STABILITY ANALYSIS OF NONLINEAR DYNAMICAL SYSTEMS WITH LÉVY TYPE PERTURBATIONS

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Ege TAMCI

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İZMİR

We approve the thesis of **Ege TAMCI**

Examining Committee Members:

Prof. Dr. Şahlar MEHERREM
Department of Mathematics, Yaşar Univeristy

Assoc. Prof. Gürsoy TURAN
Department of Civil Engineering, Izmir Institute of Technology

Assoc. Prof. Özkan KARABACAK
Department of Mechatronic Engineering, Kadir Has University

Asst. Prof. Ahmet BATAL
Department of Mathematics, Izmir Institute of Technology

Asst. Prof. Faruk TEMUR
Department of Mathematics, Izmir Institute of Technology

02 June 2023

Asst. Prof. Ahmet BATAL
Supervisor, Department of Mathematics
Izmir Institute of Technology

Prof. Dr. Ferit Acar SAVACI
Co-Supervisor, Department of Electrical
Electronics
Izmir Institute of Technology

Prof. Dr. Şirin ATILGAN
Head of the Department of
Mathematics

Prof. Dr. Mehtap EANES
Dean of the Graduate School of
Engineering and Sciences

To my family...

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ABSTRACT

STABILITY ANALYSIS OF NONLINEAR DYNAMICAL SYSTEMS WITH LÉVY TYPE PERTURBATIONS

In order to model the noise in power networks, generally, normal distribution is used. However, normal distribution is not convenient in modelling noise which has sudden peaks. Instead, combination of a continuous process and a jump processes is much more suitable. With this idea in mind, in this thesis, the stability and control of two equations used in modeling power grids is analyzed, under the assumption that they are exposed to Lévy process noise which includes jumps. These equations are the swing equation and the Kuramoto Model. The swing equation is used to model the single machine infinite bus system (SMIBS). Kuramoto Model is used when a large number of generators are considered as a network of coupled oscillators with their own natural frequencies.

In our stability control study in the SMIBS, the noise in the system has sudden and finite changes is assumed and therefore should be modelled with a modified tempered α-stable process obtained by adding a finite jump condition on the tempered α-stable process when $\alpha < 1$. The control functions depending on the mechanical power input and damping parameters are designed in order to make the system stable in probability and exponential stable at its equilibrium point. These theoretical results are supported by numerical studies.

For Kuromato model, assuming that the power network consists of two layers, namely oscillator, and control layers and that is affected with a general Lévy process which has finite jumps, functions which provide the stability of phase and frequencies are obtained, depending on oscillator and coupling strengths. In the light of the numerical studies, the control of frequency and phase synchronization up to a certain noise intensity level can be evaluated, but it is not possible beyond that level is concluded.

ÖZET

LÉVY TİP PERTÜRBASYONLU DOĞRUSAL OLMAYAN DİNAMİK SİSTEMLERİN KARARLILIK ANALİZİ

Güç sistemlerindeki gürültünün modellenmesinde sıklıkla normal dağılım kullanılır. Halbuki normal dağılım, ani sıçramaları da içeren gürültüyü modellemek için uygun değildir. Sistemdeki gürültünün sürekli bir süreç yanında, sıçramalı süreçler ile modellenmesi çok daha uygundur. Bu tezde, güç sistemlerini modelleyen temel iki denklemin sonlu sıçramaları içeren Lévy süreçlerle etkilendiği zamandaki kararlılık analizi ve kontrolünü çalışıldı. Denklemlerden ilki tek makine sonsuz bara güç sistemini (TMSBS) modelleyen salıncak denklemi, diğeri ise çok sayıdaki jenarötörün, kritik bir bağlantı değeri ile birbirine bağlı doğal frekanslara sahip osilatörler olarak kabul edildiği Kuramoto Modelidir.

TMSBS'in kararlılık kontrolü için yapılan çalışmada sistemde yer alan gürültünün ani ve sonlu değişikliğe sahip olduğu, bu nedenle $\alpha < 1$ için temperlenmiş α-stabil süreci üzerinde sonlu sıçrama şartı eklenerek elde ettiğimiz modifiye temperlenmiş α-stabil süreçi ile modellenmesi gerektiği kabul edildi. Sistemin denge noktasının bu tip bir gürültü altında temel kararlıklık çeşitlerinden olasılıksal kararlılık ve olasılıksal üstel kararlığa sahip olması için mekanik güç ve sönüm parametrelerine bağlı olarak kontrol fonksiyonları oluşturuldu. Elde edilen teorik sonuçları nümerik çalışmalar ile desteklendi. Kuramoto modeli için güç sisteminin osilatör ve kontrol katmanlarından oluşan çift katlı bir yapıda olduğunu ve bu yapının sonlu sıçramalara sahip genel bir Lévy süreci ile etkilendiği varsayımı altında bu ağın faz açılarının ve sıklığının kararlılığını sağlayan, osilatör ve kontrol katmanlarındaki bağlantı kuvvetine bağlı fonksiyonlar elde edildi. Nümerik çalışmalar ışığında Lévy sürecinin belirli bir gürültü şiddeti seviyesine kadar frekans ve faz senkronizasyonunun kontrolünün yapıldığı, sonrasında ise bunun mümkün olmadığı sonucuna varıldı.

