \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i)F(\bar{\mathbf{x}}_i) \\ \\ \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i)F(\bar{\mathbf{x}}_i) \end{bmatrix} \tag{1.24}$$

$$\begin{bmatrix} \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i)f_1(\bar{\mathbf{x}}_i) & \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i)f_2(\bar{\mathbf{x}}_i) & \ldots\ldots\ldots & \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i)f_m(\bar{\mathbf{x}}_i) \\ \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i)f_1(\bar{\mathbf{x}}_i) & \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i)f_2(\bar{\mathbf{x}}_i) & \ldots\ldots\ldots & \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i)f_m(\bar{\mathbf{x}}_i) \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i)f_1(\bar{\mathbf{x}}_i) & \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i)f_2(\bar{\mathbf{x}}_i) & \ldots\ldots\ldots & \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i)f_m(\bar{\mathbf{x}}_i) \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \ldots\ldots \\ m_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i)F(\bar{\mathbf{x}}_i) \\ \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i)F(\bar{\mathbf{x}}_i) \\ \ldots\ldots\ldots \\ \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i)F(\bar{\mathbf{x}}_i) \end{bmatrix} \tag{1.25}$$

$$\begin{bmatrix} \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i)f_1(\bar{\mathbf{x}}_i) & \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i)f_2(\bar{\mathbf{x}}_i) & \ldots\ldots\ldots & \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i)f_m(\bar{\mathbf{x}}_i) \\ \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i)f_1(\bar{\mathbf{x}}_i) & \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i)f_2(\bar{\mathbf{x}}_i) & \ldots\ldots\ldots & \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i)f_m(\bar{\mathbf{x}}_i) \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i)f_1(\bar{\mathbf{x}}_i) & \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i)f_2(\bar{\mathbf{x}}_i) & \ldots\ldots\ldots & \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i)f_m(\bar{\mathbf{x}}_i) \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ \ldots\ldots \\ n_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} f_1(\bar{\mathbf{x}}_i)F(\bar{\mathbf{x}}_i) \\ \sum_{i=1}^{n} f_2(\bar{\mathbf{x}}_i)F(\bar{\mathbf{x}}_i) \\ \ldots\ldots \\ \sum_{i=1}^{n} f_m(\bar{\mathbf{x}}_i)F(\bar{\mathbf{x}}_i) \end{bmatrix} \tag{1.26}$$

1.4.3. Chebyshev Approximation

In this section a review of the polynomial approximation methods first studied by Chebyshev (1854, 1859) are presented. Let f(x) be a continuous function defined on $x \in [a, b]$. A polynomial $P_n(x)$ of degree n is called the best approximation of f(x) if $L = \max|f(x) – P(x)|$ is minimum. The alternation theorem of Chebyshev states that for a given function f(x) and order n the best approximation is unique and the extremum values L are attained n + 2 times on [a, b] alternately with opposite signs (Figure 1.7).

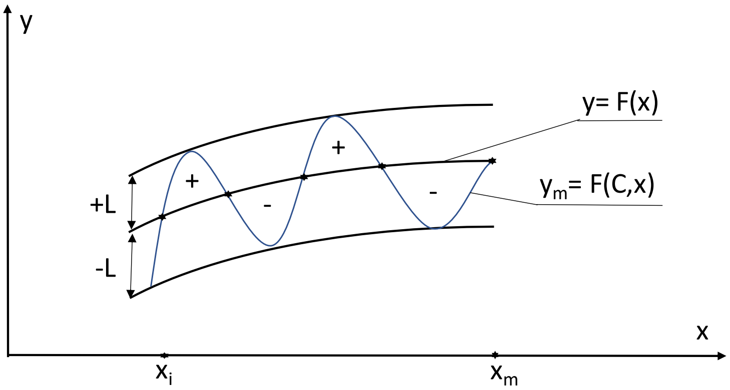


Figure 1.7. Best (Chebyshev) Approximation

Although this theorem guaranties the unique existence of the best approximation of a function, neither it, nor its proof leads to a method to find the best approximation. An iterative method is proposed by Remez (1932). The Remez algorithm is as follows (Temes, 1967):

1. Select design points $x_i^0 \in [a, b]$, i = 0, .., n + 1 (usually $x_0 = a$, $x_n+1 = b$) and linearly solve for the coefficients of an approximation polynomial P_n^0 (x) and L from the n + 2 equations $P_n^0(x_i^0) – f(x_i^0) = (–1)^iL$
2. Find the n + 2 local extrema x_i^1 of $E^0(x) = P_n^0(x_i^0) – f(x_i^0)$ in [a, b].
3. Repeat steps 1 and 2 by replacing x_i^j by x_i^{j+1} until the design points stabilize.

Convergence is not guaranteed in this iterative algorithm. In function synthesis of mechanisms, it is necessary to derive the input/output (I/O) relationship which is a function of the construction parameters (link dimensions) of the mechanism and input and output joint variables. The I/O equation is written in a polynomial form, where P_j are functions of the construction parameters and f_j are the functions of the input and output variables represented by **x**. If the number of coefficients P_j (= n) is equal to the number of construction parameters, then given n + 1 many design points $\{\mathbf{x}_i\}_1^{n+1}$, the coefficients $\{P_j\}_1^n$ and the Chebyshev error L are linearly solved from

$$\sum_{j=1}^{n} P_j f_j(\mathbf{x_i}) - F(\mathbf{x_i}) = (-1)^{i+1} L \text{ for } i = 1, ..., n+1 \tag{1.27}$$

Once $\{P_j\}_1^n$ are solved, the construction parameters are determined from $\{P_j\}_1^n$. If the number of coefficients P_j (= n) is greater than the number of construction parameters, it means that the coefficients P_j are interrelated. For illustration, consider the case where there are n – 1 construction parameters, but there are n P_j's. Then a coefficient is chosen, say, P_n, which is expressed in terms of the others. Let $P_n = \lambda$, $P_j = \ell_j + m_j\lambda$ for j = 1, …, n – 1 and $L = \ell + m\lambda$. Eq. (1.27) can be reformulated as

$$\sum_{j=1}^{n-1} (\ell_j + m_j\lambda) f_j(\mathbf{x_i}) + \lambda f_n(\mathbf{x_i}) - F(\mathbf{x_i}) = (-1)^{i+1}(\ell + m\lambda) \text{ for } i = 1, \ldots, n \tag{1.28}$$

The multipliers of λ may be collected in Eq. (1.28):

$$\sum_{j=1}^{n-1} \ell_j f_j(\mathbf{x_i}) - F(\mathbf{x_i}) + \lambda\left[\sum_{j=1}^{n-1} m_j f_j(\mathbf{x_i}) + f_n(\mathbf{x_i})\right] = (-1)^{i+1}\ell + \lambda(-1)^{i+1} m, \ i = 1, \ldots, n \tag{1.29}$$

Equating the coefficients of λ and the remaining parts in Eq. (1.29)

$$\sum_{j=1}^{n-1} \ell_j f_j(\mathbf{x_i}) + (-1)^i \ell = F(\mathbf{x_i}) \text{ and } \sum_{j=1}^{n-1} m_j f_j(\mathbf{x_i}) + (-1)^i m = -f_n(\mathbf{x_i}), \ i = 1, \ldots, n \tag{1.30}$$

Eq. (1.30) is a set of 2n linear equations with unknowns $\{\ell_j\}_1^{n-1}$, $\{m_j\}_1^{n-1}$, ℓ and m. After the unknowns are determined, since there is a relationship between $P_n = \lambda$ and some other coefficients, λ is solved from this relationship. $P_i = \ell_j + m_j\lambda$ for j = 1, …, n – 1 and $L = \ell + m\lambda$ are determined. The n – 1 construction parameters are solved from $\{P_j\}_1^{n-1}$.

If there are n – 2 construction parameters for n many P_j's, let $P_{n-1} = \lambda_1$, $P_n = \lambda_2$, $P_j = \ell_j + m_j\lambda_1 + n_j\lambda_2$ for j = 1, ..., n – 2 and $L = \ell + m\lambda_1 + n\lambda_2$. P_{n-1} and P_n depend on other coefficients. Eq. (1.27) becomes

$$\sum_{j=1}^{n-2}\left(\ell_j + m_j\lambda_1 + n_j\lambda_2\right)f_j(\mathbf{x_i}) + \lambda_1 f_{n-1}(\mathbf{x_i}) + \lambda_2 f_n(\mathbf{x_i}) - F(\mathbf{x_i}) = (-1)^{i+1}\left(\ell + m\lambda_1 + n\lambda_2\right) \qquad (1.31)$$
$$\text{for } i = 1, \ldots, n-1$$

Equating the coefficients of λ_1, λ_2 and the remaining parts in Eq. (1.31):

$$\sum_{j=1}^{n-2}\ell_j f_j(\mathbf{x_i}) + (-1)^i \ell = F(\mathbf{x_i}),\ \sum_{j=1}^{n-2} m_j f_j(\mathbf{x_i}) + (-1)^i m = -f_{n-1}(\mathbf{x_i}) \text{ and}$$
$$\sum_{j=1}^{n-2} n_j f_j(\mathbf{x_i}) + (-1)^i n = -f_n(\mathbf{x_i}) \quad i = 1, \ldots, n-1 \qquad (1.32)$$

$\{\ell_j\}_1^{n-2}$, $\{m_j\}_1^{n-2}$, $\{n_j\}_1^{n-2}$, ℓ, m and n are solved linearly from Eq. (1.32). Then λ_1 and λ_2 are solved from the relationship in between P_{n-1}, P_n and the other coefficients. $P_j = \ell_j + m_j\lambda_1 + n_j\lambda_2$ for j = 1,...,n – 2 and $L = \ell + m\lambda_1 + n\lambda_2$ are determined. The n – 2 construction parameters are solved from $\{P_j\}_1^{n-1}$.

1.5. Aim of the Thesis

Kinematically deficient manipulators are popular in applications because of structural simplicity, low cost, and ease of use and there are not many studies of 2-dof function generation synthesis in the literature. The aim of this thesis is to examine the function synthesis of 2-loop and 2-dof mechanisms using least squares and Chebyshev approximation methods.

1.6. Outline of the Thesis

This thesis comprises 7 chapters: Introduction, Literature Survey, Function Synthesis of the Planar 5R Mechanism using Least-Square Approximation, Function

Synthesis of Bennet 6R Mechanisms Using Chebyshev Approximation, Function Generation Synthesis with a 2-DoF Over-constrained Double-Spherical 7R Mechanism Using the Method of Decomposition and Least Square Approximation, Function Synthesis of a Family Of 2-Dof Planar Linkages Using Least Squares Approximation, and Conclusions. In Chapter 2, literature review and explanatory information on kinematics and calculations related to kinematic synthesis. Chapters 3-to-6 present approximate function generation solutions for different types of planar mechanisms. In Chapter 7 the results of the thesis are summarized and possible problems for future studies are discussed.

CHAPTER 2

LITERATURE SURVEY

In this Chapter, review on kinematic synthesis of mechanisms is presented briefly. Then, literature survey on function generation synthesis of planar and spherical 1- and 2-dof mechanisms is presented.

2.1. Over-Constrained Bennet 6R Mechanisms

The first example of a $\lambda = 5$ (motion space dimension) mechanism is the Sarrus (1853) linkage. The Sarrus linkage is a spatial 6R linkage (R: revolute joint) obtained by assembling two planar dyads in perpendicular planes. Later, Bennett (1905) generalized the Sarrus linkage such that the angle between the planes of the dyads is arbitrary (Figure 2.1a). For this linkage, no link has a rotational motion about the axis along the intersection of the planes of the dyads, so the linkage belongs to an RRPPP (P: prismatic) type of $\lambda = 5$ subspace. The generalized Sarrus linkage can be considered as the combination of two planar slider-crank mechanisms in intersecting planes such that the fixed link and the slider link is common to both of mechanisms. The sliding direction is along the intersection of the planes. In such an assembly, the linkage remains mobile with single dof even if the prismatic joint is removed. Hence a single loop 6R linkage is obtained. The removed prismatic joint is defined as a passive joint (Selvi, 2012).

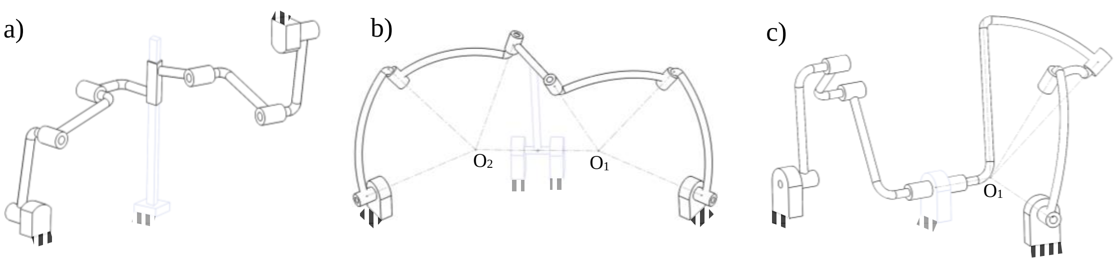


Figure 2.1. a) Double-planar 6R (generalized Sarrus) linkage, b) double-spherical 6R linkage, c) plano-spherical 6R linkage with their passive joints

Along with the generalized Sarrus linkage, Bennett (1905) introduced two more $\lambda = 5$ mechanisms. One of these mechanisms is the double-spherical 6R mechanism which is obtained by merging two spherical four-bar linkages with two common links and then removing the joint connecting the common links. The removed revolute joint is the passive joint of this linkage and its axis is along the line connecting the centers of the spherical four-bars (Figure 2.1b). It is an RRRPP type of $\lambda = 5$ subspace, where none of the links can translate along the line connecting the two spherical centers. The other mechanism is the plano-spherical 6R mechanism, which is obtained by merging a planar four-bar linkage and a spherical four-bar linkage with two common links. Once again, the revolute joint connecting the common links is removed to obtain the 6R linkage (Figure 2.1c). The axis of this passive joint passes through the center of the spherical four-bar and it is in the same direction as all the axes of the planar four-bar. No link has a translational motion along the direction of axes of the planar four-bar, hence the linkage belongs to an RRRPP type of $\lambda = 5$ subspace.

2.2. Kinematic Synthesis of Mechanisms

Due to constructional simplicity, low cost, ease of use and stiffness capabilities, kinematically deficient manipulators are popular in applications (Huang and Ding, 2012) (as an example see (Vaida et al., 2014)). Some researchers use the deficient term as a substitute to under-actuated, however what we mean by a deficient manipulator is a manipulator with less dof than the motion space dimension (λ). Although analytical synthesis methods for single dof mechanisms are widely studied (Sandor and Erdman, 1984, 1997; McCarthy and Soh, 2010). Synthesis of spherical four-bar mechanism is also widely studied. Hartenberg and Denavit (1964) and Zimmerman (1967) have respectively presented the three and four precision-point function generation of the synthesis of spherical four-bar mechanism. Rao et al. (1973), Farhang et al. (1988, 1999), Alizade et al. (1994, 2005) and Murray and McCarthy (1995) used polynomial approximation method for three, four and five precision points for the function synthesis of the spherical four bar mechanism. Cervantes et al. (2009a) worked on formulation of the function synthesis problem of the spherical four-bar mechanism for three and four precision points and further presented formulations for five and six precision points (Cervantes et al. (2009b). Suixian et al. (2009) worked on the optimal selection of precision points. Alizade and Gezgin (2011) have applied interpolation, least squares and Chebyshev approximation methods to solve the six-precision-point function synthesis of the spherical four-bar mechanism with Chebyshev spacing. Maaroof and Dede (2013, 2014) worked on the synthesis of the double-spherical 6R linkage using interpolation approximation. Levitskii and Sarkisian (1968) and Alizade and Kilit (2013) applied least-square approximation model with Chebyshev spacing.

Although analytical synthesis methods for single-dof mechanisms are widely studied, mostly optimization methods are utilized for determining link length dimensions of multi-dof mechanisms (or instance see (Alizade et al., 1975)). An exceptional study is analytical motion synthesis of a 3-RPS manipulator (Kim, 2003). There are a few studies on analytical synthesis methods for multi-dof systems. Two major studies are by Svoboda (1965) and Davitashvili (2000), who have worked on synthesis of 2-dof mechanisms. Kiper et al. (2013) worked on function synthesis of a planar 5R mechanism. Kiper and Bilgincan (2013) worked on a spherical 5R mechanism with Chebyshev approximation, where one of the fixed joints is an input, the mid-joint is the second input and the remaining fixed joint is the output. The reason of choosing the mid-joint as an input instead of a joint adjacent to a fixed joint is that this selection leads to linear set of equations.

Numerical optimization techniques are mostly used to design planar 2-dof 7-link planar mechanisms. According to Svoboda (1965) the synthesis of planar 7-bar mechanisms is done in two steps. At each stage, the two loops of the 7-bar mechanism are fixed together, in other words a joint angle is fixed. The resulting 1-dof 6 bar mechanism is discussed. Svoboda (1965) developed geometric tools for designing planar mechanisms to perform functions such as simple two-input addition, multiplication, and division. Balli and Chand (2003) synthesized motion between two dead-center positions with a 7-bar mechanism. Daivagna and Balli (2010) studied a 2-dof planar 6R1P mechanism for function synthesis. Gadad et al. (2012) discussed the synthesis of 2-dof planar 7-bar mechanisms using dyads. Lakshminarayana and Ramaiyan (1970, 1973, 1976) worked on higher-order synthesis with 7-bar and 9-bar planar mechanisms for two-inputs with position and velocity zero-defect points. Mruthyunjaya (1972a, 1972b) developed a graphical method, which they call "point position reduction" with six zero-defect points using rotary input/output and sliding input/output. Kohli and Soni (1973) studied the synthesis of function, trajectory and motion of 2 dof 7-bar mechanisms, where the equation set obtained by subtracting the loop closure equations is solved by numerical methods. They used a 2RRR-RR planar parallel mechanism with 13 zero-defect points for the function synthesis problem.

In the following Chapter, novel function generation synthesis methods for 2-dof and 2-loop mechanisms are presented.

CHAPTER 3

FUNCTION SYNTHESIS OF THE PLANAR 5R MECHANISM USING LEAST SQUARE APPROXIMATION[1]

In this section, the problem of function generation synthesis of planar 5R mechanism is studied using the least square approximation method. The study represents a case study for function generation of multi-dof systems. The studied planar 5R mechanism is designed with a fixed input joint and a moving input joint adjacent to the first input, whereas the remaining fixed joint is the output joint. The objective function of the planar 5R mechanism is expressed in polynomial form with four unknown construction parameters. The objective function involves nonlinearities hence the synthesis problem is solved semi-analytically. Finally, the construction parameters of the mechanism are determined. The present study differs from (Kiper and Bilgincan, 2013) in selection of one of the inputs, and also a different synthesis method is utilized.

3.1. Formulation

In this study, the input variables θ and ϕ for the planar 5R mechanism are associated with one of the fixed joints and the adjacent floating joint (Figure 3.1). The output variable ψ is associated with the remaining fixed joint. In practice, an extra parallelogram loop can be employed in order to actuate the mechanism at fixed joints. Since the scale of the mechanism does not affect the I/O relationship, without loss of generality it can be assumed that the fixed link length is 1. The construction parameters are a, b, d and e.

The I/O relationship for the mechanism is obtained as follows:

$$\left|\overrightarrow{CD}\right| = \left|\overrightarrow{AE} + \overrightarrow{ED} - \overrightarrow{AB} - \overrightarrow{BC}\right| \Rightarrow \left(ac\theta + bc\phi - 1 - ec\psi\right)^2 + \left(as\theta + bs\phi - es\psi\right)^2 = d^2 \quad (3.1)$$

[1] The main content of this chapter is published by Kiper et al. (2014).

