ORIGINAL ARTICLE

Local well-posedness of the higher-order nonlinear Schrödinger equation on the half-line: Single-boundary condition case

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Dedicated to the 70th birthday of Professor Athanassios S. Fokas

Abstract

We establish local well-posedness in the sense of Hadamard for a certain third-order nonlinear Schrödinger equation with a multiterm linear part and a general power nonlinearity, known as higher-order nonlinear Schrödinger equation, formulated on the half-line $\{x > 0\}$. We consider the scenario of associated coefficients such that only one boundary condition is required and hence assume a general nonhomogeneous boundary datum of Dirichlet type at $x = 0$. Our functional framework centers around fractional Sobolev spaces $H_x^s(\mathbb{R}_+)$ with respect to the spatial variable. We treat both high regularity ($s > \frac{1}{2}$) and low regularity ($s < \frac{1}{2}$) solutions: in the former setting, the relevant nonlinearity can be handled via the Banach algebra property; in the latter setting, however, this is no longer the case and, instead, delicate Strichartz estimates must be established. This task is especially challenging in the framework of nonhomogeneous initial-boundary value problems, as it involves proving boundary-type Strichartz estimates that are not common in the study of Cauchy (initial value) problems. The linear analysis, which forms the core of this work, crucially relies on a weak solution formulation defined through the novel

solution formulae obtained via the Fokas method (also known as the unified transform) for the associated forced linear problem. In this connection, we note that the higher-order Schrödinger equation comes with an increased level of difficulty due to the presence of more than one spatial derivatives in the linear part of the equation. This feature manifests itself via several complications throughout the analysis, including (i) analyticity issues related to complex square roots, which require careful treatment of branch cuts and deformations of integration contours; (ii) singularities that emerge upon changes of variables in the Fourier analysis arguments; and (iii) complicated oscillatory kernels in the weak solution formula for the linear initial-boundary value problem, which require a subtle analysis of the dispersion in terms of the regularity of the boundary data. The present work provides a first, complete treatment via the Fokas method of a nonhomogeneous initial-boundary value problem for a partial differential equation associated with a multiterm linear differential operator.

KEYWORDS
Fokas method, higher-order nonlinear Schrödinger equation, initial-boundary value problem, Korteweg–de Vries equation, low regularity solutions, nonzero boundary conditions, power nonlinearity, Strichartz estimates, unified transform, well-posedness in Sobolev spaces

1 | INTRODUCTION AND MAIN RESULTS

1.1 | Mathematical model

We consider the nonhomogeneous initial-boundary value problem for the higher-order nonlinear Schrödinger (HNLS) equation on the half-line

$$
\begin{aligned}
&iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x = f(u), \quad (x,t) \in \mathbb{R}_+ \times (0,T), \\
&u(x,0) = u_0(x), \quad x \in \mathbb{R}_+, \\
&u(0,t) = g(t), \quad t \in (0,T),
\end{aligned}
\tag{1}
$$

where $\alpha, \delta \in \mathbb{R}$, $\beta > 0$, $f(z) = \kappa |z|^p z$ with $z \in \mathbb{C}$, $\kappa \in \mathbb{C}$, $p > 0$, and $T > 0$. The reason why we only need to supplement one boundary condition at $x = 0$ is the assumption $\beta > 0$. On the other hand, if $\beta < 0$, then two boundary conditions are required at $x = 0$; this scenario will be considered in a future work. Furthermore, here we consider the case of a Dirichlet boundary datum; the case of a Neumann datum can be handled via entirely analogous ideas and techniques.

We establish local well-posedness of the nonlinear, nonhomogeneous initial-boundary value problem (1) in the sense of Hadamard, namely, we prove existence of a unique local-in-time solution that depends continuously on the initial and boundary data (see Theorems 1 and 2 below). It will be shown (see Theorem 3 below) that the evolution operator associated with the free higher-order Schrödinger operator enjoys the following regularity property:

$$\left\| e^{(-\beta \partial_x^3 + i\alpha \partial_x^2 - \delta \partial_x)t} \right\|_{H_x^s(\mathbb{R}) \to L_x^\infty(\mathbb{R}; H_t^{\frac{s+1}{3}}(-T,T))} \leq c < \infty.$$

Note that, in the case of the classical second-order Schrödinger operator (i.e., for $\alpha = 1$ and $\beta = \delta = 0$), the time regularity of the solution is described by the Sobolev exponent $\frac{2s+1}{4}$ (see Ref. 1). Hence, for $s < \frac{1}{2}$ (which implies $\frac{2s+1}{4} < \frac{s+1}{3}$), the above result for the higher-order Schrödinger operator can be regarded as a kind of smoothing. Due to this smoothing, one anticipates that the local well-posedness for the initial-boundary value problem (1) should be studied with initial data $u_0 \in H_x^s(\mathbb{R}_+)$ and boundary data $g \in H_{t,\text{loc}}^{\frac{s+1}{3}}(\mathbb{R}_+)$. In addition, for s large enough and, more precisely, for $s > \frac{1}{2}$, the relevant traces make sense in the aforementioned spaces and one has to also impose compatibility conditions between the initial and the boundary data to obtain solutions that are continuous at $t = 0$ (see Section 2.3 for more details).

Our treatment of the nonlinear problem (1) is crucially based on a contraction mapping argument applied to a weak solution formula for the associated forced linear initial-boundary value problem. This novel solution formula is derived in Section 2 via the Fokas method (also known as the unified transform). Showing that the map obtained by replacing the forcing with the power nonlinearity of (1) in the Fokas method solution is a contraction requires further assumptions on the smoothness and growth of the nonlinearity in relation to the Sobolev regularity exponent s; see (3) and (4) for the precise assumptions used in this work.

1.2 | Physical significance and motivation

For $\alpha = 1$, $\beta = \delta = 0$, and $p = 2$, the HNLS equation in (1) reduces to the celebrated cubic nonlinear Schrödinger equation (NLS) equation. Cubic NLS is a ubiquitous model in mathematical physics, with a broad spectrum of applications ranging from optics to water waves to plasmas to Bose–Einstein condensates. However, for pulses in the femtosecond regime, the classical NLS model is not precise enough and a correction involving a higher-order dispersive term is necessary (see Ref. 2 for a detailed discussion of the higher order effects upon the propagation of an optical pulse). This need for a more accurate model led to the introduction of the HNLS equation, originally in the form

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u + i\epsilon \left(\beta_1 u_{xxx} + \beta_2 (|u|^2 u)_x + \beta_3 u |u|_x^2 \right) = 0, \tag{2}$$

for modeling the femtosecond pulse propagation in nonlinear optical fibers.[3,4] Note that in (1), we have a power nonlinearity of general order $p > 0$, whereas (2) involves cubic nonlinearities that, nevertheless, contain derivatives.

Furthermore, beyond physical considerations, it should be noted that the cubic NLS equation is a prototypical example of a completely integrable system.[5] As such, in addition to analysis techniques, it has been studied extensively via the inverse scattering transform and related methods. This is not possible, however, for general nonlinearities like the one of the HNLS equation in (1), as the corresponding models are not completely integrable. In those general cases, which nevertheless remain very relevant when it comes to applications, rigorous results can be established via harmonic analysis techniques. In particular, the well-posedness of the Cauchy (initial value) problem for HNLS has been treated in a number of articles.[6–13] In addition, numerical solutions were obtained in Ref. 14. Moreover, there are some results concerning the controllability properties of this equation; see Ref. 15 for exact boundary controllability, Refs. 16 and [17] for internal feedback stabilization, and Refs. 18 and [19] for boundary feedback stabilization.

The goal of this work is to establish the local well-posedness theory for the nonlinear initial-boundary value problem (1) at the level of $H_x^s(\mathbb{R}_+)$ spatial regularity for the initial data. We are interested in both the high regularity ($s > \frac{1}{2}$) and the low regularity ($s < \frac{1}{2}$) settings. The main distinction between the two is that, in the low regularity setting, the well-known Banach algebra property of $H_x^s(\mathbb{R}_+)$ is no longer available. Instead, handling the nonlinearity $|u|^p u$ when $s < \frac{1}{2}$ requires use of more advanced tools that revolve around the celebrated Strichartz estimates. Estimates of this type measure the size and temporal decay of solutions in space-time Lebesgue norms and have played a crucial role in the treatment of the Cauchy problem of nonlinear dispersive equations since their introduction in 1977.[20] On the other hand, the use of Strichartz estimates in the analysis of initial-boundary value problems is a more recent advancement. For the Cauchy problem, Strichartz estimates involve certain norms of initial and/or interior data, whereas for initial-boundary value problems, these estimates additionally depend on information related to boundary data, for which temporal regularity also plays a key role. Such boundary-type Strichartz estimates have been recently established for some initial-boundary value problems associated with dispersive equations, see, for instance, Refs. 21–23 for the one-dimensional NLS, Refs. 24 and [25] for the two-dimensional NLS, Refs. 26–28 for NLS in n dimensions, Refs. 29 and [30] for the one-dimensional biharmonic NLS, and Ref. 31 for a fourth-order NLS in one dimension.

1.3 │ Challenges, methodology, and main results

The first contribution of the present paper is the development of a sharp linear theory through the analysis of the solutions of the relevant forced linear initial-boundary value problem (see problem (5) below). This is accomplished by decomposing this linear problem into three simpler component problems: (i) a homogeneous Cauchy problem associated with (an appropriate extension of) the initial datum; (ii) a nonhomogeneous Cauchy problem associated with (an appropriate extension of) the forcing; and (iii) a reduced initial-boundary value problem involving the original boundary datum and the spatial traces of the two aforementioned Cauchy problems.

The homogeneous Cauchy problem of item (i) is studied in Section 2.1 via classical Fourier analysis. However, the multiterm nature of the spatial differential operator introduces certain difficulties in the proofs of the temporal estimates, due to the changes of variables performed to extract the desired Sobolev norms. These difficulties are overcome by introducing a proper cutoff function that depends on the polynomial structure of the spatial differential operator.

The nonhomogeneous Cauchy problem of item (ii) is analyzed in Section 2.2 by expressing its solution in Duhamel form and then estimating it via the nonlocal (i.e., in the physical space) definition of fractional Sobolev spaces. We note that when the equation involves a multiterm spatial differential operator, the analysis of the corresponding nonhomogeneous Cauchy problem becomes quite involved when carried out via other approaches such as the Riemann–Liouville fractional integral method that was successfully applied to the Korteweg–de Vries equation (without the first-order derivative) and the NLS equation [21,32,33]. In contrast, the physical space definition of the fractional Sobolev spaces offers a more robust and direct approach in this framework; furthermore, this approach does not require interpolation arguments.

A major emphasis in this work is placed on the regularity analysis of the solution to the reduced initial-boundary value problem of item (iii) above. This is done in Section 2.3. Weak solutions of this reduced initial-boundary value problem are defined via a boundary integral operator whose explicit form is obtained through the Fokas method.[34,35] Importantly, in the multiterm framework considered in this paper, certain analyticity issues arise in the application of the Fokas method. This is because the method relies on the construction of analytic maps that respect certain spectral invariance properties of the linear dispersion relation. However, for multiterm spatial differential operators, such a construction requires use of complex square root functions that, in many cases, cause the invariance maps to be nonanalytic on some parts of the complex spectral plane. We handle this complication via suitable contour deformations around the branch cuts associated with these maps. It is worth noting that this phenomenon also appears in the context of higher-dimensional initial-boundary value problems, see, for instance, Ref. 36, as well as in equations that involve higher-order time derivatives, for example, the "good" Boussinesq equation analyzed in Refs. 37 and [38] After constructing a suitable boundary integral operator for the reduced initial-boundary value problem, we analyze it by using the oscillatory integral theory that, in particular, requires us to establish dispersive estimates of the same type like the ones satisfied by solutions of the associated Cauchy problem.

The solution of the fully nonlinear problem (1) will be constructed as a fixed point of the solution operator formed by reunifying the respective solution formulae for the three linear problems of items (i)–(iii) above. In the high regularity setting of $s > \frac{1}{2}$, the spatiotemporal estimates established in the linear theory of Section 2 lead to a contraction mapping argument in the Hadamard-type space $C([0,T]; H_x^s(\mathbb{R}_+))$. The uniqueness in this space utilizes the Sobolev embedding $H_x^s(\mathbb{R}_+) \hookrightarrow L_x^\infty(\mathbb{R}_+)$ (which is valid for $s > \frac{1}{2}$). In the low regularity setting of $s < \frac{1}{2}$, the algebra property in $H_x^s(\mathbb{R}_+)$ and the embedding $H_x^s(\mathbb{R}_+) \hookrightarrow L_x^\infty(\mathbb{R}_+)$ are no longer valid and Strichartz estimates assume the key role instead. In that case, the solution space is refined to $C([0,T]; H_x^s(\mathbb{R}_+)) \cap L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}_+))$ with (μ, r) obeying the admissibility criterion (20) associated with the underlying evolution operator. However, this only leads to a conditional uniqueness result in the aforementioned space.

The main results of this work, which emanate from the analysis described above, establish the local well-posedness of the HNLS initial-boundary value problem (1) in the high and low regularity settings and read as follows:

Theorem 1 (High regularity well-posedness). *Let $\frac{1}{2} < s \leq 2$ and $p > 0$. In addition, if $p \notin 2\mathbb{Z}$, suppose that*

$$
\begin{aligned}
&\text{if } s \in \mathbb{Z}_+, \text{then } p \geq s \text{ if } p \in \mathbb{Z}_+ \text{ and odd; } \lfloor p \rfloor \geq s - 1 \text{ if } p \notin \mathbb{Z}_+, \\
&\text{if } s \notin \mathbb{Z}_+, \text{then } p > s \text{ if } p \in \mathbb{Z}_+ \text{ and odd; } \lfloor p \rfloor \geq \lfloor s \rfloor \text{ if } p \notin \mathbb{Z}_+.
\end{aligned}
\tag{3}
$$

Then, for initial data $u_0 \in H_x^s(\mathbb{R}_+)$ and boundary data $g \in H_{t,\text{loc}}^{\frac{s+1}{3}}(\mathbb{R}_+)$ satisfying the compatibility condition (66), there is $T = T(u_0, g) > 0$ such that the initial-boundary value problem (1) for the HNLS equation on the half-line has a unique solution $u \in C([0, T]; H_x^s(\mathbb{R}_+))$. Furthermore, this solution depends continuously on the initial and boundary data.

Theorem 2 (Low regularity well-posedness). *Suppose*

$$
0 \leq s < \frac{1}{2}, \quad 1 \leq p \leq \frac{6}{1-2s}, \quad \mu = \frac{6(p+1)}{p(1-2s)}, \quad r = \frac{2(p+1)}{1+2sp}.
\tag{4}
$$

Then, for initial data $u_0 \in H_x^s(\mathbb{R}_+)$ and boundary data $g \in H_{t,\text{loc}}^{\frac{s+1}{3}}(\mathbb{R}_+)$, with the additional assumption that if $p = \frac{6}{1-2s}$ (critical case), then $\|u_0\|_{H_x^s(\mathbb{R}_+)}$ is sufficiently small, there is $T = T(u_0, g) > 0$ such that the initial-boundary value problem (1) for the HNLS equation on the half-line has a unique solution $u \in C([0, T]; H_x^s(\mathbb{R}_+)) \cap L_t^\mu((0, T); H_x^{s,r}(\mathbb{R}_+))$. Furthermore, this solution depends continuously on the initial and boundary data.

The linear estimates that form the core of the proofs of the above two theorems are established in Section 2, and the contraction mapping arguments that complete those proofs are provided in Section 3.

Remark 1. The assumptions in above theorems offer a large class of cases in terms regularity index s and the power of nonlinearity p for which the local well-posedness of the initial-boundary value problem (1) can be established. In particular, these cover physically relevant settings. For instance, energy-level solutions ($s = 1$) and strong solutions ($s = 2$) can be obtained in the presence of a cubic nonlinear source ($p = 2$).

1.4 | The Fokas method for the rigorous treatment of initial-boundary value problems

While the Cauchy problem for nonlinear dispersive equations has been broadly explored through a variety of techniques, progress toward the rigorous study of initial-boundary value problems for these equations is more limited. In fact, problems of this latter kind can present significant challenges even at the linear level. For example, while on the whole-line linear evolution equations can be easily solved via Fourier transform in the space variable, on domains with a boundary like the half-line no classical spatial transform exists for linear equations of spatial order three or higher. Another important obstacle arises in the case of boundary conditions that are nonseparable. Moreover, even when a linear initial-boundary value problem can be solved via

classical techniques, the resulting solution formula is not always useful, especially in regard to setting up an effective iteration scheme for proving the well-posedness of associated nonlinear problems.

At the linear level, the Fokas method bridges the gap between the Cauchy problem and initial-boundary value problems by providing the direct analog of the Fourier transform in domains with a boundary. Indeed, the method provides a fundamentally novel, algorithmic way of solving any linear evolution equation formulated on a variety of domains in one or higher dimensions and supplemented with any kind of admissible boundary conditions. An alternative perspective that further establishes the Fokas method as the natural counterpart of the Fourier transform in the context of linear initial-boundary value problems stems from the nonlinear component of the method, which was developed for completely integrable nonlinear equations and corresponds to the analog of the inverse scattering transform in domains with a boundary. Then, noting that the linear limit of the inverse scattering transform is nothing but the Fourier transform, it is only reasonable that the linear limit of the nonlinear component of the Fokas method, namely, the linear Fokas method, should provide the equivalent of the Fourier transform for linear initial-boundary value problems.

The analogy between the Fokas method and the Fourier transform has been solidified by a new approach introduced in recent years by Himonas and one of the authors for the well-posedness of nonlinear initial-boundary value problems. This approach is based on treating the nonlinear problem as a perturbation of its forced linear counterpart, which is, of course, a classical idea coming from the Cauchy problem. However, the linear formulae produced via the Fourier transform in the case of the Cauchy problem are now replaced by the Fokas method solution formulae (recall that Fourier transform is no longer available). As these novel formulae involve complex contours of integration, new tools and techniques are required to obtain the various linear estimates needed for the contraction mapping argument. It should be noted that several of these estimates are specific to initial-boundary value problems and do not typically arise in the study of the Cauchy problem; they are results of particular importance, as they capture the effect of the boundary conditions on the regularity of the solution of both linear and nonlinear problems.

The Fokas-method-based approach for the rigorous study of initial-boundary value problems has already been implemented in several works in the literature: NLS on the half-line and the half-plane [1,23,24,39–41], KdV on the half-line and the finite interval [42–45], "good" Boussinesq on the half-line [38], biharmonic NLS on the half-line [29], fourth-order Schrödinger equation on the half-line [31], a higher-dispersion KdV on the half-line [46], and even nondispersive models.[47,48] It should be noted that in those cases where a problem has been previously considered in the literature, the results via the new method are consistent with the existing ones, typically obtained via the Colliander–Kenig–Holmer or the Bona–Sun–Zhang approaches, for example, see Refs. 21, 22, 25, 26, 32, 33, 49–51, and [19]. Finally, we remark that a certain third-order model with cubic nonlinearity that is related to Equations (1) and (2) and is known as the Hirota equation has also been considered in the literature in the context of initial-boundary value problems, see Refs. 52–54. However, it is important to emphasize that in the present work, we treat the case of a general power nonlinearity and, furthermore, we study the associated boundary integral operator at the low regularity level of Strichartz estimates.