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CHAPTER 1

INTRODUCTION

1.1. Overview

In this thesis, the stochastic stability analysis and control of two equations have been investigated fundamental to the modeling of power grids, under the influence of Lévy process noise. This represents an application of the modelling of nonlinear dynamical systems subject to this type of perturbation. The thesis consists of the following chapters:

In this introduction, firstly, the mathematical theory needed for an understanding of the relevant subjects is provided in a preliminary section, then introduce the three main problems, together with their historical background.

In the second chapter, the analysis of the stochastic swing equation is presented, which models the single machine infinite bus system (SMIBS) perturbed with a modified tempered stable process, and provide a derivation of a control function for this equation.

The aim of the third chapter is to develop a control function which ensures the stochastic stability of power grid systems, as modeled by the Kuramoto model under Lévy-type perturbations, in asymptotic terms.

Finally, fourth chapter presents the conclusions of the thesis.

1.2. Preliminaries

This section provides the necessary definitions and theorems related to Lévy-type stochastic differential equations and stochastic stability theory. More precisely, the summary of stochastic processes, the characteristic function of a random variable, infinite divisibility and Lévy processes is given, mainly based on [1, 10]. The Lévy-Ito decomposition formula are described given by Theorem 1.6 which is extremely important and useful since it gives a unique representation to Lévy processes as the sum of continuous and jump processes. Moreover, Poisson integration, α-stable random variables and processes, Lévy type stochastic differential equations, infinitesimal generators, and finally

Ito calculus are summarized.

We start with introducing the notation used in this chapter as well as the rest of the thesis.

1.2.1. Notations

The set of $d \times d$ real-valued matrices is denoted by $\mathcal{M}_{d\times d}(\mathbb{R})$. $\mathbb{R}^d$ stands for the d-dimensional Euclidean space. The x-centered open ball with radius r in $\mathbb{R}^d$ is denoted by $B_r(x) := \{y \in \mathbb{R}^d, \quad |x - y| < r\}$ and $\widehat{B} := B_1(0)$, i.e., the unit disk. The space of n-times differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$ with continuous derivatives is denoted by $C^n(\mathbb{R}^d)$. The functions $f^-(x)$ and $f^+(x)$ are defined passively for the function $f : \mathbb{R}^d \to \mathbb{R}$, with $f^-(x)$ as the maximum of $-f(x)$ and 0, and $f^+(x)$ as the maximum of $f(x)$ and 0. The σ-algebra, is a collection of subsets of a given set S that satisfies following three key properties and denoted by $\sigma(S)$:

- The σ-algebra includes the entire set S.
- The σ-algebra's closure under complementation ensures that if a subset A is in the σ-algebra, then its complement $S \setminus A$ is also in the σ-algebra.
- The σ-algebra is closed under countable unions. This means that if we have a countable collection of subsets $A_1, A_2, A_3, \ldots$ belonging to the σ-algebra, then their union $\bigcup_{i=1}^{\infty} A_i$ is also an element of the σ-algebra.

The Borel σ-algebra on a topological space S is defined as the smallest σ-algebra that contains all open sets in S and it is presented by $\mathcal{B}(S)$.

$\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers. The Euclidean norm is used to refer to the norm denoted by $|.|$. The inner product in d-dimensional Euclidean space is denoted by (x, y) for two d-dimensional vectors $x = [x_1\ x_2 \cdots x_d]^T, y = [y_1\ y_2 \cdots y_d]^T$ and defined by the following calculation:

$$(x, y) = \sum_{i=1}^{d} x_i \cdot y_i, \quad i \in [1, 2, \cdots, d].$$

$1_A(x)$ represents the indicator function of a set A on a given set S, such that

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

In the indicator function, x belongs to set S, and A is a subset of S.

1.2.2. Stochastic Processes

A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a mathematical construction that provides a model of a random process, called an experiment. A probability space consists of the following three elementsApplebaum (2009):

Ω: a sample space, that is, the set of all possible outcomes ω of a random experiment. The sample space of a random experiment includes all possible outcomes

$\mathcal{F}$: an event space which is a σ-algebra generated with Ω. The pair $(\Omega, \mathcal{F})$ is called a measurable space, and a set in $\mathcal{F}$ is called a measurable set. The elements in $\mathcal{F}$ are referred to as events.

$\mathbb{P}$: a probability measure, a measure that assigns each event in the event space $\mathcal{F}$ a probability, which is a number between 0 and 1 with $\mathbb{P}(\Omega) = 1$.

- A collection of sub-algebras $(\mathcal{F}_t,\ t \geq 0)$ of $\mathcal{F}$ is called a filtration if the following condition is satisfied:
$$\mathcal{F}_s \subset \mathcal{F}_t,\ \ 0 \leq s \leq t.$$

- A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a filtered probability space if it is equipped with $\mathcal{F}_t$.

- A mapping $f : S_1 \rightarrow S_2$ between measurable spaces $(S_i, \mathcal{F}_i), i = 1, 2$, is called $(\mathcal{F}_1, \mathcal{F}_2)$-measurable if $f^{-1}(A) \in \mathcal{F}_1$ for all $A \in \mathcal{F}_2$. A $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable mapping between the probability space $(\Omega, \mathcal{F})$ and the Borel measure space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is called a random variable.

- A collection of random variables $X(t) : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)$, for each $t \geq 0$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a stochastic process. A stochastic process $X(t)$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called adapted if $X(t)$ is measurable on $\mathcal{F}_t$ for each $t \geq 0$.