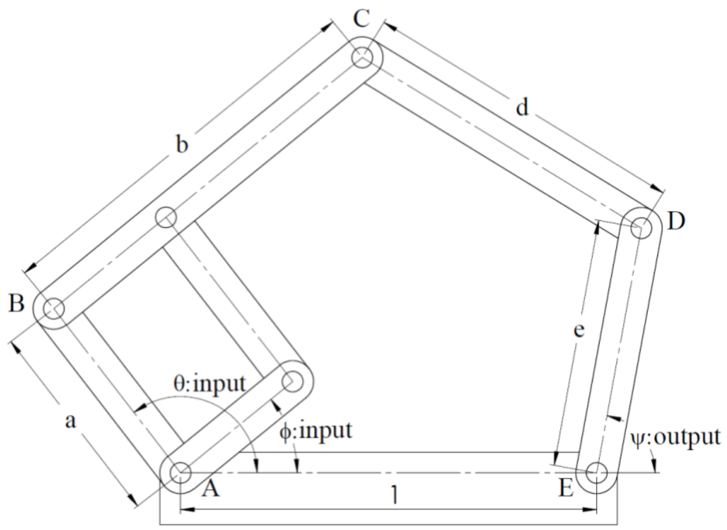


Figure 3.1. The construction and joint variables of the 5R mechanism

Rearranging Eq. (3.1) in polynomial form:

$$\sum_{j=1}^{6} P_j f_j(\mathbf{x}) - F(\mathbf{x}) = 0 \qquad (3.2)$$

where **x** represents the inputs and $\{P_j\}_1^6$, $\{f_j(\mathbf{x})\}_1^6$ and F(**x**) are defined as:

$$\begin{gathered} P_1 = \frac{-1-a^2-b^2+d^2-e^2}{2e}, P_2 = a, P_3 = b, P_4 = \frac{a}{e}, P_5 = P_3P_4 = \lambda_1 = \frac{ab}{e}, P_6 = \frac{P_5}{P_2} = \lambda_2 = \frac{b}{e}, \\ f_1(\mathbf{x}) = 1, f_2(\mathbf{x}) = \cos(\theta-\psi), f_3(\mathbf{x}) = \cos(\phi-\psi), f_4(\mathbf{x}) = \cos\theta, f_5(\mathbf{x}) = -\cos(\theta-\phi), \\ f_6(\mathbf{x}) = \cos\phi \text{ and } F(\mathbf{x}) = \cos\psi \end{gathered} \qquad (3.3)$$

There are four construction parameters, but six P_j's. Therefore, the P_5 and P_6 are defined in terms of the other P_j and two Lagrange parameters $\lambda_1 = P_5$ and $\lambda_2 = P_6$ are introduced as two more construction parameters. In order to linearize the system, let $P_j = \ell_j + m_j\lambda_1 + n_j\lambda_2$ for j = 1, 2, 3, 4. Eq. (3.2) becomes

$$\sum_{j=1}^{4} \left(\ell_j + m_j\lambda_1 + n_j\lambda_2\right) f_j(\mathbf{x}) + \lambda_1 f_5(\mathbf{x}) + \lambda_2 f_6(\mathbf{x}) - F(\mathbf{x}) = 0 \qquad (3.4)$$

Eq. (3.4)should be satisfied for all design points, hence the coefficients of λ_1, λ_2 and the rest can be dissected as follows:

$$\sum_{j=1}^{4} \ell_j f_j(\mathbf{x}) - F(\mathbf{x}) = 0 \tag{3.5}$$

$$\sum_{j=1}^{4} m_j f_j(\mathbf{x}) + f_5(\mathbf{x}) = 0 \tag{3.6}$$

$$\sum_{j=1}^{4} n_j f_j(\mathbf{x}) + f_6(\mathbf{x}) = 0 \tag{3.7}$$

In least squares approximation the number of design points, N, is necessarily greater than the number of construction parameters and the aim is to minimize the square-sum of the errors at these design points. At each design point i, the square sum of the errors corresponding to Eqs. (3.5)-(3.7) are defined as

$$S_\ell = \sum_{i=1}^{N} \left[\sum_{j=1}^{4} \ell_j f_{ji} - F(\mathbf{x}_i) \right]^2 \tag{3.8}$$

$$S_m = \sum_{i=1}^{N} \left[\sum_{j=1}^{4} m_j f_{ji} + f_{5i} \right]^2 \tag{3.9}$$

$$S_n = \sum_{i=1}^{N} \left[\sum_{j=1}^{4} n_j f_{ji} + f_{6i} \right]^2 \tag{3.10}$$

where $f_{ji} = f_j(\mathbf{x}_i), f_{5i} = f_5(\mathbf{x}_i), f_{6i} = f_6(\mathbf{x}_i)$ and $F_i = F(\mathbf{x}_i)$. In order to find the minimum of the square sums, the derivatives of Eqs. (3.8)-(3.10) with respect to ℓ_j, m_j, n_j are set to zero to obtain

$$\frac{1}{2}\frac{dS_\ell}{d\ell_j} = \sum_{i=1}^{N} \left[f_{1i}\ell_1 + f_{2i}\ell_2 + f_{3i}\ell_3 + f_{4i}\ell_4 - F_i \right] f_{ji} = 0 \qquad \text{for } j = 1,2,3,4 \tag{3.11}$$

$$\frac{1}{2}\frac{dS_m}{dm_j} = \sum_{i=1}^{N}\left[f_{1i}m_1 + f_{2i}m_2 + f_{3i}m_3 + f_{4i}m_4 + f_{5i}\right]f_{ji} = 0 \quad \text{for } j = 1,2,3,4 \tag{3.12}$$

$$\frac{1}{2}\frac{dS_n}{dn_j} = \sum_{i=1}^{N}\left[f_{1i}n_1 + f_{2i}n_2 + f_{3i}n_3 + f_{4i}n_4 + f_{6i}\right]f_{ji} = 0 \qquad \text{for } j = 1,2,3,4 \tag{3.13}$$

Eq. (3.11) is a linear set of 4 equations in unknowns ℓ_1, ℓ_2, ℓ_3, ℓ_4 and similarly Eqs. (3.12) and (3.13) are respectively linear in m_1, m_2, m_3, m_4 and n_1, n_2, n_3, n_4. Writing Eqs. (3.11)-(3.13) in matrix form:

$$\begin{gathered}\left(\sum_{i=1}^{N} f_{1i}f_{ji}\right)\ell_1 + \left(\sum_{i=1}^{N} f_{2i}f_{ji}\right)\ell_2 + \left(\sum_{i=1}^{N} f_{3i}f_{ji}\right)\ell_3 + \left(\sum_{i=1}^{N} f_{4i}f_{ji}\right)\ell_4 = \sum_{i=1}^{n} F_i f_{ji} \\ \Rightarrow \left[A_{jk}\right]\left[\ell_j\right] = \left[b_j\right]\end{gathered} \tag{3.14}$$

$$\begin{gathered}\left(\sum_{i=1}^{N} f_{1i}f_{ji}\right)m_1 + \left(\sum_{i=1}^{N} f_{2i}f_{ji}\right)m_2 + \left(\sum_{i=1}^{N} f_{3i}f_{ji}\right)m_3 + \left(\sum_{i=1}^{N} f_{4i}f_{ji}\right)m_4 = -\sum_{i=1}^{n} f_{5i}f_{ji} \\ \Rightarrow \left[A_{jk}\right]\left[m_j\right] = \left[c_j\right]\end{gathered} \tag{3.15}$$

$$\begin{gathered}\left(\sum_{i=1}^{N} f_{1i}f_{ji}\right)n_1 + \left(\sum_{i=1}^{N} f_{2i}f_{ji}\right)n_2 + \left(\sum_{i=1}^{N} f_{3i}f_{ji}\right)n_3 + \left(\sum_{i=1}^{N} f_{4i}f_{ji}\right)n_4 = -\sum_{i=1}^{n} f_{6i}f_{ji} \\ \Rightarrow \left[A_{jk}\right]\left[n_j\right] = \left[d_j\right]\end{gathered} \tag{3.16}$$

where $\left[A_{jk}\right]$ is the 4×4 coefficient matrix with

$$\begin{gathered}A_{jk} = \sum_{i=1}^{n} f_{ki}f_{ji} \quad \text{for } j \text{ and } k = 1,2,3,4 \\ \left[\ell_j\right] = \left[\ell_1 \quad \ell_2 \quad \ell_3 \quad \ell_4\right]^T \\ \left[m_j\right] = \left[m_1 \quad m_2 \quad m_3 \quad m_4\right]^T \\ \left[n_j\right] = \left[n_1 \quad n_2 \quad n_3 \quad n_4\right]^T\end{gathered} \tag{3.17}$$

$\left[b_j\right]$, $\left[c_j\right]$ and $\left[d_j\right]$ are 4×1 matrices with

$$b_j = \sum_{i=1}^{N} F_i f_{ji}, c_j = -\sum_{i=1}^{N} f_{5i} f_{ji}, d_j = -\sum_{i=1}^{N} f_{5i} f_{ji} \quad \text{for } j = 1, 2, 3, 4. \tag{3.18}$$

ℓ_j, m_j, n_j for j = 1,2,3,4 are solved from Eqs. (3.14)-(3.16) by inverting $\left[A_{jk}\right]$. As ℓ_j, m_j, n_j are determined, λ_1 and λ_2 are solved as follows:

$$\begin{aligned} &\lambda_1 = P_3 P_4 = \left(l_3 + m_3\lambda_1 + n_3\lambda_2\right)\left(l_4 + m_4\lambda_1 + n_4\lambda_2\right) \\ &\Rightarrow m_3 m_4 \lambda_1^2 + n_3 n_4 \lambda_2^2 + \left(m_3 n_4 + n_3 m_4\right)\lambda_1\lambda_2 + \left(l_3 m_4 + m_3 l_4 - 1\right)\lambda_1 \\ &\qquad + \left(l_3 n_4 + n_3 l_4\right)\lambda_2 + l_3 l_4 = 0 \end{aligned} \tag{3.19}$$

$$\lambda_2 = \frac{P_5}{P_2} = \frac{\lambda_1}{l_2 + m_2\lambda_1 + n_2\lambda_2} \Rightarrow n_2\lambda_2^2 + m_2\lambda_1\lambda_2 - \lambda_1 + l_2\lambda_2 = 0 \tag{3.20}$$

λ_1 can be solved from Eq. (3.20):

$$\lambda_1 = \frac{n_2\lambda_2^2 + l_2\lambda_2}{1 - m_2\lambda_2} \tag{3.21}$$

Substituting Eq. (3.21) in Eq. (3.19):

$$\begin{aligned} &m_3 m_4 \left(n_2\lambda_2^2 + l_2\lambda_2\right)^2 + n_3 n_4 \lambda_2^2 \left(1 - m_2\lambda_2\right)^2 \\ &\quad + \left(m_3 n_4 + n_3 m_4\right)\left(n_2\lambda_2^2 + l_2\lambda_2\right)\left(1 - m_2\lambda_2\right)\lambda_2 \\ &\quad + \left(l_3 m_4 + m_3 l_4 - 1\right)\left(n_2\lambda_2^2 + l_2\lambda_2\right)\left(1 - m_2\lambda_2\right) \\ &\quad + \left(l_3 n_4 + n_3 l_4\right)\lambda_2 \left(1 - m_2\lambda_2\right)^2 + l_3 l_4 \left(1 - m_2\lambda_2\right)^2 = 0 \end{aligned} \tag{3.22}$$

Eq. (3.22) is a degree 4 polynomial in λ_2. There may be 4, 2 or no real solutions for λ_2. If exists, once one of the solutions for λ_2 is selected, λ_1 is determined from Eq. (3.21).

$P_j = \ell_j + m_j\lambda_1 + n_j\lambda_2$ for j = 1, 2, 3, 4 are determined and the construction parameters are solved from Eq. (3.3) as

$$a = P_2, b = P_3, e = \frac{a}{P_4}, d = \sqrt{1 + a^2 + b^2 + e^2 + 2eP_1} \quad (3.23)$$

3.2. The Function Synthesis Problem

Let the function to be generated be z = f (x, y) for $x_{min} \le x \le x_{max}$ and $y_{min} \le y \le y_{max}$. The independent variables x and y should be related to the mechanism inputs θ and ϕ, and the dependent variable z should be related to the mechanism output ψ. θ, ϕ and ψ are in ranges $\theta_{min} \le \theta \le \theta_{max}$, $\phi_{min} \le \phi \le \phi_{max}$, $\psi_{min} \le \psi \le \psi_{max}$ and the limits can be arbitrarily chosen. One can linearly relate x to input θ, y to input ϕ and z to output ψ as

$$\frac{x - x_{min}}{x_{max} - x_{min}} = \frac{\phi - \phi_{min}}{\phi_{max} - \phi_{min}}, \quad \frac{y - y_{min}}{y_{max} - y_{min}} = \frac{\psi - \psi_{min}}{\psi_{max} - \psi_{min}}, \quad \frac{z - z_{min}}{z_{max} - z_{min}} = \frac{\theta - \theta_{min}}{\theta_{max} - \theta_{min}} \quad (3.24)$$

Then desired ψ and θ values for given input ϕ are found as follows:

$$\theta = \frac{\theta_{max} - \theta_{min}}{x_{max} - x_{min}}(x - x_{min}) + \theta_{min}, \; \phi = \frac{\phi_{max} - \phi_{min}}{y_{max} - y_{min}}(y - y_{min}) + \phi_{min},$$
$$\psi = \frac{\psi_{max} - \psi_{min}}{z_{max} - z_{min}}(z - z_{min}) + \psi_{min} \quad (3.25)$$

and conversely

$$x = \frac{x_{max} - x_{min}}{\theta_{max} - \theta_{min}}(\theta - \theta_{min}) + x_{min}, \; y = \frac{y_{max} - y_{min}}{\phi_{max} - \phi_{min}}(\phi - \phi_{min}) + y_{min},$$
$$z = \frac{z_{max} - z_{min}}{\psi_{max} - \psi_{min}}(\psi - \psi_{min}) + z_{min} \quad (3.26)$$

Eq. (3.25) is used when determining the design points $\{\theta_i\}_1^N$, $\{\phi_i\}_1^N$ and $\{\psi_i\}_1^N$ in terms of $\{x_i\}_1^N$, $\{y_i\}_1^N$ and $\{z_i\}_1^N = \{f(x_i, y_i)\}_1^N$. The design points ($x_i$, y_i) can be selected with equal spacing on the rectangular domain given by $x_{min} \le x \le x_{max}$ and $y_{min} \le y \le y_{max}$, i.e.

$$x_i = x_{min} + \frac{i-1}{N-1}(x_{max} - x_{min}) \text{ and } y_i = y_{min} + \frac{i-1}{N-1}(y_{max} - y_{min}) \text{ for } i = 1, 2, \ldots, N \quad (3.27)$$

Eq. (3.26) is used after the synthesis is performed, to check the error in between the desired z = f(x, y) and the generated z with the mechanism. At this step, one shall determine the output values of the mechanism loops for several given input values by solving the I/O relationship.

3.3. Case Study

The formulations in the previous sections were implemented in MS Excel and a case study was worked out for a function $z = f(x, y) = x^{1.1}y^{1.4}$ for $5 \leq x \leq 9$ and $1 \leq y \leq 4$. Limits for the mechanism input and output angles are selected as $75° \geq \theta \geq 30°$, $80° \leq \phi \leq 130°$, $120° \leq \psi \leq 170°$. Actually, several different limit values were employed, but the final selection is done according to a small maximum error and good link length ratios. The design points are selected with equal spacing of 30 intervals for both x and y. That is, there are totally 900 design points.

As a result of computations, the maximum percentage error is found as $100(\psi_{computed} - \psi_{desired})/\psi_{desired}$ = 1,33%. The variation of the percentage error over the domain of x and y is illustrated in Figure 3.2. The construction parameters were calculated as a = 2.382, b = 1.636, d = 2.671, e = 1.577.

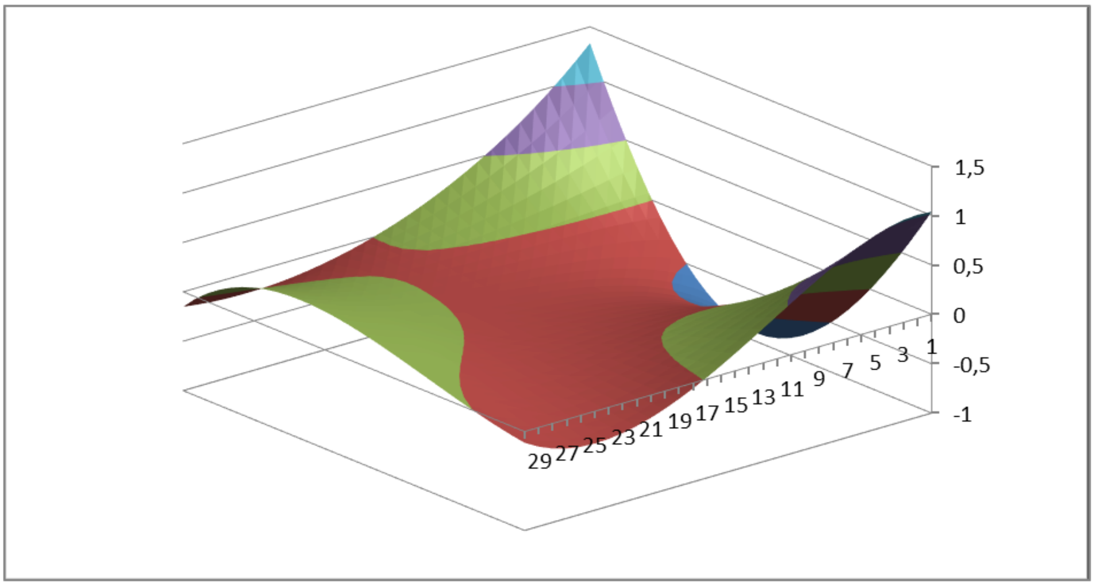


Figure 3.2. Percentage error variation

CHAPTER 4

FUNCTION SYNTHESIS OF BENNETT 6R MECHANISMS USING CHEBYSHEV APPROXIMATION[2]

This study focuses on approximate function synthesis of the three types of over-constrained Bennett 6R mechanisms using Chebyshev approximation. The three mechanisms are the double-planar, double-spherical and the plano-spherical 6R linkages. The single-loop 6R mechanisms are dissected into two imaginary loops and function synthesis is performed for both loops. First, the link lengths are employed as construction parameters of the mechanism. Then extra construction parameters for the input or output joint variables are introduced in order to increase the design points and hence enhance the accuracy of approximation. The synthesis formulations are applied computationally as case studies. The case studies illustrate how a designer can compare the three types of Bennett 6R mechanisms for the same function. Also, a comparison of the spherical four-bar with the double-spherical 6R mechanism is done and it is showed that the accuracy is improved when the 6R linkage is used.

4.1. The Objective Functions

The I/O relationships of the Bennett 6R mechanisms are derived in (Alizade et al., 2013). The objective functions are formulated based on the I/O relationships.

4.1.1. The Double-Planar 6R Mechanism with 6 Parameters

Together with the passive prismatic joint, the double-planar 6R linkage may be considered to be composed of a pair of slider-crank mechanisms. Let ϕ be the input angle, θ be the output angle and s be the passive joint variable of the double-planar 6R mechanism shown in Figure 4.1. For the time being, assume ϕ_0, s_0 and θ_0 to be zero. The relationship between the input ϕ and the output θ does not change if the mechanism

[2] The main content of this chapter is published by Alizade et al. (2014).

dimensions are scaled, so without loss of generality the fixed link length can be assumed as g = 1.

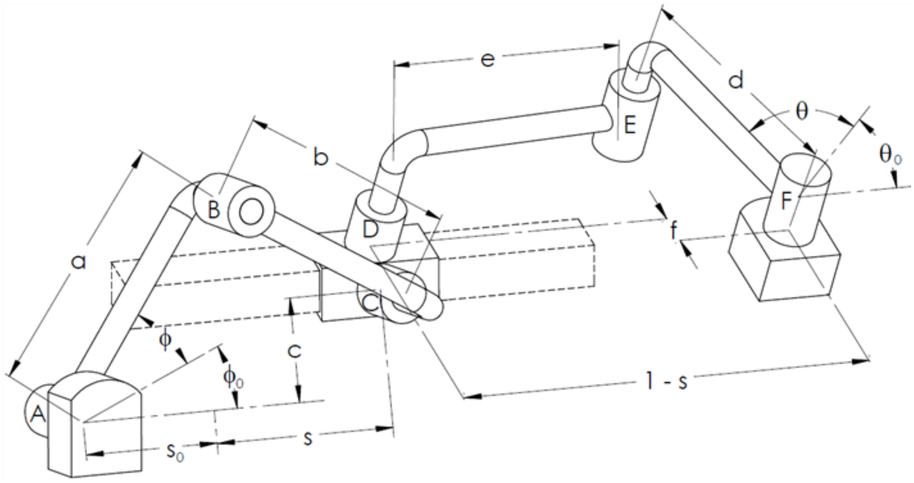


Figure 4.1. Kinematic representation of the double-planar 6R linkage

For the double planar 6R mechanism the I/O equation for the imaginary loop ABC is given by (Alizade et al., 2013)

$$b^2 = (a\cos\phi - s)^2 + (a\sin\phi - c)^2 \Rightarrow -a^2 + b^2 - c^2 + 2as\cos\phi + 2ac\sin\phi = s^2 \tag{4.1}$$

This I/O relationship contains three construction parameters, namely a, b, c. Together with the Chebyshev error L, there are four parameters to be determined. Therefore, the function synthesis may be performed for four design points. Rewriting Eq. (4.1) in polynomial form

$$\delta_i(\mathbf{x_i}) = F(\mathbf{x_i}) - \sum_{j=1}^{3} P_j f_j(\mathbf{x_i}) = (-1)^{i+1} L \text{ for } i = 1,...,4 \tag{4.2}$$

where

$$\begin{gathered} P_1 = -a^2 + b^2 - c^2,\ f_1(\mathbf{x_i}) = 1,\ P_2 = a,\ f_2(\mathbf{x_i}) = 2s_i\cos\phi_i, \\ P_3 = ac,\ f_3(\mathbf{x_i}) = 2\sin\phi_i,\ F(\mathbf{x_i}) = s_i^{\,2} \end{gathered} \tag{4.3}$$

$\mathbf{x_i} = (\phi_i, s_i)$ represents the i^{th} design point and $\delta_i(\mathbf{x_i}) = \pm L$ is the error. The reason for selecting the objective function as $F(\mathbf{x_i})=s^2$ is that this function solely comprises the output s and the error $\delta_i(\mathbf{x_i})$ better represents the error in the output. Given the design points $\mathbf{x_i}=(\phi_i, s_i)$ for i = 1,...,4, P_1, P_2, P_3 and L are determined via the Remez algorithm. Once P_1, P_2, P_3 are determined, the construction parameters a, b and c are found from Eq. (4.3) as follows:

$$a=P_2,\ c=\frac{P_3}{a},\ b=\sqrt{P_1+a^2+c^2} \tag{4.4}$$

If any of a or c turns out to be negative, or if b is not real, the limits for ϕ_i and/or s_i shall be altered. For the imaginary loop DEF (Alizade et al., 2013)

$$\begin{gathered} e^2=(1-s+d\cos\theta)^2+(-f+d\sin\theta)^2 \\ \Rightarrow d^2-e^2+f^2+(1-s)^2+2d(1-s)\cos\theta-2df\sin\theta=0 \end{gathered} \tag{4.5}$$