We conclude by noting that rigorous treatment of initial-boundary value problems through the Fokas formulae has not only played an important role in establishing well-posedness results; it has also given insight toward solving problems that stem from other related fields such as

systems theory and control, because there is a close connection between regularity theory and controllability. It is well known that initial-boundary value problems with nonhomogeneous boundary conditions can be used to model physical evolutions in which the boundary input acts as control. Such boundary control models are particularly important for governing dynamics of physical processes in which access to the interior of a medium is blocked or not feasible, whereas manipulations through the boundary remain an efficient choice. See, for instance, Refs. 55–57 for some recent applications of the Fokas method to boundary control problems related to the heat and Schrödinger equations.

2 | LINEAR THEORY

In this section, we study the forced linear initial-boundary value problem

$$
\begin{aligned}
iu_t + i\beta u_{xxx} + \alpha u_{xx} + i\delta u_x &= f, \quad (x,t) \in \mathbb{R}_+ \times (0,T), \\
u(x,0) &= u_0(x), \quad x \in \mathbb{R}_+, \\
u(0,t) &= g(t), \quad t \in (0,T),
\end{aligned}
\tag{5}
$$

where $\alpha, \delta \in \mathbb{R}$ and $\beta > 0$. The analysis of the linear problem (5) will be carried out via a *decomposition-reunification* approach. This decomposition allows us to split the problem into three simpler components, two of which are Cauchy problems on the real line with data associated with u_0 and f, respectively, and one of which is a (reduced) initial-boundary value problem with data associated with g as well as the traces of the solutions of the aforementioned Cauchy problems at $x = 0$.

2.1 | Homogeneous linear Cauchy problem

Consider the problem

$$
\begin{aligned}
iy_t + i\beta y_{xxx} + \alpha y_{xx} + i\delta y_x &= 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \\
y(x,0) &= y_0(x), \quad x \in \mathbb{R},
\end{aligned}
\tag{6}
$$

where $y_0 = E_0 u_0$ denotes an extension of u_0 with respect to a fixed bounded extension operator $E_0 : H_x^s(\mathbb{R}_+) \to H_x^s(\mathbb{R})$, namely, we have

$$
\|E_0 u_0\|_{H_x^s(\mathbb{R})} \lesssim \|u_0\|_{H_x^s(\mathbb{R}_+)}.
\tag{7}
$$

Theorem 3. *Let $s \in \mathbb{R}$. The unique solution of the Cauchy problem (6), denoted by $y = S[y_0; 0]$, belongs to $C(\mathbb{R}_t; H_x^s(\mathbb{R}))$ and satisfies the conservation law*

$$
\|y(\cdot, t)\|_{H_x^s(\mathbb{R})} = \|y_0\|_{H_x^s(\mathbb{R})}, \quad t \in \mathbb{R}.
\tag{8}
$$

Moreover, if $\alpha^2 + 3\beta\delta \geq 0$, then $y \in C(\mathbb{R}_x; H_t^{\frac{s+1}{3}}(-T,T))$ for $T > 0$ and there exists a constant $c = c(s,\alpha,\beta,\delta) \geq 0$ such that

$$\sup_{x\in\mathbb{R}} \|y(x,\cdot)\|_{H_t^{\frac{s+1}{3}}(-T,T)} \leq c(1 + T^{\frac{1}{2}})\|y_0\|_{H_x^s(\mathbb{R})}, \tag{9}$$

while if $\alpha^2 + 3\beta\delta < 0$, then $y \in C(\mathbb{R}_x; H_t^{\frac{s+1}{3}}(\mathbb{R}))$ and there is a constant $c = c(s,\alpha,\beta,\delta) \geq 0$ such that

$$\sup_{x\in\mathbb{R}} \|y(x,\cdot)\|_{H_t^{\frac{s+1}{3}}(\mathbb{R})} \leq c\|y_0\|_{H_x^s(\mathbb{R})}. \tag{10}$$

Proof. Taking the Fourier transform of (6) with respect to x, we find $\hat{y}(k,t) = e^{-\omega(k)t}\hat{y}_0(k)$ where

$$\omega(k) := -i\beta k^3 + i\alpha k^2 + i\delta k. \tag{11}$$

For $k \in \mathbb{R}$, $\omega(k)$ is purely imaginary; thus, $|\hat{y}(k,t)| = |\hat{y}_0(k)|$ and the conservation law (8) readily follows via Plancherel's theorem. The continuity of the map $t \mapsto y(t)$ from $[0,T]$ into $H_x^s(\mathbb{R})$ follows from the dominated convergence theorem and the fact that $y_0 \in H_x^s(\mathbb{R})$.

To prove the temporal estimates (9) and (10), we start from the Fourier transform solution representation

$$y(x,t) = S[y_0; 0](x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx - \omega(k)t}\hat{y}_0(k)dk. \tag{12}$$

Consider the real-valued map $\tau = i\omega(k)$. Notice that if $\alpha^2 + 3\beta\delta \leq 0$, then τ is monotone increasing and so $k = (i\omega)^{-1}(\tau)$ is well-defined. In the case of strict inequality $\alpha^2 + 3\beta\delta < 0$, we observe that $i\omega'(k) > -(\alpha^2 + 3\beta\delta)/(3\beta) > 0$, and so, by the inverse function theorem, we can change variable from k to τ to rewrite (12) as

$$y(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(i\omega)^{-1}(\tau)x + i\tau t}\hat{y}_0((i\omega)^{-1}(\tau))\frac{d\tau}{i\omega'((i\omega)^{-1}(\tau))}. \tag{13}$$

In addition, we have $i\omega(k) = \mathcal{O}(k^3)$ and $\frac{1}{i\omega'(k)} = \mathcal{O}(k^{-2})$ as $|k| \to \infty$. Using the Fourier transform characterization of the Sobolev norm, for each $x \in \mathbb{R}$, we find

$$\|y(x,\cdot)\|^2_{H_t^{\frac{s+1}{3}}(\mathbb{R})} = \int_{\mathbb{R}} (1 + \tau^2)^{\frac{s+1}{3}}|\hat{y}_0((i\omega)^{-1}(\tau))|^2 \frac{d\tau}{|i\omega'((i\omega)^{-1}(\tau))|^2}$$

$$\lesssim \int_{\mathbb{R}} (1 + k^2)^s|\hat{y}_0(k)|^2 dk = \|y_0\|^2_{H_x^s(\mathbb{R})},$$

which amounts to estimate (10).

Next, consider the case $\alpha^2 + 3\beta\delta \geq 0$. Let $\theta \in C_c^\infty(\mathbb{R})$ be a function whose additional properties will be specified below. Then, we can write $y = y_1 + y_2$, where

$$y_1(x,t) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx - \omega(k)t} \theta(k) \widehat{y}_0(k) dk,$$

$$y_2(x,t) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx - \omega(k)t} (1 - \theta(k)) \widehat{y}_0(k) dk. \tag{14}$$

Taking jth-order time derivative of y_1 and using Cauchy–Schwarz inequality, we deduce

$$|\partial_t^j y_1(x,t)| \leq \frac{1}{2\pi} \int_{\mathrm{supp}(\theta)} |\omega(k)|^j |\theta(k)| |\widehat{y}_0(k)| dk$$

$$\lesssim \left(\int_{\mathrm{supp}(\theta)} (1 + k^2)^{-s} |\omega(k)|^{2j} dk \right)^{\frac{1}{2}} \|y_0\|_{H_x^s(\mathbb{R})} = c(s,j,\theta) \|y_0\|_{H_x^s(\mathbb{R})}.$$

We note that this inequality holds for any $s \in \mathbb{R}$. Thus, by the physical space characterization of the Sobolev norm, namely,

$$\|f\|_{H_t^\mu(-T,T)} = \sum_{j=0}^\mu \|\partial_t^j f\|_{L_t^2(-T,T)}, \quad \mu \in \mathbb{N}_0, \tag{15}$$

we obtain

$$\|y_1(x,\cdot)\|_{H_t^\mu(-T,T)} \leq c(s,\mu,\theta) T^{\frac{1}{2}} \|y_0\|_{H_x^s(\mathbb{R})} \tag{16}$$

for any $\mu \in \mathbb{N}_0$ and any $x, s \in \mathbb{R}$. Then, since given any $m \in \mathbb{R}$, we can always find $\mu \in \mathbb{N} \cup \{0\}$ such that $m \leq \mu$, estimate (16) readily implies

$$\|y_1(x,\cdot)\|_{H_t^m(-T,T)} \leq c(s,m,\theta) T^{\frac{1}{2}} \|y_0\|_{H_x^s(\mathbb{R})}, \quad m,s,x \in \mathbb{R}. \tag{17}$$

To handle y_2, we note that given $\alpha, \delta \in \mathbb{R}$ and $\beta > 0$ satisfying $\alpha^2 + 3\beta\delta \geq 0$, one can find $k_j = k_j(\alpha, \delta, \beta) \in \mathbb{R}$, $j = 1, 2$, such that (i) the roots $\frac{\alpha \pm \sqrt{\alpha^2 + 3\beta\delta}}{3\beta}$ of $\omega'(k) = 0$ lie in (k_1, k_2) and (ii) the mapping $\tau = i\omega(k)$ is monotone increasing on $\mathbb{R} \setminus (k_1, k_2)$. Now, let $k_3 < k_1$ and $k_4 > k_2$ be any two numbers and fix θ so that it further satisfies the condition

$$\theta(k) = \begin{cases} 1, & k \in [k_1, k_2], \\ 0, & k \notin (k_3, k_4), \end{cases}$$

as well as the condition $0 \leq |\theta(k)| \leq 1$, $k \in \mathbb{R}$. Now, we can rewrite y_2 as

$$y_2(x,t) = \frac{1}{2\pi} \int_{\mathbb{R} \setminus [k_1, k_2]} e^{ikx - \omega(k)t} (1 - \theta(k)) \widehat{y}_0(k) dk$$

$$= \frac{1}{2\pi} \int_{(i\omega)(\mathbb{R}\setminus[k_1,k_2])} e^{i(i\omega)^{-1}(\tau)x+i\tau t}(1 - \theta((i\omega)^{-1}(\tau)))\hat{y}_0((i\omega)^{-1}(\tau)) \frac{d\tau}{i\omega'((i\omega)^{-1}(\tau))}.$$

Using the definition of the Sobolev norm, for each $x \in \mathbb{R}$, we have

$$
\begin{aligned}
\|y_2(x,\cdot)\|^2_{H_t^{\frac{s+1}{3}}(\mathbb{R})} &= \int_{\mathbb{R}} (1+\tau^2)^{\frac{s+1}{3}} |\hat{y}_2(x,\tau)|^2 d\tau \\
&= \int_{(i\omega)(\mathbb{R}\setminus[k_1,k_2])} (1+\tau^2)^{\frac{s+1}{3}} \frac{|1-\theta((i\omega)^{-1}(\tau))|^2 |\hat{y}_0((i\omega)^{-1}(\tau))|^2}{|i\omega'((i\omega)^{-1}(\tau))|^2} d\tau \\
&\lesssim \int_{\mathbb{R}\setminus[k_1,k_2]} \frac{(1+(i\omega(k))^2)^{\frac{s+1}{3}}}{|i\omega'(k)|} |\hat{y}_0(k)|^2 dk \\
&\lesssim \int_{\mathbb{R}} (1+k^2)^s |\hat{y}_0(k)|^2 dk = \|y_0\|^2_{H_x^s(\mathbb{R})},
\end{aligned}
\tag{18}
$$

where the last inequality follows from the fact that $i\omega(k) = \mathcal{O}(k^3)$ and $\frac{1}{i\omega'(k)} = \mathcal{O}(k^{-2})$ as $|k| \to \infty$. Hence, (9) follows from (17) and (18). Continuity in x once again follows from the dominated convergence theorem. ∎

Notice that the conservation law (8) allows us to control the $L_t^\infty((0,T); H_x^s(\mathbb{R}))$ norm of the solution to the homogeneous linear Cauchy problem by the $H_x^s(\mathbb{R})$ norm of the initial data. As we shall show below, this is also the case for the mixed Lebesgue norms $L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))$, where $H^{s,r}(\mathbb{R})$ is the usual Bessel potential space defined with norm

$$\|f\|_{H^{s,r}(\mathbb{R})} := \left\| \mathcal{F}^{-1}\left\{ (1+k^2)^{\frac{s}{2}} \mathcal{F}\{f\}(k) \right\} \right\|_{L^r(\mathbb{R})}, \tag{19}$$

and (μ, r) is any higher-order Schrödinger admissible pair, that is, any pair (μ, r) satisfying

$$\mu, r \geq 2, \quad \frac{3}{\mu} + \frac{1}{r} = \frac{1}{2}. \tag{20}$$

More precisely, we have the following Strichartz estimate:

Theorem 4. *Let $s \in \mathbb{R}$ and suppose (μ, r) is higher-order Schrödinger admissible in the sense of (20). Then, the solution of the homogeneous linear Cauchy problem (6) satisfies the Strichartz estimate*

$$\|y\|_{L_t^\mu((0,T);H_x^{s,r}(\mathbb{R}))} \lesssim \|y_0\|_{H_x^s(\mathbb{R})}. \tag{21}$$

Proof. By the definition (19) of the $H^{s,r}$-norm, we have

$$\|y\|_{L_t^\mu(\mathbb{R};H_x^{s,r}(\mathbb{R}))} = \left\| \mathcal{F}^{-1}\left\{ (1+k^2)^{\frac{s}{2}} \hat{y}^{(x)}(k,\cdot) \right\} \right\|_{L_t^\mu((0,T);L_x^r(\mathbb{R}))}.$$

Recalling that $\hat{y}^{(x)}(k,t) = e^{-\omega(k)t}\hat{y}_0(k)$, we have

$$\mathcal{F}^{-1}\left\{(1+k^2)^{\frac{s}{2}}\hat{y}^{(x)}(k,t)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}(1+k^2)^{\frac{s}{2}}\hat{y}_0(k)dk = S[\varphi;0](t),$$

where $\varphi(x) = \mathcal{F}^{-1}\{(1+k^2)^{\frac{s}{2}}\hat{y}_0(k)\}(x)$. So, it suffices to prove that

$$\|S[\varphi;0]\|_{L_t^\mu((0,T);L_x^r(\mathbb{R}))} \lesssim \|\varphi\|_{L_x^2(\mathbb{R})}. \tag{22}$$

For this, we note that by the definition of the Fourier transform, we can write

$$S[\varphi;0](x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} I(x,y,t)\varphi(y)dy,$$

where

$$I(x,y,t) := \int_{\mathbb{R}} e^{ik(x-y-t\delta)+i(t\beta k^3-t\alpha k^2)}dk.$$

Then, invoking the following dispersive estimate from the proof of Lemma 4.2 in Ref. 6,

$$|I(x,y,t)| \lesssim |\beta t|^{-\frac{1}{3}}, \quad t \neq 0,$$

where the inequality constant is independent of x,y,t, and proceeding along the lines of the proof of Theorem 4.1 in Ref. 6 (see also relevant proof in Ref. 13), we infer the desired estimate (22). ∎

2.2 | Nonhomogeneous linear cauchy problem

We continue our linear analysis with the problem

$$
\begin{aligned}
iz_t + i\beta z_{xxx} + \alpha z_{xx} + i\delta z_x &= F, \quad (x,t) \in \mathbb{R} \times (0,T), \\
z(x,0) &= 0, \quad x \in \mathbb{R},
\end{aligned}
\tag{23}
$$

where $F = E_0 f \in L_t^2((0,T);H_x^s(\mathbb{R}))$ is a spatial extension of $f \in L_t^2((0,T);H_x^s(\mathbb{R}_+))$. Thanks to Duhamel's principle, the solution of the nonhomogeneous problem (23), denoted by $S[0;F]$, can be expressed as

$$
\begin{aligned}
z(x,t) = S[0;F](x,t) &= -i\int_0^t S[F(\cdot,t');0](x,t-t')dt' \\
&= -\frac{i}{2\pi}\int_0^t \int_{\mathbb{R}} e^{ikx-\omega(k)(t-t')}\widehat{F}(k,t')dkdt',
\end{aligned}
\tag{24}
$$

where, for each $t' \in [0, t]$, $S[F(\cdot, t'); 0]$ denotes the solution to the homogeneous problem (6) with initial data $F(x, t')$. We then have the following result, whose proof is based on the approach that was used for the Korteweg–de Vries equation in Ref. 43.

Theorem 5. *The unique solution of (23) satisfies the space estimate*

$$\sup_{t \in [0,T]} \|z(\cdot, t)\|_{H^s_x(\mathbb{R})} \le \|F\|_{L^1_t((0,T); H^s_x(\mathbb{R}))}, \quad s \in \mathbb{R}. \tag{25}$$

Moreover, if $-1 \le s \le 2$ with $s \ne \frac{1}{2}$, then the following time estimate holds:

$$\sup_{x \in \mathbb{R}} \|z(x, \cdot)\|_{H^{\frac{s+1}{3}}_t(0,T)} \lesssim \max\{T^{\frac{1}{2}}(1 + T^{\frac{1}{2}}), T^{\sigma}\} \|F\|_{L^2_t((0,T); H^s_x(\mathbb{R}))}, \tag{26}$$

where

$$\sigma = \begin{cases} \frac{1-2s}{6}, & -1 \le s < \frac{1}{2}, \\ \frac{2-s}{3}, & \frac{1}{2} < s < 2, \\ \frac{1}{2}, & s = 2. \end{cases} \tag{27}$$

Remark 2. For $2 < s < \frac{7}{2}$, due to the fractional norm $\|\partial_t z(x, \cdot)\|_{m-1}$ (see definition (29) below), the analog of the time estimate (26) turns out to be

$$\sup_{x \in \mathbb{R}} \|z(x, \cdot)\|_{H^{\frac{s+1}{3}}_t(0,T)} \lesssim \max\{T^{\frac{1}{2}}(1 + T^{\frac{1}{2}}), T^{\sigma}\} \|F\|_{L^2_t((0,T); H^s_x(\mathbb{R}))} + \sup_{x \in \mathbb{R}} \|F(x, \cdot)\|_{H^{\frac{s+1}{3}-1}_t(0,T)}.$$

The appearance of the space $C(\mathbb{R}_x; H^{\frac{s+1}{3}-1}(0,T))$ via the relevant norm on the right-hand side has a direct impact on the analysis of the nonlinear problem, as it eventually requires one to establish an appropriate multilinear estimate for the term $\||u|^p u(x, \cdot)\|_{H^{\frac{s+1}{3}-1}_t(0,T)}$ (note that the underlying range of s implies $0 < \frac{s+1}{3} - 1 < \frac{1}{2}$ and so the algebra property is not available). For this reason, a different approach (perhaps via energy estimates) might be preferable for showing well-posedness in this higher range of s. In any case, this task lies outside the scope of the present work, which instead focuses on solutions of lower smoothness and, in particular, toward the low regularity setting $0 \le s < \frac{1}{2}$.

Proof. In view of the Duhamel representation (24), the space estimate (25) readily follows from the homogeneous conservation law (8).