The distribution or law of random variable X is a Borel probability measure p_X on $\mathbb{R}^d$, and is defined as follows:

$$p_X = \mathbb{P} \circ X^{-1}.$$

The Borel probability measure p_X is an important tool characterizing stochastic processes. It is also used in integration. The expectation operator, $\mathbb{E}$, is the integral of a random variable X with respect to the probability measure $\mathbb{P}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, defined as follows:

$$\mathbb{E}(X) = \int_{\Omega} X(\omega)\mathbb{P}(d\omega) = \int_{\mathbb{R}^d} x p_X(dx). \tag{1.1}$$

The expectation $\mathbb{E}(X)$ defines the mean (m) of the random variable X if it is finite. $\mathbb{E}(X^n)$ is called the nth moment of the random variable X and is generally represented by α_n. Another way to calculate the integral is via the density function f_X which is the Rodon-Nikodym derivative of the probability measure p_X with respect to the Lebesgue measure dx on $\mathbb{R}^d$, as defined by the following theorem:

Theorem 1.1 *(Radon Nikodym)Applebaum (2009) If a σ-finite measure μ and a finite measure ν are defined on arbitrary measure space $(S, \mathcal{F})$, and they satisfy the following condition*

$$\mu(A) = 0 \Rightarrow \nu(A) = 0, \; for \; all \; A \in \mathcal{F} \tag{1.2}$$

then there is a measurable function g, unique in the almost everywhere sense, that satisfies the following integral representation:

$$\nu(A) = \int_A g(x)\mu(dx) \tag{1.3}$$

Here, the measurable function g is called the Radon–Nikodym derivative of ν with respect to μ and denoted as $g = d\nu/d\mu$. Thus, to compute the moment in (1.1), the Radon-Nikodym derivative of the probability measure p_X with respect to the Lebesgue measure dx can be used as follows:

$$\alpha_n = \mathbb{E}(X^n) = \int_{\mathbb{R}^d} x^n f_X(x)dx.$$

$\mathbb{E}((X-m)^n)$ is called the nth central moment and denoted by μ_n. In particular, the variance is the 2nd central moment of a random variable, and is denoted by σ^2. The standard deviation is the square root of the variance of a random variable X. Moreover, when two random variables $X_1,\ X_2$ are independent, their joint probability $\mathbb{P}(X_1 = A, X_2 = B) = \mathbb{P}(X_1 = A)\mathbb{P}(X_2 = B)$, for all $A, B \in \mathbb{R}^d$.

The following Lemma 1.1 will be used in the definition of infinitely divisible processes, to be given in the following section.

Lemma 1.1 *Applebaum (2009) Let $X_1, \ldots, X_n$ be a sequence of independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Then the following equality is satisfied:*

$$\mathbb{E}(X_1 X_2 \cdots X_n) = \mathbb{E}(X_1)\mathbb{E}(X_2)\cdots\mathbb{E}(X_n), \quad i \in [1, 2, \cdots, n].$$

1.2.3. Conditional Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and the conditional expectation of a random variable X given an event $A \in \mathcal{F}$, be defined as Grigoriu (2002):

$$\mathbb{E}[X|A] = \int X(\omega)\mathbb{P}_A(d\omega) \tag{1.4}$$

If $\mathbb{P}(A) > 0$ then the conditional probability measure is defined as follows:

$$\mathbb{P}_A(\cdot) = \frac{\mathbb{P}(A \cap (\cdot))}{\mathbb{P}(A)} \tag{1.5}$$

The conditional expectation of a random variable can be considered as the average value of X over the set A. Furthermore, the conditional expectation of a random variable on a σ-algebra is a random variable defined such that; Let $\{A_n\}$ be a countable collection of measurable sets A_n that partition Ω such that $\sum_n A_n = \Omega,\ A_n \cap A_n^{'} = 0$ for $n \neq n^{'}$, and $A_n \in \mathcal{F}$. Let $\mathcal{G}$ be the sub σ-field of $\mathcal{F}$, generated by $\{A_n\}$. The conditional expectation of a random variable X on $\mathcal{G}$, denoted as $\mathbb{E}[X|\mathcal{G}] : \Omega \to \mathbb{R}$, is evaluated when $\mathbb{P}(A_n) > 0$ as follows:

$$\mathbb{E}[X|\mathcal{G}] = \sum_n \mathbb{E}[X|A_n]1_{A_n}(\omega). \tag{1.6}$$

Processes can also be defined using their characteristic functions. In the preliminary section, one important transformation is made where the linearly time-varying state structure for the process (bt, At, tv) is achieved. One of the issues that must be explained in the preliminary section is how the Lévy-Ito decomposition can be obtained, demonstrating that processes can be decomposed into continuous and discontinuous processes using their characteristic functions.

1.2.4. Characteristic Function

Here briefly describe the characteristic function and the related Kac's Theorem are described.