Eq. (4.5), once again, contains three construction parameters: d, e, f. Note that Eq. (4.1) and Eq. (4.5) have similar form. Again, there are four parameters to be determined and hence four design points are required. The polynomial form in Eq. (4.2) can be used, but this time

$$\begin{gathered} P_1=\frac{d^2-e^2+f^2}{2df},\ f_1(\mathbf{x_i})=1,\ P_2=\frac{1}{f},\ f_2(\mathbf{x_i})=(1-s_i)\cos\theta_i,\ P_3=\frac{1}{2df},\ f_3(\mathbf{x_i})=(1-s_i)^2 \\ F(\mathbf{x_i})=\sin\theta_i,\ \mathbf{x_i}=(s_i,\theta_i) \end{gathered} \tag{4.6}$$

Given the design points $\mathbf{x_i}=(s_i,\theta_i)$ for i = 1,...,4, P_1, P_2, P_3 and L are solved via Remez algorithm. The construction parameters d, e and f are determined from Eq. (4.6) as follows:

$$f=\frac{1}{P_2},\ d=\frac{1}{2P_3 f},\ e=\sqrt{d^2+f^2-2P_1 df} \tag{4.7}$$

4.1.2. The Double-Planar 6R Mechanism with 7, 8 and 9 Parameters

When the input and output angles are not measured from the same reference (x-axis in Figure 4.1), extra construction parameters can be defined as the location of the reference geometry with respect to the fixed frame. In this sense the input angle ϕ may be measured from an initial angle ϕ_0. In this case, Eq. (4.1) can be modified as

$$-a^2+b^2-c^2+2as\cos\phi_0\cos\phi-2as\sin\phi_0\sin\phi+2ac\cos\phi_0\sin\phi+2ac\sin\phi_0\cos\phi=s^2 \quad (4.8)$$

rewriting Eq. (4.8) in polynomial form

$$\delta_i(\mathbf{x_i})=F(\mathbf{x_i})-\sum_{j=1}^{5}P_j f_j(\mathbf{x_i})=(-1)^{i+1}L \text{ for i = 1,...,5} \quad (4.9)$$

where

$$\begin{gathered} P_1=-a^2+b^2-c^2,\ f_1(\mathbf{x_i})=1,\ P_2=a\cos\phi_0,\ f_2(\mathbf{x_i})=2s_i\cos\phi, \\ P_3=a\sin\phi_0,\ f_3(\mathbf{x_i})=-2s_i\sin\phi_i,\ P_4=ac\cos\phi_0,\ f_4(\mathbf{x_i})=2\sin\phi_i, \\ P_5=ac\sin\phi_0,\ f_5(\mathbf{x_i})=2\cos\phi_i,\ F(\mathbf{x_i})=s_i^2 \text{ and } \mathbf{x_i}=(\phi_i,s_i) \end{gathered} \quad (4.10)$$

P_2, P_3, P_4 and P_5 are dependent such that

$$P_3P_4 = P_2P_5 \quad (4.11)$$

Let $P_5 = \lambda$, $P_j=\ell_j\lambda+m_j$ for j = 1,...,4 and $L=\ell\lambda+m$. After finding ℓ_j and m_j as explained in Section 1.4.3, λ is determined as follows:

$$\begin{aligned} P_3P_4-P_2P_5 &=(\ell_3\lambda+m_3)(\ell_4\lambda+m_4)-\lambda(\ell_2\lambda+m_2) \\ &=(\ell_3\ell_4-\ell_2)\lambda^2+(\ell_3m_4+m_3\ell_4-m_2)\lambda+m_3m_4=0 \\ \Rightarrow\lambda &=\frac{-(\ell_3m_4+m_3\ell_4-m_2)\mp\sqrt{(\ell_3m_4+m_3\ell_4-m_2)^2-4m_3m_4(\ell_3\ell_4-\ell_2)}}{2(\ell_3\ell_4-\ell_2)} \end{aligned} \quad (4.12)$$

Note that there are two λ solutions. The designer can choose the solution, which yields the less error in the end. After determining λ, $P_j = \ell_j \lambda + m_j$ for j = 1,...,4 and $L = \ell\lambda + m$ are determined. Using Eq. (4.10) the construction parameters a, b, c and ϕ_0 in terms of P_j are found as follows:

$$\phi_0 = \operatorname{atan2}(P_2, P_3),\ a = \sqrt{P_2^2 + P_3^2},\ c = \frac{\sqrt{P_4^2 + P_5^2}}{a},\ b = \sqrt{P_1 + a^2 + c^2} \tag{4.13}$$

Note that all the construction parameters are determined uniquely and the solution is guaranteed provided that $P_1 + a^2 + c^2 \geq 0$. Similarly, one can introduce s_0 in Eq. (4.1):

$$\begin{gathered} b^2 = (a\cos\phi - s - s_0)^2 + (a\sin\phi - c)^2 \\ \Rightarrow -a^2 + b^2 - c^2 - s_0^2 + 2ac\sin\phi + 2as\cos\phi - 2s_0 s + 2as_0\cos\phi = s^2 \end{gathered} \tag{4.14}$$

Eq. (4.14) can be written in the polynomial form given in Eq. (4.9) where

$$\begin{gathered} P_1 = -a^2 + b^2 - c^2 - s_0^2,\ f_1(\mathbf{x_i}) = 1,\ P_2 = ac,\ f_2(\mathbf{x_i}) = 2\sin\phi_i,\ P_3 = a,\ f_3(\mathbf{x_i}) = 2s_i\cos\phi_i, \\ P_4 = s_0,\ f_4(\mathbf{x_i}) = -2s_i,\ P_5 = as_0,\ f_5(\mathbf{x_i}) = 2\cos\phi_i,\ F(\mathbf{x_i}) = s_i^2 \end{gathered} \tag{4.15}$$

Note that

$$P_5 = P_3P_4 \tag{4.16}$$

Once again, let $P_5 = \lambda$, $P_j = \ell_j\lambda + m_j$ for j = 1,...,4 and $L = \ell\lambda + m$. After finding ℓ_j and m_j as explained in Section 1.4.3, λ is determined as follows:

$$\begin{gathered} P_3P_4 - P_5 = (\ell_3\lambda + m_3)(\ell_4\lambda + m_4) - \lambda = \ell_3\ell_4\lambda^2 + (\ell_3 m_4 + m_3\ell_4 - 1)\lambda + m_3m_4 = 0 \\ \Rightarrow \lambda = \frac{-(\ell_3 m_4 + m_3\ell_4 - 1) \mp \sqrt{(\ell_3 m_4 + m_3\ell_4 - 1)^2 - 4\ell_3\ell_4 m_3 m_4}}{2\ell_3\ell_4} \end{gathered} \tag{4.17}$$

There are two λ solutions. Having found λ, $P_j = \ell_j\lambda + m_j$ for j = 1,...,4 and

$L=\ell\lambda+m$. The construction parameters are solved uniquely from Eq. (4.15) as:

$$a=P_3,\ s_0=P_4,\ c=P_2/a,\ b=\sqrt{P_1+a^2+c^2+s_0^{\ 2}} \tag{4.18}$$

The solution is feasible if P_2, P_3, $P_4 \geq 0$ and $P_1 + a^2 + c^2 + s_0^2 \geq 0$.

Similar to introducing ϕ_0, the output θ can be measured from a different reference with an initial angle of θ_0 as shown in Figure 4.1. Then Eq. (4.5) yields

$$\begin{aligned} &d^2-e^2+f^2+(1-s)^2+2d(1-s)\cos\theta_0\cos\theta \\ &\quad -2d(1-s)\sin\theta_0\sin\theta-2df\sin\theta_0\cos\theta-2df\cos\theta_0\sin\theta=0 \end{aligned} \tag{4.19}$$

Eq. (4.19) can be written in the polynomial form given in Eq. (4.9) where

$$\begin{aligned} &P_1=\frac{d^2-e^2+f^2}{2df\cos\theta_0},\ f_1(\mathbf{x_i})=1,\ P_2=\frac{1}{2df\cos\theta_0},\ f_2(\mathbf{x_i})=(1-s_i)^2,\ P_3=\frac{1}{f},\ f_3(\mathbf{x_i})=(1-s_i)\cos\theta_i, \\ &P_4=\tan\theta_0,\ f_4(\mathbf{x_i})=-\cos\theta_i,\ P_5=\frac{\tan\theta_0}{f},\ f_5(\mathbf{x_i})=-(1-s_i)\sin\theta_i,\ F(\mathbf{x_i})=\sin\theta_i \end{aligned} \tag{4.20}$$

Again $P_5 = P_3P_4$ and $P_5 = \lambda$ can be solved using Eq. (4.17). $P_j=\ell_j\lambda+m_j$ for j = 1,...,4 and $L=\ell\lambda+m$ are determined and the construction parameters are solved from Eq. (4.20) as:

$$\theta_0=\tan^{-1}P_4,\ f=\frac{1}{P_3},\ d=\frac{1}{2P_2 f\cos\theta_0},\ e=\sqrt{d^2+f^2-2df\cos\theta_0 P_1} \tag{4.21}$$

Also it is possible to consider inclusion of s_0 to loop DEF. However, inclusion of s_0 can be done to either of the loops, not both. The next step is to include of both ϕ_0 and s_0 to loop ABC. In that case, Eq. (4.8) can be modified as

$$\begin{aligned} &-a^2+b^2-c^2-s_0^{\ 2}-2s_0 s+2as\cos\phi_0\cos\phi-2as\sin\phi_0\sin\phi \\ &\quad +2a(s_0\cos\phi_0+c\sin\phi_0)\cos\phi-2a(s_0\sin\phi_0-c\cos\phi_0)\sin\phi=s^2 \end{aligned} \tag{4.22}$$

Eq. (4.22) can be written in polynomial form as

$$\delta_i(\mathbf{x_i}) = F(\mathbf{x_i}) - \sum_{j=1}^{6} P_j f_j(\mathbf{x_i}) = (-1)^{i+1} L \text{ for } i = 1,\ldots,6 \tag{4.23}$$

where

$$\begin{gathered} P_1 = -a^2 + b^2 - c^2 - s_0{}^2,\ f_1(\mathbf{x_i}) = 1,\ P_2 = s_0,\ f_2(\mathbf{x_i}) = -2s_i,\ P_3 = a\cos\phi_0,\ f_3(\mathbf{x_i}) = 2s_i\cos\phi_i, \\ P_4 = a\sin\phi_0,\ f_4(\mathbf{x_i}) = -2s_i\sin\phi_i,\ P_5 = a(s_0\cos\phi_0 + c\sin\phi_0),\ f_5(\mathbf{x_i}) = 2\cos\phi_i, \\ P_6 = a(s_0\sin\phi_0 - c\cos\phi_0),\ f_6(\mathbf{x_i}) = -2\sin\phi_i,\ F(\mathbf{x_i}) = s_i{}^2 \end{gathered} \tag{4.24}$$

There are 5 construction parameters, but 6 polynomial coefficients. The dependency in between P_j is obtained by eliminating c from;

$$P_5 = P_2 P_3 + c P_4 \quad \text{and} \quad P_6 = P_2 P_4 - c P_3 \tag{4.25}$$

$$P_3 P_5 + P_4 P_6 = P_2\left(P_3{}^2 + P_4{}^2\right) \tag{4.26}$$

Let $P_2 = \lambda$ (any of P_3, ..., P_6 may also be selected as λ), $P_j = \ell_j \lambda + m_j$ for j = 1, 3, .., 6 and $L = \ell\lambda + m$ $L = \ell\lambda + m$. ℓ_j and m_j can be solved linearly as explained in Section 1.4.3. Then, Eq. (4.26) is a cubic equation in terms of λ and the three solutions for λ can be found analytically. It is guaranteed that at least one of the solutions is real. In case of three real solutions, the designer can pick the solution which yields lesser error.

As an alternative, it is possible to introduce s_0 and θ_0 simultaneously to loop DEF. This case is similar to loop ABC and again it is necessary to solve a cubic equation.

4.1.3. The Double-Spherical 6R Mechanism with 8 Parameters

A double-spherical 6R mechanism is shown in Figure 4.2. The imaginary spherical 4-bar loops ABCD and AEFG share the common passive joint axis A, which is along the line connecting the spherical loop centers O_1 and O_2. The x-axis of the fixed coordinate system is chosen along O_1O_2 and the y-axis is selected such that the O_1B revolute joint axis remains on the xy-plane. ϕ, ψ and θ are the respective input, passive joint and output angle. O_1D and O_2E axes are in general skew with a twist angle of γ. The radii of the spheres do not affect the I/O relationship, so without loss of generality, assume both radii as 1. Also, notice that the distance $|O_1O_2|$ has no effect on the I/O relationship. For the time being assume $\phi_0 = \gamma = \theta_0 = 0$.

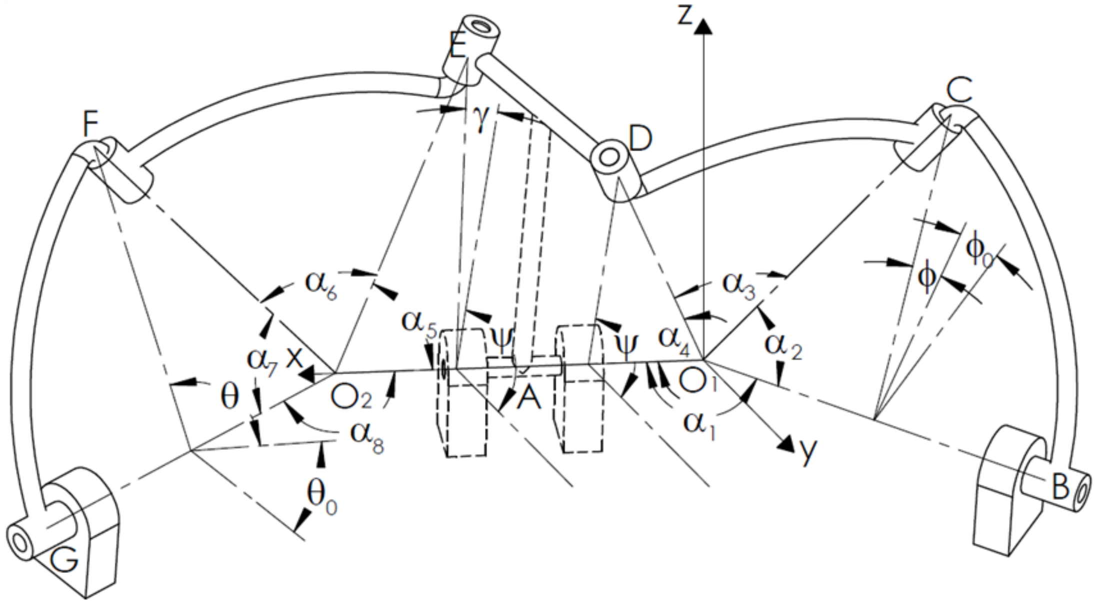


Figure 4.2. Kinematic representation of the double-spherical 6R linkage

For the imaginary loop ABCD the I/O equation reads (Alizade et al., 2013)

$$\begin{gathered}\cos\alpha_1\cos\alpha_2\cos\alpha_4-\cos\alpha_3-\sin\alpha_1\sin\alpha_2\cos\alpha_4\cos\phi+\sin\alpha_1\cos\alpha_2\sin\alpha_4\cos\psi \\ +\cos\alpha_1\sin\alpha_2\sin\alpha_4\cos\phi\cos\psi+\sin\alpha_2\sin\alpha_4\sin\phi\sin\psi=0\end{gathered} \tag{4.27}$$

Eq. (4.27) contains four construction parameters: α_1, α_2, α_3, α_4. Together with the Chebyshev error L, there are five parameters to be determined. The function synthesis may be performed for five design points. Writing Eq. (4.27) in polynomial form:

$$\delta_i(\mathbf{x_i})=F(\mathbf{x_i})-\sum_{j=1}^{4}P_j f_j(\mathbf{x_i})=(-1)^{i+1}L \text{ for } i = 1,\ldots,5 \text{ for} \tag{4.28}$$

Where

$$P_1=\frac{\cos\alpha_3-\cos\alpha_1\cos\alpha_2\cos\alpha_4}{\sin\alpha_1\cos\alpha_2\sin\alpha_4},\ f_1(\mathbf{x_i})=1,\ P_2=\frac{\tan\alpha_2}{\tan\alpha_4},\ f_2(\mathbf{x_i})=\cos\phi_i,$$
$$P_3=\frac{\tan\alpha_2}{\tan\alpha_1},\ f_3(\mathbf{x_i})=-\cos\phi_i\cos\psi_i,\ P_4=\frac{\tan\alpha_2}{\sin\alpha_1},\ f_4(\mathbf{x_i})=-\sin\phi_i\sin\psi_i,\ F(\mathbf{x_i})=\cos\psi_i \tag{4.29}$$

$\mathbf{x_i}=(\phi_i,\psi_i)$ represents the i[th] design point. P_j are solved linearly from Eq. (4.28) using the Remez algorithm and then the construction parameters are solved from Eq. (4.29) as follows:

$$\alpha_1=\cos^{-1}\frac{P_3}{P_4},\ \alpha_2=\tan^{-1}(P_3\tan\alpha_1),\ \alpha_4=\tan^{-1}\frac{\tan\alpha_2}{P_2},$$
$$\alpha_3=\cos^{-1}(\cos\alpha_1\cos\alpha_2\cos\alpha_4+P_1\sin\alpha_1\cos\alpha_2\sin\alpha_4) \tag{4.30}$$

Notice that the solution for the construction parameters is not unique. For example, there are two alternative solutions for α_1: $\cos^{-1}(P_3/P_4)$ and $-\cos^{-1}(P_3/P_4)$. In Eq. (4.30) three are 16 possible solutions. The solution/solutions which yield the desired function generation should be determined by either constructing a virtual/actual model of the mechanism or checking the input/output values.

For loop AEFG the I/O equation for $\gamma = 0$ is given by (Alizade et al., 2013)

$$\cos\alpha_5\cos\alpha_7\cos\alpha_8-\cos\alpha_6+\sin\alpha_5\cos\alpha_7\sin\alpha_8\cos\psi-\sin\alpha_5\sin\alpha_7\cos\alpha_8\cos\theta\cos\psi$$
$$+\sin\alpha_5\sin\alpha_7\sin\theta\sin\psi+\cos\alpha_5\sin\alpha_7\sin\alpha_8\cos\theta=0 \tag{4.31}$$

Eq. (4.31) can be written in polynomial form given in Eq. (4.28) where

$$P_1=\frac{\cos\alpha_6-\cos\alpha_5\cos\alpha_7\cos\alpha_8}{\cos\alpha_5\sin\alpha_7\sin\alpha_8},\ f_1(\mathbf{x_i})=1,\ P_2=-\frac{\tan\alpha_5}{\tan\alpha_7},\ f_2(\mathbf{x_i})=\cos\psi_i,$$
$$P_3=\frac{\tan\alpha_5}{\tan\alpha_8},\ f_3(\mathbf{x_i})=\cos\theta_i\cos\psi_i,\ P_4=-\frac{\tan\alpha_5}{\sin\alpha_8},\ f_4(\mathbf{x_i})=\sin\theta_i\sin\psi_i, \tag{4.32}$$
$$F(\mathbf{x_i})=\cos\theta_i \text{ and } \mathbf{x_i}=(\psi_i,\theta_i)$$

For given 5 design points, $\mathbf{x_i}=(\phi_i,\psi_i)$, P_j are solved linearly from Eq. (4.28) using the Remez algorithm and then, the construction parameters are solved from Eq. (4.32) as follows:

$$\alpha_8=\cos^{-1}\left(-\frac{P_3}{P_4}\right),\ \alpha_5=\tan^{-1}\left(P_3\tan\alpha_8\right),\ \alpha_7=\tan^{-1}-\frac{\tan\alpha_5}{P_2},$$
$$\alpha_6=\cos^{-1}\left(\cos\alpha_5\cos\alpha_7\cos\alpha_8+P_1\cos\alpha_5\sin\alpha_7\sin\alpha_8\right) \tag{4.33}$$

4.1.4. The Double-Spherical 6R Mechanism with 9, 10 and 11 Parameters

When extra construction parameter ϕ_0 is added into Eq. (4.27):

$$\begin{gathered}\cos\alpha_1\cos\alpha_2\cos\alpha_4-\cos\alpha_3-\sin\alpha_1\sin\alpha_2\cos\alpha_4\cos\phi_0\cos\phi\\+\sin\alpha_1\sin\alpha_2\cos\alpha_4\sin\phi_0\sin\phi+\sin\alpha_1\cos\alpha_2\sin\alpha_4\cos\psi\\+\cos\alpha_1\sin\alpha_2\sin\alpha_4\cos\phi_0\cos\phi\cos\psi-\cos\alpha_1\sin\alpha_2\sin\alpha_4\sin\phi_0\sin\phi\cos\psi\\+\sin\alpha_2\sin\alpha_4\sin\phi_0\cos\phi\sin\psi+\sin\alpha_2\sin\alpha_4\cos\phi_0\sin\phi\sin\psi=0\end{gathered} \tag{4.34}$$

Eq. (4.34) can be written in polynomial form as

$$\delta_i(\mathbf{x_i})=F(\mathbf{x_i})-\sum_{j=1}^{7}P_j f_j(\mathbf{x_i})=(-1)^{i+1}L\ \text{ for } i=1,\ldots,6 \tag{4.35}$$

where

$$\begin{gathered}P_1=\frac{\cos\alpha_1\cos\alpha_2\cos\alpha_4-\cos\alpha_3}{\cos\alpha_1\sin\alpha_2\sin\alpha_4\sin\phi_0},\ f_1(\mathbf{x_i})=1,\ P_2=\frac{\tan\alpha_1}{\tan\alpha_4},\ f_2(\mathbf{x_i})=\sin\phi_i,\\P_3=\frac{\tan\alpha_1}{\tan\alpha_2\sin\phi_0},\ f_3(\mathbf{x_i})=\cos\psi_i,\ P_4=\frac{1}{\tan\phi_0},\ f_4(\mathbf{x_i})=\cos\phi_i\cos\psi_i,\\P_5=\frac{1}{\cos\alpha_1},\ f_5(\mathbf{x_i})=\cos\phi_i\sin\psi_i,\ P_6=-\frac{\tan\alpha_1}{\tan\alpha_4\tan\phi_0},\ f_6(\mathbf{x_i})=\cos\phi_i,\\P_7=\frac{1}{\cos\alpha_1\tan\phi_0},\ f_7(\mathbf{x_i})=\sin\phi_i\sin\psi_i,\ F(\mathbf{x_i})=\sin\phi_i\cos\psi_i\end{gathered} \tag{4.36}$$

There are five construction parameters, i.e. α_1, α_2, α_3, α_4 and ϕ_0, however there are seven P_j. Indeed, two of the P_j can be represented in terms of other as

$$P_6=-P_2P_4 \text{ and } P_7=P_4P_5 \tag{4.37}$$

Let $P_6 = \lambda_1$, $P_7 = \lambda_2$, $P_j = \ell_j + m_j\lambda_1 + n_j\lambda_2$ for j = 1,...,5 and $L = \ell + m\lambda_1 + n\lambda_2$. ℓ_j, m_j and n_j are found as explained in Section 1.4.3. Writing Eq. (4.37) in terms of λ_1 and λ_2:

$$
\begin{aligned}
P_2P_4 + P_6 &= m_2m_4\lambda_1^2 + n_2n_4\lambda_2^2 + (m_2n_4 + n_2m_4)\lambda_1\lambda_2 \\
&\qquad + (\ell_2m_4 + m_2\ell_4 + 1)\lambda_1 + (\ell_2n_4 + n_2\ell_4)\lambda_2 + \ell_2\ell_4 = 0 \\
P_4P_5 - P_7 &= m_4m_5\lambda_1^2 + n_4n_5\lambda_2^2 + (m_4n_5 + n_4m_5)\lambda_1\lambda_2 \\
&\qquad + (\ell_4m_5 + m_4\ell_5 - 1)\lambda_1 + (\ell_4n_5 + n_4\ell_5)\lambda_2 + \ell_4\ell_5 = 0
\end{aligned}
\tag{4.38}
$$

As shown in (Alizade, Kilit, 2005), when λ_2 is eliminated from Eq. (4.38) a degree 3 polynomial equation in terms of λ_1 is obtained and it can be solved analytically. At this point, it should be emphasized that the P_j in Eq. (4.36) are selected as such on purpose so that Eq. (4.37) is obtained and hence, there are three solutions. For other cases, the degree 4 term, which appears after elimination in Eq. (4.38), does not vanish, and it is possible that there are no real solutions.