We proceed to the time estimate (26). Restricting $s \ge -1$ allows us to employ the physical space characterization of the Sobolev norm since then the exponent $\frac{s+1}{3}$ is nonnegative. In particular, for $-1 \le s < 2$, setting $m := \frac{s+1}{3}$ and observing that $0 \le m < 1$, we have

$$\|z(x, \cdot)\|_{H^m_t(0,T)} = \|z(x, \cdot)\|_{L^2_t(0,T)} + \|z(x, \cdot)\|_m, \tag{28}$$

where the fractional part of the Sobolev norm is zero for $m = 0$ and for $0 < m < 1$ is given by

$$\|z(x, \cdot)\|_m^2 = 2 \int_0^T \int_0^{T-t} \frac{|z(x, t+l) - z(x, t)|^2}{l^{1+2m}} \, dl \, dt. \tag{29}$$

For each $x \in \mathbb{R}$, employing Minkowski's integral inequality and subsequently using the homogeneous time estimates (9) and (10) for $\alpha^2 + 3\beta\delta \geq 0$ and $\alpha^2 + 3\beta\delta < 0$, respectively, along with the Cauchy–Schwarz inequality, we obtain

$$\|z(x, \cdot)\|_{L_t^2(0,T)} \leq \int_0^T \|S[F(\cdot, t'); 0](x, \cdot - t')\|_{L_t^2(0,T)} \, dt'$$

$$\lesssim (1 + T^{\frac{1}{2}}) \int_0^T \|F(\cdot, t')\|_{H_x^{-1}(\mathbb{R})} \, dt' \tag{30}$$

$$\lesssim T^{\frac{1}{2}} (1 + T^{\frac{1}{2}}) \|F\|_{L_t^2((0,T); H_x^s(\mathbb{R}))}.$$

For the fractional norm, noting that

$$|z(x, t+l) - z(x, t)|^2 \leq \left| \int_0^t S[F(\cdot, t'); 0](x, t+l-t') - S[F(\cdot, t'); 0](x, t-t') \, dt' \right|^2$$

$$+ \left| \int_t^{t+l} S[F(\cdot, t'); 0](x, t+l-t') \, dt' \right|^2,$$

we have $\|z(x, \cdot)\|_m^2 \lesssim I + J$, where

$$I := \int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \left(\int_0^T \left| S[F(\cdot, t'); 0](x, t+l-t') - S[F(\cdot, t'); 0](x, t-t') \right| \, dt' \right)^2 \, dl \, dt, \tag{31}$$

$$J := \int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \left| \int_t^{t+l} S[F(\cdot, t'); 0](x, t+l-t') \, dt' \right|^2 \, dl \, dt. \tag{32}$$

For I, we proceed as follows. First, we multiply the integrand by the characteristic function $\chi_{[0,T-t]}(l)$ so that $\chi_{[0,T-t]}(l) = 1$ for $0 \leq l \leq T - t$ and $\chi_{[0,T-t]}(l) = 0$ otherwise. This allows us to replace $T - t$ by T in the upper limit of the integral with respect to l. Then, we use Minkowski's inequality for the triple integral, and finally, we use the definition of $\chi_{[0,T-t]}(l)$ once again to switch T by $T - t$ in the limit of the integral taken with respect to l. Performing these steps and employing the homogeneous time estimates (9) and (10), we find

$$I \leq \left(\int_0^T \left(\int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \left| S[F(\cdot, t'); 0](x, t+l-t') - S[F(\cdot, t'); 0](x, t-t') \right|^2 \, dl \, dt \right)^{\frac{1}{2}} \, dt' \right)^2 \tag{33}$$

$$\simeq \left(\int_0^T \left\| S[F(\cdot,t');0](x,\cdot-t') \right\|_m dt' \right)^2 \lesssim \left(\int_0^T \left\| F(\cdot,t') \right\|_{H_x^s(\mathbb{R})} dt' \right)^2 \lesssim T \int_0^T \left\| F(\cdot,t) \right\|_{H_x^s(\mathbb{R})}^2 dt.$$

To estimate J, we consider the cases $0 < m < \frac{1}{2}$ and $\frac{1}{2} < m < 1$ separately. The range $\frac{1}{2} < m < 1$ corresponds to $\frac{1}{2} < s < 2$, and hence, we can employ the Sobolev embedding theorem in x. In particular, substituting for $S[F(\cdot,t');0](x,t+l-t')$ via (12) and then using the Sobolev embedding, the Fourier transform characterization of the Sobolev norm, and the fact that $\omega(k)$ is imaginary for $k \in \mathbb{R}$, we have

$$J \leq \int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \left\| \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx-\omega(k)(t+l)} \int_t^{t+l} e^{\omega(k)t'} \widehat{F}(k,t')dt'dk \right\|_{L_x^\infty(\mathbb{R})}^2 dldt$$

$$\lesssim \int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \left\| \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx-\omega(k)(t+l)} \int_t^{t+l} e^{\omega(k)t'} \widehat{F}(k,t')dt'dk \right\|_{H_x^s(\mathbb{R})}^2 dldt$$

$$= \int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \int_{\mathbb{R}} \left(1+k^2\right)^s \left| \int_t^{t+l} e^{\omega(k)t'} \widehat{F}(k,t')dt' \right|^2 dkdldt.$$

Thus, by Minkowski's integral inequality between the integrals with respect to t' and k, Cauchy–Schwarz inequality in the t'-integral, and Fubini's theorem between the integrals with respect to t and t',

$$J \lesssim \int_0^T \int_0^{T-t} \frac{1}{l^{1+2m}} \left(\int_t^{t+l} \left\| F(\cdot,t') \right\|_{H_x^s(\mathbb{R})} dt' \right)^2 dldt$$

$$\leq \int_0^T \int_0^{T-t} \int_t^{t+l} \left\| F(\cdot,t') \right\|_{H_x^s(\mathbb{R})}^2 l^{-2m} dt'dldt$$

$$= \int_0^T \left\| F(\cdot,t') \right\|_{H_x^s(\mathbb{R})}^2 \int_0^T l^{-2m} \int_{t'-l}^{t'} dtdldt'$$ (34)

$$\simeq \frac{T^{2-2m}}{2-2m} \int_0^T \left\| F(\cdot,t') \right\|_{H_x^s(\mathbb{R})}^2 dt'.$$

The range $0 < m < \frac{1}{2}$ corresponds to $-1 < s < \frac{1}{2}$, and hence, Sobolev embedding is no longer available. However, the fact that $m < \frac{1}{2}$ allows to proceed via the Cauchy–Schwarz inequality in

t' as follows:

$$
\begin{aligned}
J &\lesssim \int_0^T \frac{1}{l^{2m}} \int_0^{T-l} \int_t^{t+l} |S[F(\cdot,t');0](x,t+l-t')|^2 dt' dt dl \\
&= \int_0^T \frac{1}{l^{2m}} \int_l^T \int_{t-l}^t |S[F(\cdot,t');0](x,t-t')|^2 dt' dt dl \\
&\lesssim \left(\int_0^T \frac{1}{l^{2m}} dl \right) \int_0^T \left\| S[F(\cdot,t');0](x,\cdot - t') \right\|_{L_t^2(t',T)}^2 dt' \\
&\lesssim \frac{T^{1-2m}}{1-2m} \int_0^T \left\| F(\cdot,t') \right\|_{H_x^{-1}(\mathbb{R})}^2 dt'.
\end{aligned}
\tag{35}
$$

Note that the equality above is due to the change of variable $t \mapsto t - l$, and the inequality succeeding it follows by extending the range of the integrals with respect to t' and t and then interchanging the resulting integrals. The final inequality is thanks to Theorem 3.

Estimates (30), (33), (34), and (35) combined with the Sobolev norm definition (28) imply the desired time estimate (26) in the range $-1 \leq s < 2$ with $s \neq \frac{1}{2}$.

Finally, we consider $2 \leq s < \frac{7}{2}$. As this range corresponds to $1 \leq m < \frac{3}{2}$, the Sobolev norm (28) must be modified to

$$
\|z(x,\cdot)\|_{H_t^m(0,T)}^2 = \|z(x,\cdot)\|_{H_t^1(0,T)}^2 + \|\partial_t z(x,\cdot)\|_{m-1}^2.
$$

Differentiating (24) in t, we have

$$
\partial_t z(x,t) = -iS[F(\cdot,t);0](x,0) - i \int_0^t \partial_t \left[S[F(\cdot,t');0](x,t-t') \right] dt'.
\tag{36}
$$

We begin by observing that $S[F(\cdot,t);0](x,0) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} \widehat{F}(k,t) dk = F(x,t)$. Moreover, by using the Fourier transform property for derivatives, we note that

$$
\begin{aligned}
\partial_t \left[S[F(\cdot,t');0](x,t-t') \right] &= \partial_t \left[\frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx-\omega(k)(t-t')} \widehat{F}(k,t') dk \right] \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} [-\omega(k)] e^{ikx-\omega(k)(t-t')} \widehat{F}(k,t') dk \\
&= S\left[(-\beta\partial_x^3 + i\alpha\partial_x^2 - \delta\partial_x) F(\cdot,t');0 \right](x,t-t').
\end{aligned}
\tag{37}
$$

Therefore, (36) can be rewritten as

$$
\partial_t z(x,t) = -iF(x,t) - i \int_0^t S\left[(-\beta\partial_x^3 + i\alpha\partial_x^2 - \delta\partial_x) F(\cdot,t');0 \right](x,t-t') dt'
\tag{38}
$$

and so,

$$\|\partial_t z(x, \cdot)\|_{L_t^2(0,T)} \leq \|F(x, \cdot)\|_{L_t^2(0,T)} + \left\|\int_0^t S\big[(-\beta\partial_x^3 + i\alpha\partial_x^2 - \delta\partial_x)F(\cdot, t'); 0\big](x, t - t')dt'\right\|_{L_t^2(0,T)}.$$

The first term on the right-hand side can be handled as follows:

$$\|F(x, \cdot)\|_{L_t^2(0,T)} \leq \sup_{x \in \mathbb{R}} \|F(x, \cdot)\|_{L_t^2(0,T)} \leq \|F\|_{L_t^2((0,T); H_x^{\frac{1}{2}+}(\mathbb{R}))}. \tag{39}$$

For the second term, extending the range of integration in t' and then applying Minkowski's integral inequality in combination with Theorem 3, we have

$$\left\|\int_0^t S\big[(-\beta\partial_x^3 + i\alpha\partial_x^2 - \delta\partial_x)F(\cdot, t'); 0\big](x, t - t')dt'\right\|_{L_t^2(0,T)}$$

$$\lesssim (1 + T^{\frac{1}{2}}) \int_0^T \left\|(-\beta\partial_x^3 + i\alpha\partial_x^2 - \delta\partial_x)F(\cdot, t')\right\|_{H_x^{-1}(\mathbb{R})} dt'$$

$$\lesssim (1 + T^{\frac{1}{2}}) \left[\beta \int_0^T \left\|\partial_x^3 F(\cdot, t')\right\|_{H_x^{-1}(\mathbb{R})} dt' + |\alpha| \int_0^T \left\|\partial_x^2 F(\cdot, t')\right\|_{H_x^{-1}(\mathbb{R})} dt' + |\delta| \int_0^T \left\|\partial_x F(\cdot, t')\right\|_{H_x^{-1}(\mathbb{R})} dt'\right]$$

$$\lesssim (1 + T^{\frac{1}{2}}) \left[\beta \int_0^T \left\|F(\cdot, t')\right\|_{H_x^2(\mathbb{R})} dt' + |\alpha| \int_0^T \left\|F(\cdot, t')\right\|_{H_x^1(\mathbb{R})} dt' + |\delta| \int_0^T \left\|F(\cdot, t')\right\|_{L_x^2(\mathbb{R})} dt'\right]$$

$$\lesssim (1 + T^{\frac{1}{2}}) \int_0^T \left\|F(\cdot, t')\right\|_{H_x^2(\mathbb{R})} dt'. \tag{40}$$

Together, estimates (39) and (40) imply the bound

$$\|\partial_t z(x, \cdot)\|_{L_t^2(0,T)} \lesssim T^{\frac{1}{2}}(1 + T^{\frac{1}{2}})\|F\|_{L_t^2((0,T); H_x^2(\mathbb{R}))}, \tag{41}$$

which corresponds to the desired estimate (26) in the case $s = 2$. ∎

Regarding $L_t^\mu L_x^r$ Strichartz-type estimates for the nonhomogeneous linear Cauchy problem (23), we have the following result that is a consequence of the homogeneous Strichartz estimates given in Theorem 4.

Theorem 6. *Let $s \in \mathbb{R}$ and suppose (μ, r) is higher-order Schrödinger admissible in the sense of (20). Then, the solution of the nonhomogeneous linear Cauchy problem (23) satisfies the Strichartz estimate*

$$\|z\|_{L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))} \lesssim \|F\|_{L_t^1((0,T); H_x^s(\mathbb{R}))}. \tag{42}$$

Proof. Letting $H(x, t, t') := \chi_{\{t' \le t\}}(t') S[F(\cdot, t'); 0](x, t - t')$, we rewrite $\|z\|_{L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))}$ as

$$\|z\|_{L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))} = \left\| \int_0^T H(\cdot, \cdot, t') dt' \right\|_{L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))}. \tag{43}$$

Therefore, in view of the homogeneous Strichartz estimate (21), we readily infer

$$\|z\|_{L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))} \le \int_0^T \left\| H(\cdot, \cdot, t') \right\|_{L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))} dt'$$

$$\le \int_0^T \left\| S[F(\cdot, t'); 0](\cdot, \cdot - t') \right\|_{L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))} dt' \lesssim \int_0^T \left\| F(\cdot, t') \right\|_{H_x^s(\mathbb{R})} dt'.$$

∎

2.3 | Reduced initial-boundary value problem

We consider the reduced initial-boundary value problem

$$iq_t + i\beta q_{xxx} + \alpha q_{xx} + i\delta q_x = 0, \quad (x, t) \in \mathbb{R}_+ \times (0, T'),$$

$$q(x, 0) = 0, \quad x \in \mathbb{R}_+, \tag{44}$$

$$q(0, t) = g_0(t) := E_b[g - y(0, \cdot) - z(0, \cdot)](t), \quad t \in (0, T'),$$

where $T' > T$, $y(0, t)$ and $z(0, t)$ are the solutions to the homogeneous and nonhomogeneous Cauchy problems (6) and (23) evaluated at $x = 0$, and $E_b : H_t^{(s+1)/3}(0, T) \to H_t^{(s+1)/3}(\mathbb{R})$ is a fixed bounded extension operator satisfying the additional property that supp $g_0 \subset [0, T']$. The construction of such an extension is analogous to the one provided in detail in Section 3 of Ref. 40 in the context of the linear Schrödinger equation. In particular, we note that, for continuous Sobolev data, a compactly supported extension can be constructed thanks to the compatibility between the initial and boundary data at $(x, t) = (0, 0)$ (see also discussion above Theorem 7). In this connection, observe that the traces $y(0, t)$ and $z(0, t)$ are well defined and belong to $H_t^{(s+1)/3}(0, T)$ in view of Theorems 3 and 5.

2.3.1 | Solution formula

We obtain a formula to represent weak solutions of the reduced initial-boundary value problem (44) via Fokas's unified transform method. To this end, we first assume that q is sufficiently smooth up to the boundary of $\mathbb{R}_+ \times (0, T')$ and decays sufficiently fast as $x \to \infty$, uniformly in $[0, T']$.

The definition of the standard Fourier transform on $\mathbb{R}$ applied on the piecewise-defined function

$$F(x) = \begin{cases} f(x), & x > 0, \\ 0, & x < 0, \end{cases} \quad f \in L^2(0, \infty),$$

gives rise to the half-line Fourier transform pair

$$\widehat{f}(k) = \int_0^\infty e^{-ikx} f(x) dx, \quad \operatorname{Im}(k) \leq 0,$$

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} \widehat{f}(k) dk, \quad x > 0. \tag{45}$$

Note that the above half-line Fourier transform makes sense for all $\operatorname{Im}(k) \leq 0$ and not just for $k \in \mathbb{R}$ as its whole-line counterpart. Taking the half-line Fourier transform (45) of (44) and integrating over $(0, t)$, we obtain the following spectral identity known as the global relation:

$$e^{\omega(k)t} \widehat{q}(k, t) = (-\beta k^2 + \alpha k + \delta) \widetilde{g}_0(\omega(k), t) + (i\beta k - i\alpha) \widetilde{g}_1(\omega(k), t) + \beta \widetilde{g}_2(\omega(k), t), \quad \operatorname{Im} k \leq 0, \tag{46}$$

where ω is given by (11) and the temporal transforms $\widetilde{g}_j(\omega(k), t)$ are defined by

$$\widetilde{g}_j(k, t) = \int_0^t e^{kt'} \partial_x^j q(0, t') dt', \quad k \in \mathbb{C}, \quad j = 0, 1, 2. \tag{47}$$

Then, by the inversion formula in (45),

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ikx - \omega(k)t} \left[(-\beta k^2 + \alpha k + \delta) \widetilde{g}_0(\omega(k), t) + (i\beta k - i\alpha) \widetilde{g}_1(\omega(k), t) + \beta \widetilde{g}_2(\omega(k), t) \right] dk. \tag{48}$$

The transforms $\widetilde{g}_1$ and $\widetilde{g}_2$ involve the unknown boundary values $q_x(0, t)$ and $q_{xx}(0, t)$. To eliminate them from (48), we proceed as follows. For $D := \{k \in \mathbb{C} : \operatorname{Re}(\omega(k)) < 0\}$, consider the region

$$D^+ := D \cap \{\operatorname{Im}(k) > 0\} = \left\{ \operatorname{Im}(k) > 0 : 3\left(\operatorname{Re}(k) - \frac{\alpha}{3\beta}\right)^2 - \operatorname{Im}(k)^2 - \frac{\alpha^2 + 3\beta\delta}{3\beta^2} < 0 \right\}, \tag{49}$$

which is depicted in Figure 1 for the various signs of the quantity $\alpha^2 + 3\beta\delta$. Then, thanks to analyticity (Cauchy's theorem) and exponential decay, it follows that (e.g., see Appendix A in Ref. 24 for a detailed explanation in the context of the linear Schrödinger equation)

$$q(x, t) = \frac{1}{2\pi} \int_{\partial D^+} e^{ikx - \omega(k)t} \left[(-\beta k^2 + \alpha k + \delta) \widetilde{g}_0(\omega(k), t) + (i\beta k - i\alpha) \widetilde{g}_1(\omega(k), t) + \beta \widetilde{g}_2(\omega(k), t) \right] dk, \tag{50}$$

where the contour ∂D^+ is positively oriented, that is, it is traversed in the direction such that D^+ stays to the left of the contour, as shown in Figure 1.
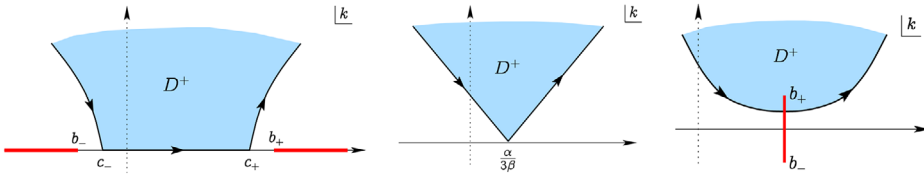
FIGURE 1 The region D^+ defined by (49) for $\alpha^2 + 3\beta\delta > 0$ (left), $\alpha^2 + 3\beta\delta = 0$ (center), and $\alpha^2 + 3\beta\delta < 0$ (right). In the first case, the square root branch cut $\mathcal{B} = (-\infty, b_-] \cup [b_+, \infty)$ with branch points $b_\pm = \frac{1}{3\beta}(\alpha \pm 2\sqrt{\alpha^2 + 3\beta\delta})$ (shown in red) stays outside the region D^+, whereas in the second case, there is no branching. On the other hand, in the third case, the branch cut $\widetilde{B}$ (shown in red) is taken along the vertical line segment connecting b_+ to b_- and so part of it lies in D^+; thus, a local deformation around $\widetilde{B}$ is performed as shown in Figure 2 below.