Definition 1.1 *Applebaum (2009); Soong (1973) Let X be a random variable defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *and its range takes the values in* $\mathbb{R}^d$*. The Fourier transform of the probability law/probability measure* p_X *is called the characteristic function of X, and is defined by*

$$\phi_X(u) = E(e^{i(u,X)}) = \int_\Omega e^{i(u,X(\omega)}\mathbb{P}(d\omega) = \int_{\mathbb{R}^d} e^{i(u,y))}p_X(dy),$$

for each $u \in \mathbb{R}^d$*.*

If $X = (X_1, \cdots, X_d)$ and $\mathbb{E}(|X_j^n|)$ is finite, the nth moments of components of the random variable can be generated via its characteristic function as follows:

$$\mathbb{E}(X_j^n) = (-i)^n \frac{\partial^n}{\partial u_j^n}\phi_X(u)|_{u=0}, \quad 1 \le j \le d.$$

Theorem 1.2 (Kac's Theorem) *Applebaum (2009) The random variables* $X_1, \ldots, X_n$ *are independent if and only if the following equality is satisfied for all* $u_1, \ldots, u_n \in \mathbb{R}^d$*:*

$$\mathbb{E}\left(\exp\left[i\sum_{j=1}^n \left(u_j, X_j\right)\right]\right) = \phi_{X_1}(u_1)\cdots\phi_{X_n}(u_n).$$

If the X_i are independent then the e^{iX_i} are independent for each $i \in \mathbb{N}$, so Kac's theorem is a natural consequence of Lemma 1.1.

1.2.5. Infinite Divisibility

Infinite divisibility means having a nth root, in the convolutional sense of a probability measure for each $n \in \mathbb{N}$. Here some related definitions and tools are given.

Definition 1.2 *[Convolution of Measures]Applebaum (2009) The convolution of two probability measures $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}^d)$ is defined by the following integral:*

$$(\mu_1 * \mu_2)(A) = \int\int_{\mathbb{R}^d \times \mathbb{R}^d} 1_A(x+y)\mu_1(dy)\mu_2(dx). \tag{1.7}$$

Proposition 1.1 *Applebaum (2009)By Fubini's Theorem, this definition 1.2 is equivalent to the following equality:*

$$(\mu_1 * \mu_2)(A) = \int_{\mathbb{R}^d} \mu_1(A-x)\mu_2(dx) = \int_{\mathbb{R}^d} \mu_2(A-x)\mu_1(dx).$$

Proof By fixing x in the first integral, the following is obtained:

$$\int_{\mathbb{R}^d} 1_A(x+y)\mu_1(dy) = \int_{\mathbb{R}^d} 1_{A-x}(y)\mu_1(dy) = \mu(A-x). \tag{1.8}$$

□

Proposition 1.2 *Applebaum (2009) Let $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{R}^d)$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$. Then the following equality is satisfied:*

$$\int_{\mathbb{R}^d} f(y)(\mu_1 * \mu_2)(dy) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x+y)\mu_1(dy)\mu_2(dy).$$

Corollary 1.1 *For two random variables X_1 and X_2 defined on $(\Omega, \mathcal{F}, \mathbb{P})$, the following equality is satisfied:*

$$\mathbb{E}(f(X_1+X_2)) = \int_{\mathbb{R}^d} f(y)(\mu_1 * \mu_2)(dy),$$

for each $f \in \mathcal{B}_b(\mathbb{R}^d)$.

Definition 1.3 *Applebaum (2009) A probability measure is called a convolution of the nth root of* $\mu \in \mathcal{M}$ *if* μ *is represented as follows:*

$$\mu = \mu^{1/n} * \mu^{1/n} \cdots * \mu^{1/n},$$

where $\mu^{1/n} \in \mathcal{M}$ *is the nth root in the convolution sense.*

Definition 1.4 (Infinite divisibility) *Applebaum (2009) A random variable X is called infinitely divisible if its distribution has the following representation for any given* $n \in \mathbb{N}$*, with i.i.d. random variables* $Y_1, ..., Y_n$*:*

$$X \stackrel{d}{=} Y_1 + ... + Y_n, \ , \ n \geq 2. \tag{1.9}$$

With the help of this definition, it can be seen that the characteristic function of the random variable X provides the following equation:

$$E(e^{i(u,X)}) = E(e^{i(u,Y_1+...+Y_n)}) = E(e^{i(u,Y_1)})^n.$$

Since each $E(e^{i(u,Y_1)})$ defines a characteristic function, the characteristic function of X is called infinitely divisible. According to the following proposition, if a probability measure is infinitesimal divisible, then any nth root of its characteristic function is also a characteristic function.

Proposition 1.3 *Applebaum (2009) The probability law* $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ *is infinitely divisible if and only if, for each* $n \in \mathbb{N}$ *there exists* $\mu^{1/n} \in \mathcal{M}\left(\mathbb{R}^d\right)$ *for which*

$$\phi_\mu(u) = \left[\phi_{\mu^{1/n}}(u)\right]^n, \ \ \phi_\mu(u) = \int_{\mathbb{R}^d} e^{i(u,x)}\mu(dx),$$

for each $u \in \mathbb{R}^d$.

In the light of the information given above, the following proposition is reached.

Proposition 1.4 *Applebaum (2009) For a given random variable X with probability measure* μ_X*, the following expressions are equivalent:*

1) X is an infinitely divisible random variable, such that $X \stackrel{d}{=} Y_1 + ... + Y_n$, $n \geq 2$, *i.i.d. random variables* $Y_1, ..., Y_n$.