After determining λ_1 and λ_2, $P_j = \ell_j + m_j\lambda_1 + n_j\lambda_2$ for j = 1,...,5 and $L = \ell + m\lambda_1 + n\lambda_2$ are determined. From Eq. (4.36) the construction parameters α_1, α_2, α_3, α_4 and ϕ_0 are found as follows:

$$
\begin{gathered}
\phi_0 = \tan^{-1}\frac{1}{P_4},\ \alpha_1 = \cos^{-1}\frac{1}{P_5},\ \alpha_2 = \tan^{-1}\frac{\tan\alpha_1}{P_3\sin\phi_0},\ \alpha_4 = \tan^{-1}\frac{\tan\alpha_1}{P_2}, \\
\alpha_3 = \cos^{-1}(\cos\alpha_1\cos\alpha_2\cos\alpha_4 - P_1\cos\alpha_1\sin\alpha_2\sin\alpha_4\sin\phi_0)
\end{gathered}
\tag{4.39}
$$

When the twist angle γ in between O_1D and O_2E in Figure 4.2 is taken into account Eq. (4.27) becomes

$$
\begin{gathered}
\cos\alpha_1\cos\alpha_2\cos\alpha_4 - \cos\alpha_3 - \sin\alpha_1\sin\alpha_2\cos\alpha_4\cos\phi \\
+ \sin\alpha_1\cos\alpha_2\sin\alpha_4\cos\gamma\cos\psi - \sin\alpha_1\cos\alpha_2\sin\alpha_4\sin\gamma\sin\psi \\
+ \cos\alpha_1\sin\alpha_2\sin\alpha_4\cos\gamma\cos\phi\cos\psi - \cos\alpha_1\sin\alpha_2\sin\alpha_4\sin\gamma\cos\phi\sin\psi \\
+ \sin\alpha_2\sin\alpha_4\sin\gamma\sin\phi\cos\psi + \sin\alpha_2\sin\alpha_4\cos\gamma\sin\phi\sin\psi = 0
\end{gathered}
\tag{4.40}
$$

Eq. (4.40) can be written in polynomial form in Eq. (4.35) where

$$
\begin{gathered}
P_1=\frac{\cos\alpha_1\cos\alpha_2\cos\alpha_4-\cos\alpha_3}{\cos\alpha_1\sin\alpha_2\sin\alpha_4\sin\gamma},\ f_1(\mathbf{x_i})=1,\ P_2=\frac{\tan\alpha_1}{\tan\alpha_4\sin\gamma},\ f_2(\mathbf{x_i})=-\cos\phi_i,\\
P_3=\frac{\tan\alpha_1}{\tan\alpha_2},\ f_3(\mathbf{x_i})=-\sin\psi_i,\ P_4=\frac{1}{\tan\gamma},\ f_4(\mathbf{x_i})=\cos\phi_i\cos\psi_i,\\
P_5=\frac{1}{\cos\alpha_1},\ f_5(\mathbf{x_i})=\sin\phi_i\cos\psi_i,\ P_6=-\frac{\tan\alpha_1}{\tan\alpha_2\tan\gamma},\ f_6(\mathbf{x_i})=-\cos\psi_i,\\
P_7=\frac{1}{\cos\alpha_1\tan\gamma},\ f_7(\mathbf{x_i})=\sin\phi_i\sin\psi_i,\ F(\mathbf{x_i})=\cos\phi_i\sin\psi_i
\end{gathered}
\tag{4.41}
$$

P_6 and P_7 depend on other P_j's as

$$P_6=-P_3P_4 \text{ and } P_7=P_4P_5 \tag{4.42}$$

Which has the same form as in Eq. (4.37) and yield 3 solutions similar to the previous case. The construction parameters are obtained from Eq. (4.41) as follows:

$$
\begin{gathered}
\gamma=\tan^{-1}\frac{1}{P_4},\ \alpha_1=\cos^{-1}\frac{1}{P_5},\ \alpha_2=\tan^{-1}\frac{\tan\alpha_1}{P_3},\ \alpha_4=\tan^{-1}\frac{\tan\alpha_1}{P_2\sin\gamma},\\
\alpha_3=\cos^{-1}(\cos\alpha_1\cos\alpha_2\cos\alpha_4-P_1\cos\alpha_1\sin\alpha_2\sin\alpha_4\sin\gamma)
\end{gathered}
\tag{4.43}
$$

The twist angle γ can also be included in Eq. (4.31) to replace ψ by $\gamma+\psi$:

$$
\begin{gathered}
\cos\alpha_5\cos\alpha_7\cos\alpha_8-\cos\alpha_6+\cos\alpha_5\sin\alpha_7\sin\alpha_8\cos\theta\\
+\sin\alpha_5\cos\alpha_7\sin\alpha_8\cos\gamma\cos\psi-\sin\alpha_5\cos\alpha_7\sin\alpha_8\sin\gamma\sin\psi\\
-\sin\alpha_5\sin\alpha_7\cos\alpha_8\cos\gamma\cos\theta\cos\psi+\sin\alpha_5\sin\alpha_7\cos\alpha_8\sin\gamma\cos\theta\sin\psi\\
+\sin\alpha_5\sin\alpha_7\cos\gamma\sin\theta\sin\psi+\sin\alpha_5\sin\alpha_7\sin\gamma\sin\theta\cos\psi=0
\end{gathered}
\tag{4.44}
$$

Writing Eq. (4.44) in polynomial form Eq. (4.35), the P_j are selected as

$$
\begin{gathered}
P_1=\frac{\cos\alpha_5\cos\alpha_7\cos\alpha_8-\cos\alpha_6}{\sin\alpha_5\sin\alpha_7\cos\alpha_8\cos\gamma},\ f_1(\mathbf{x_i})=1,\ P_2=\frac{\tan\alpha_8}{\tan\alpha_7},\ f_2(\mathbf{x_i})=\cos\psi_i, P_3=\tan\gamma\\
f_3(\mathbf{x_i})=\cos\theta_i\sin\psi_i,\ P_4=\frac{\tan\alpha_8}{\tan\alpha_5\cos\gamma},\ f_4(\mathbf{x_i})=\cos\theta_i, P_5=\frac{1}{\cos\alpha_8}, f_5(\mathbf{x_i})=\sin\theta_i\sin\psi_i\\
P_6=-\frac{\tan\alpha_8\tan\gamma}{\tan\alpha_7},\ f_6(\mathbf{x_i})=\sin\psi_i, P_7=\frac{\tan\gamma}{\cos\alpha_8}, f_7(\mathbf{x_i})=\sin\theta_i\cos\psi_i,\ F(\mathbf{x_i})=\cos\theta_i\cos\psi_i
\end{gathered}
\tag{4.45}
$$

P_6 and P_7 depend on other P_j's as

$$P_6 = -P_2P_3 \text{ and } P_7 = P_3P_5 \tag{4.46}$$

which has the same form as in Eq. (4.37) and yield 3 solutions. The construction parameters are obtained from Eq. (4.45) as follows:

$$\begin{gathered}\gamma=\tan^{-1}P_3,\ \alpha_8=\cos^{-1}\frac{1}{P_5},\ \alpha_7=\tan^{-1}\left(-\frac{\tan\alpha_8\tan\gamma}{P_6}\right),\ \alpha_5=\tan^{-1}\frac{\tan\alpha_8}{P_4\cos\gamma},\\ \alpha_6=\cos^{-1}\left(\cos\alpha_5\cos\alpha_7\cos\alpha_8-P_1\sin\alpha_5\sin\alpha_7\cos\alpha_8\cos\gamma\right)\end{gathered} \tag{4.47}$$

Next, the extra parameter θ_0 can be introduced in order to replace θ by $\theta_0 + \theta$ in Eq. (4.31)

$$\begin{gathered}\cos\alpha_5\cos\alpha_7\cos\alpha_8-\cos\alpha_6+\sin\alpha_5\cos\alpha_7\sin\alpha_8\cos\psi\\ -\sin\alpha_5\sin\alpha_7\cos\alpha_8\cos\theta_0\cos\theta\cos\psi+\sin\alpha_5\sin\alpha_7\cos\alpha_8\sin\theta_0\sin\theta\cos\psi\\ +\sin\alpha_5\sin\alpha_7\cos\theta_0\sin\theta\sin\psi+\sin\alpha_5\sin\alpha_7\sin\theta_0\cos\theta\sin\psi\\ +\cos\alpha_5\sin\alpha_7\sin\alpha_8\cos\theta_0\cos\theta-\cos\alpha_5\sin\alpha_7\sin\alpha_8\sin\theta_0\sin\theta=0\end{gathered} \tag{4.48}$$

Writing Eq. (4.48) in polynomial form Eq. (4.35), the P_j's are selected as

$$\begin{gathered}P_1=\frac{\cos\alpha_5\cos\alpha_7\cos\alpha_8-\cos\alpha_6}{\sin\alpha_5\sin\alpha_7\cos\alpha_8\cos\theta_0},\ f_1(\mathbf{x_i})=1,\ P_2=\frac{\tan\alpha_8}{\tan\alpha_7\cos\theta_0},\ f_2(\mathbf{x_i})=\cos\psi_i,\\ P_3=\tan\theta_0,\ f_3(\mathbf{x_i})=\cos\psi_i\sin\theta_i,\ P_4=\frac{1}{\cos\alpha_8},\ f_4(\mathbf{x_i})=\sin\psi_i\sin\theta_i,\\ P_5=\frac{\tan\alpha_8}{\tan\alpha_5},\ f_5(\mathbf{x_i})=\cos\theta_i,\ P_6=-\frac{\tan\alpha_8\tan\theta_0}{\tan\alpha_5},\\ f_6(\mathbf{x_i})=\sin\theta_i,\ P_7=\frac{\tan\theta_0}{\cos\alpha_8},\ f_7(\mathbf{x_i})=\sin\psi_i\cos\theta_i,\ F(\mathbf{x_i})=\cos\psi_i\cos\theta_i\end{gathered} \tag{4.49}$$

P_6 and P_7 depend on other P_j's as

$$P_6 = -P_3P_5 \text{ and } P_7 = P_3P_4 \tag{4.50}$$

Which has the same form as in Eq. (4.37) and yield 3 solutions. The construction parameters are obtained from Eq. (4.49) as follows:

$$\theta_0 = \tan^{-1} P_3,\ \alpha_8 = \cos^{-1}\frac{1}{P_4},\ \alpha_7 = \tan^{-1}\frac{\tan\alpha_8}{P_2\cos\theta_0},\ \alpha_5 = \tan^{-1}\frac{\tan\alpha_8}{P_5},$$
$$\alpha_6 = \cos^{-1}\left(\cos\alpha_5\cos\alpha_7\cos\alpha_8 - P_1\sin\alpha_5\sin\alpha_7\cos\alpha_8\cos\theta_0\right) \tag{4.51}$$

It is also possible to include both of ϕ_0 and γ in loop ABCD and both of γ and θ_0 in loop AEFG. This case is already studied in (Alizade, Gezgin, 2011), where it is necessary to define six nonlinear parameters λ_k for k = 1,...,6. The solution for λ_k necessitates numerical solution.

4.1.5. Plano-Spherical 6R Mechanism with 7 Parameters

A plano-spherical 6R linkage is depicted in Figure 4.3, where ϕ is the input and θ is the output, or vice versa. The link length definitions for the spherical and planar parts are the same as for the corresponding imaginary loops of the double-planar and double-spherical linkages. In general, there is a twist angle γ in between O_1D' and AE directions. The x-axis direction of the fixed coordinate system is selected to be parallel to the joint axis directions of the planar loop. The y-axis is selected such that the xy-plane includes the O_1B axis of the spherical loop and the z-axis is along the common normal of O_1B and G rotation axes. For the time being assume $\phi_0 = \gamma = \theta_0 = 0$.

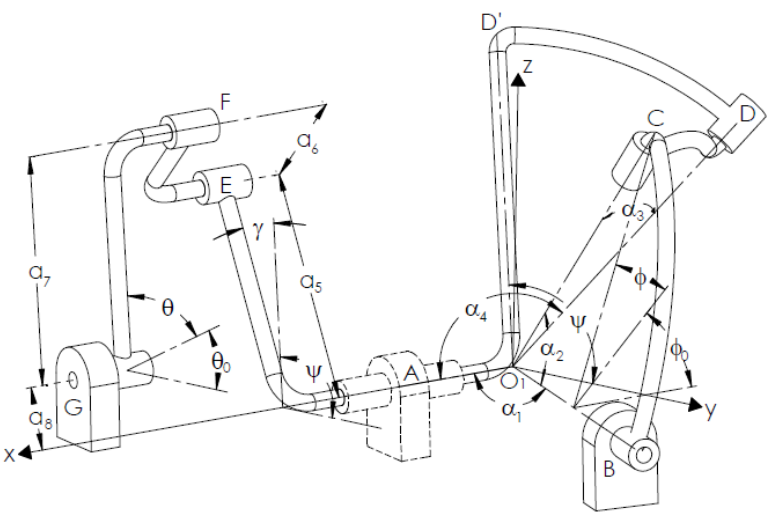


Figure 4.3. Kinematic representation of the plano-spherical 6R linkage

The I/O relationship for the spherical loop is the same as for the first loop of double-spherical 6R linkage and is given by Eq. (4.27). All links of the planar loop move parallel to the yz-plane, so the x coordinates are irrelevant when analyzing this loop. Scaling all the link lengths of the planar four bar mechanism has no effect on the function synthesis task, so without loss of generality one can assume $a_5 = 1$. For $\gamma = 0$, the I/O equation for the planar loop can be derived as (Alizade et al., 2013):

$$1 - a_6^2 + a_7^2 + a_8^2 - 2a_8 \sin\psi - 2a_7 \cos(\psi - \theta) + 2a_7 a_8 \sin\theta = 0 \tag{4.52}$$

This I/O relationship contains three construction parameters: a_6, a_7, a_8. Together with the Chebyshev error L, there are four parameters to be determined. Therefore, the function synthesis may be performed for four design points. Eq. (4.52) can be written in the polynomial form of Eq. (4.2), where

$$\begin{gathered} P_1 = \frac{1 - a_6^2 + a_7^2 + a_8^2}{2a_7 a_8},\ f_1(\mathbf{x_i}) = 1,\ P_2 = \frac{1}{a_7},\ f_2(\mathbf{x_i}) = -\sin\psi_i, \\ P_3 = \frac{1}{a_8},\ f_3(\mathbf{x_i}) = -\cos(\psi_i - \theta_i),\ F(\mathbf{x_i}) = -\sin\theta_i \end{gathered} \tag{4.53}$$

P_j are solved linearly from Eq. (4.2) using the Remez algorithm and then the construction parameters are solved from Eq. (4.53) as follows:

$$a_7 = \frac{1}{P_2},\ a_8 = \frac{1}{P_3},\ a_6 = \sqrt{1 + a_7^2 + a_8^2 - 2P_1 a_7 a_8} \tag{4.54}$$

4.1.6. Plano-Spherical 6R Mechanism with 8, 9 and 10 Parameters

When we include the twist angle γ is included to the planar loop Eq. (4.52) becomes

$$\begin{gathered} 1 - a_6^2 + a_7^2 + a_8^2 - 2a_8 \cos\gamma \sin\psi - 2a_8 \sin\gamma \cos\psi - 2a_7 \cos\gamma \cos(\psi - \theta) \\ +2a_7 \sin\gamma \sin(\psi - \theta) + 2a_7 a_8 \sin\theta = 0 \end{gathered} \tag{4.55}$$

Eq. (4.55) can be written in polynomial form in Eq. (4.9) where

$$P_1=\frac{1-a_6^2+a_7^2+a_8^2}{2a_7a_8},\ f_1(\mathbf{x_i})=1,\ P_2=\frac{\cos\gamma}{a_7},\ f_2(\mathbf{x_i})=-\sin\psi_i,\ P_3=\frac{\sin\gamma}{a_7},\ f_3(\mathbf{x_i})=-\cos\psi_i$$
$$P_4=\frac{\cos\gamma}{a_8},\ f_4(\mathbf{x_i})=-\cos(\psi_i-\theta_i),\ P_5=\frac{\sin\gamma}{a_8},\ f_5(\mathbf{x_i})=\sin(\psi_i-\theta_i),\ F(\mathbf{x_i})=-\sin\theta_i \tag{4.56}$$

The dependency in between P_j's is the same as Eq. (4.11). Let $P_5 = \lambda$, $P_j=\ell_j\lambda+m_j$ for j = 1,...,4 and $L=\ell\lambda+m$. After finding ℓ_j and m_j as explained in Section 1.4.3, λ is determined as in Eq. (4.12). The construction parameters are determined as

$$\gamma=\operatorname{atan2}(P_2,P_3),\ a_7=\frac{\cos\gamma}{P_2},\ a_8=\frac{\cos\gamma}{P_4},\ a_6=\sqrt{1+a_7^2+a_8^2-P_1 2a_7a_8} \tag{4.57}$$

When $\theta_0 + \theta$ is used instead of θ in Eq. (4.52), one obtains

$$\begin{gathered}1-a_6^2+a_7^2+a_8^2-2a_8\sin\psi-2a_7\cos\theta_0\cos(\psi-\theta)-2a_7\sin\theta_0\sin(\psi-\theta)\\ +2a_7a_8\sin\theta_0\cos\theta+2a_7a_8\cos\theta_0\sin\theta=0\end{gathered} \tag{4.58}$$

Eq. (4.55) can be written in polynomial form in Eq. (4.9) where

$$P_1=\frac{1-a_6^2+a_7^2+a_8^2}{2a_7a_8\cos\theta_0},\ f_1(\mathbf{x_i})=1,\ P_2=\frac{1}{a_7\cos\theta_0},\ f_2(\mathbf{x_i})=-\sin\psi_i,\ P_3=\tan\theta_0,\ f_3(\mathbf{x_i})=\cos\theta_i$$
$$P_4=\frac{1}{a_8},\ f_4(\mathbf{x_i})=-\cos(\psi_i-\theta_i),P_5=\frac{\tan\theta_0}{a_8},\ f_5(\mathbf{x_i})=-\sin(\psi_i-\theta_i),\ F(\mathbf{x_i})=-\sin\theta_i \tag{4.59}$$

The dependency in between P_j's is the same as Eq. (4.16). Let $P_5 = \lambda$, $P_j=\ell_j\lambda+m_j$ for j = 1,...,4 and $L=\ell\lambda+m$. After finding ℓ_j and m_j as explained in Section 1.4.3, λ is determined as in Eq. (4.17). The construction parameters are determined as

$$\theta_0=\begin{cases}\tan^{-1}P_3 & \text{if } P_2>0\\ \tan^{-1}P_3+\pi & \text{if } P_2<0\end{cases},\ a_8=\frac{1}{P_4},\ a_7=\frac{1}{P_2\cos\theta_0},\ a_6=\sqrt{1+a_7^2+a_8^2-2P_1a_7a_8\cos\theta_0} \tag{4.60}$$