The fact that the integral (50) is taken along the deformed contour ∂D^+ will allow us to eliminate the unknown transforms $\widetilde{g}_1$ and $\widetilde{g}_2$ from (50) by employing two additional spectral identities emanating from the global relation (46) through suitable transformations that keep the spectral function $\omega(k)$ invariant. In particular, both of these identities are valid along ∂D^+, and so, we will be able to use them simultaneously. It is important to emphasize that the two additional identities are not valid along $\mathbb{R}$, which is the reason why the deformation from $\mathbb{R}$ to ∂D^+ that leads to (50) is necessary.

To determine the symmetry transformations, we solve the equation $\omega(\nu) = \omega(k)$ for $\nu = \nu(k)$.

(i) If $\alpha^2 + 3\beta\delta > 0$, then the two nontrivial symmetries are

$$\nu_\pm(k) = -\frac{1}{2}\left(k - \frac{\alpha}{\beta}\right) \pm \frac{\sqrt{3}\,i}{2}\left[\left(k - \frac{\alpha}{3\beta}\right)^2 - \frac{4(\alpha^2 + 3\beta\delta)}{9\beta^2}\right]^{\frac{1}{2}}. \tag{51}$$

The square root term in (51) is defined as follows. Denoting the two branch points by $b_\pm := \frac{1}{3\beta}(\alpha \pm 2\sqrt{\alpha^2 + 3\beta\delta})$, we write $k - b_\pm = |k - b_\pm|e^{i\theta_\pm}$ with $-\pi < \theta_- \leq \pi$ and $0 \leq \theta_+ < 2\pi$, which correspond to branch cuts along $[b_+, \infty)$ for $(k - b_+)^{\frac{1}{2}}$ and along $(-\infty, b_-]$ for $(k - b_-)^{\frac{1}{2}}$. Then, we associate the square root in (51) with the single-valued function

$$\left[\left(k - \frac{\alpha}{3\beta}\right)^2 - \frac{4(\alpha^2 + 3\beta\delta)}{9\beta^2}\right]^{\frac{1}{2}} = \sqrt{|k - b_+||k - b_-|}\,e^{i(\theta_+ + \theta_-)/2}, \tag{52}$$

which is analytic for all $k \notin \mathcal{B} := (-\infty, b_-] \cup [b_+, \infty)$. In turn, this definition ensures that $\nu_\pm$ are analytic for all $k \in \mathbb{C} \setminus \mathcal{B}$. Importantly, as shown in Figure 1, $\mathcal{B} \cap \overline{D^+} = \infty$.

(ii) $\alpha^2 + 3\beta\delta = 0$. In this case, the symmetries are the two entire functions

$$\nu_\pm(k) = -\frac{1}{2}\left(k - \frac{\alpha}{\beta}\right) \pm \frac{\sqrt{3}\,i}{2}\left(k - \frac{\alpha}{3\beta}\right), \tag{53}$$

as shown in Figure 1.

FIGURE 2 Deformation of ∂D^+ to $\partial \widetilde{D}^+$ for $\alpha^2 + 3\beta\delta = 0$ (left) and $\alpha^2 + 3\beta\delta < 0$ (right). The left deformation is carried out to stay away from the point $\frac{\alpha}{3\beta}$, which is a zero of the quantity $\nu_-(k) - \nu_+(k)$. The right deformation is done to avoid crossing the branch cut $\widetilde{B}$ (shown in red).
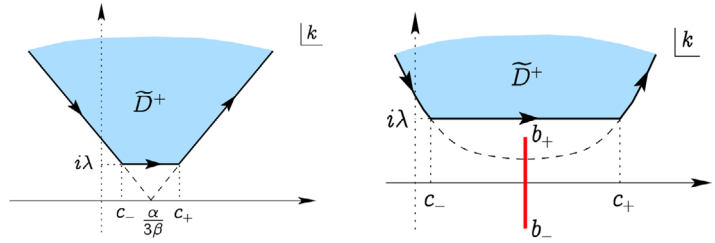
(iii) $\alpha^2 + 3\beta\delta < 0$. In that case, the symmetries are again given by (51); however, as the branch points $b_\pm$ are now complex conjugates along the line $\mathrm{Re}(k) = \frac{\alpha}{3\beta}$, we write $k - b_\pm = |k - b_\pm| e^{i(\theta_\pm - \pi/2)}$ with $0 \leq \theta_\pm < 2\pi$ and corresponding branch cuts along the vertical half-lines from $b_\pm$ to $\frac{\alpha}{3\beta} - i\infty$, so that

$$\left[\left(k - \frac{\alpha}{3\beta}\right)^2 - \frac{4(\alpha^2 + 3\beta\delta)}{9\beta^2}\right]^{\frac{1}{2}} = \sqrt{|k - b_+||k - b_-|}\, e^{i(\theta_+ + \theta_- - \pi)/2} \tag{54}$$

is single-valued and analytic for all $k \in \mathbb{C} \setminus \widetilde{B}$, where $\widetilde{B}$ is the finite vertical segment connecting b_+ and b_-, as shown in Figure 1. Note that $\widetilde{B} \cap \overline{D^+} \neq \infty$ as part of the branch cut $\widetilde{B}$ lies inside the region D^+. For this reason, *before* employing the symmetries $\nu_\pm$ for the elimination of the unknown transforms $\widetilde{g}_2$ and $\widetilde{g}_1$ from (50), we use Cauchy's theorem to deform the contour ∂D^+ in (50) to the modified contour $\partial \widetilde{D}^+$, which corresponds to the positively oriented boundary of the region $\widetilde{D}^+$ shown in Figure 2. This way, the branch cut $\widetilde{B}$ is avoided *prior to* the use of the symmetries $\nu_\pm$, allowing us to take advantage of analyticity inside the region $\widetilde{D}^+$ later.

In view of the above discussion, we rewrite (50) as

$$q(x,t) = \frac{1}{2\pi} \int_\Gamma e^{ikx - \omega(k)t} \left[(-\beta k^2 + \alpha k + \delta)\widetilde{g}_0(\omega(k),t) + i(\beta k - \alpha)\widetilde{g}_1(\omega(k),t) + \beta\,\widetilde{g}_2(\omega(k),t)\right] dk, \tag{55}$$

where the integration contour Γ is given by

$$\Gamma = \begin{cases} \partial D^+, & \alpha^2 + 3\beta\delta > 0, \\ \partial \widetilde{D}^+, & \alpha^2 + 3\beta\delta \leq 0. \end{cases} \tag{56}$$

Replacing k by $\nu_\pm(k)$ in the global relation (46) and using the fact that $\omega(\nu_\pm(k)) = \omega(k)$, we get the spectral identities

$$e^{\omega(k)t}\,\widehat{q}(\nu_\pm(k), t) = \left(-\beta\nu_\pm^2(k) + \alpha\nu_\pm(k) + \delta\right)\widetilde{g}_0(\omega(k),t) + i\left(\beta\nu_\pm(k) - \alpha\right)\widetilde{g}_1(\omega(k),t)$$
$$+ \beta\,\widetilde{g}_2(\omega(k),t), \quad \mathrm{Im}(\nu_\pm(k)) \leq 0. \tag{57}$$

We emphasize that the above identities are valid only for k such that $\mathrm{Im}(\nu_\pm(k)) \leq 0$. Thus, to employ them for the elimination of the unknown boundary values from (55), we need to ensure that $\Gamma \subseteq \{\mathrm{Im}(\nu_\pm(k)) \leq 0\}$. This is proved in the following lemma.

Lemma 1. *Let $\nu_\pm = \nu_\pm(k)$ be the nontrivial (i.e., $\nu_\pm \not\equiv k$) solutions of the equation $\omega(\nu) = \omega(k)$ as given by (51) or (53), depending on the value of $\alpha^2 + 3\beta\delta$. If $k \in \overline{D^+}$, then $\mathrm{Im}(\nu_\pm) \leq 0$.*

Proof. For all $k = k_R + ik_I \in \overline{D^+}$ such that $\nu(k) \neq k$ satisfies $\omega(\nu) = \omega(k)$, we must have $\beta(\nu^2 + k\nu + k^2) - \alpha(\nu + k) - \delta = 0$. Writing $\nu = \nu_R + i\nu_I$ and taking real and imaginary parts, this equation is equivalent to the system

$$\widetilde{\nu}_R \widetilde{\nu}_I = ck_I, \quad \widetilde{\nu}_R^2 - \widetilde{\nu}_I^2 = d, \tag{58}$$

where $\widetilde{\nu}_R = \nu_R + \frac{k_R}{2} - \frac{\alpha}{2\beta}$, $\widetilde{\nu}_I = \nu_I + \frac{k_I}{2}$, $c = \frac{\alpha}{4\beta} - \frac{3k_R}{4}$, and $d = -\frac{3}{4}k_R^2 + \frac{3}{4}k_I^2 + \frac{\alpha}{2\beta}k_R + \frac{\alpha^2}{4\beta^2} + \frac{\delta}{\beta}$. If $\widetilde{\nu}_I = 0$, then $\nu_I = -\frac{k_I}{2} \leq 0$ as $k \in \overline{D^+}$ and we are done. So, let us assume $\widetilde{\nu}_I \neq 0$. Then, combining the two equations in (58), we obtain $\widetilde{\nu}_I^4 + d\widetilde{\nu}_I^2 - c^2 k_I^2 = 0$, which can be solved for $\widetilde{\nu}_I^2$ to yield $\widetilde{\nu}_I^2 = \frac{-d \pm \sqrt{d^2 + 4c^2 k_I^2}}{2} = -\frac{d}{2} \pm \sqrt{\frac{d^2}{4} + c^2 k_I^2}$. Note that only the positive sign is acceptable since $\widetilde{\nu}_I \in \mathbb{R} \Rightarrow \widetilde{\nu}_I^2 \geq 0$. That is, $\widetilde{\nu}_I^2 = -\frac{d}{2} + \sqrt{\frac{d^2}{4} + c^2 k_I^2}$ implying $\widetilde{\nu}_I = \pm\sqrt{-\frac{d}{2} + \sqrt{\frac{d^2}{4} + c^2 k_I^2}}$. In turn, from the first of Equations (58), we get $\widetilde{\nu}_R = \pm \dfrac{ck_I}{\sqrt{-\frac{d}{2} + \sqrt{\frac{d^2}{4} + c^2 k_I^2}}}$ and so

$$\nu_R = \pm \frac{ck_I}{\sqrt{-\frac{d}{2} + \sqrt{\frac{d^2}{4} + c^2 k_I^2}}} - \frac{k_R}{2} + \frac{\alpha}{2\beta}, \quad \nu_I = -\frac{k_I}{2} \pm \sqrt{-\frac{d}{2} + \sqrt{\frac{d^2}{4} + c^2 k_I^2}}. \tag{59}$$

Observe that the radicand of the outer square root involved in the above expressions is a nonnegative number, and hence, that square root is a real (nonnegative) number. In addition, note that expressions (59) are consistent with Equations (51) and (53); however, their dependence on k_R and k_I (as opposed to k) is not suitable for discussing the analyticity of the associated expressions for ν, which is why (51) and (53) were used earlier for that purpose. On the other hand, (59) are the forms convenient for proving Lemma 1.

The case of the negative square root sign in (59) is straightforward as then $\nu_I \leq 0$ for all $k_I \geq 0$ and, in particular, for $k \in \overline{D^+}$ as desired. On the other hand, the case of positive square root sign in (59) requires more work. More specifically, by definition (49), for $k \in \overline{D^+}$, we have

$$3\left(k_R - \frac{\alpha}{3\beta}\right)^2 - k_I^2 - \frac{\alpha^2 + 3\beta\delta}{3\beta^2} \leq 0, \tag{60}$$

which can be rearranged to $-\frac{3}{4}k_R^2 + \frac{1}{4}k_I^2 + \frac{\alpha}{2\beta}k_R + \frac{\delta}{4\beta} \geq 0$. For $k_I \neq 0$ (note that $k_I = 0$ implies $\nu_I = 0$ and we are done), this is equivalent to $\frac{k_I^4}{16} + k_I^2 \frac{d}{4} \geq c^2 k_I^2$ or, after completing the square, $(\frac{k_I^2}{4} + \frac{d}{2})^2 \geq \frac{d^2}{4} + c^2 k_I^2$. Hence, $\frac{k_I^2}{4} \geq -\frac{d}{2} + \sqrt{\frac{d^2}{4} + c^2 k_I^2}$ or $\frac{k_I^2}{4} \leq -\frac{d}{2} - \sqrt{\frac{d^2}{4} + c^2 k_I^2}$ and, as the second inequality is not possible because it would imply that $k_I^2 \leq 0$, taking the square root of the first

inequality and using the fact that $k_I \geq 0$ for $k \in \overline{D^+}$, we obtain $0 \geq -\frac{k_I}{2} + \sqrt{-\frac{d}{2} + \sqrt{\frac{d^2}{4} + c^2 k_I^2}} = \nu_I$ as desired.

The proof so far has been under the assumption that $\nu(k) \neq k$; however, although $\nu \not\equiv k$ by hypothesis, there could still be points in $\overline{D^+}$ where $\nu(k) = k$, and hence, this scenario must also be considered. In that case, recalling that $\nu_{\pm}$ satisfy $\beta(\nu_{\pm}^2 + k\nu_{\pm} + k^2) - \alpha(\nu_{\pm} + k) - \delta = 0$, we infer that if $k \in \mathbb{C}$ is such that $\nu_{\pm}(k) = k$, then $3\beta k^2 - 2\alpha k - \delta = 0$. If $\alpha^2 + 3\beta\delta \geq 0$, then $k = k_{\pm} = \frac{\alpha}{3\beta} \pm \frac{\sqrt{\alpha^2 + 3\beta\delta}}{3\beta} \in \mathbb{R}$, that is, $k_I = \text{Im}(\nu_{\pm}) = 0$ and we are done. If $\alpha^2 + 3\beta\delta < 0$, then $k = k_{\pm} = \frac{\alpha}{3\beta} \pm i\frac{\sqrt{-(\alpha^2 + 3\beta\delta)}}{3\beta}$. Note that $k_- \notin \overline{D^+}$ since $\text{Im}(k_-) < 0$. Also, $k_+ \notin \overline{D^+}$ because if $\alpha^2 + 3\beta\delta < 0$ and $k \in \overline{D^+}$, then by (60), we must have $k_I \geq \sqrt{-\frac{\alpha^2 + 3\beta\delta}{3\beta^2}} = \frac{\sqrt{-(\alpha^2 + 3\beta\delta)}}{\sqrt{3}\beta} > \frac{\sqrt{-(\alpha^2 + 3\beta\delta)}}{3\beta} = \text{Im}(k_+)$. This completes the proof of Lemma 1. ∎

Thanks to Lemma 1, both of the identities (57) are valid for $k \in \overline{D^+}$ and hence can be solved simultaneously as a system for the unknown transforms $\widetilde{g}_1(\omega(k), t)$ and $\widetilde{g}_2(\omega(k), t)$ to yield

$$\widetilde{g}_1(\omega(k), t) = \frac{e^{\omega(k)t}}{i\beta[\nu_+(k) - \nu_-(k)]}[\widehat{q}(\nu_+(k), t) - \widehat{q}(\nu_-(k), t)] + ik\widetilde{g}_0(\omega(k), t), \tag{61}$$

$$\widetilde{g}_2(\omega(k), t) = \frac{e^{\omega(k)t}}{\beta^2[\nu_-(k) - \nu_+(k)]}[(\beta\nu_-(k) - \alpha)\widehat{q}(\nu_+(k), t) - (\beta\nu_+(k) - \alpha)\widehat{q}(\nu_-(k), t)] \tag{62}$$

$$- k^2 \widetilde{g}_0(\omega(k), t).$$

Substituting these expressions in the integral representation (55), we obtain

$$q(x, t) = \frac{1}{2\pi} \int_{\Gamma} e^{ikx - \omega(k)t} \left(-3\beta k^2 + 2\alpha k + \delta\right) \widetilde{g}_0(\omega(k), t) dk$$

$$+ \frac{1}{2\pi} \int_{\Gamma} e^{ikx} \left[\frac{\nu_-(k) - k}{\nu_-(k) - \nu_+(k)} \widehat{q}(\nu_+(k), t) - \frac{\nu_+(k) - k}{\nu_-(k) - \nu_+(k)} \widehat{q}(\nu_-(k), t)\right] dk. \tag{63}$$

Note that the definition (56) of Γ in conjunction with the choices of the contour $\partial \widetilde{D}^+$, as shown in Figure 2, ensures that $\nu_-(k) - \nu_+(k)$ stays away from zero. Indeed, for $\alpha^2 + 3\beta\delta > 0$, the solutions of $\nu_-(k) - \nu_+(k) = 0$ occur at the branch points $b_{\pm}$, which lie on the real axis and outside segment $[\frac{1}{3\beta}(\alpha - \sqrt{\alpha^2 + 3\beta\delta}), \frac{1}{3\beta}(\alpha + \sqrt{\alpha^2 + 3\beta\delta})]$ forming the base of $\Gamma = \partial D^+$ (see left panel of Figure 1). Moreover, for $\alpha^2 + 3\beta\delta = 0$, the quantity $\nu_-(k) - \nu_+(k)$ vanishes at $\frac{\alpha}{3\beta}$, which is bypassed by $\Gamma = \partial D^+$ as shown on the left panel of Figure 2. Finally, for $\alpha^2 + 3\beta\delta < 0$, the roots of $\nu_-(k) - \nu_+(k) = 0$ are again at the branch points $b_{\pm}$, and so, they stay below the contour $\Gamma = \partial D^+$ depicted on the right panel of Figure 2.

Therefore, using analyticity (Cauchy's theorem) along with exponential decay as $|k| \to \infty$ inside D^+ or $\widetilde{D}^+$, as appropriate, we conclude that the second k-integral on the right-hand side of (63) is equal to zero. (To see the decay, note that $|e^{ikx - i\nu_{\pm}y}| = e^{-\text{Im}(k)x + \text{Im}(\nu_{\pm})y}$ and use Lemma 1 together with the fact that $x, y > 0$.) Consequently, we deduce the solution formula

$$q(x, t) = -\frac{i}{2\pi} \int_{\Gamma} e^{ikx - \omega(k)t} \omega'(k) \widetilde{g}_0(\omega(k), t) dk. \tag{64}$$

In fact, noting that $|e^{-\omega(k)(t-t')}| = e^{\mathrm{Re}(\omega(k))(t'-t)}$ and recalling that, by definition (49), $\mathrm{Re}(\omega(k)) < 0$ inside D^+, we see that the exponential $e^{ikx-\omega(k)(t-t')}$ decays as $|k| \to \infty$ inside D^+ for all $x > 0$, $t' > t$. Thus, combining this decay with analyticity, in the second argument of the time transform $\widetilde{g}_0$, we can replace t by any fixed $T' > t$ and thereby obtain the following equivalent version of the solution formula (64), which is more convenient for the purpose of linear estimates as we will see below:

$$q(x,t) = -\frac{i}{2\pi} \int_\Gamma e^{ikx-\omega(k)t} \omega'(k) \widetilde{g}_0(\omega(k), T') dk. \tag{65}$$

2.3.2 | Compatibility between the data

Recall that the initial and boundary data of the initial-boundary value problem (5) belong to the L^2-based Sobolev spaces $H_x^s(\mathbb{R}_+)$ and $H_t^{(s+1)/3}(0,T)$, respectively. Moreover, in view of the range of validity of Theorem 5 for the nonhomogeneous Cauchy problem established earlier, as well as of Theorem 7 for the reduced initial-boundary value problem proved below, we will restrict our attention to the range $0 \leq s \leq 2$ with $s \neq \frac{1}{2}$.