2) The probability measure of X is infinitely divisible, such that $\mu = \mu^{1/n} * \mu^{1/n} \cdots * \mu^{1/n}$, *where* $\mu^{1/n} \in \mathcal{M}$ *is the convolution of the nth root.*

3) The characteristic function of X is infinitely divisible, such that $E(e^{i(u,Y_1)}) = E(e^{i(u,X)})^{1/n}$, *where* $E(e^{i(u,X)})^{1/n}$ *is a characteristic function.*

Proof

- (1) $\rightarrow$ (2): If $X \stackrel{d}{=} Y_1 + ... + Y_n,\ t \geq 0,\ n \geq 2$ then the probability measure of $X(t)$ can be evaluated as $\mu_X = \mu_{Y_1} * \mu_{Y_2^n} * \cdots * \mu_{Y_n}$. Therefore, the desired nth root $\mu^{1/n}$ can be taken as the common probability measure μ_{Y_i} of Y_i.

- (2) $\rightarrow$ (3): If the $f(x) = \exp(iux)$ for each $u \in \mathbb{R}^d$ in Proposition1.2 are taken, this can be evaluated as

$$\phi_X = \int \cdots \int e^{i(u,y_1+...+y_n)} \mu_X^{1/n}(dy_1) \cdots \mu_X^{1/n}(dy_n) = \Psi_Y(u)^n,$$

  where $\Psi_Y(u) = \int_{\mathbb{R}^d} e^i u, y(\mu_X)^{1/n}(dy)$. This implies that the nth root of ϕ_X defines a characteristic function, $\Psi_Y(u)$.

- (3) $\rightarrow$ (1): If ϕ_X is infinitely divisible, then it can be represented as follows:

$$E(e^{i(u,X)}) = E(e^{i(u,Y_1)})^n.$$

  For independent copies of $Y_1,\ Y_2, \cdots Y_n$, by Theorem 1.2 the following equality holds:

$$E(e^{i(u,Y_1)})^n = E(e^{i(u,Y_1+...+Y_n)}).$$

  Equivalently

$$E(e^{i(u,X)}) = E(e^{i(u,Y_1+...+Y_n)}). \tag{1.10}$$

  Using (1.10), the definition of infinite divisibility of a random variable are obtained, as follows:

$$X \stackrel{d}{=} Y_1 + ... + Y_n, \quad n \geq 2.$$

□

An n-dimensional vector which consists of random variables is called an n-dimensional random vector. The following are some examples of infinitely divisible random variables important for our thesis:

Definition 1.5 (Gaussian Random Vector) *Grigoriu (2002) A random vector $X \in \mathbb{R}^d$ is called a Gaussian / normal random vector if its probability density function $f(x)$ can be written in the following general form:*

$$f_X(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det(A)}} \exp\left[-\frac{1}{2}\left(x - m, A^{-1}(x - m)\right)\right],$$

where the mean $m \in \mathbb{R}^d$ is an n-dimensional vector, and the covariance matrix $A = \mathbb{E}\left((X - m)(X - m)^{\mathrm{T}}\right)$ is strictly positive definite symmetric $d \times d$ for all $x \in \mathbb{R}^d$.

The characteristic function of a Gaussian random vector can be represented as

$$\phi_X(u) = \exp\left[i(m, u) - \frac{1}{2}(u, Au)\right], \quad u \in \mathbb{R}^d. \tag{1.11}$$

The random vector X is infinitely divisible with $Y_j \sim N(m/n, (1/n)A)$ for each $1 \leq j \leq n$, as can be seen from the equation above. Here, $N(m, A)$ represents a normal distribution of X, that is, a distribution whose density function is f_X.

The nth root of (1.11),

$$[\phi_X(u)]^{1/n} = \exp\left[i\left(\frac{m}{n}, u\right) - \frac{1}{2}\left(u, \frac{1}{n}Au\right)\right],$$

has a similar distribution. That is, taking the nth root of a characteristic function also gives a characteristic function Y_j. If $m = \vec{0}$ and $A = I$, then X is called a standard normal distribution, and $X \stackrel{d}{=} N\left(0, \sigma^2 I\right)$ are shown for some $\sigma > 0$. Another definition of standard normal distribution is constructed via the central limit theorem.

Theorem 1.3 (Central Limit Theorem) *Grigoriu (2002) Let X_is be a i.i.d sequence with mean m and variance σ^2, the sum of the X_is define the random variable X, i.e. $X = \sum_{i=1}^n X_i$. If the normalized random variable Y is defined as*

$$Y = (X - nm)/\sigma n^{1/2},$$

then the distribution of the random variable Y converges to the standard normal distribution as $n \to \infty$.

Definition 1.6 (Poisson Random Variable) *Applebaum (2009) A nonnegative integer valued random variable X is called a Poisson random variable if there exists $\lambda > 0$ such that*

$$P(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}.$$

The distribution of the Poisson random variable X is denoted by $\pi(\lambda)$, with the parameter λ representing the average or expected frequency of occurrences. i.e. $X \sim \pi(\lambda)$. Also $\mathbb{E}(X) = \mathrm{Var}(X) = \lambda$, and the characteristic function of the Poisson random variable is given as follows:

$$\phi_X(u) = \exp\left[\lambda\left(e^{iu} - 1\right)\right]. \tag{1.12}$$

As can be seen from the equation above, this is infinitely divisible, with $Y_j \sim \pi(\lambda/n)$, where $1 \leq j \leq n, n \in \mathbb{N}$.

Definition 1.7 (Compound Poisson Random Variable) *Applebaum (2009) X is called a compound Poisson random variable if it can be represented as*

$$X = \sum_{n=0}^{N} D_n,$$

where $\{D_n\}_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables taking values in $\mathbb{R}^d$, with the probability measure μ_{D_n}, and $N \sim \pi(\lambda)$ is a Poisson random variable that is independent of the D_n. Distribution of the compound process X is presented as $X \sim \pi(\lambda, \mu_D)$.