Finally, both of γ and θ_0 can be added to Eq. (4.52) to obtain

$$\begin{aligned}&1-a_6^2+a_7^2+a_8^2-2a_8\cos\gamma\sin\psi-2a_8\sin\gamma\cos\psi-2a_7\cos(\gamma-\theta_0)\cos(\psi-\theta)\\&+2a_7\sin(\gamma-\theta_0)\sin(\psi-\theta)+2a_7a_8\sin\theta_0\cos\theta+2a_7a_8\cos\theta_0\sin\theta=0\end{aligned} \tag{4.61}$$

Eq. (4.61) can be written in polynomial form in Eq. (4.23) where

$$\begin{aligned}&P_1=\frac{1-a_6^2+a_7^2+a_8^2}{2a_7a_8\cos\theta_0},\ f_1(\mathbf{x_i})=1,\ P_2=\frac{\cos\gamma}{a_7\cos\theta_0},\ f_2(\mathbf{x_i})=-\sin\psi_i,\ P_3=\tan\theta_0,\\&f_3(\mathbf{x_i})=\cos\theta_i, P_4=\frac{\sin\gamma}{a_7\cos\theta_0},\ f_4(\mathbf{x_i})=-\cos\psi_i,\ P_5=\frac{\cos(\gamma-\theta_0)}{a_8\cos\theta_0},\\&f_5(\mathbf{x_i})=-\cos(\psi_i-\theta_i), P_6=\frac{\sin(\gamma-\theta_0)}{a_8\cos\theta_0},\ f_6(\mathbf{x_i})=\sin(\psi_i-\theta_i),\ F(\mathbf{x_i})=-\sin\theta_i\end{aligned} \tag{4.62}$$

There are five construction parameters, whereas there are six P_j's. The relationship between P_j's can be found to be

$$P_3(P_2P_5+P_4P_6)=P_4P_5-P_2P_6 \tag{4.63}$$

Let $P_6 = \lambda$, $P_j=\ell_j\lambda+m_j$ for j = 1,...,5 and $L=\ell\lambda+m$. ℓ_j and m_j can be solved linearly as explained in Section 1.4.3. Eq. (4.63) is a cubic equation in terms of λ and can be solved analytically. The construction parameters are solved from Eq. (4.62) as

$$\begin{aligned}&\theta_0=\tan^{-1}P_3,\ \gamma=\mathrm{atan2}(P_2\cos\theta_0,P_4\cos\theta_0),\ a_7=\frac{\cos\gamma}{P_2\cos\theta_0},\ a_8=\frac{\cos(\gamma-\theta_0)}{P_5\cos\theta_0},\\&a_6=\sqrt{1+a_7^2+a_8^2-2P_1a_7a_8\cos\theta_0}\end{aligned} \tag{4.64}$$

4.2. The Function Synthesis Problem

Let the function to be generated be z = f(x) for $x_{min} \le x \le x_{max}$ and $z_{min} \le z \le z_{max}$. The independent variable x should be related to the mechanism input ϕ and the dependent variable z should be related to the mechanism output θ. However, since the method of decomposition is used, there is an intermediate joint variable ψ for the double-spherical and plano-spherical mechanisms and s for the double-planar mechanism. Let $y = g(x)$ for $y_{min} \le y \le y_{max}$ such that $z = h(y)$ and hence $f(x) = h(g(x))$. Depending on the application some or all of $z = f(x)$, x_{min}, x_{max}, z_{min} and z_{max} may be demanded by the specific task. However, the designer can freely select $y = g(x)$, y_{min} and y_{max}. The design may be enhanced via different selection of the function g and also the boundaries of y. One can linearly relate x to ϕ, y to ψ (or s) and z to θ as

$$\frac{x-x_{min}}{x_{max}-x_{min}}=\frac{\phi-\phi_{min}}{\phi_{max}-\phi_{min}} \, , \, \frac{y-y_{min}}{y_{max}-y_{min}}=\frac{\psi-\psi_{min}}{\psi_{max}-\psi_{min}} \, , \, \frac{z-z_{min}}{z_{max}-z_{min}}=\frac{\theta-\theta_{min}}{\theta_{max}-\theta_{min}} \tag{4.65}$$

Then desired ψ and θ values for given input ϕ are found as follows:

$$\begin{gathered}\phi=\frac{\phi_{max}-\phi_{min}}{x_{max}-x_{min}}\left(x-x_{min}\right)+\phi_{min} \, , \, \psi=\frac{\psi_{max}-\psi_{min}}{y_{max}-y_{min}}\left(g(x)-y_{min}\right)+\psi_{min} \, , \\ \theta=\frac{\theta_{max}-\theta_{min}}{z_{max}-z_{min}}\left(f(x)-z_{min}\right)+\theta_{min}\end{gathered} \tag{4.66}$$

and conversely

$$\begin{gathered}x=\frac{x_{max}-x_{min}}{\phi_{max}-\phi_{min}}\left(\phi-\phi_{min}\right)+x_{min} \, , \, y=\frac{y_{max}-y_{min}}{\psi_{max}-\psi_{min}}\left(\psi-\psi_{min}\right)+y_{min} \, , \\ z=\frac{z_{max}-z_{min}}{\theta_{max}-\theta_{min}}\left(\theta-\theta_{min}\right)+z_{min}\end{gathered} \tag{4.67}$$

Eq. (4.66) is used when determining the design points $\{\phi_i\}_1^n$, $\{\psi_i\}_1^n$ and $\{\theta_i\}_1^n$ from $\{x_i\}_1^n$, $\{y_i\}_1^n=\{g(x_i)\}_1^n$ and $\{z_i\}_1^n=\{f(x_i)\}_1^n$. Selection of $\{x_i\}_1^n$ may be done with equal spacing, Chebyshev spacing, or any other type of spacing depending of the application. The experience of the author is that the Remez algorithm converges faster (in 3 or 4 iterations) when the initial spacing is selected as the Chebyshev spacing.

Eq. (4.67) is used after the synthesis is done and one needs to check the error in between the desired $z = f(x)$ and the generated z with the mechanism. At this step, one shall determine the output values of the mechanism loops for several given input values, say 100 values, by solving the I/O relationship.

4.3. Numerical Examples

All the formulations in the previous sections are implemented in MS Excel® and several different function synthesis tasks are performed. Here some of the results are presented for illustration.

First, power functions of type $z = x^p$ are worked out. As an example, consider $z = x^{0.5}$ for $1 \le x \le 5$, which is decomposed as $y = x^{0.6}$, $z = y^{5/6}$. For $130° \ge \phi \ge 50°$ input range and $210° \le \theta \le 270°$ output range, synthesis is performed for all of the double-planar, double-spherical and plano-spherical 6-R linkages with no extra parameters ϕ_0, θ_0, etc. in order to compare the resultant error. The intermediate joint variable limits are chosen as $0.3 \le s \le 0.9$ for the double-planar linkage and $110° \le \psi \le 200°$ for the double-spherical and the plano-spherical linkages. The initial design points in between the limits are determined using Chebyshev spacing. The percentage error variation in the z values defined as

$$\%Error = \frac{z_{desired} - z_{mechanism}}{z_{desired}} \times 100 \tag{4.68}$$

is given in Figure 4.4.

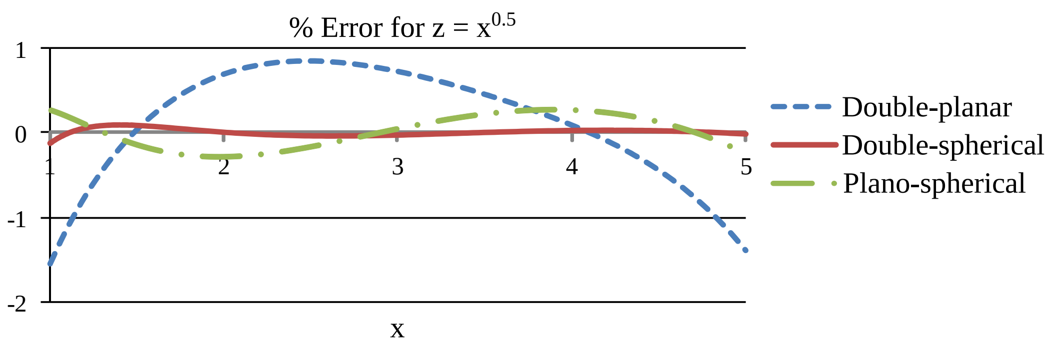


Figure 4.4. Percentage error variations for $z = x^{0.5}$

As seen from Figure 4.4, the lowest percentage error values are obtained with the double-spherical linkage. The maximum error magnitude is 0.123% for the double-spherical linkage, 0.291% for the plano-spherical linkage and 1.54% for the double-planar linkage. So, for $z = x^{0.5}$ it seems like the double-spherical linkage provides the best approximation. Of course, it is possible to change the intermediate function $y = f(x)$ and the boundaries for the input, output and intermediate joint variables. For different selections, another linkage may be better than the others. The designer similarly may test several different conditions for the same function in order to determine the best-suited mechanism for that specific function.

It is important to note that not every function can be generated with all of the Bennett linkages for arbitrary selections of the joint variable limits. In the above example, a proper set of joint variable limits are found so that the synthesis can be performed for all of the three linkages. If one is to work on just one of the linkages for a specific function, it is possible to decrease the maximum error to the levels of 0.01%. For example, for the same function as above, i.e. for $z = x^{0.5}$ for $1 \leq x \leq 5$, if $y(x) = x^{0.8}$, $126° \geq \phi \geq 59°$, $193° \leq \theta \leq 260°$ and $94° \leq \psi \leq 199°$, the maximum absolute error is as low as 0.0074%.

The second function that is worked on is $z = \sin(x)$ function for $45° \leq x \leq 60°$, which is decomposed as $y = \tan(x/2)$ and $z = 2y/(1+y^2)$. The reason for such a short range for x is to be able to compare the three linkages for the same function and same joint variable limits. In this case, the extra parameters ϕ_0 and θ_0 are employed for the input and output angles, respectively. The same joint variable limits as above are used: $130° \geq \phi \geq 50°$, $210° \leq \theta \leq 270°$, $0.6 \leq s \leq 0.9$ and $110° \leq \psi \leq 200°$. In this case, the error values for all mechanisms are very low, so it is not necessary to present the error variation graph.

The results are summarized in

Table 4.1. The maximum error magnitude is 9.3×10^{-4}% for the double-spherical linkage, 9.2×10^{-3}% for the plano-spherical linkage and 0.042% for the double-planar linkage. As expected, as the number of design points increases, the accuracy of approximation increases. The construction parameters designed for the function generation of $z = x^{0.5}$ and $z = \sin(x)$ are listed in

Table 4.1. Note that some link lengths for the plano-spherical linkage turn out to be negative. This does not mean that the construction of the mechanism is not possible. Referring to Figure 4.4, a_8 is a directed dimension, so it can be negative, anyway. On the other hand, a_7 being computed negative means that the angle of this link should not be measured from +y direction to GF direction as in Figure 4.4, but in the opposite direction. Practically this does not cause any problems. One can simply extend this link and get the measurement from the extension.

Table 4.1. Designed construction parameters for generation of $z=x^{0.5}$ and z=sin(x)

Linkage	Function	ϕ_0	θ_0	Link lengths	$\lvert\%\text{Error}\rvert_{max}$
Double-planar	$z = x^{0.5}$	-	-	a = 0.45044, b = 0.6757, c = 0.65565, d = 0.32562, e = 0.575, f = 0.23706, g = 1	1.54%
Double-spherical	$z = x^{0.5}$	-	-	α_1 = 158.40°, α_2 = 129.13°, α_3 = 65.34°, α_4 = 94.45°, α_5 = 150.67°, α_6 = 82.36°, α_7 = 93.03°, α_8 = 159.25°	0.123%
Plano-spherical	$z = x^{0.5}$	-	-	α_1 = 158.40°, α_2 = 129.13°, α_3 = 65.34°, α_4 = 94.45°, a_5 = 1, a_6 = 1.1770, a_7 = –0.5488, a_8 = –0.1790	0.291%
Double-spherical*	$z = x^{0.5}$	-	-	α_1 = 156.20°, α_2 = 274.79°, α_3 = 77.03°, α_4 = 323.18°, α_5 = 351.84°, α_6 = 93.15°, α_7 = 279.21°, α_8 = 172.37°	0.0074%
Double-planar	z = sin(x)	90.96°	–13.84°	a = 0.3854, b = 1.4708, c = 1.0943, d = 0.0708, e = 4.0310, f = 3.9615, g = 1	0.042%

Double-spherical	z = sin(x)	172.88°	43.25°	α_1 = 194.54°, α_2 = 62.24°, α_3 = 126.61°, α_4 = 51.52°, α_5 = 174.46°, α_6 = 171.44°, α_7 = 172.04°, α_8 = 175.94°	9.3×10^{-4}%
Plano-spherical	z = sin(x)	172.88°	135.52°	α_1 = 194.54°, α_2 = 62.24°, α_3 = 126.61°, α_4 = 51.52°, a_5 = 1, a_6 = 1.3908, a_7 = 0.4618, a_8 =-0.0899	9.2×10^{-3}%

In

Table 4.1 the first three rows correspond to synthesis of different linkages for the same range of input/output variables for comparison reasons, whereas row 4 corresponds to synthesis of a double-spherical linkage for further minimized error by changing the input/output ranges.

The last case study is a function used in Alizade and Kilit (2005) for comparison. In the paper, the authors generate $z = x^{0.6}$ for $1 \le x \le 5$ by means of a spherical four-bar mechanism with five construction parameters: the four link lengths α_1, α_2, α_3, α_4 and the extra parameter θ_0 for the output angle. They used interpolation approximation with equally spaced five precision points and the limits are selected as $8° \le \phi \le 80°$, $5° \le \theta \le 160°$ for the input and the output joint variables, respectively. The variation in the percentage error is depicted in Figure 4.5 In this case, the maximum absolute percentage error is 2.229%.

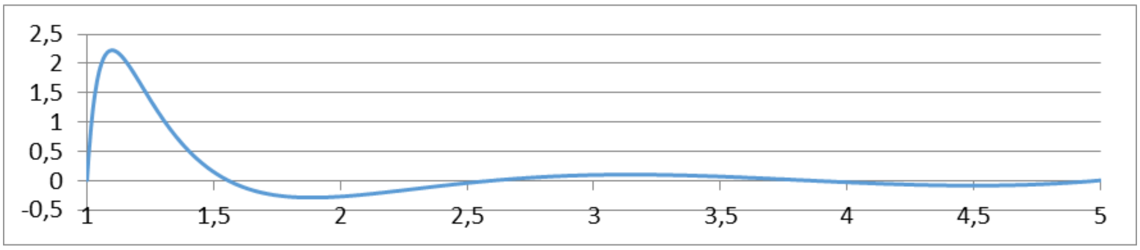


Figure 4.5. Percentage error variation for $z = x^{0.6}$ for spherical four-bar mechanism with interpolation approximation

When Chebyshev approximation is used for the same function $z = x^{0.6}$ for $1 \leq x \leq 5$ with the spherical four-bar mechanism with same joint variable limits and five construction parameters, relatively lower error values are obtained as shown in Figure 4.6. In this case, the maximum absolute error is 1.28%.

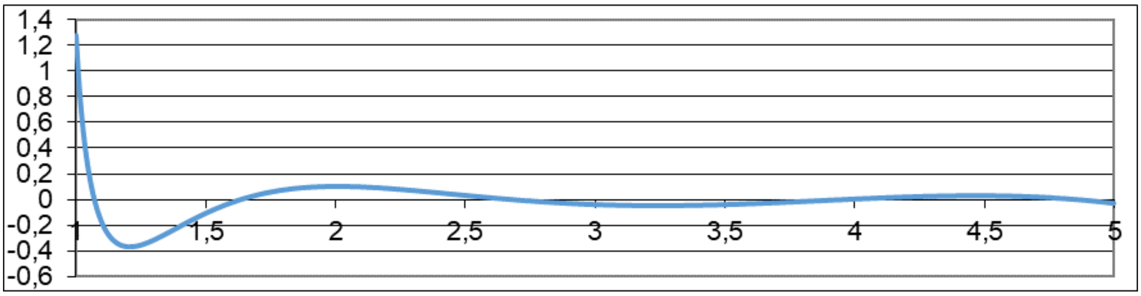


Figure 4.6. Percentage error variation for $z = x^{0.6}$ for spherical four-bar mechanism with Chebyshev approximation

When the Chebyshev approximation is applied with the double spherical mechanism for the same function with the same input-output joint limits and for $y = x^{0.75}$ and $75° \leq \psi \leq 160°$, the error variation in Figure 4.7 is obtained. The maximum absolute error is 0.016%.

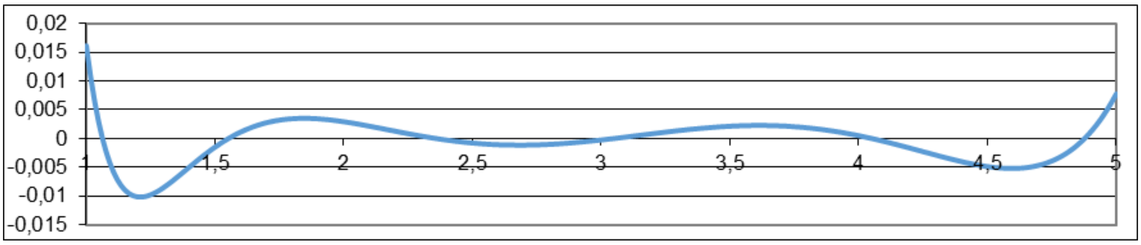


Figure 4.7. Percentage error variation for $z = x^{0.6}$ for double-spherical 6R mechanism with Chebyshev approximation

The designed construction parameters for the spherical four-bar and double-spherical 6R linkages are listed in Table 4.2.

Table 4.2. Designed construction parameters for generation of $z = x^{0.6}$

Linkage	Approximation	ϕ_0	θ_0	Link lengths	$\|\%Error\|_{max}$

Spherical 4R	Interpolation	-	11.03°	$\alpha_1 = 10.75°$, $\alpha_2 = 294.90°$, $\alpha_3 = 30.46°$, $\alpha_4 = 270.47°$	2.23%
Spherical 4R	Chebyshev	-	9.96°	$\alpha_1 = 271.92°$, $\alpha_2 = 87.00°$, $\alpha_3 = 34.01°$, $\alpha_4 = 142.30°$	1.28%
Double-spherical	Chebyshev	52°	–4.91°	$\alpha_1 = 10.75°$,$\alpha_2 = 114.9°$,$\alpha_3 = 28.34°$,$\alpha_4 = 93.17°$, $\alpha_5 = 337.74°$,$\alpha_6 = 203.92°$,$\alpha_7 = 355.24°$,$\alpha_8 = 185.03°$	0.016%

The construction parameter values in

Table 4.1 and Table 4.2 are used to construct solid models of the mechanisms and these models are tested to satisfy the input/output joint values.

CHAPTER 5

FUNCTION GENERATION SYNTHESIS WITH A 2-DOF OVER-CONSTRAINED DOUBLE-SPHERICAL 7R MECHANISM USING THE METHOD OF DECOMPOSITION AND LEAST SQUARES APPROXIMATION[3]

This study addresses the approximate function generation synthesis with an over-constrained 2-dof double spherical 7R mechanism using least squares approximation with equal spacing of the design points on the input domain. The 7R mechanism is a con-structed by combining a spherical 5R mechanism with a spherical 4R mechanism with distant centers and a common moving link and then removing the common link. This construction allows the analysis and synthesis of the resulting single-loop mechanism by decomposing it into fictitious 5R and 4R loops. The two inputs to the mechanism are provided in the 5R loop and the output is in the 4R loop. The fictitious output of the 5R loop is an input to the 4R loop. This intermediate variable is used to also decompose the function to be generated. This decomposition provides the designer extra freedom in synthesis and enables decreasing the error of approximation. A case study is presented at the end of the study where the 7R design is com-pared with an equivalent spherical 5R mechanism; hence the advantage of the 7R mechanism is demonstrated.

5.1. The Double-Spherical 7R Mechanism

The double spherical 7R mechanism in Figure 5.1 is constructed by combining the spherical 5-bar ABCDE and the spherical 4-bar AFGH and then removing the common joint A. The inputs of the mechanism are the angles θ and ϕ in the fictitious 5-bar and the output is the angle η associated to link 8. In applying the method of decomposition, the output ψ of the 5-bar is treated as the input to the 4-bar. For a function synthesis problem,

[3] The main content of this chapter is published by Kiper and Bağdadioğlu (2015).

without loss of generality, the radii of the spheres can be taken as 1. The construction parameters are the spherical link lengths α_1, …, α_9 and the twist angle γ. To simplify the formulation let us assume $\gamma = 0$. See (Alizade et al., 2014) for the general case.