For $\frac{1}{2} < s \leq 2$, continuity becomes relevant to our analysis and it turns out that we need to impose a compatibility condition between the initial and the boundary data. More specifically, note that if $\frac{1}{2} < s \leq 2$, then $\frac{1}{2} < \frac{s+1}{3} \leq 1$. Therefore, both of the traces $u_0(0)$ and $g(0)$ are well defined. Furthermore, since $y(0, \cdot)$ and $z(0, \cdot)$ belong to $H_t^{(s+1)/3}(0,T)$ by Theorems 3 and 5, the traces $y(0,0)$ and $z(0,0)$ are well defined and equal to $u_0(0)$ and 0, respectively, due to continuity and the initial conditions in problems (6) and (23). Thus, using continuity at zero for the function $g_0 \in H_t^{(s+1)/3}(\mathbb{R})$ defined in (44), we have

$$g_0(0) = \lim_{t\to 0^+} g_0(t) = \lim_{t\to 0^+} [g(t) - y(0,t) - z(0,t)] = g(0) - y(0,0) - z(0,0) = g(0) - u_0(0),$$

which, upon imposing the (natural) compatibility condition,

$$u_0(0) = g(0), \quad \frac{1}{2} < s \leq 2, \tag{66}$$

implies that the boundary datum of the reduced problem (44) vanishes at $t = 0$, that is,

$$g_0(0) = 0, \quad \frac{1}{2} < s \leq 2. \tag{67}$$

This feature will turn out to be convenient in the proof of Theorem 7 that follows next.

2.3.3 | Sobolev-type estimates

We now establish the basic space estimate in the initial-boundary value problem setting. More precisely, we prove the following.

Theorem 7. *Let $s \geq 0$. Then, the unique solution of the reduced initial-boundary value problem* (44) *satisfies*

$$\|q(\cdot,t)\|_{H^s_x(\mathbb{R}_+)} \leq c \left(1 + \sqrt{T'}e^{cT'}\right)\|g_0\|_{H^{\frac{s+1}{3}}_t(0,T')} \tag{68}$$

uniformly for $t \in [0,T']$, where $c > 0$ is a constant that only depends on α, β, δ, and s.

Proof. We employ the Fokas method solution formula (65). First, recalling the definition (49) of D^+ and the various scenarios depending on the sign of $\alpha^2 + 3\beta\delta$ that are shown in Figures 1 and 2, we parameterize the integration contour in (65) as $\Gamma = (-\gamma_1) \cup \gamma_2 \cup \gamma_3$ with

$$\gamma_1(m) = \frac{\alpha - \sqrt{3\beta^2 m^2 + \alpha^2 + 3\beta\delta}}{3\beta} + im, \qquad\qquad \lambda \leq m < \infty,$$

$$\gamma_2(m) = m + i\lambda, \qquad\qquad c_- < m < c_+, \tag{69}$$

$$\gamma_3(m) = \frac{\alpha + \sqrt{3\beta^2 m^2 + \alpha^2 + 3\beta\delta}}{3\beta} + im, \qquad\qquad \lambda \leq m < \infty,$$

where, as depicted in Figures 1 and 2, $c_\pm = \frac{\alpha \pm \sqrt{3\beta^2\lambda^2 + \alpha^2 + 3\beta\delta}}{3\beta}$ and $\lambda > 0$ is a fixed nonnegative real number such that

$$\begin{cases} \lambda = 0, & \alpha^2 + 3\beta\delta > 0, \quad \text{(first panel in Figure 1)} \\ \lambda > \frac{2\sqrt{-(\alpha^2+3\beta\delta)}}{3\beta}, & \alpha^2 + 3\beta\delta \leq 0. \quad \text{(Figure 2)}. \end{cases} \tag{70}$$

In view of the above parameterization, for any $j \in \mathbb{N}_0$, we have

$$\partial^j_x q(x,t) = -\frac{1}{2\pi}\int_\infty^\lambda (i\gamma_1(m))^j e^{i\gamma_1(m)x - (\omega(\gamma_1(m))t} \widetilde{g}_0(\omega(\gamma_1(m)), T')\frac{d[i\omega(\gamma_1(m))]}{dm}\, dm \tag{71}$$

$$-\frac{1}{2\pi}\int_{c_-}^{c_+} (i\gamma_2(m))^j e^{i\gamma_2(m)x - (\omega(\gamma_2(m))t} \widetilde{g}_0(\omega(\gamma_2(m)), T')\frac{d[i\omega(\gamma_2(m))]}{dm}\, dm \tag{72}$$

$$-\frac{1}{2\pi}\int_\lambda^\infty (i\gamma_3(m))^j e^{i\gamma_3(m)x - (\omega(\gamma_3(m))t} \widetilde{g}_0(\omega(\gamma_3(m)), T')\frac{d[i\omega(\gamma_3(m))]}{dm}\, dm \tag{73}$$

$$=: q_1(x,t) + q_2(x,t) + q_3(x,t).$$

As the terms q_1 and q_3 are analogous, they can be handled in a similar fashion, and hence, we only provide the details for the estimation of q_1 given by (71). Since

$$\|q_1(\cdot,t)\|^2_{L^2_x(\mathbb{R}_+)} = \frac{1}{4\pi^2}\int_0^\infty \left|\int_\lambda^\infty (i\gamma_1(m))^j e^{i\gamma_1(m)x - (\omega(\gamma_1(m))t} \widetilde{g}_0(\omega(\gamma_1(m)), T')\frac{d[i\omega(\gamma_1(m))]}{dm}\, dm\right|^2 dx$$

$$\lesssim \int_0^\infty \left(\int_0^\infty e^{-mx}|\gamma_1(m)|^j |\widetilde{g}_0(\omega(\gamma_1(m)), T')|\left|\frac{d[i\omega(\gamma_1(m))]}{dm}\right| \chi_{(\lambda,\infty)}(m)dm\right)^2 dx,$$

by the boundedness of the Laplace transform in $L^2(\mathbb{R}_+)$ (e.g., see Lemma 3.2 in [1]), we have

$$\|q_1(\cdot, t)\|^2_{L^2_x(\mathbb{R}_+)} \lesssim \int_\lambda^\infty |\gamma_1(m)|^{2j} |\widetilde{g}_0(\omega(\gamma_1(m)), T')|^2 \left|\frac{d[i\omega(\gamma_1(m))]}{dm}\right|^2 dm. \tag{74}$$

Let $\tau(m) = i\omega(\gamma_1(m))$, $m \in [\lambda, \infty)$. Note that $\tau(m) \in \mathbb{R}$ since $\gamma_1(m) \in \partial D^+$ and $\mathrm{Re}(\omega(k)) = 0$ for $k \in \partial D^+$ and, more precisely, $\mathrm{Range}(\tau) = [i\omega(c_- + i\lambda), \infty)$. Furthermore, since $\tau'(m) \neq 0$ on (λ, ∞) and $\tau \to \infty$ as $m \to \infty$, it follows that $\tau : [\lambda, \infty) \to [i\omega(c_- + i\lambda), \infty)$ is monotone increasing and so $\tau'(m) > 0$. Then, (74) becomes

$$\|q_1(\cdot, t)\|^2_{L^2_x(\mathbb{R}_+)} \lesssim \int_\lambda^\infty |\gamma_1(m)|^{2j} |\widetilde{g}_0(-i\tau(m), T')|^2 [\tau'(m)]^2 dm$$
$$= \int_\lambda^\infty |\gamma_1(m)|^{2j} |\widehat{g}_0(\tau(m))|^2 [\tau'(m)]^2 dm \tag{75}$$

after observing that the time transform (47) of g_0 at T' is, in fact, the Fourier transform of g_0 thanks to the fact that g_0 has compact support inside $(0, T')$, namely,

$$\widetilde{g}_0(-i\tau(m), T') = \widehat{g}_0(\tau(m)). \tag{76}$$

Next, we have the following auxiliary result.

Lemma 2. *There is a constant $c > 0$ depending only on α, β, δ such that*

$$\sup_{m \in [\lambda, \infty)} \frac{|\gamma_1(m)|^{2j} \tau'(m)}{[1 + \tau^2(m)]^{\frac{j+1}{3}}} \leq c < \infty.$$

We prove Lemma 2 after the end of the current proof. Employing it in combination with (75), we obtain

$$\|q_1(\cdot, t)\|^2_{L^2_x(\mathbb{R}_+)} \lesssim \int_\lambda^\infty [1 + \tau^2(m)]^{\frac{j+1}{3}} |\widehat{g}_0(\tau(m))|^2 \tau'(m) dm$$
$$= \int_{i\omega(c_- + i\lambda)}^\infty (1 + \tau^2)^{\frac{j+1}{3}} |\widehat{g}_0(\tau)|^2 d\tau = \|g_0\|^2_{H_t^{\frac{j+1}{3}}(\mathbb{R})} \tag{77}$$

uniformly for $t \in [0, T']$, completing the estimation of q_1.

We proceed to the estimation of q_2 given by (72).

Case 1: $\alpha^2 + 3\beta\delta > 0$. Then, $\lambda = 0$ and by the definition of γ_2, we can rewrite q_2 as

$$q_2(x, t) = -\frac{i}{2\pi} \int_{c_-}^{c_+} (im)^j e^{imx - \omega(m)t} \widetilde{g}_0(\omega(m), T') \omega'(m) dm, \tag{78}$$

so that $q_2(\cdot, t)$ can be regarded as the inverse spatial Fourier transform of the function

$$Q_2(m, t) = \begin{cases} 0, & m \notin (c_-, c_+), \\ -i(im)^j e^{-\omega(m)t} \widetilde{g}_0(\omega(m), T') \omega'(m), & m \in (c_-, c_+). \end{cases} \tag{79}$$

Note that $|e^{\omega(m)\rho}| = 1$ for $m \in (c_-, c_+)$, $\rho \in \mathbb{R}$. Hence, using the definition of the t-transform (47) and the Cauchy–Schwarz inequality, we have

$$|\widetilde{g}_0(\omega(m), T')| \leq \sqrt{T'} \, \|g_0\|_{L^2_t(0,T')}, \tag{80}$$

implying via Plancherel's theorem that

$$\|q_2(\cdot, t)\|^2_{L^2_x(\mathbb{R}_+)} = \int_{-\infty}^{\infty} |Q_2(m,t)|^2 dm \leq T' \|g_0\|^2_{L^2_t(0,T')} \int_{c_-}^{c_+} |m|^{2j} |\omega'(m)|^2 dm$$

$$= cT' \|g_0\|^2_{L^2_t(0,T')} \lesssim T' \|g_0\|^2_{H_t^{\frac{j+1}{3}}(0,T')} \tag{81}$$

with the various constants depending on α, β, δ, and j.

Case 2: $\alpha^2 + 3\beta\delta \leq 0$. Then, $\lambda > \frac{2\sqrt{-(\alpha^2 + 3\beta\delta)}}{3\beta} > 0$ and, by the definition of γ_2,

$$|q_2(x,t)| \leq \frac{1}{2\pi} e^{-\lambda x} \int_{c_-}^{c_+} |m^2 + \lambda^2|^{\frac{j}{2}} \left| e^{-(\omega(m+i\lambda)t} \, \widetilde{g}_0(\omega(m+i\lambda), T') \, \omega'(m+i\lambda) \right| dm. \tag{82}$$

Recall that for $k \in D^+$, we have $\mathrm{Re}(\omega(k)) < 0$, which implies $|e^{\omega(m+i\lambda)\rho}| \leq 1$ for $m \in (c_-, c_+)$, $\rho \in [0, T']$. Therefore, similarly to (80),

$$|\widetilde{g}_0(\omega(m+i\lambda), T')| \leq \sqrt{T'} \, \|g_0\|_{L^2_t(0,T')}. \tag{83}$$

Combining (82) and (83), we deduce

$$|q_2(x,t)| \leq c\sqrt{T'} e^{cT'} \|g_0\|_{L^2_t(0,T')} e^{-\lambda x}.$$

Taking the square of the above inequality, integrating with respect to $x \in (0, \infty)$ (for this step, recall that $\lambda > 0$), and then taking square roots, we obtain

$$\|q_2(\cdot, t)\|_{L^2_x(\mathbb{R}_+)} \leq \frac{c}{\sqrt{2\lambda}} \sqrt{T'} e^{cT'} \|g_0\|_{L^2_t(0,T')} \lesssim \sqrt{T'} e^{cT'} \|g_0\|_{H_t^{\frac{j+1}{3}}(0,T')},$$

where the constant of the last inequality depends only on α, β, δ, and j.

The desired estimate (68) has been established for $s \in \mathbb{N}_0$. The proof for $s \geq 0$ follows by interpolation, for example, see Theorem 5.1 in Ref. 58. Make a remark about the possibility of using the fractional norm along the lines of the nonhomogeneous Cauchy problem and Ref. 1. ∎

Proof of Lemma 2. First, we make a few observations. From the definition (11) of ω and the triangle inequality,

$$|\omega(k)| \geq \beta|k|^3 - |\alpha k^2 + \delta k| \geq \beta|k|^3 - (|\alpha||k|^2 + |\delta||k|).$$

In addition, for $|k| \geq \frac{|\alpha| + \sqrt{\alpha^2 + 2\beta|\delta|}}{\beta}$, we have $|\alpha||k|^2 + |\delta||k| \leq \frac{1}{2}\beta|k|^3$, and so, noting also that $\mathrm{Re}(\omega(k)) = 0$ along ∂D^+,

$$|i\omega(k)| \geq \frac{\beta}{2}|k|^3 \Rightarrow \frac{1}{1 + [i\omega(k)]^2} \leq \frac{1}{1 + \frac{\beta^2}{4}|k|^6} \simeq \frac{1}{1 + |k|^6}. \tag{84}$$

Observe further that $|\gamma_1(m)| \geq m$ thus $|\gamma_1(m)|$ can be made as large as we wish by taking $m \in [\lambda, \infty)$ large enough. Therefore, using (84), for large enough m, we have

$$\frac{1}{1 + \tau^2(m)} = \frac{1}{1 + [i\omega(\gamma_1(m)]^2} \lesssim \frac{1}{1 + |\gamma_1(m)|^6}.$$

On the other hand, for $|k| \geq 1$, we have $|k|^2 \geq |k|$, and so, by the triangle inequality,

$$|\omega'(k)| \leq 3\beta|k|^2 + 2|\alpha||k| + |\delta| \leq (3\beta + 2|\alpha| + |\delta|)(1 + |k|^2).$$

From the definition of γ_1, there exist nonnegative constants c_1, c_2 depending on α, β, δ such that

$$|\gamma_1(m)| \leq c_1 m \text{ and } |\gamma_1'(m)| \leq c_2, \quad m \in [\lambda, \infty).$$

Hence, there are some constants $c_3 > 0$, $M \geq \lambda$ depending on α, β, δ such that

$$\frac{|\gamma_1(m)|^{2j}\left|\omega'(\gamma_1(m))\gamma_1'(m)\right|}{[1 + \tau^2(m)]^{\frac{j+1}{3}}} \leq c_3 \frac{(|\gamma_1(m)|^2)^j (1 + |\gamma_1(m)|^2)}{(1 + |\gamma_1(m)|^2)^{j+1}} \leq c_3, \quad m > M.$$

However, by continuity of the function on the left-hand side on the compact interval $[\lambda, M]$, there is also some constant $c_4 > 0$ depending on α, β, δ such that

$$\frac{|\gamma_1(m)|^{2j}\left|\omega'(\gamma_1(m))\gamma_1'(m)\right|}{[1 + \tau^2(m)]^{\frac{j+1}{3}}} \leq c_4, \quad m \in [\lambda, M].$$

Combining the last two inequalities yields the desired estimate with $c = \max\{c_3, c_4\} < \infty$. ∎

2.3.4 | Strichartz-type estimates

It turns out convenient to reparameterize the contour of integration in the solution formula (65) of the reduced initial-boundary value problem (44) as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with

$$\Gamma_1(m) = m + i\sqrt{3\left(m - \frac{\alpha}{3\beta}\right)^2 - \frac{\alpha^2 + 3\beta\delta}{3\beta^2}}, \qquad -\infty < m \leq c_-,$$

$$\Gamma_2(m) = m + i\lambda, \qquad c_- < m < c_+, \qquad (85)$$

$$\Gamma_3(m) = m + i\sqrt{3\left(m - \frac{\alpha}{3\beta}\right)^2 - \frac{\alpha^2 + 3\beta\delta}{3\beta^2}}, \qquad c_+ \leq m < \infty,$$

where, as before, $c_\pm = \frac{\alpha \pm \sqrt{3\beta^2\lambda^2 + \alpha^2 + 3\beta\delta}}{3\beta}$ and $\lambda > 0$ satisfies (70). With this parameterization, formula (65) can be expressed as the sum

$$q(x, t) = -\frac{i}{2\pi} \sum_{j=1}^{3} \int_{\Gamma_j} e^{ikx - \omega(k)t} \, \widetilde{g}_0(\omega(k), T')\omega'(k)dk =: \sum_{j=1}^{3} q_j(x, t).$$

We first consider q_1, which after recalling also (76) takes the form

$$q_1(x,t) = -\frac{i}{2\pi} \int_{-\infty}^{c_-} e^{i\Gamma_1(m)x - \omega(\Gamma_1(m))t} \, \widehat{g}_0(i\omega(\Gamma_1(m))) \, \omega'(\Gamma_1(m))\Gamma_1'(m)dm$$

$$= \frac{1}{2\pi} \int_{-\infty}^{c_-} e^{i\Gamma_1(m)x - \omega(\Gamma_1(m))t} \left(\int_{-\infty}^{\infty} e^{-imy}\Psi_1(y)dy \right) dm, \tag{86}$$

where Ψ_1 is the inverse Fourier transform of

$$\widehat{\Psi}_1(m) := \begin{cases} -i\widehat{g}_0(i\omega(\Gamma_1(m))) \, \omega'(\Gamma_1(m))\Gamma_1'(m), & m \le c_-, \\ 0, & m > c_-. \end{cases}$$

Then, introducing the kernel

$$\mathcal{K}(y;x,t) = \int_{-\infty}^{c_-} e^{i\phi(m;x,y,t)}p(m;x)dm \tag{87}$$

with amplitude

$$p(m;x) = e^{-x\sqrt{3\left(m - \frac{\alpha}{3\beta}\right)^2 - \frac{\alpha^2 + 3\beta\delta}{3\beta^2}}} \tag{88}$$

and phase

$$\phi(m;x,y,t) = m(x - y) + i\omega(\Gamma_1(m))t$$

$$= m(x - y) + t\left[-8\beta m^3 + 8\alpha m^2 + 2\left(\delta - \frac{\alpha^2}{\beta} \right)m - \frac{\alpha\delta}{\beta} \right], \tag{89}$$

we can rearrange (86) in the form

$$q_1(x,t) = [K_1(t)\Psi_1](x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{K}(y;x,t)\Psi_1(y)dy. \tag{90}$$

This writing provides the starting point for proving the following central estimate of Strichartz type.