Proposition 1.5 *Applebaum (2009) The characteristic function of the compound Poisson random variable X given in 1.12 for each $u \in \mathbb{R}^d$ is represented as follows:*

$$\phi_X(u) = \exp\left[\int_{\mathbb{R}^d} \left(e^{i(u,y)} - 1\right) \lambda\mu_D(dy)\right].$$

Proof Let ϕ_X be the characteristic function of X. D_n and N are independent random variables, so

$$
\begin{aligned}
\phi_X(u) &= \sum_{n=0}^{\infty} \mathbb{E}(\exp[i(u, D(1) + \cdots + D(N))] \mid N = n)P(N = n) \\
&= \sum_{n=0}^{\infty} \mathbb{E}(\exp[i(u, D(1) + \cdots + D(n))])e^{-\lambda}\frac{\lambda^n}{n!} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{[\lambda \phi_D(u)]^n}{n!} \\
&= \exp\left[\lambda\left(\phi_D(u) - 1\right)\right]
\end{aligned}
$$

where $\phi_D(u) = \int_{\mathbb{R}^d} e^{i(u,y)} \mu_D(dy)$. □

This proposition implies that the characteristic function of the compound Poisson random variable is infinitely divisible. This can be shown by considering the random variables as $Y_j \overset{d}{=} \pi(\lambda/n, \mu_D)$, where $1 \leq j \leq n$ and $n \in \mathbb{N}$. Consequently, we can represent the compound process as $X = Y_1, Y_2, ..., Y_n$.
Lévy-Khintchine theorem is essential to this thesis. It gives information about the conditions for a probability measure to be infinitely divisible.

Theorem 1.4 (Lévy-Khintchine formula) *Applebaum (2009) If a measure* $\mu \in \mathcal{M}(\mathbb{R}^d)$ *has a Fourier transform with the following integral representation*

$$
\phi_\mu(u) = \exp\left\{ i(b, u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d - \{0\}} (e^{i(u,y)} - 1 - i(u, y)1_{\hat{\mathcal{B}}}(y))\nu(dy) \right\}, \tag{1.13}
$$

then it is infinitely divisible.

Formula (1.13) is called the Lévy-Khintchine formula. In this, $b \in \mathbb{R}^d$ is a constant vector, A is a positive definite symmetric matrix and ν is a measure satisfying the following condition:

$$
\int_{\mathbb{R}^d - \{0\}} (1 \wedge ||y||^2)\nu(dy) < \infty. \tag{1.14}
$$

A measure ν satisfying condition (1.14) is called the Lévy measure. Also, (b, A, ν) is called a Lévy triplet.

We will now give some important infinitely divisible processes. In the following sections, a combination of these processes gives the general form of the Lévy process, with the help of the Lévy-Khintchine theorem, will be shown.

Definition 1.8 (Brownian Motion) *An adapted process is called a standard Brownian motion, denoted $B(t)$, if it has the following properties:*

1) $B(t) \stackrel{d}{=} N(0, tI)$ for each $t \geq 0$,

2) $B_t - B_s$ is independent of $\mathcal{F}_s$, with stationary increments $B(t) - B(s) \stackrel{d}{=} B(t - s)$, for $0 \leq s < t < \infty$,

3) $B(t)$ has almost surely everywhere continuous sample paths or continuous trajectories.

Here $N(0, tI)$ is a normal distribution. A Gaussian processes $C(t)$, with distribution $N(tb, tA)$ in $\mathbb{R}^d$, can be constructed from Brownian motion in $\mathbb{R}^d$ as follows:

$$C(t) = bt + \sigma B(t),$$

where A is the $d \times d$ positive definite symmetric matrix, $\sigma \times \sigma^T = A$, and σ and b are d-dimensional vectors.

Definition 1.9 (Martingale) *Applebaum (2009) An adapted, right continuous with left limits (cadlag) process $X(t)$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a martingale if it satisfies the following conditions:*

1) $\mathbb{E}(|X(t)|) < \infty$, $0 \leq t < \infty$,

2) $\mathbb{E}(X(t)|\mathcal{F}_s) = X(s), \quad a.s.\ 0 \leq s \leq t.$

Definition 1.10 (Poisson Process) *Applebaum (2009) A counting process $\{N(t), t \in [0, \infty)\}$ is called a Poisson process with rate λ, if the following conditions hold:*

1) $N(0) = 0$,

2) $N(t)$ *has independent increments,*

3) the count in any interval of length $t > 0$ has the following distribution:

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad \mathbb{E}(N(1)) = \lambda.$$

The compensated Poisson process $\tilde{N}(t)$ is a martingale obtained over the Poisson process, and is defined as follows:

$$\tilde{N}(t) = N(t) - \lambda t.$$

Here λ can be considered as the probability of making a jump at the moment, as can be seen from the following equation:

$$\lim_{\Delta t \to 0} \frac{P(N(\Delta t) = 1)}{\Delta t} = \frac{(\lambda \Delta t)}{\Delta t} = \lambda, \quad \lambda \Delta t e^{-\lambda \Delta t} \simeq \lambda \Delta t$$

Definition 1.11 (Compound Poisson Process) *Grigoriu (2002) A process $C(t)$ is called a compound Poisson process if it is defined as follows:*

$$C(t) = \sum_{n=1}^{N(t)} Y_n, \quad t > 0, \tag{1.15}$$

where $N(t)$ is a Poisson process with intensity λ and Y_n is a i.i.d. sequence of random variables with common probability law μ_Y.