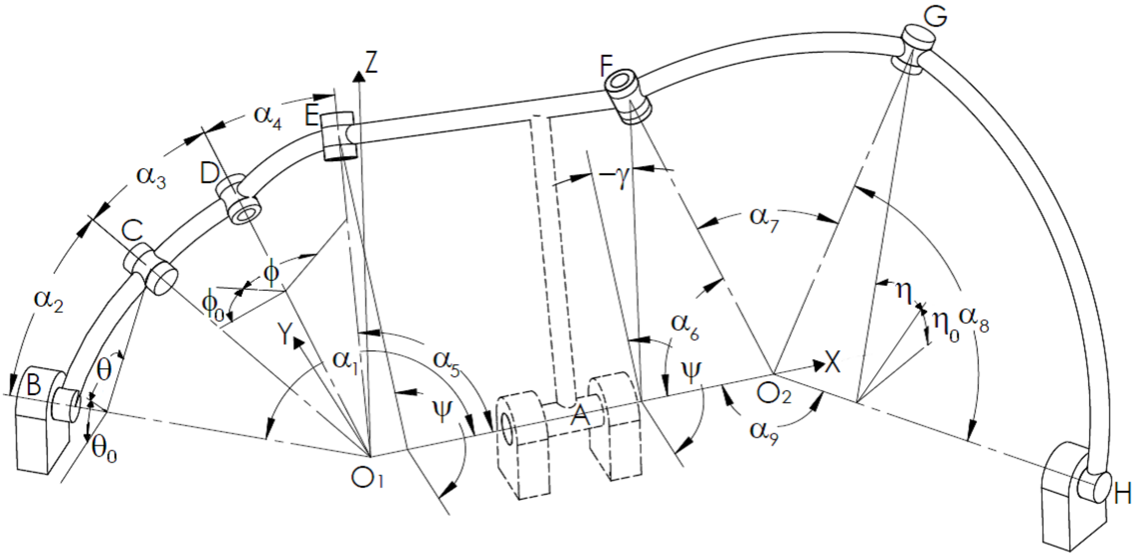


Figure 5.1. The double-spherical 7R mechanism

The starting point is deriving the input/output (I/O) relationship for the loops. Coordinates of B, C and E:

$$\overline{C} = Z(\alpha_1)X(\theta)Z(\alpha_2)\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}c_1c_2 - s_1s_2c\theta\\ s_1c_2 + c_1s_2c\theta\\ s_2s\theta\end{bmatrix},\ \overline{E} = X(-\psi)Z(-\alpha_5)\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}c_5\\ -s_5c\psi\\ s_5s\psi\end{bmatrix} \quad (5.1)$$

where X(.) and Z(.) are 3×3 rotation matrices about x- and z-axes, respectively, $c_k = \cos\alpha_k$, $s_k = \sin\alpha_k$, $c\theta = \cos\theta$, etc. Let $\angle CE = \beta$. β depends on α_3, α_4 and the joint variable ϕ via the spherical cosine theorem for triangle CDE:

$$c_3c_4 + s_3s_4c\phi = c\beta \quad (5.2)$$

On the other hand, the scalar product of $\overline{C}$ and $\overline{E}$ from Eq. (5.1) yields

$$\overline{C}\cdot\overline{E} = c\beta \Rightarrow c_1c_2c_5 - s_1s_2c_5c\theta - s_1c_2s_5c\psi - c_1s_2s_5c\theta c\psi + s_2s\theta s_5s\psi = c\beta \quad (5.3)$$

Combining Eqs. (5.2)-(5.3) and rearranging

$$c_1c_2c_5 - c_3c_4 - s_3s_4c\phi - s_1s_2c_5c\theta + s_2s_5s\theta s\psi - c_1s_2s_5c\theta c\psi - s_1c_2s_5c\psi = 0 \qquad (5.4)$$

Eq. (5.4) can be written in the following polynomial form

$$\sum_{j=1}^{m} P_j f_j(\mathbf{x}) - F(\mathbf{x}) = 0 \qquad (5.5)$$

where m = 5, $\mathbf{x} = \{\theta, \phi, \psi\}$ and

$$\begin{gathered} P_1 = \frac{c_1c_2c_5 - c_3c_4}{s_1c_2s_5},\ P_2 = \frac{s_3s_4}{s_1c_2s_5},\ P_3 = \frac{t_2}{t_5},\ P_4 = \frac{t_2}{s_1},\ P_5 = \frac{t_2}{t_1} \\ f_1(\mathbf{x}) = 1,\ f_2(\mathbf{x}) = -c\phi,\ f_3(\mathbf{x}) = -c\theta,\ f_4(\mathbf{x}) = s\theta s\psi,\ f_5(\mathbf{x}) = -c\theta c\psi \\ \text{and } F(\mathbf{x}) = c\psi \end{gathered} \qquad (5.6)$$

for $t_k = \tan\alpha_k$. P_j's are determined using least squares approximation. After P_j's are solved, the construction parameters of the mechanism are determined from Eq. (5.6) as

$$\alpha_1 = \pm\cos^{-1}\frac{P_5}{P_4},\ \alpha_2 = \tan^{-1}\left(P_4 s_1\right),\ \alpha_5 = \tan^{-1}\frac{t_2}{P_3},\ \alpha_3 = \frac{A \pm B}{2}, \alpha_4 = \frac{A \mp B}{2} \qquad (5.7)$$

$$A = \cos^{-1}\left[c_2\left(c_1c_5 - \left(P_1 + P_2\right)s_1s_5\right)\right] \text{ and } B = \cos^{-1}\left[c_2\left(c_1c_5 - \left(P_1 - P_2\right)s_1s_5\right)\right] \qquad (5.8)$$

Notice from Figure 5.1 and Eq. (5.2) that interchanging α_3 and α_4 does not affect the I/O relationship. For the spherical 4-bar AFGH, the coordinates of joints F and G are

$$\overline{F} = X(-\psi)Z(\alpha_6)\begin{bmatrix}-1\\0\\0\end{bmatrix} = \begin{bmatrix}-c_6\\-s_6c\psi\\s_6s\psi\end{bmatrix},\ \overline{G} = Z(\alpha_9)X(-\eta)Z(\alpha_8)\begin{bmatrix}-1\\0\\0\end{bmatrix} = -\begin{bmatrix}c_8c_9 - s_8s_9c\eta\\c_8s_9 + s_8c_9c\eta\\-s_8s\eta\end{bmatrix} \qquad (5.9)$$

Evaluating the scalar product of $\overline{F}$ and $\overline{G}$ and manipulating:

$$c_6c_8c_9 - c_7 + s_6c_8s_9c\psi + s_6s_8c_9c\eta c\psi + s_6s_8s\eta s\psi - c_6s_8s_9c\eta = 0 \qquad (5.10)$$

Eq. (5.10) can be written in polynomial form of Eq. (5.5), but m = 4, $\mathbf{x} = \{\psi, \eta\}$ and

$$P_1 = \frac{c_6c_8c_9 - c_7}{c_6s_8s_9},\ P_2 = \frac{t_6}{t_8},\ P_3 = \frac{t_6}{t_9},\ P_4 = \frac{t_6}{s_9},$$
$$f_1(\mathbf{x}) = 1,\ f_2(\mathbf{x}) = c\psi,\ f_3(\mathbf{x}) = c\eta c\psi,\ f_4(\mathbf{x}) = s\eta s\psi \text{ and } F(\mathbf{x}) = c\eta \qquad (5.11)$$

After P_j's are linearly solved, the construction parameters are determined from Eq. (5.11) as

$$\alpha_9 = \pm\cos^{-1}\frac{P_3}{P_4}, \alpha_6 = \tan^{-1}(P_4s_9), \alpha_8 = \tan^{-1}\frac{t_6}{P_2}, \alpha_7 = \pm\cos^{-1}(c_6c_8c_9 - c_6s_8s_9P_1) \qquad (5.12)$$

5.2. Function Generation Synthesis

Let the function to be generated be z = f(x, y) for $x_{min} \le x \le x_{max}$ and $y_{min} \le y \le y_{max}$. The independent variables x and y should be related to the mechanism inputs θ and ϕ and the dependent variable z should be related to the mechanism output η. Via method of decomposition the intermediate joint angle ψ should be related to an intermediate variable w such that w = g(x, y) and z = h(w) = f(x, y). θ, ϕ, ψ and η can be chosen in arbitrary ranges $\theta_{min} \le \theta \le \theta_{max}$, $\phi_{min} \le \phi \le \phi_{max}$, $\psi_{min} \le \psi \le \psi_{max}$, $\eta_{min} \le \psi \le \eta_{max}$. One can linearly relate x, y, w and z to θ, ϕ, ψ and η as

$$\frac{x - x_{min}}{x_{max} - x_{min}} = \frac{\theta - \theta_{min}}{\theta_{max} - \theta_{min}},\ \frac{y - y_{min}}{y_{max} - y_{min}} = \frac{\phi - \phi_{min}}{\phi_{max} - \phi_{min}},$$
$$\frac{w - w_{min}}{w_{max} - w_{min}} = \frac{\psi - \psi_{min}}{\psi_{max} - \psi_{min}} \text{ and } \frac{z - z_{min}}{z_{max} - z_{min}} = \frac{\eta - \eta_{min}}{\eta_{max} - \eta_{min}} \qquad (5.13)$$

Then desired ψ and η values for given inputs θ and ϕ are found as follows:

$$\theta = \frac{\theta_{max} - \theta_{min}}{x_{max} - x_{min}}\left(x - x_{min}\right) + \theta_{min}, \ \phi = \frac{\phi_{max} - \phi_{min}}{y_{max} - y_{min}}\left(y - y_{min}\right) + \phi_{min},$$
$$\psi = \frac{\psi_{max} - \psi_{min}}{w_{max} - w_{min}}\left(g\left(x, y\right) - w_{min}\right) \text{ and } \eta = \frac{\eta_{max} - \eta_{min}}{z_{max} - z_{min}}\left(f\left(x\right) - z_{min}\right) + \eta_{min} \tag{5.14}$$

and conversely

$$x = \frac{\theta - \theta_{min}}{\theta_{max} - \theta_{min}}\left(x_{max} - x_{min}\right) + x_{min}, \ y = \frac{\phi - \phi_{min}}{\phi_{max} - \phi_{min}}\left(y_{max} - y_{min}\right) + y_{min},$$
$$w = \frac{\psi - \psi_{min}}{\psi_{max} - \psi_{min}}\left(w_{max} - w_{min}\right) + w_{min} \text{ and } z = \frac{\eta - \eta_{min}}{\eta_{max} - \eta_{min}}\left(z_{max} - z_{min}\right) + z_{min} \tag{5.15}$$

Eq. (5.14) is used for determining the design points θ_i, ϕ_i, ψ_i and η_i from x_i, y_i, w_i is equal $g(x_i, y_i)$ and $z_i = f(x_i, y_i)$. Selection of x_i and y_i may be done with equal spacing, Chebyshev spacing, or any other type of spacing.

In least squares approximation the number of design points, n, should be more than the number of construction parameters m (= 5 for the 5-bar and 4 for the 4-bar mechanism) and the aim is to minimize the square sum of the errors at the design points $\mathbf{x}_i$ for i = 1, …, n. Due to the generation error, Eq. (5.5) is not exactly satisfied, but there is an error δ_i. In order to find the minimum of the square sum, the square sum is differentiated with respect to coefficients P_j and equated to zero:

$$\frac{d}{dP_j}\left\{\sum_{i=1}^{n}\left[\sum_{j=1}^{m} P_j f_j\left(\mathbf{x}_i\right) - F\left(\mathbf{x}_i\right)\right]^2\right\} = 0 \text{ for } j = 1, \ldots, m \tag{5.16}$$

Eqs. (5.16) are linear in P_j’s, hence P_j’s can be determined uniquely. However, there are some restrictions on P_j’s in order to obtain a mechanism. For instance, from Eq. (5.7) it is seen that $|P_5| \leq |P_4|$ in order to be able to compute $\cos^{-1}$.

The maximum percentage error is defined as

$$\%|E|_{max} = \max\left(\left|\frac{z_{desired} - z_{generated}}{z_{desired}}\right| \times 100\right) \tag{5.17}$$

During the computations $\%|E|_{max}$ is monitored and the freely chosen parameters that are associated with selection of the intermediate function w and the limits of the input/output joint variables are tuned in order to minimize the maximum error.

5.3. Case Study

As an example, consider the function $z = x^{0.6}y^{0.2}$ for $5 \le x \le 10$ and $14 \le y \le 17$. Let the intermediate variable as $w = x^a y^b$ such that $z = w^c$, where c can be chosen freely, a = 0.6/c and b = 0.2/c. 25 design points are employed as equally spaced 5 by 5 grid for the inputs x and y. The limits of the inputs θ and ϕ, the passive joint variable ψ and the output η of the mechanism are also free to choose. Therefore, there are 9 free parameters in this synthesis problem. After several trials on the free parameters, a solution with relatively low error is determined for c = 0.9, $145° \le \theta \le 300°$, $100° \ge \phi \ge 80°$, $105° \le \psi \le 185°$ and $250° \ge \eta \ge 185°$. The maximum percentage error is found as 0.656%. The maximum percentage error can be further decreased; however, Percentage error variation is depicted in Figure 5.2. For comparison, also the synthesis with a spherical 5R mechanism is worked out for the same function and maximum percentage error is found as 0.834%.

The construction parameters of the designed 7R mechanism are found as α_1 = 126.13°, α_2 = 31.61°, α_3 = 127.69°, α_4 = 17.89°, α_5 = 86.65°, α_6 = 28.47°, α_7 = 171.52°, α_8 = 35.52° and α_9 = 166.74°. It is verified that the mechanism successfully generates the desired function by means of a CAD model which is given in Figure 5.3.

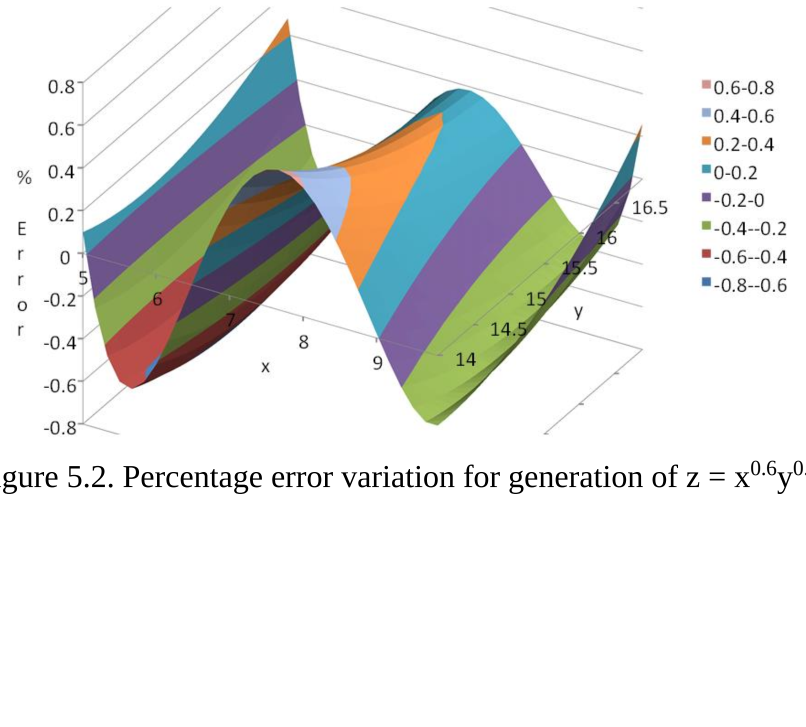


Figure 5.2. Percentage error variation for generation of $z = x^{0.6}y^{0.2}$

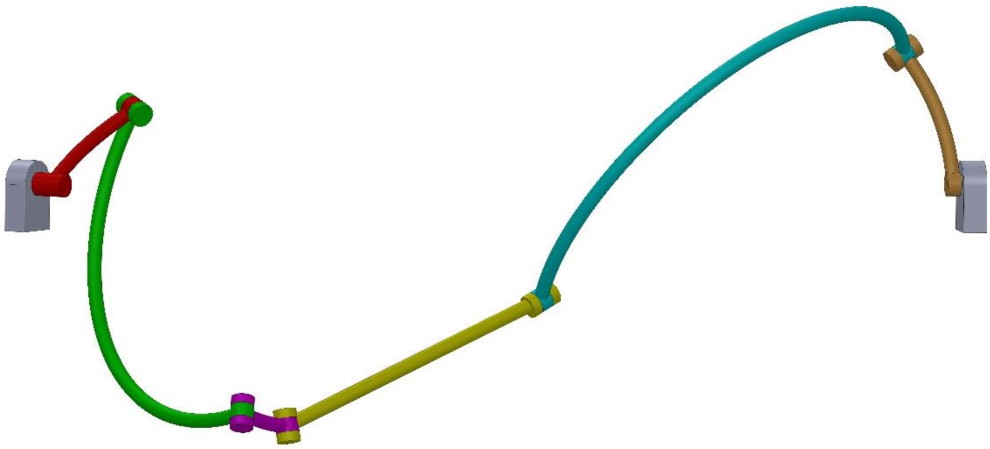


Figure 5.3. CAD model of the designed 7R mechanism

CHAPTER 6

FUNCTION SYNTHESIS OF A FAMILY OF 2-DOF PLANAR LINKAGES USING LEAST SQUARES APPROXIMATION[4]

In this study, function synthesis problem with planar 2-dof five-link and seven-link linkages by using least squares approximation method is addressed in this study. While five-link linkages have single loop, seven link mechanisms have two independent loops. $\underline{\mathrm{RP}}$RRR, $\underline{\mathrm{RR}}$RRR, 2R$\underline{\mathrm{P}}$R-RRR, R$\underline{\mathrm{P}}$R-$\underline{\mathrm{R}}$RR-RRR, 2$\underline{\mathrm{R}}$RR-RRR, $\underline{\mathrm{PR}}$RRR, $\underline{\mathrm{PP}}$RRR, $\underline{\mathrm{P}}$RR-$\underline{\mathrm{R}}$RR-RRR, $\underline{\mathrm{P}}$RR-$\underline{\mathrm{P}}$RR-RRR mechanisms are examined, where R stands for a revolute joint and P stands for a prismatic joint. The formulations are implemented in computer and many computational examples are worked out. One of the computational studies is presented as an example.

6.1. Problem Definition and General Formulation

It is required to generate a continuous function z = f(x, y) via a 2-dof planar linkage which includes an RRR chain. The RRR part of the mechanism is illustrated in Figure 6.1. The rest of the mechanism is to be attached to the revolute joint designated by point P.

The following alternative link groups are considered for the rest of the mechanism: a) an $\underline{\mathrm{RP}}$ chain, b) an $\underline{\mathrm{RR}}$ chain, c) an R$\underline{\mathrm{P}}$R$\underline{\mathrm{P}}$R loop, d) an R$\underline{\mathrm{P}}$RR$\underline{\mathrm{R}}$ loop and e) an $\underline{\mathrm{R}}$RRR$\underline{\mathrm{R}}$ loop, where _ represents an input joint. These additions result in the following mechanisms (Figure 6.2): a) $\underline{\mathrm{RP}}$RRR, b) $\underline{\mathrm{RR}}$RRR, c) 2R$\underline{\mathrm{P}}$R-RRR, d) R$\underline{\mathrm{P}}$R-$\underline{\mathrm{R}}$RR-RRR and e) 2$\underline{\mathrm{R}}$RR-RRR. All these mechanisms will have their respective two inputs as prismatic and/or revolute inputs, however all problems will be united such that the lower part of the mechanism somehow positions point P so that the problem can be recast as find output ψ for given polar coordinates $Se^{i\phi}$.

[4] The main content of this chapter is published by Kiper and Bağdadioğlu (2015).

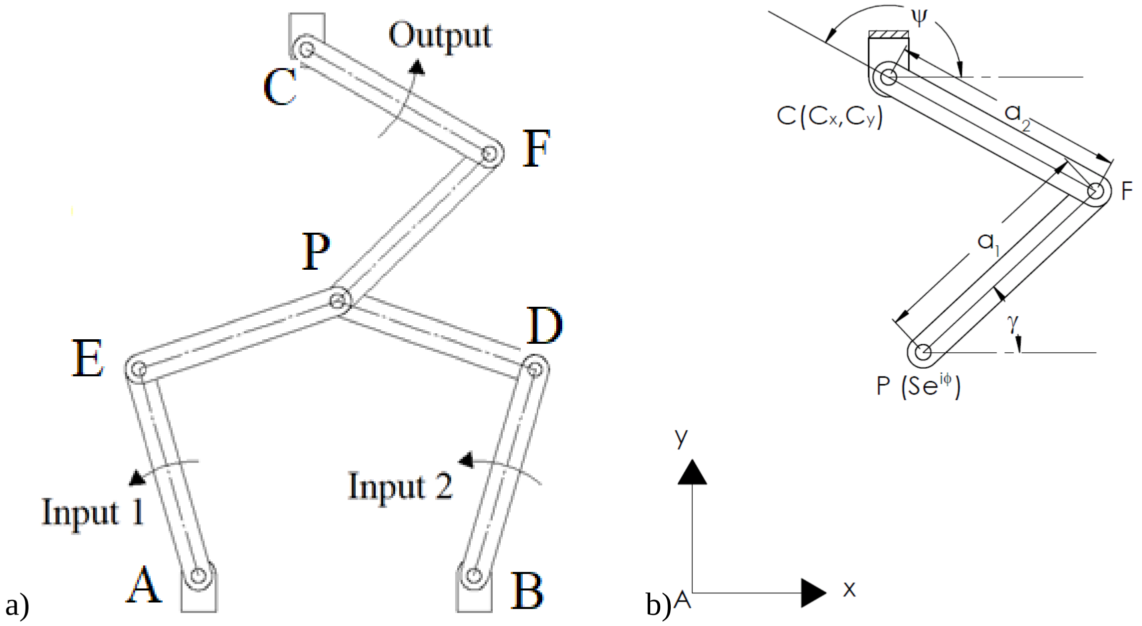


Figure 6.1. a) General view of 2 dof mechanism b) RRR part of the mechanism

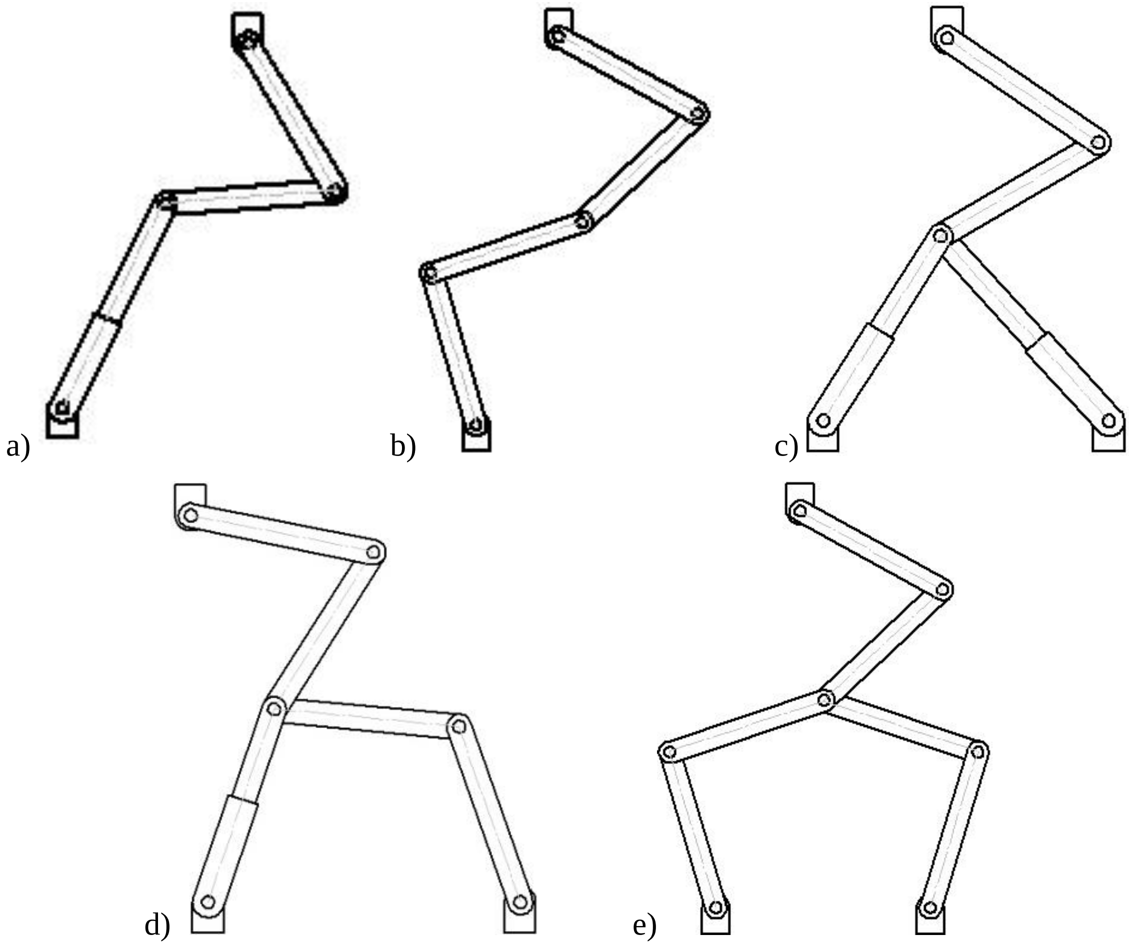


Figure 6.2. a) RPRRR, b) RRRRR, c) 2RPR-RRR, d) RPR-RRR-RRR and e) 2RRR-RRR mechanisms