Theorem 8. *Let $s \ge 0$ and (μ, r) be higher-order Schrödinger admissible in the sense of (20). Then,*

$$\|q\|_{L_t^\mu((0,T');H_x^{s,r}(\mathbb{R}_+))} \lesssim \left(1 + (T')^{\frac{1}{\mu} + \frac{1}{2}} \right) \|g_0\|_{H_t^{\frac{s+1}{3}}(0,T')}, \tag{91}$$

where $H_x^{s,r}(\mathbb{R}_+)$ is the restriction on $\mathbb{R}_+$ of the Bessel potential space $H_x^{s,r}(\mathbb{R})$ defined by (19) and the inequality constant depends only r, s.

Proof. We will use a standard duality argument. Let $\eta \in C_c([0, T']; \mathcal{D}(\mathbb{R}_+))$ be an arbitrary function. Then,

$$2\pi \left| \int_0^{T'} \langle K_1(t)\Psi_1, \eta(\cdot, t) \rangle_{L_x^2(\mathbb{R}_+)} dt \right| = \left| \int_0^{T'} \int_0^\infty \left(\int_{-\infty}^\infty \mathcal{K}(y; x, t)\Psi_1(y)dy \right) \overline{\eta(x, t)} dx dt \right|$$

$$= \int_{-\infty}^\infty \Psi_1(y) \overline{\int_0^{T'} \int_0^\infty \overline{\mathcal{K}(y; x, t)} \, \eta(x, t) dx dt} \, dy \qquad (92)$$

$$\leq \|\Psi_1\|_{L^2(\mathbb{R})} \left\| \int_0^{T'} \int_0^\infty \overline{\mathcal{K}(y; x, t)} \, \eta(x, t) dx dt \right\|_{L_y^2(\mathbb{R})}.$$

Set $K_2(y) := \int_0^{T'} \int_0^\infty \overline{\mathcal{K}(y; x, t)} \eta(x, t) dx dt$. By the definition of the L^2-norm, we have

$$\|K_2\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^\infty \left(\int_0^{T'} \int_0^{T'} \int_0^\infty \int_0^\infty \overline{\mathcal{K}(y; x, t)} \eta(x, t) \mathcal{K}(y; x', t') \overline{\eta(x', t')} dx dx' dt dt' \right) dy$$

$$= \int_0^{T'} \int_0^{T'} \int_0^\infty \int_0^\infty \eta(x, t) \overline{\eta(x', t')} K_3(x, x'; t, t') dx dx' dt dt'$$

$$= \int_0^{T'} \int_0^\infty \eta(x, t) \left(\int_0^{T'} \int_0^\infty \overline{\eta(x', t')} K_3(x, x'; t, t') dx' dt' \right) dx dt,$$

where $K_3(x, x'; t, t') := \int_{-\infty}^\infty \overline{\mathcal{K}(y; x, t)} \mathcal{K}(y; x', t') dy$. Then, by Hölder's inequality in (x, t) and then Minkowski's integral inequality between x and t', we deduce

$$\|K_2\|_{L^2(\mathbb{R})}^2 \leq \|\eta\|_{L_t^{\mu'}((0,T'); L_x^{r'}(\mathbb{R}_+))} \left\| \int_0^{T'} \int_0^\infty \overline{\eta(x', t')} K_3(x, x'; t, t') dx' dt' \right\|_{L_t^\mu((0,T'); L_x^r(\mathbb{R}_+))}$$

$$\leq \|\eta\|_{L_t^{\mu'}((0,T'); L_x^{r'}(\mathbb{R}_+))} \left\| \int_0^{T'} \left\| \int_0^\infty \overline{\eta(x', t')} K_3(x, x'; t, t') dx' \right\|_{L_x^r(\mathbb{R}_+)} dt' \right\|_{L_t^\mu(0,T')}. \qquad (93)$$

We begin with the estimation of the interior $L_x^r(\mathbb{R}_+)$-norm. Using the definition (87) of $\mathcal{K}$, we rewrite K_3 in the form of an oscillatory integral:

$$K_3(x, x'; t, t') = \int_{-\infty}^\infty \left(\int_{-\infty}^{c_-} e^{-i\phi(m; x, y, t)} p(m; x) dm \right) \left(\int_{-\infty}^{c_-} e^{i\phi(m'; x', y, t')} p(m'; x') dm' \right) dy$$

$$= \int_{-\infty}^{c_-} p(m; x) \int_{-\infty}^\infty e^{-i\phi(m; x, y, t)} \left(\int_{-\infty}^{c_-} e^{i\phi(m'; x', y, t')} p(m'; x') dm' \right) dy dm.$$

Recalling the definition (89) of the phase function ϕ and introducing the function

$$Q(m'; x', t') := \begin{cases} e^{im'x' - \omega(\Gamma_1(m'))t'} p(m'; x'), & m' \in (-\infty, c_-], \\ 0, & m' \in (c_-, \infty), \end{cases}$$

we have via the Fourier inversion theorem

$$\int_{-\infty}^{\infty} e^{-i\phi(m;x,y,t)}\left(\int_{-\infty}^{c_-} e^{i\phi(m';x',y,t')} p(m';x')dm'\right)dy$$

$$= 2\pi e^{-imx+\omega(\Gamma_1(m))t} \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{imy}\left(\int_{-\infty}^{\infty} e^{-im'y}Q(m';x',t')dm'\right)dy$$

$$= 2\pi e^{-imx+\omega(\Gamma_1(m))t}\, Q(m;x',t').$$

Thus, for $m \in (-\infty, c_-]$, we deduce

$$\int_{-\infty}^{\infty} e^{-i\phi(m;x,y,t)}\left(\int_{-\infty}^{c_-} e^{i\phi(m';x',y,t')} p(m';x')dm'\right)dy = 2\pi e^{-i\phi(m;x,x',t-t')} p(m;x'),$$

and consequently,

$$K_3(x,x';t,t') = 2\pi \int_{-\infty}^{c_-} e^{-i\phi(m;x,x',t-t')} p(m;x+x')dm.$$

Next, we employ the following fundamental result.

Lemma 3. *Let* $K(x,y,z,t) = \int_{-\infty}^{c_-} e^{i\phi(m;x,y,t)} p(m;z)dm$, *where* $x, z \in \mathbb{R}_+$ *and* $y, t \in \mathbb{R}$. *Then,*

$$|K(x,y,z,t)| \lesssim |t|^{-\frac{1}{3}}, \quad t \neq 0, \tag{94}$$

where the constant of the inequality is independent of x, y, z, t.

The proof of Lemma 3 relies on the classical van der Corput lemma and is provided after the end of the current proof. Observe that Lemma 3 with y, z, t replaced, respectively, by $x', x + x', t - t'$ yields

$$|K_3(x,x',t,t')| \lesssim |t-t'|^{-\frac{1}{3}}, \quad t \neq t',$$

with inequality constant independent of x, x', t, and t'. This dispersive estimate implies

$$\left\|\int_0^\infty \overline{\eta(x',t')}K_3(x,x';t,t')dx'\right\|_{L_x^\infty(\mathbb{R}_+)} \lesssim |t-t'|^{-\frac{1}{3}}\|\eta(t')\|_{L_x^1(\mathbb{R}_+)}. \tag{95}$$

On the other hand, we also have

$$\left\|\int_0^\infty \overline{\eta(x',t')}K_3(x,x';t,t')dx'\right\|_{L_x^2(\mathbb{R}_+)} \lesssim \|\eta(t')\|_{L_x^2(\mathbb{R}_+)}. \tag{96}$$

Indeed, we have

$$\left\|\int_0^\infty \overline{\eta(x',t')}K_3(x,x';t,t')dx'\right\|_{L_x^2(\mathbb{R}_+)}^2 = \int_0^\infty \left|\int_0^\infty \overline{\eta(x',t')}K_3(x,x';t,t')dx'\right|^2 dx$$

$$= (2\pi)^2 \int_0^\infty \left| \int_0^\infty \overline{\eta(x',t')} \left(\int_{-\infty}^{c_-} e^{-i\phi(m;x,x',t-t')} p(m; x+x')dm \right) dx' \right|^2 dx$$

$$\leq (2\pi)^2 \int_0^\infty \left(\int_{-\infty}^{c_-} e^{-xs(m)} \left(\int_0^\infty e^{-x's(m)} |\eta(x',t')|dx' \right) dm \right)^2 dx,$$

where $s(m) = \sqrt{3(m - \frac{\alpha}{3\beta})^2 - \frac{\alpha^2+3\beta\delta}{3\beta^2}}$. The claimed estimate (96) then directly follows by invoking the following lemma, which provides a generalization of the $L^2(\mathbb{R}_+)$-boundedness of the Laplace transform given in Ref. 1 and is established after the end of the current proof.

Lemma 4. *The estimates in* (i) *and* (ii) *below hold true for* $f \in L_m^2(-\infty, c_-)$ *and* $f \in L_x^2(\mathbb{R}_+)$, *respectively.*

$$\text{(i)} \quad \left\| \int_{-\infty}^{c_-} e^{-xs(m)} f(m)dm \right\|_{L_x^2(\mathbb{R}_+)} \lesssim \|f\|_{L_m^2(-\infty, c_-)},$$

$$\text{(ii)} \quad \left\| \int_0^\infty e^{-xs(m)} f(x)dx \right\|_{L_m^2(-\infty, c_-)} \lesssim \|f\|_{L_x^2(\mathbb{R}_+)}.$$

Now, (95) and (96) together with Riesz–Thorin interpolation theorem yield for any $r \geq 2$ that

$$\left\| \int_0^\infty \overline{\eta(x',t')} K_3(x, x'; t, t')dx' \right\|_{L_x^r(\mathbb{R}_+)} \lesssim |t - t'|^{-\frac{2}{\mu}} \|\eta(t')\|_{L_x^{r'}(\mathbb{R}_+)}, \tag{97}$$

where $\frac{1}{r'} = 1 - \frac{1}{r}$ and we have also used (20). Hence, for any $\eta \in L_t^{\mu'}((0, T'); L_x^{r'}(\mathbb{R}_+))$, we obtain

$$\int_0^{T'} \left\| \int_0^\infty \overline{\eta(x',t')} K_3(x, x'; t, t')dx' \right\|_{L_x^r(\mathbb{R})} dt' \lesssim \int_0^{T'} |t - t'|^{-\frac{2}{\mu}} \|\eta(t')\|_{L_x^{r'}(\mathbb{R})} dt'.$$

Handling the right-hand side via Hardy–Littlewood–Sobolev fractional integration (e.g., see Theorem 1 on page 119 of Ref. 59) and combining the resulting inequality with (93), we infer

$$\|K_2\|_{L^2(\mathbb{R})} \lesssim \|\eta\|_{L_t^{\mu'}((0, T'); L_x^{r'}(\mathbb{R}_+))},$$

which can be combined with (92) to yield

$$\|q_1\|_{L_t^\mu((0, T'); L_x^r(\mathbb{R}_+))} \lesssim \|\Psi_1\|_{L^2(\mathbb{R})}. \tag{98}$$

Differentiating the expression (86) j times in x and repeating the above arguments, for any $j \in \mathbb{N}_0$, we conclude that

$$\|\partial_x^j q_1\|_{L_t^\mu((0, T'); L_x^r(\mathbb{R}_+))} \lesssim \|\partial_x^j \Psi_1\|_{L^2(\mathbb{R})} \lesssim \|g_0\|_{H_t^{\frac{j+1}{3}}(\mathbb{R})}. \tag{99}$$

Observe that the left-hand side of estimate (99) is simply the $L_t^\mu((0, T'); W^{j,r}(\mathbb{R}_+))$-norm of q_1. In this connection, note that, according to a classical result by Calderón [60], for any $j \in \mathbb{N}_0$, $1 < r < \infty$ the Sobolev space $W^{j,r}(\mathbb{R})$ and the Bessel potential space $H^{j,r}(\mathbb{R})$ coincide (i.e., they are equal as

sets). Indeed, it is not hard to show that $W^{j,r}(\mathbb{R}) \subseteq H^{j,r}(\mathbb{R})$. On the other hand, showing that $H^{j,r}(\mathbb{R}) \subseteq W^{j,r}(\mathbb{R})$ is more involved; see page 22 of §2.3 in Ref. 61, where the result is proved with the help of the Mikhlin–Hörmander theorem for L^p multipliers (Theorem 6.2.7 in Ref. 62). Thus, for any $j \in \mathbb{N}_0$, $1 < r < \infty$, we have $\| \cdot \|_{H^{j,r}(\mathbb{R})} \simeq \| \cdot \|_{W^{j,r}(\mathbb{R})}$, and so,

$$\|q_1\|_{W^{j,r}(\mathbb{R}_+)} := \inf_{\substack{\tilde{q}_1 \in W^{j,r}(\mathbb{R}) \\ \tilde{q}_1|_{\mathbb{R}_+} = q_1}} \|\tilde{q}_1\|_{W^{j,r}(\mathbb{R})} \simeq \inf_{\substack{\tilde{q}_1 \in H^{j,r}(\mathbb{R}) \\ \tilde{q}_1|_{\mathbb{R}_+} = q_1}} \|\tilde{q}_1\|_{H^{j,r}(\mathbb{R})} =: \|q_1\|_{H^{s,r}(\mathbb{R}_+)}. \tag{100}$$

Observing that the left-hand side of estimate (99) is simply the $W^{j,r}(\mathbb{R}_+)$-norm of q_1, in view of (100), we see that (99) is, in fact, equivalent to

$$\|q_1\|_{L_t^{\mu}((0,T');H_x^{j,r}(\mathbb{R}_+))} \lesssim \|g_0\|_{H_t^{\frac{j+1}{3}}(\mathbb{R})}. \tag{101}$$

Finally, by interpolation (e.g., see Theorem 5.1 in Ref. 58), we deduce

$$\|q_1\|_{L_t^{\mu}((0,T');H_x^{s,r}(\mathbb{R}_+))} \lesssim \|g_0\|_{H_t^{\frac{s+1}{3}}(\mathbb{R})}, \quad s \geq 0, \tag{102}$$

completing the estimation of q_1.

To estimate $\|q_2\|_{L_t^{\mu}((0,T');H_x^{s,r}(\mathbb{R}_+))}$, we use (78)–(80) (note the difference in notation, as q_2 in those expressions now corresponds to $\partial_x^j q_2$) as the portions γ_2 and Γ_2 of the two parameterizations (69) and (85) coincide. In particular,

$$\|\partial_x^j q_2(\cdot,t)\|_{L_x^r(\mathbb{R}_+)} = \left(\int_{-\infty}^{\infty} |Q_2(m,t)|^r dm \right)^{\frac{1}{r}}$$

$$\leq \left((T')^{\frac{r}{2}} \|g_0\|_{L_t^2(0,T')}^r \int_{c_-}^{c_+} m^{jr} |\omega'(m)|^r dm \right)^{\frac{1}{r}} = c_{j,r} \sqrt{T'} \|g_0\|_{L_t^2(0,T')}.$$

Therefore, for any $j \in \mathbb{N}_0$, we find

$$\|\partial_x^j q_2\|_{L_t^{\mu}((0,T');L_x^r(\mathbb{R}_+))} = c_{j,r}(T')^{\frac{1}{\mu}+\frac{1}{2}} \|g_0\|_{L_t^2(0,T')} \leq c_{j,r}(T')^{\frac{1}{\mu}+\frac{1}{2}} \|g_0\|_{H_t^{\frac{j+1}{3}}(0,T')},$$

and, using again the equivalence of the Bessel potential and Sobolev norms (100) along with interpolation, we conclude that

$$\|q_2\|_{L_t^{\mu}((0,T');H_x^{s,r}(\mathbb{R}_+))} \lesssim (T')^{\frac{1}{\mu}+\frac{1}{2}} \|g_0\|_{H_t^{\frac{s+1}{3}}(0,T')}, \quad s \geq 0. \tag{103}$$

As the estimation of q_3 is similar to that of q_1, the proof of Theorem 8 is complete. ∎

Proof of Lemma 3. By the Fundamental Theorem of Calculus, K can be rewritten as

$$K(x,y,z,t) = -\int_{-\infty}^{c_-} \frac{dI}{dm}(m;x,y,t)\, p(m;z) dm,$$

where $I(m; x, y, t) := \int_m^{c_-} e^{i\phi(\xi; x, y, t)} d\xi$. Integrating by parts using the fact that $I(c_-; x, y, t) = 0$ and $p(m; z) \to 0$ as $m \to -\infty$, and noting also that $\frac{dp}{dm}(m; z) > 0$, we get

$$|K(x, y, z, t)| \le \int_{-\infty}^{c_-} |I(m; x, y, t)| \frac{dp}{dm}(m; z) \, dm.$$

According to van der Corput's lemma (e.g., see page 370 in Ref. 63), if $\eta(\xi)$ is a real function such that $|\eta^{(j)}(\xi)| \ge \alpha$ on $[a, b]$, with the additional condition that $\eta'(\xi)$ is monotone if $j = 1$, then

$$\left| \int_a^b e^{i\eta(\xi)} d\xi \right| \le c_j \alpha^{-\frac{1}{j}}.$$

Noting that $|\phi^{(3)}(\xi; x, y, t)| = 48\beta|t|$, we can employ this classical result for I with $\eta(\xi) = \phi(\xi; x, y, t)$ to infer that $|I(m; x, y, t)| \lesssim |t|^{-\frac{1}{3}}$, $t \ne 0$, where the constant of inequality is independent of m, x, y, t. In turn, for any $t \ne 0$ and $z > 0$, we obtain

$$|K(x, y, z, t)| \lesssim |t|^{-\frac{1}{3}} \int_{-\infty}^{c_-} \frac{dp}{dm}(m; z) dm = |t|^{-\frac{1}{3}} e^{-\lambda z} \le |t|^{-\frac{1}{3}},$$

which is the desired estimate. ∎

Proof of Lemma 4. First, we prove part (i). By definition of $s(m)$, we have

$$\frac{ds(m)}{dm} = \frac{3(m - \frac{\alpha}{\beta})}{s(m)} = -\sqrt{3} \frac{\sqrt{s^2(m) + c_{\alpha,\beta,\delta}}}{s(m)}$$

with $c_{\alpha,\beta,\delta} = \frac{\alpha^2 + 3\beta\delta}{3\beta^2}$. Therefore, upon change of variable $s = s(m)$, we get

$$\int_{-\infty}^{c_-} e^{-xs(m)} f(m) dm = \frac{1}{\sqrt{3}} \int_\lambda^\infty e^{-xs} f(m(s)) \frac{s}{\sqrt{s^2 + c_{\alpha,\beta,\delta}}} ds = \int_0^\infty e^{-xs} f_\lambda(s) ds,$$

where

$$f_\lambda(s) := \begin{cases} \frac{1}{\sqrt{3}} f(m(s)) \frac{s}{\sqrt{s^2 + c_{\alpha,\beta,\delta}}} & s \in (\lambda, \infty), \\ 0, & s \notin (\lambda, \infty). \end{cases}$$

Using the L^2-boundedness of the Laplace transform (see Lemma 3.2 in Ref. 1), we get

$$\left\| \int_0^\infty e^{-xs} f_\lambda(s) ds \right\|_{L_x^2(\mathbb{R}_+)} \lesssim \|f_\lambda\|_{L_s^2(\mathbb{R}_+)}.$$

Finally, note that

$$\|f_\lambda\|_{L_s^2(\mathbb{R}_+)}^2 = \frac{1}{3} \int_\lambda^\infty |f(m(s))|^2 \frac{s^2}{s^2 + c_{\alpha,\beta,\delta}} ds$$

$$= \frac{1}{\sqrt{3}} \int_{-\infty}^{c_-} |f(m)|^2 \frac{s(m)}{\sqrt{s^2(m) + c_{\alpha,\beta,\delta}}} dm \lesssim \int_{-\infty}^{c_-} |f(m)|^2 dm.$$