1.2.6. Lévy Processes

An adapted, cadlag stochastic process defined on a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$ is called a Lévy process if the following conditions are satisfied Applebaum (2009):

1) $X(0) = 0$ a.s.,
2) Random variables of the sequence $\{X(t_{i+1}) - X(t_i)\}$, $0 \leq t_1 < t_2, ..., t_n < \infty$, $n \in \mathbb{N}$ are independent,
3) The distribution of $X(t)$ satisfies $X(t+s) - X(s) \overset{d}{=} X(t)$ (this property is called having stationary increments),
4) $X(t)$ satisfies the following property such that for each pair $\epsilon > 0$ and $s \geq 0$

$$\lim_{t \to s} \mathbb{P}(\ \omega \mid \ \|X(t)(\omega) - X(s)(\omega)\| > \epsilon) = 0 \tag{1.16}$$

this property is called stochastic continuity.

Definition 1.12 (Poisson Random Measure) *Applebaum (2009) Let $A \in \mathcal{B}(S)$ for $S \in \mathbb{R}^d - \{0\}$ and $U = \mathbb{R}^+ \times S$. If an integer-valued stochastic process $N(t, A)$ satisfies the* *following conditions, it is called a Poisson random measure on U:*

1) *$N(t, A)$ is a Poisson process distributed with intensity measure defined as $\lambda(A) = \mathbb{E}(N(1, A))$ for each $A \in \mathcal{B}(S)$,*

2) *$N(t, A_1), ..., N(t, A_n)$ are independent for mutually disjoint sets $A_1, ..., A_n \in \mathcal{B}(\mathbb{R}^d - \{0\})$.*

The jump behavior of Lévy processes $X(t)$ is modeled with a related Poisson random measure. The following Poisson random measure $N(t, A)$ gives the number of jumps of Lévy process $X(t)$, whose jumps are located in each $A \in \mathbb{R}^d - \{0\}$ and $t \geq 0$:

$$N(t, A)(\omega) = \sharp(0 < s < t,\ X(s) - X(s-) \in A) = \sum_{0 \leq s \leq t} 1_A(\Delta X(s)),$$

where the jump process related to the Lévy process is given by

$$\Delta X(t) = X(t) - X(t-).$$

A Borel measure, $\nu(.)$ which is defined via random measure as $\nu(.) = \mathbb{E}N(1, .)$, defines the distribution of the jumps of Lévy process $X(t)$, and is also the intensity measure of the Poisson random measure $N(t, A)$. The compensated Poisson process obtained over the Poisson process, which models the small jumps in Lévy processes, is a martingale whenever $\nu(A) < \infty$, and is defined as:

$$\tilde{N}(t, A) = N(t, A) - \nu(A)t. \tag{1.17}$$

Below, the general definition of the Poisson integral is given. This will be useful in the calculation of the total distance jumped by the Lévy process.

1.2.7. Poisson Integration

Let A be bounded below, $N(t, A)$ be a Poisson random measure connected to a Lévy process $X = (X(t))$ and f be a Borel measurable function from $\mathbb{R}^d$ to $\mathbb{R}^d$. The Poisson integral of f can be defined as a random finite sum for $t > 0$ as follows Applebaum (2009):

$$\int_A f(x)N(t,dx) = \sum_{0\le s\le t} f(\Delta X(s))1_{(\Delta X(u)\in A)},$$

where $0 \le s \le t$.

Here, x defines the length of the jump and $N(t,dx)$ gives the number of jumps falling inside the interval $[x, x+dx]$ up to time t. In particular, if $f(x) = x$ is defined, the distance $X(t)$ is found takes up to time t.

Theorem 1.5 *Applebaum (2009) The characteristic function of the Poisson integral* $\int_A f(x)N(t,dx)$ *has the following form for each* $u \in \mathbb{R}^d$*:*

$$\mathbb{E}\left(\exp\left[i\left(u, \int_A f(x)N(t,dx)\right)\right]\right) = \exp\left[t\int_{\mathbb{R}^d}\left(e^{i(u,x)} - 1\right)\nu_{f,A}(dx)\right],$$

where $\nu_{f,A}(B) = \nu\left(A \cap f^{-1}(B)\right)$ *and* $f \in L^1(A, \nu(A))$ *, for each* $B \in \mathcal{B}\left(\mathbb{R}^d\right)$.

As this is an important theorem in our studies, its proof is provided below.

Proof Let f be a simple function defined by $f = \sum_{j=1}^n c_j\chi_{A_j}$, for each $c_j \in \mathbb{R}^d$ and disjoint Borel subsets $A_j \in A$; then:

$$\mathbb{E}\left(\exp\left[i\left(u, \int_A f(x)N(t,dx)\right)\right]\right) = \mathbb{E}\left(\exp\left[i\left(u, \sum_{j=1}^n c_jN\left(t, A_j\right)\right)\right]\right),$$

which, by Theorem 1.2,

$$= \prod_{j=1}^n \mathbb{E}\left(\exp\left[i\left(u, c_jN\left(t, A_j\right)\right)\right]\right),$$

and by considering the characteristics of the compound random variable and $\mathbb{E}(N(t, A_j)) = t\nu(A_j)$

$$= \prod_{j=1}^n \exp\left\{t\left[\exp\left(i\left(u, c_j\right)\right) - 1\right]\nu\left(A_j\right)\right\}$$

$$= \exp\left[t\int_A \{\exp[i(u, f(x))] - 1\}\nu(dx)\right].$$

For each given $f \in L^1(A, \nu(A))$, a series of simple functions that converge to f in L_1 may be constructed, using the dominated convergence theorem.