For further simplification of the problem, it is assumed that the link length dimensions of the lower part of the mechanism are already selected and it is needed to design the remaining four of the link lengths shown in Figure 6.1: a_1, a_2, C_x and C_y.

$$\begin{aligned}
&\overrightarrow{PA}+\overrightarrow{AC}+\overrightarrow{CF}=\overrightarrow{AC}-\overrightarrow{FC}-\overrightarrow{AP}=\overrightarrow{PF}\\
&\Rightarrow\left|\overrightarrow{FC}-\overrightarrow{AC}+\overrightarrow{AP}\right|=\left|C_x+iC_y-a_2e^{i\psi}-Se^{i\phi}\right|=a_1\\
&\Rightarrow\left(C_x-a_2c\psi-Sc\phi\right)^2+\left(C_y-a_2s\psi-Ss\phi\right)^2=a_1{}^2\\
&\Rightarrow a_1{}^2-a^2{}_2-C_x{}^2-C_y{}^2-S^2+2a_2C_xc\psi+2C_xSc\phi\\
&\quad+2a_2C_ys\psi-2a_2Sc\left(\psi-\phi\right)+2C_ySs\phi=0
\end{aligned}\tag{6.1}$$

where c and s stand for cosine and sine, respectively. Eq. (6.1) can be written in polynomial form:

$$\sum_{j=1}^{6}P_jf_j\left(\overline{x}\right)-F\left(\overline{x}\right)=0\tag{6.2}$$

where $\overline{x}$ represents the input and output variables (S, ϕ, ψ) and

$$\begin{gathered}
P_1=a^2{}_1-a^2{}_2-C_x{}^2-C_y{}^2,\ P_2=C_y,\ P_3=C_x,\ P_4=a_2,\ P_5=a_2C_x,\ P_6=-a_2C_y,\ F(\overline{x})=S^2\\
f_1(\overline{x})=1,\ f_2(\overline{x})=2Ss\phi,\ f_3(\overline{x})=2Sc\phi,\ f_4(\overline{x})=-2Sc(\psi-\phi),\ f_5(\overline{x})=2c\psi,\ f_6(\overline{x})=-2s\psi,
\end{gathered}\tag{6.3}$$

In Eq. (6.2) there are four construction parameters (a_1, a_2, C_x and C_y), but six P_j's. P_5 and P_6 can be represented in terms of the other P_j's as

$$P_5=P_3P_4\quad\text{and}\quad P_6=-P_2P_4\tag{6.4}$$

Let $P_5 = \lambda_1$ and $P_6 = \lambda_2$ and $P_j = \lambda_j + m_j\lambda_1 + n_j\lambda_2$ for j = 1,..,4. Eq. (6.2) becomes

$$\sum_{j=1}^{4}\left(\ell_j+m_j\lambda_1+n_j\lambda_2\right)f_j\left(\overline{x}\right)+\lambda_1f_5\left(\overline{x}\right)+\lambda_2f_6\left(\overline{x}\right)-F\left(\overline{x}\right)=0\tag{6.5}$$

Coefficients of 1, λ_1 and λ_2 in Eq. (6.5) can be separated as

$$\sum_{j=1}^{4} \ell_j f_j(\overline{x}) - F(\overline{x}) = 0 \tag{6.6}$$

$$\sum_{j=1}^{4} m_j f_j(\overline{x}) + f_5(\overline{x}) = 0 \tag{6.7}$$

$$\sum_{j=1}^{4} n_j f_j(\overline{x}) + f_6(\overline{x}) = 0 \tag{6.8}$$

For least squares approximation with N (>4) design points $\overline{x}_i$ for i = 1,..,N, let

$$S_\ell = \sum_{i=1}^{N} \left[\sum_{j=1}^{4} \ell_j f_{ji} - F_i \right]^2 \tag{6.9}$$

$$S_m = \sum_{i=1}^{N} \left[\sum_{j=1}^{4} m_j f_{ji} + f_{5i} \right]^2 \tag{6.10}$$

$$S_n = \sum_{i=1}^{N} \left[\sum_{j=1}^{4} n_j f_{ji} + f_{6i} \right]^2 \tag{6.11}$$

where $f_{ji} = f_j(\overline{x}_i), f_{5i} = f_5(\overline{x}_i), f_{6i} = f_6(\overline{x}_i)$ and $F_i = F(\overline{x}_i)$. Partial derivatives of Eqs. (6.9)-(6.11) with respect to λ_j, m_j and n_j are to be equated to zero to find the minimum of the sum of squares:

$$\frac{1}{2}\frac{dS_\ell}{d\ell_k} = \sum_{i=1}^{N} \left[f_{1i}\ell_1 + f_{2i}\ell_2 + f_{3i}\ell_3 + f_{4i}\ell_4 - F_i \right] f_{ki} = 0 \qquad \text{for } k = 1,2,3,4 \tag{6.12}$$

$$\frac{1}{2}\frac{dS_m}{dm_k} = \sum_{i=1}^{N} \left[f_{1i}m_1 + f_{2i}m_2 + f_{3i}m_3 + f_{4i}m_4 + f_{5i} \right] f_{ki} = 0 \quad \text{for } k = 1,2,3,4 \tag{6.13}$$

$$\frac{1}{2}\frac{dS_n}{dn_k} = \sum_{i=1}^{N} \left[f_{1i}n_1 + f_{2i}n_2 + f_{3i}n_3 + f_{4i}n_4 + f_{6i} \right] f_{ki} = 0 \qquad \text{for } k = 1,2,3,4 \tag{6.14}$$

Eqs. (6.12)-(6.14) each represent four linear equations in four unknowns: λ_1, λ_2, λ_3, λ_4; m_1, m_2, m_3, m_4 and n_1, n_2, n_3, n_4, respectively. Eqs. (6.12)-(6.14) in matrix form:

$$\left(\sum_{i=1}^{N} f_{1i}f_{ki}\right)\ell_1+\left(\sum_{i=1}^{N} f_{2i}f_{ki}\right)\ell_2+\left(\sum_{i=1}^{N} f_{3i}f_{ki}\right)\ell_3+\left(\sum_{i=1}^{N} f_{4i}f_{ki}\right)\ell_4=\sum_{i=1}^{N} F_i f_{ki}$$
$$\Rightarrow\left[A_{kj}\right]\left[\ell_j\right]=\left[b_k\right] \quad \text{for } k=1,2,3,4 \tag{6.15}$$

$$\left(\sum_{i=1}^{N} f_{1i}f_{ki}\right)m_1+\left(\sum_{i=1}^{N} f_{2i}f_{ki}\right)m_2+\left(\sum_{i=1}^{N} f_{3i}f_{ki}\right)m_3+\left(\sum_{i=1}^{N} f_{4i}f_{ki}\right)m_4=-\sum_{i=1}^{N} f_{5i}f_{ki}$$
$$\Rightarrow\left[A_{kj}\right]\left[m_j\right]=\left[c_k\right] \quad \text{for } k=1,2,3,4 \tag{6.16}$$

$$\left(\sum_{i=1}^{N} f_{1i}f_{ki}\right)n_1+\left(\sum_{i=1}^{N} f_{2i}f_{ki}\right)n_2+\left(\sum_{i=1}^{N} f_{3i}f_{ki}\right)n_3+\left(\sum_{i=1}^{N} f_{4i}f_{ki}\right)n_4=-\sum_{i=1}^{N} f_{6i}f_{ki}$$
$$\Rightarrow\left[A_{kj}\right]\left[n_j\right]=\left[d_k\right] \quad \text{for } k=1,2,3,4 \tag{6.17}$$

where $\left[A_{kj}\right]$ is a 4×4 coefficient matrix with

$$\begin{gathered}A_{kj}=\sum_{i=1}^{N} f_{ki}f_{ji}\\ \left[\ell_j\right]=\left[\ell_1 \quad \ell_2 \quad \ell_3 \quad \ell_4\right]^T\\ \left[m_j\right]=\left[m_1 \quad m_2 \quad m_3 \quad m_4\right]^T\\ \left[n_j\right]=\left[n_1 \quad n_2 \quad n_3 \quad n_4\right]^T\end{gathered} \tag{6.18}$$

$\left[b_k\right]$, $\left[c_k\right]$ and $\left[d_k\right]$ are 4×1 column matrices with

$$b_k=\sum_{i=1}^{N} F_i f_{ki},\ c_k=-\sum_{i=1}^{N} f_{5i}f_{ki},\ d_k=-\sum_{i=1}^{N} f_{5i}f_{ki} \quad \text{for } j,\ k\ =\ 1,..,4 \tag{6.19}$$

λ_j, m_j and n_j can be linearly solved from Eqs. (6.15)-(6.17). λ_1 and λ_2 are determined in terms of λ_j, m_j and n_j as follows:

$$\begin{aligned}\lambda_1 = P_3P_4 &= \left(\ell_3 + m_3\lambda_1 + n_3\lambda_2\right)\left(\ell_4 + m_4\lambda_1 + n_4\lambda_2\right)\\ &\Rightarrow m_3m_4\lambda_1^2 + n_3n_4\lambda_2^2 + \left(m_3n_4 + n_3m_4\right)\lambda_1\lambda_2\\ &\quad + \left(\ell_3m_4 + m_3\ell_4 - 1\right)\lambda_1 + \left(l_3n_4 + n_3l_4\right)\lambda_2 + \ell_3\ell_4 = 0\end{aligned} \tag{6.20}$$

$$\begin{aligned}\lambda_2 = -P_2P_4 &= -\left(\ell_2 + m_2\lambda_1 + n_2\lambda_2\right)\left(\ell_4 + m_4\lambda_1 + n_4\lambda_2\right)\\ &\Rightarrow m_2m_4\lambda_1^2 + n_2n_4\lambda_2^2 + \left(m_2n_4 + n_2m_4\right)\lambda_1\lambda_2\\ &\quad + \left(\ell_2m_4 + m_2\ell_4 + 1\right)\lambda_1 + \left(\ell_2n_4 + n_2\ell_4\right)\lambda_2 + \ell_2\ell_4 = 0\end{aligned} \tag{6.21}$$

λ_2 is eliminated from Eqs. (6.20)-(6.21) and a degree 3 polynomial equation in terms of λ_1 is obtained. The details of this equation are presented in (Alizade and Kilit, 2005). When solving this equation, there may be one or three real solutions. In case of three real solutions, one should take the solution which provides less error and also feasible link dimensions. Once λ_1 and λ_2 are found, $P_j = \lambda_j + m_j\lambda_1 + n_j\lambda_2$ for j = 1,..,4 are determined and the construction parameters are solved from Eq. (6.3) as

$$C_x = P_3,\ C_y = P_2,\ a_2 = P_4,\ a_1 = \sqrt{P_1 + a_2^2 + C_x^2 + C_y^2} \tag{6.22}$$

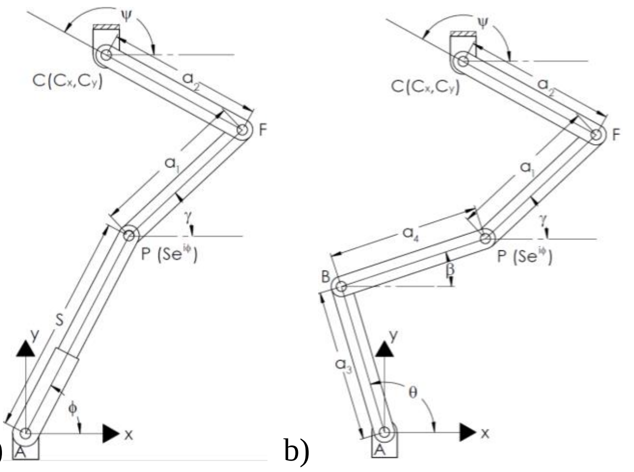
Notice that $P_1 + a_2^2 + C_x^2 + C_y^2$ should be non-negative in order to get a real solution. The following sections present how S and ϕ are found for the five different mechanisms in Figure 6.2 and explain how the function synthesis problem is formulated.

6.2. <u>RP</u>RRR and <u>RR</u>RRR Mechanisms

For the <u>RP</u>RRR mechanism illustrated in Figure 6.3a the input joint parameters ϕ and S are directly equal to the polar coordinates of point P, hence the formulations above directly apply. For this mechanism there is no need to assume any link dimensions.

For the <u>RR</u>RRR mechanism illustrated in Figure 6.3b, the inputs are the angles θ and β. The function synthesis of the <u>RR</u>RRR mechanism with least squares approximation is presented in (Kiper et al., 2014), where it is assumed that |AC| = 1 (due to scalability of the mechanism without affecting the input/output (I/O) relationship) and all of a_1, a_2, a_3, a_4 are solved for. Due to selection of |AC| = 1, only one of C_x or C_y is independently found, but two more construction parameters are designed (a_3 and a_4).

Notice that the angle between the x axis in Figure 6.3b and the AC direction serves as an initial angle θ_0 which can be assumed or determined depending on the formulation.



a) b)

Figure 6.3. a) RPRRR and b) RRRRR mechanisms

For the RRRRR mechanism, alternatively one can keep C_x and C_y as design parameters and assume a_3 and a_4 values. One of a_3 and a_4 may be selected as 1 due to the scalability of the mechanism. In this case, for given inputs θ and β, the polar coordinates S and ϕ can be simply determined as follows:

$$
\begin{gathered}
\overrightarrow{AB}+\overrightarrow{BP}=\overrightarrow{AP}\Rightarrow a_3e^{i\theta}+a_4e^{i\beta}=Se^{i\phi} \\
\Rightarrow S=\sqrt{(a_3c\theta+a_4c\beta)^2+(a_3s\theta+a_4s\beta)^2}=\sqrt{a_3^{\,2}+a_4^{\,2}+2a_3a_4c(\theta-\beta)}\ , \\
\phi=\text{atan2}(a_3c\theta+a_4c\beta, a_3s\theta+a_4s\beta)
\end{gathered} \tag{6.23}
$$

6.3. 2RPR-RRR Mechanism

For the 2RPR-RRR mechanism shown in Figure 6.4 the inputs are the two prismatic joint variables S and S_2. The link length a_3 needs to be assumed. The first prismatic input directly gives S, but it is necessary to determine ϕ in terms of both of the inputs S and S_2. Using cosine theorem in triangle PAB:

$$
S_2^{\,2}=a_3^{\,2}+S^2-2a_3Sc\phi\Rightarrow\phi=\cos^{-1}\frac{a_3^{\,2}+S^2-S_2^{\,2}}{2a_3S} \tag{6.24}
$$

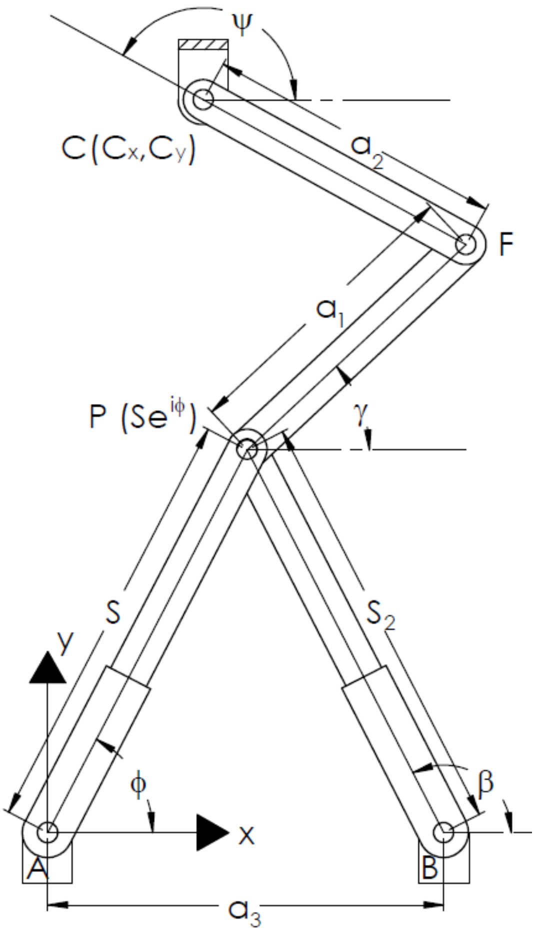


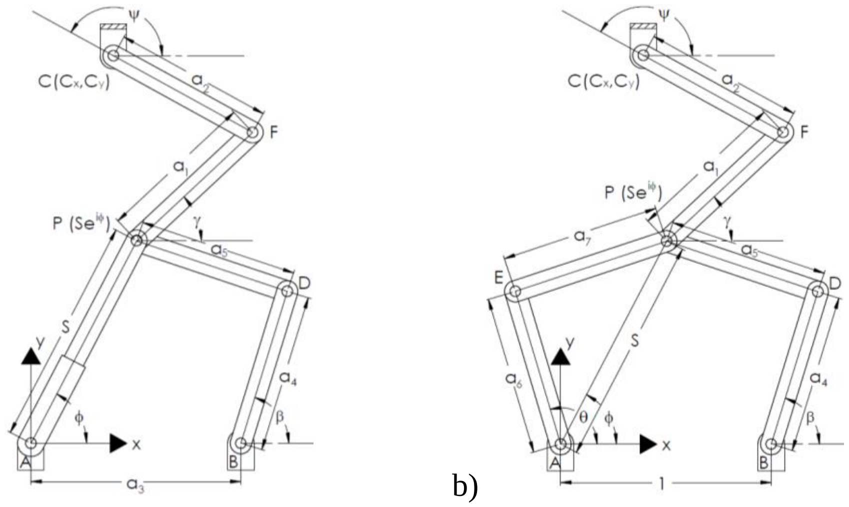
Figure 6.4. The 2R<u>P</u>R-RRR Mechanism

6.4. R<u>P</u>R-<u>R</u>RR-RRR Mechanism

For the R<u>P</u>R-<u>R</u>RR-RRR mechanism shown in Figure 6.5a the inputs are the prismatic joint variable S and the revolute joint variable β. The link lengths a_3, a_4 and a_5 need to be assumed. Once again, the prismatic input directly gives S, but it is necessary to determine ϕ in terms of both S and β. This relationship can be obtained as follows:

$$
\begin{aligned}
-\overrightarrow{BD}-\overrightarrow{AB}+\overrightarrow{AP}=\overrightarrow{DP} \Rightarrow \left|\overrightarrow{AP}-\overrightarrow{BD}-\overrightarrow{AB}\right| = \left|Se^{i\phi}-a_3-a_4e^{i\beta}\right| = a_5 \\
\Rightarrow (Sc\phi - a_3 - a_4c\beta)^2 + (Ss\phi - a_4s\beta)^2 = a_5^2 \\
\Rightarrow -2(a_3+a_4c\beta)Sc\phi - 2a_4Ss\beta s\phi = a_5^2 - a_3^2 - a_4^2 - S^2 - 2a_3a_4c\beta
\end{aligned} \tag{6.25}
$$

Eq. (6.25) is in the form $Ac\phi + Bs\phi = c$ and can be solved for ϕ using the tangent of the half angle substitution. Note that there will be two solutions and either of the solutions can be selected in favor of the designer.



a) b)