Next, we establish part (ii). Setting $F(m) := \int_0^\infty e^{-xs(m)} f(x) dx$ and using the Cauchy–Schwarz inequality along the lines of the proof of Lemma 3.2 in Ref. 1, we have

$$|F(m)|^2 = \left| \int_0^\infty e^{-xs(m)} f(x) dx \right|^2 = \left| \int_0^\infty e^{-\frac{xs(m)}{2}} f(x) x^{\frac{1}{4}} \cdot x^{-\frac{1}{4}} e^{-\frac{xs(m)}{2}} dx \right|^2$$

$$\leq \left(\int_0^\infty e^{-xs(m)} |f(x)|^2 x^{\frac{1}{2}} dx \right) \left(\int_0^\infty e^{-xs(m)} x^{-\frac{1}{2}} dx \right).$$

Then, since the second integral on the right-hand side is equal to $\frac{1}{\sqrt{s(m)}} \int_0^\infty e^{-u} u^{-\frac{1}{2}} du = \frac{\sqrt{\pi}}{\sqrt{s(m)}}$,

$$\int_{-\infty}^{c_-} |F(m)|^2 dm = \sqrt{\pi} \int_{-\infty}^{c_-} \left(\int_0^\infty \frac{1}{\sqrt{s(m)}} e^{-xs(m)} |f(x)|^2 x^{\frac{1}{2}} dx \right) dm.$$

Finally, noting that

$$\int_{-\infty}^{c_-} \frac{1}{\sqrt{s(m)}} x^{\frac{1}{2}} e^{-xs(m)} dm = \frac{1}{\sqrt{3}} \int_\lambda^\infty e^{-xs} s^{-\frac{1}{2}} x^{\frac{1}{2}} \frac{s}{\sqrt{s^2 + c_{\alpha,\beta,\delta}}} ds \lesssim \int_\lambda^\infty e^{-xs} s^{-\frac{1}{2}} x^{\frac{1}{2}} ds \leq c\sqrt{\pi},$$

we arrive at the desired estimate

$$\int_{-\infty}^{c_-} |F(m)|^2 dm \lesssim \int_0^\infty |f(x)|^2 dx$$

completing the proof of the lemma. ∎

3 | NONLINEAR ANALYSIS

The various linear estimates established in Section 2 will now be combined with a contraction mapping argument in order to establish local well-posedness in the sense of Hadamard for the nonlinear initial-boundary value problem (1). In view of these linear results, the solution space will change as we transition from the setting of high regularity ($\frac{1}{2} < s \leq 2$) to the one of low regularity ($0 \leq s < \frac{1}{2}$). More specifically, in the former case, well-posedness will be established in the space $C([0, T]; H_x^s(\mathbb{R}_+))$ for a appropriate choice of $T > 0$ (see Theorem 1), while in the latter case, that space will be refined by intersecting it with the Strichartz-inspired space $L_t^\mu((0, T); H_x^{s,r}(\mathbb{R}_+))$ for an admissible choice of exponents (μ, r) in terms of the nonlinearity order p and the Sobolev exponent s according to (20) (see Theorem 2).

3.1 | Linear reunification

The nonlinear analysis will be performed by using a solution operator $u \mapsto \Phi u$ associated with the original forced linear initial-boundary value problem (5). To this end, thanks to the superposition principle, we reunify the solution representation formulae corresponding to (i) the homogeneous Cauchy problem (6), (ii) the nonhomogeneous Cauchy problem (23), and (iii) the reduced initial-boundary value problem (44). More precisely, given u, we formally define the map

$$
\begin{aligned}
\Phi u &:= y|_{Q_T} + z^u|_{Q_T} + q^u|_{(0,T)} \\
&\equiv S[E_0 u_0; 0]\big|_{Q_T} + S[0; f(Eu)]\big|_{Q_T} - \frac{i}{2\pi} \int_\Gamma e^{ikx-\omega(k)t} \omega'(k) \widetilde{g}_0^u(\omega(k), T') dk \bigg|_{(0,T)},
\end{aligned}
\tag{104}
$$

where $Q_T = \mathbb{R}_+ \times (0, T)$ for some $T > 0$ to be determined and

$$
g_0^u(t) := E_b \big\{ g(\cdot) - S[E_0 u_0; 0](0, \cdot) - S[0; f(Eu)](0, \cdot) \big\}(t)
\tag{105}
$$

with the temporal transform $\widetilde{g}_0^u(\omega(k), T')$ defined according to (47). The extension operators E_0 and E_b were defined below problems (6) and (44) respectively; importantly, E_0 satisfies inequality (7) and E_b induces compact support on g_0, namely, $\text{supp} g_0 \subset [0, T'], T' > T$. Moreover, the operator E is a similar bounded fixed extension operator. In particular, for $s > \frac{1}{2}$, we take $E = E_0$, while for $0 \le s < \frac{1}{2}$, we take E from $H_x^s(\mathbb{R}_+) \cap H_x^{s,r}(\mathbb{R}_+)$ into $H_x^s(\mathbb{R}) \cap H_x^{s,r}(\mathbb{R})$ for a certain $r > 2$ to be specified later.

In view of (104), we define the solutions of the nonlinear problem (1) as the fixed points of the operator Φ. Thus, our goal will be to prove the existence of a unique such fixed point in a suitable function space. Throughout our analysis, we assume $u_0 \in H_x^s(\mathbb{R}_+)$ and $g \in H_{t,\text{loc}}^{\frac{s+1}{3}}(\mathbb{R}_+)$ with $s \in [0, 2] \setminus \{\frac{1}{2}\}$ and the compatibility conditions (66) in place as necessary. We first treat the high regularity case $\frac{1}{2} < s \le 2$ in which we are able to employ the algebra property of $H_x^s(\mathbb{R}_+)$, and then move on to the low regularity case $0 \le s < \frac{1}{2}$ in which we address the lack of the algebra property by refining our solution space motivated by the linear Strichartz estimates.

3.2 | High regularity solutions: Proof of Theorem 1

In the high regularity setting, we suppose that $\frac{1}{2} < s \le 2$ and $p > 0$ with the additional assumptions (3) as necessary. Our goal is to establish local well-posedness in the space $X_T := C([0, T]; H_x^s(\mathbb{R}_+))$ for some $T > 0$ to be determined. We consider X_T as a metric space with the metric

$$
d_{X_T}(u_1, u_2) := \|u_1 - u_2\|_{X_T}, \quad u_1, u_2 \in X_T.
$$

Note that any closed ball in X_T is a complete subspace.

Showing that Φ is into. The conservation law (8) in Theorem 3 and the boundedness (7) of the spatial extension operator E_0 imply

$$
\|y|_{Q_T}\|_{X_T} \le \|S[E_0 u_0; 0]\|_{C([0,T];H_x^s(\mathbb{R}))} = \|E_0 u_0\|_{H_x^s(\mathbb{R})} \lesssim \|u_0\|_{H_x^s(\mathbb{R}_+)},
\tag{106}
$$

which takes care of the first term in (104). For the second term in (104), let $u \in X_T$ and combine the nonhomogeneous estimate (25) in Theorem 5 with the algebra property in $H_x^s(\mathbb{R})$ to yield

$$
\|z^u|_{Q_T}\|_{X_T} \leq \|S[0; f(E_0 u)]\|_{C([0,T];H_x^s(\mathbb{R}))} \lesssim \int_0^T \|f(E_0 u(\cdot, t))\|_{H_x^s(\mathbb{R})} dt
$$

$$
\lesssim \int_0^T \|E_0 u(\cdot, t)\|_{H_x^s(\mathbb{R})}^{p+1} dt \lesssim \int_0^T \|u(\cdot, t)\|_{H_x^s(\mathbb{R}_+)}^{p+1} dt \lesssim T \|u\|_{X_T}^{p+1}. \tag{107}
$$

Regarding the third term in (104), using estimate (68) in Theorem 7 and the boundedness of the temporal extension operator E_b (see Section 3 of Ref. 40 for more details), we get (say with $T' = 2T$)

$$
\|q^u|_{(0,T)}\|_{X_T} \leq \|q\|_{X_{T'}} \lesssim \left(1 + \sqrt{T'} e^{cT'}\right) \|g_0^u\|_{H_t^{\frac{s+1}{3}}(0,T')}
$$

$$
\lesssim \left(1 + \sqrt{T'} e^{cT'}\right) \|g_0^u\|_{H_t^{\frac{s+1}{3}}(0,T)} \lesssim \left(1 + \sqrt{T} e^{cT}\right) \|g_0^u\|_{H_t^{\frac{s+1}{3}}(0,T)}. \tag{108}
$$

By using the definition of g_0 in (44) and temporal trace estimates (9), (10), and (26), we obtain

$$
\|g_0^u\|_{H_t^{\frac{s+1}{3}}(0,T)} \lesssim \|g\|_{H_t^{\frac{s+1}{3}}(0,T)} + (1 + T^{\frac{1}{2}}) \|u_0\|_{H_x^s(\mathbb{R}_+)}
$$

$$
+ \max\{T^{\frac{1}{2}}(1 + T^{\frac{1}{2}}), T^\sigma\} \|f(E_0 u)\|_{L_t^2((0,T);H_x^s(\mathbb{R}))}, \tag{109}
$$

with σ given by (27). By using the definition of the solution space X_T and the boundedness (7) of the spatial extension operator E_0, we have

$$
\|f(E_0 u)\|_{L_t^2((0,T);H_x^s(\mathbb{R}))} \lesssim T^{\frac{1}{2}} \|u\|_{X_T}^{p+1}. \tag{110}
$$

Using the definition (104) of Φ and combining estimates (106)–(110), we deduce

$$
\|\Phi(u)\|_{X_T} \leq c_0 \left(c_1(T) \|u_0\|_{H_x^s(\mathbb{R}_+)} + c_2(T) \|g\|_{H_t^{\frac{s+1}{3}}(0,T)} + c_3(T) \|u\|_{X_T}^{p+1} \right), \tag{111}
$$

where the positive constants c_1, c_2, c_3 are given by $c_1(T) = (1 + \sqrt{T} e^{cT})(1 + T^{\frac{1}{2}})$, $c_2(T) = (1 + \sqrt{T} e^{cT})$, $c_3(T) = T + (1 + \sqrt{T} e^{cT}) T^{\frac{1}{2}} \max\{T^{\frac{1}{2}}(1 + T^{\frac{1}{2}}), T^\sigma\}$ and c_0 is a nonnegative constant independent of T and only depending on fixed parameters such as α, β, δ, and s.

In view of estimate (111), we set $R(T) := 2A(T)$ with

$$
A(T) := c_0 \left(c_1(T) \|u_0\|_{H_x^s(\mathbb{R}_+)} + c_2(T) \|g\|_{H_t^{\frac{s+1}{3}}(0,T)} \right),
$$

and choose T small enough so that $A(T) + c_0 c_3(T) R(T)^{p+1} \leq R(T)$ or, equivalently, $c_0 c_3(T) R^p(T) \leq \frac{1}{2}$. We note that such a choice is possible because $c_3(T) \to 0^+$ and $R(T)$ remains bounded as $T \to 0^+$. Then, for that choice of T, the map Φ takes the closed ball $\overline{B_{R(T)}(0)} \subset X_T$ into itself. It remains to show that Φ is a contraction on $\overline{B_{R(T)}(0)}$.

Showing that Φ is a contraction. Let $u_1, u_2 \in \overline{B_{R(T)}(0)}$. Then,

$$
\begin{aligned}
\|\Phi(u_1) - \Phi(u_2)\|_{X_T} = {} & \left\| z^{u_1}|_{Q_T} - z^{u_2}|_{Q_T} \right\|_{X_T} + \left\| q^{u_1}|_{(0,T)} - q^{u_2}|_{(0,T)} \right\|_{X_T} \\
\lesssim {} & \left\| S[0; f(E_0 u_1) - f(E_0 u_2)] \right\|_{C([0,T];H_x^s(\mathbb{R}))} \\
& + \left(1 + \sqrt{T} e^{cT} \right) \left\| g_0^{u_1} - g_0^{u_2} \right\|_{H_t^{\frac{s+1}{3}}(0,T)}.
\end{aligned}
\tag{112}
$$

We then recall the following difference estimate (e.g., see Ref. 50).

Lemma 5. *Let $s > \frac{1}{2}$, $p > 0$ satisfy (3) and $\varphi, \varphi_1, \varphi_2 \in H^s(\mathbb{R})$. Then,*

$$
\left\| |\varphi_1|^p \varphi_1 - |\varphi_2|^p \varphi_2 \right\|_{H^s(\mathbb{R})} \lesssim \left(\|\varphi_1\|_{H^s(\mathbb{R})}^p + \|\varphi_2\|_{H^s(\mathbb{R})}^p \right) \|\varphi_1 - \varphi_2\|_{H^s(\mathbb{R})}.
$$

Employing Lemma 5 and the arguments used earlier in (107), we deduce

$$
\left\| S[0; f(E_0 u_1) - f(E_0 u_2)] \right\|_{C([0,T];H_x^s(\mathbb{R}))} \lesssim T \left(\|u_1\|_{X_T}^p + \|u_2\|_{X_T}^p \right) \|u_1 - u_2\|_{X_T}.
\tag{113}
$$

Moreover, for the difference of boundary data, we have, similarly to (109),

$$
\begin{aligned}
\left\| g_0^{u_1} - g_0^{u_2} \right\|_{H_t^{\frac{s+1}{3}}(0,T)} \lesssim {} & \max\{T^{\frac{1}{2}}(1 + T^{\frac{1}{2}}), T^\sigma\} \| f(E_0 u_1) - f(E_0 u_2) \|_{L_t^2((0,T);H_x^s(\mathbb{R}))} \\
\lesssim {} & \max\{T^{\frac{1}{2}}(1 + T^{\frac{1}{2}}), T^\sigma\} T^{\frac{1}{2}} \left(\|u_1\|_{X_T}^p + \|u_2\|_{X_T}^p \right) \|u_1 - u_2\|_{X_T},
\end{aligned}
\tag{114}
$$

where σ is given by (27). Combining (113) and (114) with (112), we obtain

$$
\|\Phi(u_1) - \Phi(u_2)\|_{X_T} \lesssim c_3(T) \left(\|u_1\|_{X_T}^p + \|u_2\|_{X_T}^p \right) \|u_1 - u_2\|_{X_T} \lesssim c_3(T) R^p(T) \|u_1 - u_2\|_{X_T}.
\tag{115}
$$

Note that $c_3(T) \to 0^+$ and $R(T)$ remains bounded as $T \to 0^+$. Therefore, for sufficiently small $T > 0$, the map Φ is a contraction on $\overline{B_{R(T)}(0)}$, and hence, Φ has a unique fixed point in $\overline{B_{R(T)}(0)}$ that, as noted earlier, amounts to local existence of a unique solution to the HNLS initial-boundary value problem (1) on $\overline{B_{R(T)}(0)}$.

Extending uniqueness to X_T. To prove uniqueness over the entire space X_T and not just the closed ball $\overline{B_{R(T)}(0)}$, we suppose that $u_1, u_2 \in X_T$ are two solutions associated with the same pair of initial and boundary data (u_0, g). At first, we consider the case of u_1, u_2 being sufficiently smooth and, along with their derivatives, decaying sufficiently fast as $x \to \infty$. This allows us to proceed via energy estimates. In particular, we note that the difference $w := u_1 - u_2$ solves the following problem:

$$
\begin{aligned}
& i w_t + i \beta w_{xxx} + \alpha w_{xx} + i \delta w_x = f(u_1) - f(u_2), \quad (x,t) \in \mathbb{R}_+ \times (0,T), \\
& w(x,0) = 0, \quad x \in \mathbb{R}_+, \\
& w(0,t) = 0, \quad t \in (0,T).
\end{aligned}
\tag{116}
$$

Multiplying the main equation by $\overline{w}$, integrating in x, taking imaginary parts, and using Lemma 5 and the embedding $H_x^s(\mathbb{R}_+) \hookrightarrow L_x^\infty(\mathbb{R}_+)$, which is valid for $s > \frac{1}{2}$, we find

$$
\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L_x^2(\mathbb{R}_+)}^2 = -\frac{\beta}{2}|w_x(0,t)|^2 + \text{Im}\int_0^\infty [f(u_1(x,t)) - f(u_2(x,t))]\overline{w}(x,t)dx
$$

$$
\lesssim \int_0^\infty (|u_1(x,t)|^p + |u_2(x,t)|^p)|w(x,t)|^2 dx
$$

$$
\lesssim \left(\|u_1(t)\|_{H_x^s(\mathbb{R}_+)}^p + \|u_2(t)\|_{H_x^s(\mathbb{R}_+)}^p\right)\|w(t)\|_{L_x^2(\mathbb{R}_+)}^2
$$

$$
\lesssim \left(\|u_1\|_{X_T}^p + \|u_2(t)\|_{X_T}^p\right)\|w(t)\|_{L_x^2(\mathbb{R}_+)}^2.
$$

Setting $y(t) := \|w(t)\|_{L_x^2(\mathbb{R}_+)}$, the above energy estimate is satisfied, provided that $y'(t) - cy(t) \leq 0$, $t \in (0, T)$ for some nonnegative constant c. Solving this differential inequality alongside the condition $y(0) = \|w(0)\|_{L_x^2(\mathbb{R}_+)} = 0$ (note that $w(x, 0) \equiv 0$), we obtain $y \equiv 0$, that is, $w = u_1 - u_2 \equiv 0$. The case of rough u_1, u_2 can be treated via mollification along the lines of the arguments used in the proof of Proposition 1.4 in Ref. 21.

Continuous dependence on the data. For $(u_0, g) \in H_x^s(\mathbb{R}_+) \times H_{t,\text{loc}}^{\frac{s+1}{3}}(\mathbb{R}_+)$, let

$$
T_{\max} := \sup\{T > 0 \,|\, \text{there is a solution associated to the data } (u_0, g) \text{ on } [0, T]\}.
$$

Then, either $T_{\max} = \infty$ or else $T_{\max} < \infty$ and there is no solution $u \in X_{T_{\max}}$ since otherwise the lifespan of u could be extended beyond $T_{\max}$ by starting with initial datum equal to $u(T_{\max})$. Therefore, we may let $u \in C([0, T_{\max}); H_x^s(\mathbb{R}_+))$ be the maximal solution associated with the data (u_0, g); then, for $T < T_{\max}$, in particular, $u|_{[0,T]}$ is the unique solution in X_T established above.

Let $T < T_{\max}$ be small enough that Φ is a contraction on $\overline{B_{R(T)}(0)}$ for any solution associated with data $(v_0, h) \in H_x^s(\mathbb{R}_+) \times H_{t,\text{loc}}^{\frac{s+1}{3}}(\mathbb{R}_+)$ and satisfying

$$
\|v_0\|_{H_x^s(\mathbb{R}_+)} + \|h\|_{H_t^{\frac{s+1}{3}}(0,T)} \leq 2\left(\|u_0\|_{H_x^s(\mathbb{R}_+)} + \|g\|_{H_t^{\frac{s+1}{3}}(0,T)}\right).
$$

If follows that if $\delta > 0$ is small enough, for (v_0, h) satisfying

$$
\|v_0 - u_0\|_{H_x^s(\mathbb{R}_+)} + \|g - h\|_{H_t^{\frac{s+1}{3}}(0,T)} < \delta,
$$

the associated solution v belongs to $\overline{B_{R(T)}(0)}$. Therefore, u and v are both fixed points of Φ on $\overline{B_{R(T)}(0)}$ associated with the pairs of data (u_0, g) and (v_0, h), respectively. Then, the corresponding nonlinear estimates from the contraction argument imply

$$
\|u - v\|_{X_T} = \|\Phi u - \Phi v\|_{X_T} \lesssim c(T)\left(\|u_0 - v_0\|_{H_x^s(\mathbb{R}_+)} + \|g - h\|_{H_t^{\frac{s+1}{3}}(0,T)}\right) \lesssim \delta c(T),
$$

which amounts to continuity of the data-to-solution map. The proof of Theorem 1 for well-posedness in the high regularity setting is complete.