Figure 6.5. a) R<u>P</u>R-<u>R</u>RR-RRR and b) 2<u>R</u>RR-RRR mechanisms

6.5. 2<u>R</u>RR-RRR Mechanism

For the 2<u>R</u>RR-RRR mechanism shown in Figure 6.5b the inputs are the two revolute joint variables θ and β. Notice that for this mechanism scale of the link lengths does not affect the I/O relationship and without loss of generality let $|AB| = 1$. The other four link lengths of the AEPDB loop, i.e. a_4, a_5, a_6 and a_7 are to be assumed. Given the inputs θ and β, the coordinates of E and D are $a_6e^{i\theta}$ and $1 + a_4e^{i\beta}$, respectively. From link lengths a_5 and a_7 one can write

$$
\begin{gathered}
\left|\overrightarrow{PE}\right| = \left|P_x + iP_y - a_6e^{i\theta}\right| = a_7 \Rightarrow \left(P_x - a_6c\theta\right)^2 + \left(P_y - a_6s\theta\right)^2 = a_7^2 \\
\Rightarrow P_x^2 + P_y^2 + a_6^2 - a_7^2 - 2a_6c\theta P_x - 2a_6s\theta P_y = 0
\end{gathered}
\tag{6.26}
$$

$$
\begin{gathered}
\left|\overrightarrow{PD}\right| = \left|P_x + iP_y - 1 - a_4e^{i\beta}\right| = a_5 \Rightarrow \left(P_x - 1 - a_4c\beta\right)^2 + \left(P_y - a_4s\beta\right)^2 = a_5^2 \\
\Rightarrow P_x^2 + P_y^2 + 1 + a_4^2 + 2a_4c\beta - a_5^2 - 2\left(1 + a_4c\beta\right)P_x - 2a_4s\beta P_y = 0
\end{gathered}
\tag{6.27}
$$

P_x and P_y should be solved from Eqs. (6.26)-(6.27). Subtracting Eq. (6.26) from Eq. (6.27)

$$a_6^2 - a_7^2 - 1 - a_4^2 - 2a_4 c\beta + a_5^2 + 2(1 + a_4 c\beta - a_6 c\theta)P_x + 2(a_4 s\beta - a_6 s\theta)P_y = 0$$
$$\Rightarrow P_y = mP_x + n \tag{6.28}$$

where

$$m = -\frac{1 + a_4 c\beta - a_6 c\theta}{a_4 s\beta - a_6 s\theta} \quad \text{and} \quad n = -\frac{a_6^2 - a_7^2 - 1 - a_4^2 - 2a_4 c\beta + a_5^2}{2(a_4 s\beta - a_6 s\theta)} \tag{6.29}$$

Substituting Eq. (6.28) in Eq. (6.26):

$$P_x^2 + (mP_x + n)^2 + a_6^2 - a_7^2 - 2a_6 c\theta P_x - 2a_6 s\theta(mP_x + n) = 0$$
$$\Rightarrow (1 + m^2)P_x^2 + 2(mn - a_6 c\theta - ma_6 s\theta)P_x + n^2 + a_6^2 - a_7^2 - 2na_6 s\theta = 0 \tag{6.30}$$
$$\Rightarrow P_x = \frac{-(mn - a_6 c\theta - ma_6 s\theta) \pm \sqrt{(mn - a_6 c\theta - ma_6 s\theta)^2 - (1 + m^2)(n^2 + a_6^2 - a_7^2 - 2na_6 s\theta)}}{1 + m^2}$$

The ± sign in Eq. (6.30) is due to the two alternative assembly modes of the dyad EPD. Both signs can be selected in favor of the designer. Once P_x is determined, P_y is found from Eq. (6.28). S and ϕ in terms of P_x and P_y requires conversion from Cartesian coordinates to polar coordinates:

$$S = \sqrt{P_x^2 + P_y^2} \quad \text{and} \quad \phi = \operatorname{atan2}(P_x, P_y) \tag{6.31}$$

6.6. <u>PR</u>RRR and <u>PP</u>RRRR Mechanisms

<u>PR</u>RRR mechanism is illustrated in Figure 6.6a the joint variables are S_1 and β. In this mechanism link length a_3 can be chosen arbitrarily.

S and ϕ can be found in terms of S_1 and β as

$$S_1 + a_3 e^{i\theta} = S_1 + a_3 \cos\theta + i a_3 \sin\theta = S e^{i\phi}$$
$$\Rightarrow S = \sqrt{(S_1 + a_3 \cos\theta)^2 + a_3^2 \sin^2\theta} \quad \text{and} \quad \phi = \operatorname{atan2}(S_1 + a_3 \cos\theta,\ a_3 \sin\theta) \tag{6.32}$$

For the <u>PP</u>RRR mechanism shown in Figure 6.6b the two prismatic joints are inputs and

$$S_1 + iS_2 = Se^{i\phi} \Rightarrow S = \sqrt{S_1^{\,2} + S_2^{\,2}} \quad \text{and} \quad \phi = \operatorname{atan2}(S_1, S_2) \tag{6.33}$$

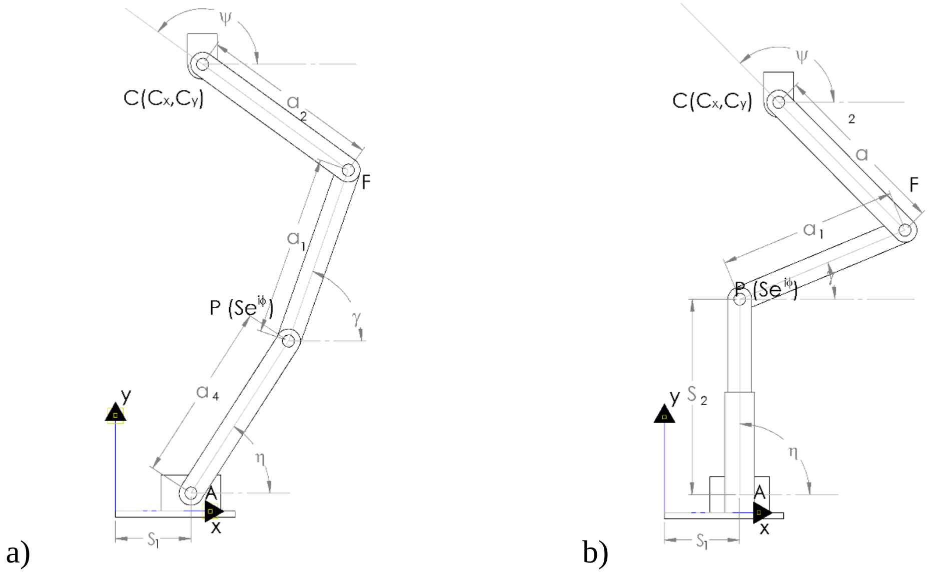


Figure 6.6. a) PRRRR and b) PPRRR mechanisms

6.7. PRR-RRR-RRR Mechanisms

The prismatic joint variable S_1 and revolute joint variable β are the inputs for PRR-RRR-RRR mechanism shown in Figure 6.7. In this mechanism a_3, a_4, a_5 and a_6 can be chosen arbitrarily.

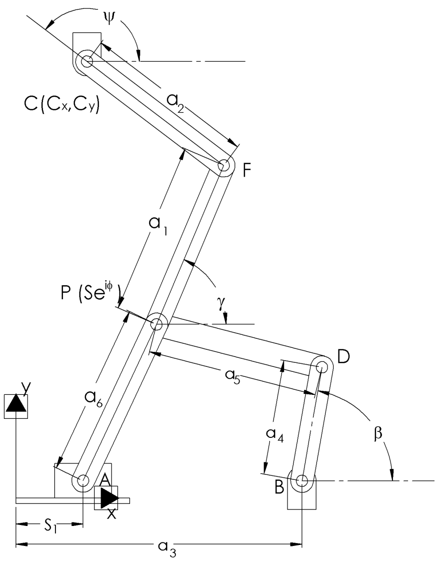


Figure 6.7. PRR-RRR-RRR mechanism

When S_1 and β are given, the joint locations of A and D will be known, hence

$$\left|\overrightarrow{AP}\right|=\left|P_x+iP_y-S_1\right|=a_6 \Rightarrow (P_x-S_1)^2+(P_y)^2=a_6^2$$
$$\Rightarrow P_x^2+P_y^2-2S_1P_x+S_1^2-a_6^2=0 \tag{6.34}$$

$$\left|\overrightarrow{DP}\right|=\left|P_x+iP_y-a_3-a_4e^{i\beta}\right|=a_5 \Rightarrow (P_x-a_3-a_4c\beta)^2+(P_y-a_4s\beta)^2=a_5^2$$
$$\Rightarrow P_x^2+P_y^2-2(a_3+a_4c\beta)P_x-2a_4s\beta P_y+a_3^2+a_4^2-a_5^2+2a_3a_4c\beta=0 \tag{6.35}$$

In order to solve for P_x and P_y, Eq.(6.35) is subtracted from Eq.(6.34)

$$a_3^2+a_4^2-a_5^2+a_6^2-S_1^2+2a_3a_4c\beta-2(a_3+a_4c\beta-S_1)P_x-2a_4s\beta P_y=0 \tag{6.36}$$

P_y can be obtained in terms of P_x from Eq. (6.36) as

$$P_y=mP_x+n \tag{6.37}$$

where

$$m=-\frac{a_3+a_4c\beta-S_1}{a_4s\beta} \quad \text{and} \quad n=\frac{a_3^2+a_4^2-a_5^2+a_6^2-S_1^2+2a_3a_4c\beta}{2a_4s\beta} \tag{6.38}$$

Substituting Eq. (6.37) in Eq. (6.34) and solving for P_x:

$$P_x^2+(mP_x+n)^2-2S_1P_x+S_1^2-a_6^2=0 \Rightarrow P_x=\frac{S_1-mn\pm\sqrt{(S_1-mn)^2-(1+m^2)(S_1^2-a_6^2+n^2)}}{1+m^2} \tag{6.39}$$

P_y is found via Eq. (6.37). S and ϕ in terms of P_x and P_y are found via Eq. (6.31)

6.8. <u>P</u>RR-<u>P</u>RR-RRR and <u>P</u>RR-R<u>P</u>R-RRR Mechanisms

For the <u>P</u>RR-<u>P</u>RR-RRR mechanism shown in Figure 6.8a, prismatic joint variables S_1 and S_2 are the inputs. The link lengths a_3 and a_4 can be chosen freely.

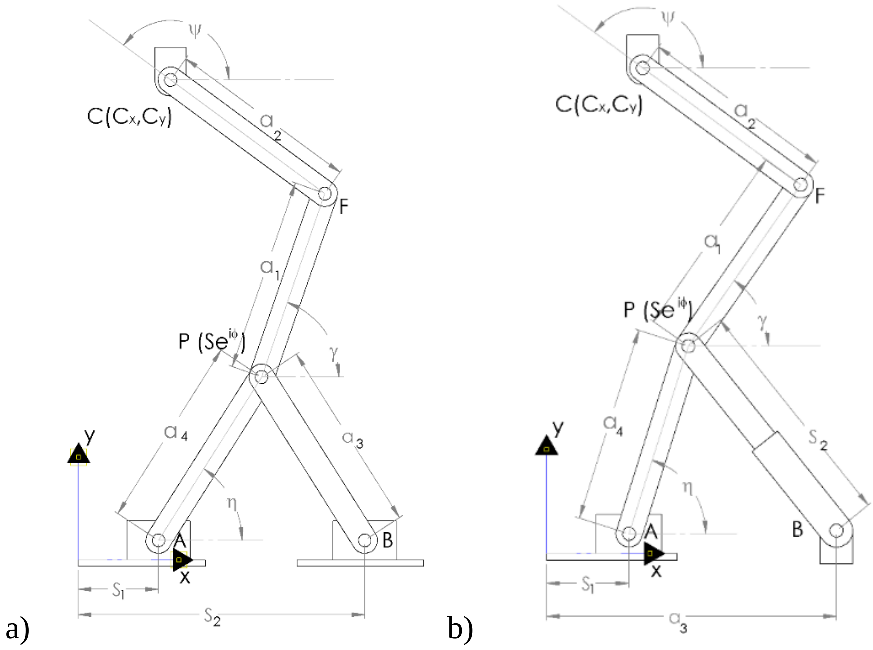


Figure 6.8. a) 2<u>P</u>RR-RRR and b) <u>P</u>RR-R<u>P</u>R-RRR mechanisms

Writing cosine theorem in triangle PAB:

$$a_3^2 = a_4^2 + (S_2 - S_1)^2 - 2a_4(S_2 - S_1)c\eta \Rightarrow \eta = \cos^{-1}\left(\frac{a_4^2 + (S_2 - S_1)^2 - a_3^2}{2a_4(S_2 - S_1)}\right) \qquad (6.40)$$

$$\begin{aligned} &S_1 + a_3 e^{i\eta} = S_1 + a_3\cos\eta + i a_3 \sin\eta = Se^{i\phi} \\ &\Rightarrow S = \sqrt{(S_1 + a_3\cos\eta)^2 + a_3^2\sin^2\eta} \quad \text{and} \quad \phi = \text{atan2}(S_1 + a_3\cos\eta,\ a_3\sin\eta) \end{aligned} \qquad (6.41)$$

For the <u>P</u>RR-R<u>P</u>R-RRR mechanism shown in Figure 6.8b, prismatic joint variables S_1 and S_2 are the inputs. The link lengths a_3 and a_4 can be chosen freely. Writing cosine theorem in triangle PAB:

$$S_2^2 = a_4^2 + (a_3 - S_1)^2 - 2a_4(a_3 - S_1)c\eta \Rightarrow \eta = \cos^{-1}\left(\frac{a_4^2 + (a_3 - S_1)^2 - S_2^2}{2a_4(a_3 - S_1)}\right) \qquad (6.42)$$

$$\begin{aligned} &S_1 + a_4 e^{i\eta} = S_1 + a_4\cos\eta + i a_4 \sin\eta = Se^{i\phi} \\ &\Rightarrow S = \sqrt{(S_1 + a_4\cos\eta)^2 + a_4^2\sin^2\eta} \quad \text{and} \quad \phi = \text{atan2}(S_1 + a_4\cos\eta,\ a_4\sin\eta) \end{aligned} \qquad (6.43)$$

6.9. The Function Synthesis Problem

The function synthesis problem for any of the above-mentioned mechanisms is formulated as follows:

1. Given the design points x_i, y_i and z_i for i = 1,..,N > 4 the design points are linearly related to the inputs q and r and the output ψ of the mechanism:

$$\frac{x_i - x_1}{x_N - x_1} = \frac{q_i - q_1}{q_N - q_1} \quad , \quad \frac{y_i - y_1}{y_N - y_1} = \frac{r_i - r_1}{r_N - r_1} \quad \text{and} \quad \frac{z_i - z_1}{z_N - z_1} = \frac{\psi_i - \psi_1}{\psi_N - \psi_1} \tag{6.44}$$

   q and r are S and ϕ for RPRRR; θ and β for RRRRR; S and S_2 for 2RPR-RRR; S and β for RPR-RRR-RRR and θ and β for 2RRR-RRR mechanism. Which of the inputs is q and which one is r does not matter. The limits q_1, q_N, r_1, r_N, ψ_1 and ψ_N are arbitrarily chosen and the designer can play with them to decrease the amount of error and/or find a mechanism with more feasible link lengths.

2. S_i and ϕ_i are determined in terms of the inputs as explained in the above sections. Table 6.1 summarizes the dependencies of S and ϕ to the inputs.
3. The synthesis of the mechanism is performed with least squares approximation as explained in the first section. The percentage error variation between the desired and generated z values is plotted and the error is decreased by changing the assumed link lengths and/or input/output joint limits.

Table 6.1. Dependency of S and ϕ to the inputs

Mechanism	**Input**	**S**	**ϕ**
RPRRR	S and ϕ	S	ϕ
RRRRR	θ and β	$\sqrt{a_3^2 + a_4^2 + 2a_3a_4c(\theta-\beta)}$	$\mathrm{atan2}(a_3c\theta + a_4c\beta, a_3s\theta + a_4s\beta)$
2RPR-RRR	S and S_2	S	$\cos^{-1}\left[\left(a_3^2 + S^2 - S_2^2\right)/\left(2a_3S\right)\right]$
RPR-RRR-RRR	S and β	S	Equation (6.25)
2RRR-RRR	θ and β	Equation (6.28)-(6.31)	
PRRRR	S_1 and β	$\sqrt{(S_1 + a_3\cos\theta)^2 + a_3^2\sin^2\theta}$	$\mathrm{atan2}(S_1 + a_3\cos\theta,\ a_3\sin\theta)$
PPRRR	S_1 and S_2	$\sqrt{S_1^2 + S_2^2}$	$\mathrm{atan2}(S_1, S_2)$
PRR-RRR-RRR	S_1 and β	Equation (6.39) and Equation (6.31)	
2PRR-RRR	S_1 and S_2	$\sqrt{(S_1 + a_3\cos\eta)^2 + a_3^2\sin^2\eta}$	$\mathrm{atan2}(S_1 + a_3\cos\eta,\ a_3\sin\eta)$
PRR-RPR-RRR	S_1 and S_2	$\sqrt{(S_1 + a_4\cos\eta)^2 + a_4^2\sin^2\eta}$	$\mathrm{atan2}(S_1 + a_4\cos\eta,\ a_4\sin\eta)$

6.10. Case Study

All the formulations in the previous sections are implemented in MS Excel® and several different function synthesis tasks are performed. Here some of the results are presented for illustration. As an example, in function generation of P̲RR-R̲RR-RRR mechanism, consider the function $z = x^{1.2}y^{0.2}$ for $3 \leq x \leq 6$ and $4 \leq y \leq 5$. The choice of this function does not imply an application and has been studied purely for academic purposes. x and y definition intervals were divided into 30 equal intervals, that is, the synthesis was made with a total of 900 design points. In the P̲RR-R̲RR-RRR mechanism, the size of the four links can be selected freely. Therefore, by changing the limits of these link dimensions and input/output parameters, a mechanism design with a low maximum error value has been made. The absolute percent error used as a design criterion is shown below;

$$\%\text{Error} = \left| \frac{z_{desired} - z_{computed}}{z_{desired}} \right| \times 100 \qquad (6.45)$$

After several trials on the free parameters, a solution with relatively low error is determined for $a_3 = 6$, $a_4 = 4.5$, $a_5 = 5$, $a_6 = 4$, $1 \leq S \leq 5$, $75° \leq \beta \leq 110°$ and $110° \leq \psi \leq 165°$. The maximum percentage error is found as 2.436%. Percentage error variation is depicted in Figure 6.9. The construction parameters of the designed mechanisms are $a_1 = 3.827$, $a_2 = 6.649$, $C_x = 6.022$ and $C_y = 4.083$.

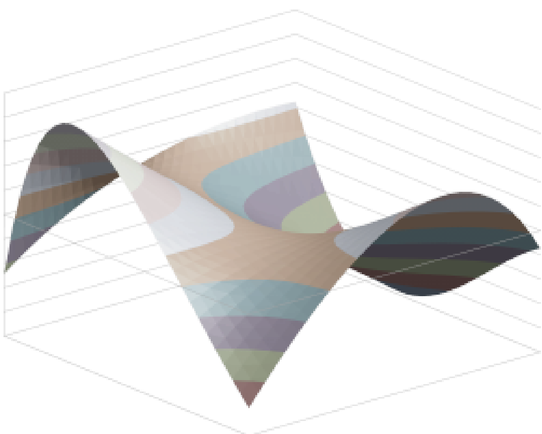


Figure 6.9. The percentage of the error variation for generation of $z = x^{1.2}y^{0.2}$

CHAPTER 7

CONCLUSIONS

In this thesis, studies with different approaches to different function generation synthesis problems for various mechanisms are issued. The synthesis problems are mathematically formulated and analytical and semi-analytical solutions of the problems are presented.

In CHAPTER 3, function generation problem for a planar 5R mechanism is addressed using semi analytical solution for given design point set. The problem is formulated starting with analytically defining the objective function from the I/O relationship and solving for the Lagrange terms with numerical techniques. A computational example is presented and the error variation is given.

In CHAPTER 4, the single-loop Bennett 6R mechanisms possess much more construction parameters than the single-loop planar four-bar, spherical four-bar or planar slider-crank mechanisms and hence, they may be used for function approximation purposes with a relatively better accuracy. The method of decomposition makes it possible to analytically formulate the function synthesis problem for mechanisms with many construction parameters. The easy computer implementation of the formulation enables the designer to quickly work on several alternative designs and come up with an accurate function generator mechanism. The case studies illustrate how a designer can compare the three types of Bennett 6R mechanisms for the same function. Also, the comparison of the spherical four-bar mechanisms with the double-spherical 6R mechanism clearly shows that the accuracy is improved when the 6R linkage is used for generating the same function.

CHAPTER 5 focuses on formulation of function generation synthesis of an over-constrained 2-dof 7R mechanism using least squares approximation and method of decomposition. The inputs of the mechanism are chosen such that the resulting synthesis equations are linear. Several case studies are performed and one of them is presented. The case study shows that the maximum error may be decreased by using a 7R mechanism instead of a spherical 5R mechanism.

In CHAPTER 6, a common design method is presented for the synthesis of two-input functions using many different planar 2-dof mechanisms involving only rotary and sliding joints. For the synthesis formulation to be based on analytical calculation instead of time-consuming numerical optimization methods, some limb dimensions were freely chosen and the design of the remaining limb dimensions was studied. The least squares approximation method was chosen as the synthesis method, which allows to select as many design points as desired on the domain. In practice, the methods in this study can be applied for a particular mechanism, or if the designer has the freedom to choose the mechanism to fulfill a particular function, it is also possible to select the mechanism that gives the best result by working on all mechanisms.

Future studies involve applying these approximation methods for other multi-loop and multi-dof mechanisms and comparing the results.

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