3.3 | Low regularity solutions: Proof of theorem 2

In this setting, we work under the assumptions (4). The lack of the algebra property brings in the need for the various Strichartz estimates established in Section 2 and hence motivates the solution space

$$Y_T := C([0,T]; H_x^s(\mathbb{R}_+)) \cap L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}_+)).$$

It is convenient to also consider the associated space on the whole spatial line, namely,

$$\widetilde{Y}_T := C([0,T]; H_x^s(\mathbb{R})) \cap L_t^\mu((0,T); H_x^{s,r}(\mathbb{R})).$$

The following lemma will serve as the low regularity analog of the algebra property and Lemma 5.

Lemma 6. *Let (s,p), (μ,r) satisfy (4) and suppose $\varphi, \varphi_1, \varphi_2 \in L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))$. Then,*

$$\left\| |\varphi|^p \varphi \right\|_{L_t^1((0,T); H_x^s(\mathbb{R}))} \lesssim T^{\frac{\mu-p-1}{\mu}} \|\varphi\|_{L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))}^{p+1}, \tag{117}$$

$$\left\| |\varphi_1|^p \varphi_1 - |\varphi_2|^p \varphi_2 \right\|_{L_t^1((0,T); H_x^s(\mathbb{R}))} \lesssim T^{\frac{\mu-p-1}{\mu}} \left(\|\varphi_1\|_{L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))}^p + \|\varphi_2\|_{L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))}^p \right)$$

$$\cdot \|\varphi_1 - \varphi_2\|_{L_t^\mu((0,T); H_x^{s,r}(\mathbb{R}))}. \tag{118}$$

Lemma 6 is proved after the end of the current proof and corresponds to the one-dimensional analog of inequality (6.17) for the two-dimensional NLS equation proved in Ref. 24. Note, importantly, that the admissibility conditions (4) are different than those in Ref. 24 due to the third-order dispersion of the HNLS equation. Thus, the proof of Lemma 6 does not follow from Ref. 24. Now, we are ready to prove Theorem 2 for low regularity solutions.

Existence. First, we consider the *subcritical case* $p \neq \frac{6}{1-2s}$ so that $\frac{\mu-p-1}{\mu} > 0$. We work again with the solution operator (104), which was obtained via linear reunification. Theorems 3 and 4 imply

$$\|y|_{Q_T}\|_{Y_T} \leq \|y\|_{\widetilde{Y}_T} \lesssim \|E_0 u_0\|_{H_x^s(\mathbb{R})} \lesssim \|u_0\|_{H_x^s(\mathbb{R}_+)}, \tag{119}$$

while Theorems 5 and 6 along with inequality (117) and the same argument that was used in (107) yield

$$\|z^u|_{Q_T}\|_{Y_T} \leq \|z^u\|_{\widetilde{Y}_T} \lesssim \left(T + T^{\frac{\mu-p-1}{\mu}} \right) \|u\|_{Y_T}^{p+1}. \tag{120}$$

Furthermore, Theorem 8 (with say $T' = 2T$) and the same arguments that led to (108) imply

$$\|q^u|_{(0,T)}\|_{Y_T} \lesssim \left(1 + \sqrt{T}e^{cT} + T^{\frac{1}{\mu}+\frac{1}{2}} \right) \|g_0^u\|_{H_t^{\frac{s+1}{3}}(0,T)}. \tag{121}$$

Combining (119)–(121) and proceeding along the lines of the arguments that resulted in (109) and (110), we obtain

$$\|\Phi(u)\|_{Y_T} \leq c_0 \left(c_1(T)\|u_0\|_{H_x^s(\mathbb{R}_+)} + c_2(T)\|g\|_{H_t^{\frac{s+1}{3}}(0,T)} + c_3(T)\|u\|_{Y_T}^{p+1} \right),$$

where the positive constants c_1, c_2, c_3 are given by $c_1(T) = (1 + \sqrt{T}e^{cT} + T^{\frac{1}{\mu}+\frac{1}{2}})(1 + T^{\frac{1}{2}})$, $c_2(T) = (1 + \sqrt{T}e^{cT} + T^{\frac{1}{\mu}+\frac{1}{2}})$, $c_3(T) = (T + T^{\frac{\mu-p-1}{\mu}}) + (1 + \sqrt{T}e^{cT} + T^{\frac{1}{\mu}+\frac{1}{2}})T^{\frac{1}{2}}\max\{T^{\frac{1}{2}}(1 + T^{\frac{1}{2}}), T^{\sigma}\}$ and c_0 is a nonnegative constant independent of T and only depending on fixed parameters such as α, β, δ, and s.

For the contraction, given $u_1, u_2 \in Y_T$, we employ inequality 118 together with the same arguments that led to (115) to infer

$$\|\Phi(u_1) - \Phi(u_2)\|_{Y_T} \lesssim c_3(T)\left(\|u_1\|_{Y_T}^p + \|u_2\|_{Y_T}^p\right)\|u_1 - u_2\|_{Y_T}. \tag{122}$$

This estimate implies the existence of a fixed point in Y_T for sufficiently small $T > 0$ via the same arguments that were used in the proof of Theorem 1.

Next, we consider the *critical case* $p = \frac{6}{1-2s}$. The difference here compared to the subcritical case is that the limit $c_3(T) \to 0^+$ as $T \to 0^+$ is no longer true; however, Φ is still a contraction, provided that the data (and, correspondingly, the radius of the closed ball that depends on the size of the data) are chosen sufficiently small.

Uniqueness. We adapt the method used for the Cauchy problem in the proof of Proposition 4.2 of Ref. 64 to the framework of initial-boundary value problems.

First, consider the subcritical case $p \neq \frac{6}{1-2s}$. Let $u_1 = \Phi(u_1), u_2 = \Phi(u_2) \in Y_T$ be two solutions associated with the same pair of initial and boundary data. Suppose to the contrary that there is $t \in [0, T]$ for which $u_1(t) \neq u_2(t)$, and let

$$t_{\inf} := \inf\{t \in [0, T] \mid u_1(t) \neq u_2(t)\}.$$

Taking $t_n < t_{\inf}$ such that $t_n \to t_{\inf}^-$ as $n \to \infty$, we see that $u_1(t_n) = u_2(t_n)$ by definition of $t_{\inf}$. Thus, in view of the fact that u_1, u_2 are both continuous from $[0, T]$ into $H_x^s(\mathbb{R}_+)$, taking the limit $n \to \infty$, we deduce that $u_1(t_{\inf}) = u_2(t_{\inf}) =: \varphi \in H_x^s(\mathbb{R}_+)$ makes sense. Set $U_1(t) = u_1(t + t_{\inf})$ and $U_2(t) = u_2(t + t_{\inf})$. Then, U_1 and U_2 are both solutions on the temporal interval $[0, T - t_{\inf}]$ that satisfy the same initial and boundary conditions, namely,

$$U_1(0) = U_2(0) = \varphi, \quad U_1|_{x=0} = U_2|_{x=0} = g(\cdot + t_{\inf}) =: g_{\inf}.$$

Since U_1 and U_2 are continuous in t, by the definition of $t_{\inf}$, there is a $\delta > 0$ such that $U_1 \neq U_2$ for $t \in (0, \delta)$. Let $t = t_{\inf} + \varepsilon$ with $\varepsilon \in (0, \delta)$ fixed and to be specified below. We have

$$\|U_1 - U_2\|_{L_t^\mu((0,\varepsilon);H_x^{s,r}(\mathbb{R}_+))} \lesssim c_{\inf}(\varepsilon)\left(\|U_1\|_{L_t^\mu((0,\varepsilon);H_x^{s,r}(\mathbb{R}_+))}^p + \|U_2\|_{L_t^\mu((0,\varepsilon);H_x^{s,r}(\mathbb{R}_+))}^p\right)$$
$$\cdot \|U_1 - U_2\|_{L_t^\mu((0,\varepsilon);H_x^{s,r}(\mathbb{R}_+))}, \tag{123}$$

where $c_{\inf}(\varepsilon) := \varepsilon^{\frac{\mu-p-1}{\mu}} + \varepsilon^{\frac{1}{\mu}+\frac{1}{2}}\varepsilon^{\frac{1}{2}}\max\{\varepsilon^{\frac{1}{2}}(1 + \varepsilon^{\frac{1}{2}}), \varepsilon^{\sigma}\}$. Let $\varepsilon \in (0, \delta)$ be small enough so that

$$c_{\inf}(\varepsilon)\left(\|U_1\|_{L_t^\mu((0,\varepsilon);H_x^{s,r}(\mathbb{R}_+))}^p + \|U_2\|_{L_t^\mu((0,\varepsilon);H_x^{s,r}(\mathbb{R}_+))}^p\right) < 1, \tag{124}$$

which is possible because $c_{\inf}(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$. Then, (123) implies that $U_1 = U_2$ on $(0, \varepsilon) \subset (0, \delta)$, leading to a contradiction. Hence, uniqueness follows.

In the critical case $p = \frac{6}{1-2s}$, although the limit $c_{\inf}(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$ is no longer true, the uniqueness argument remains valid as (124) still holds due to the fact that, due to the dominated convergence theorem, the norms $\|U_1\|_{L_t^\mu((0,\varepsilon);H_x^{s,r}(\mathbb{R}_+))}$ and $\|U_2\|_{L_t^\mu((0,\varepsilon);H_x^{s,r}(\mathbb{R}_+))}$ can be made arbitrarily small by taking ε small enough.

Finally, the continuous dependence of the unique solution in Y_T on the initial and boundary data can be proved as in the high regularity setting, thereby completing the proof of Theorem 2.

Proof of Lemma 6. By Hölder's inequality,

$$\||\varphi|^p\varphi\|_{L_t^1((0,T);H^s(\mathbb{R}))} \leq T^{\frac{\mu-p-1}{\mu}}\left(\int_0^T \||\varphi(t)|^p\varphi(t)\|_{H_x^s(\mathbb{R})}^{\frac{\mu}{p+1}}dt\right)^{\frac{p+1}{\mu}}.$$

On the other hand, $\|\varphi\|_{L_t^\mu(0,T_0;H_x^{s,r}(\mathbb{R}))}^{p+1} = (\int_0^T \|\varphi(t)\|_{H_x^{s,r}(\mathbb{R})}^\mu dt)^{\frac{p+1}{\mu}}$. Hence, in order to establish (117), it suffices to prove that

$$\|D^\theta(|\varphi(t)|^p\varphi(t))\|_{L_x^2(\mathbb{R})} \lesssim \|\varphi(t)\|_{H_x^{\theta,r}(\mathbb{R})}^{p+1}, \quad t \in (0,T), \tag{125}$$

for $\theta = 0$ and $\theta = s$. To this end, we set $F(z) := |z|^p z, z \in \mathbb{C}$. If $s \neq 0$, by using the chain rule for fractional derivatives (e.g., see Proposition 3.1 in [65]), we have

$$\|D^s F(\varphi(t))\|_{L_x^2(\mathbb{R})} \lesssim \|F'(\varphi(t))\|_{L_x^{\frac{\mu}{3}}(\mathbb{R})} \|D^s\varphi(t)\|_{L_x^r(\mathbb{R})} \tag{126}$$

with $\frac{1}{2} = \frac{3}{\mu} + \frac{1}{r}$. Noting that $|F'(\varphi(t))| \leq (p+1)|\varphi(t)|^p$, we further find

$$\|F'(\varphi(t))\|_{L_x^{\frac{\mu}{3}}(\mathbb{R})} \lesssim \|\varphi(t)\|_{L_x^{\frac{p\mu}{3}}(\mathbb{R})}^p, \tag{127}$$

while for $\frac{3}{p\mu} = \frac{1}{r} - s$, we also have the embedding

$$\|\varphi(t)\|_{L_x^{\frac{p\mu}{3}}(\mathbb{R})} \lesssim \|\varphi(t)\|_{H_x^{s,r}(\mathbb{R})}. \tag{128}$$

Combining (127) and (128) with (126), we obtain (125) for $\theta = s \neq 0$. Notice that $r = 2(p+1)$ for $s = 0$. Therefore,

$$\||\varphi(t)|^p\varphi(t)\|_{L_x^2(\mathbb{R})} = \|\varphi(t)\|_{L_x^{2(p+1)}(\mathbb{R})}^{p+1} = \|\varphi(t)\|_{L_x^r(\mathbb{R})}^{p+1},$$

which corresponds to (125) for $\theta = 0$.

Regarding inequality (118) for the differences, we first consider the case $s = 0$, which implies $r = 2(p+1)$. Using the standard pointwise difference estimate for the power-type nonlinearity and then applying Hölder's inequality in x, we get

$$\||\varphi_1|^p\varphi_1 - |\varphi_2|^p\varphi_2\|_{L_t^1((0,T);L_x^2(\mathbb{R}))} \lesssim \int_0^T \left(\int_{-\infty}^\infty (|\varphi_1(x,t)|^p + |\varphi_2(x,t)|^p)^2 |\varphi_1(x,t) - \varphi_2(x,t)|^2 dx\right)^{\frac{1}{2}} dt$$

$$\lesssim \int_0^T \left(\|\varphi_1(t)\|_{L_x^r(\mathbb{R})}^p + \|\varphi_2(t)\|_{L_x^r(\mathbb{R})}^p \right) \|\varphi_1(t) - \varphi_2(t)\|_{L_x^r(\mathbb{R})} dt,$$

and the desired estimate (118) for $s = 0$ follows via Hölder's inequality in t.

Next, let us consider the case $s \neq 0$, in which $r = \frac{2(p+1)}{1+2sp}$. First, observe that for $z_1, z_2 \in \mathbb{C}$ and $\xi(\rho) = (1 - \rho)z_2 + \rho z_1$, $\rho \in [0, 1]$, we have $\xi(0) = z_1$, $\xi(1) = z_2$, $\xi'(\rho) = z_1 - z_2$. Moreover,

$$|z_2|^p z_2 - |z_1|^p z_1 = \int_0^1 \frac{d}{d\rho}(|\xi(\rho)|^p \xi(\rho)) d\rho$$

$$= \frac{(p+2)}{2}(z_1 - z_2) \int_0^1 |\xi(\rho)|^p d\rho + \frac{p}{2}(\bar{z}_1 - \bar{z}_2) \int_0^1 |\xi(\rho)|^{p-2} \xi^2(\rho) d\rho.$$

Combining this writing with the fractional product rule (see Proposition 3.3 in Ref. 65), we find

$$\|D^s F(\varphi_1(t)) - D^s F(\varphi_2(t))\|_{L_x^2(\mathbb{R})} \lesssim \|D^s(\varphi_1(t) - \varphi_2(t))\|_{L_x^r(\mathbb{R})} \sup_{\rho \in [0,1]} \||w(t)|^p\|_{L_x^{\frac{2r}{r-2}}(\mathbb{R})}$$

$$+ \|\varphi_1(t) - \varphi_2(t)\|_{L_x^{\frac{p\mu}{3}}(\mathbb{R})} \left(\sup_{\rho \in [0,1]} \left\{ \|D^s(G(w(t)))\|_{L_x^{r_1}(\mathbb{R})} \right\} \right),$$

where $\frac{1}{r_1} = \frac{1}{2} - \frac{3}{p\mu}$, $w(t) = (1 - \rho)\varphi_2(t) + \rho \varphi_1(t)$, and $G(z) = F'(z) = \frac{p+2}{2}|z|^p + \frac{p}{2}|z|^{p-2}z^2$, $z \in \mathbb{C}$.

Observing that $|G'(w(t))| \leq p(p + 1)|w(t)|^{p-1}$ for $p > 1$, we use the fractional chain rule (Proposition 3.1 in Ref. 65) to infer that, for $p > 1$,

$$\|D^s(G(w(t)))\|_{L_x^{r_1}(\mathbb{R})} \lesssim \||w(t)|^{p-1}\|_{L_x^{r_2}(\mathbb{R})} \|D^s w(t)\|_{L_x^r(\mathbb{R})}$$

$$\lesssim \|w(t)\|_{L_x^{\frac{\mu p}{3}}(\mathbb{R})}^{p-1} \|D^s w(t)\|_{L_x^r(\mathbb{R})} \lesssim \|w(t)\|_{H_x^{s,r}(\mathbb{R})}^p,$$

where $\frac{1}{r_2} = \frac{1}{r_1} - \frac{1}{r}$. In the above, the second inequality is due to the fact that, in view of (4), $r_2 = \frac{2(p+1)}{(p-1)(1-2s)} = \frac{\mu p}{3(p-1)}$, and the third inequality follows from the embedding (128). Furthermore, notice that $\frac{2r}{r-2} = \frac{\mu}{3}$ and so, using once again, the embedding (128),

$$\||w(t)|^p\|_{L_x^{\frac{2r}{r-2}}(\mathbb{R})} = \|w(t)\|_{L_x^{\frac{\mu p}{3}}(\mathbb{R})}^p \lesssim \|w(t)\|_{H_x^{s,r}(\mathbb{R})}^p.$$

Combining the last three estimates, we deduce

$$\|D^s F(\varphi_1(t)) - D^s F(\varphi_2(t))\|_{L_x^2(\mathbb{R})} \lesssim \left(\|\varphi_1(t)\|_{H_x^{s,r}(\mathbb{R})}^p + \|\varphi_2(t)\|_{H_x^{s,r}(\mathbb{R})}^p \right) \|\varphi_1(t) - \varphi_2(t)\|_{H_x^{s,r}(\mathbb{R})}.$$

Then, integrating over $(0, T)$, applying Hölder's inequality in t, and combining the resulting estimate with the case of $s = 0$, we obtain (118) for $s \neq 0$ and $p > 1$.

Finally, for $p = 1$, we note that $\frac{1}{2} = \frac{1+2s}{4} + \frac{1-2s}{4} = \frac{1}{r} + \frac{1-2s}{4} = \frac{1}{r} + \frac{3}{\mu}$. Therefore,

$$\|D^s F(\varphi_1(t)) - D^s F(\varphi_2(t))\|_{L_x^2(\mathbb{R})} \lesssim \|D^s(\varphi_1(t) - \varphi_2(t))\|_{L_x^r(\mathbb{R})} \sup_{\rho \in [0,1]} \|w(t)\|_{L_x^{\frac{\mu}{3}}(\mathbb{R})}$$

$$+ \|\varphi_1(t) - \varphi_2(t)\|_{L_x^{\frac{\mu}{3}}(\mathbb{R})} \left(\sup_{\rho \in [0,1]} \left\{ \|D^s(G(w(t)))\|_{L_x^r(\mathbb{R})} \right\} \right)$$

$$\lesssim \left(\|\varphi_1(t)\|_{H_x^{s,r}(\mathbb{R})}^p + \|\varphi_2(t)\|_{H_x^{s,r}(\mathbb{R})}^p \right) \|\varphi_1(t) - \varphi_2(t)\|_{H_x^{s,r}(\mathbb{R})}$$

with the last step thanks to the embedding (128). ∎

ACKNOWLEDGMENTS

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current study.

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