□

Definition 1.13 (Characteristic Function of Lévy Process) *Applebaum (2009) The characteristic function of the Lévy process is given as follows:*

$$\phi_{X(t)}(u) = \exp t \left\{ i(b,u) - \frac{1}{2}(u,Au) + \int_{\mathbb{R}^d - \{0\}} (e^{i(u,y)} - 1 - i(u,y)1_{\hat{B}}(y))\nu(dy) \right\}.$$

Here, vector $b \in \mathbb{R}^d$ describes the deterministic part of the motion, called the drift, and A is a positive definite $d \times d$ matrix.

When $A \neq 0$ and $\nu = 0$, this is equivalent to a Gaussian process with mean vector bt and covariance matrix At. The Lévy measure ν, which characterizes the behaviour of the discontinuous part in Lévy processes, gives their average number of jumps. A Lévy process is determined by its "Lévy-Khintchine triplet" (a, A, ν). From the characteristic function of the Lévy process above, it can easily be seen that this expression represents the characteristic function of the sum of the independent drift vector bt, Gaussian process B_A, compensated Poisson integral $\int_{0<\|x\|<1} x\tilde{N}(t,dx)$ and Poisson integral $\int_{\|x\|>1} xN(t,dx)$, obtained from definition1.5 and Theorem 1.5.

The existence of the Lévy-Ito Decomposition, described below, is clearly apparent, given the previous definition.

Theorem 1.6 (Lévy-Ito Decomposition) *Sato (2000) Let $X(t)$ be a Lévy Process. Then $X(t)$ has the following representation for each $t \geq 0$:*

$$X(t) = bt + B_A(t) + \int_{0<\|x\|<1} x\tilde{N}(t,dx) + \int_{\|x\|>1} xN(t,dx), \tag{1.18}$$

where $b \in \mathbb{R}^d$, B_A is a Brownian motion with covariance matrix A, and N is an independent Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$.

Theorem 1.6 allows us to present the Lévy process as the sum of continuous and discontinuous parts; the discontinuous part consists of a mixture of small and large jumps.

1.2.8. α-Stable Random Variables and Processes

In order to understand α-stable processes, it is necessary to begin with the diffusion equation; the idea of the α-stable process evolved from this. First, transition probability related to the diffusion equation is defined, then shows that the solution of the diffusion equation has the structure of a transient density function.

Definition 1.14 (Transition probabilities) *Gawarecki and Mandrekar (2011)Let $X(t)$ be a stochastic process; its transition probability density functions are defined as the probability function $X(t+s)$, with that $X(t) = y$ for given time variables $t, s \in \mathbb{R}_{\geq 0}$.*

If the process is Brownian motion, the density function of the increment of the process, denoted G, is defined as follows:

$$G(x, y, s) = \frac{1}{\sqrt{2\pi s}} e^{(x-y)^2/2s} \tag{1.19}$$

Here, G represents the Green's function, and the increment process $X(t+s) - X(t)$ is a Gaussian with variance s and mean y. The function G can also be defined as the solution of the diffusion process with initial value $\phi(x)$ denoted by p, as follows Samoradnitsky and Taqqu (1994); Baeumer (2010):

$$\begin{cases} \partial_t p = \partial_x^2 p, & \text{if } t > 0 \\ p(0, x) = \phi(x) \end{cases}$$

The diffusion equation $\partial_t p = \partial_x^2 p$ defines the transition densities of a Brownian motion $B(t)$; its solutions spread at the rate $t^{1/2}$ in time. The solution of the space fractional equation $\partial_t p = \partial_x^\alpha p$ for $0 < \alpha < 2$ is called the density function of totally skewed α-stable Lévy motion $S(t)$. The evolution of process $S(t)$ in time is given by $S(ct) = c^{1/\alpha} S(t)$ for any time scale c. Another definition of totally skewed α-stable Lévy motion $S(t)$ is given by the general central limit theorem, as follows: the scaling limit of a random walk with power-law jumps (Lévy Flight), $c^{1/\alpha}(X_1 + \cdots + X_{[ct]}) \to S(t)$ in distribution as time scale ($c \to \infty$). The distribution of independent jumps X_i satisfies $P(X > r) \approx r^{-\alpha}$. Therefore, moments of $S(t)$ larger than α do not exist. In order to overcome this, Mantegna and Stanley proposed a modification of α-stable Lévy motion, truncated Lévy flights. Tempered Lévy motion takes a different approach, however, applying exponential tempering to the probability of large jumps Mantegna and Stanley (1994, 1995). This modification to the α stable process guarantees the existence of all moments, unlike α-stable processes which lack moments larger than α.

Definition 1.15 (Stable Random Variable) *Samoradnitsky and Taqqu (1994) A random variable X is called a stable random variable in $\mathbb{R}^d$ if for any given positive numbers A, B, there exists a positive number C and a real number D that satisfy the following distribution.*

$$AX_1 + BX_2 = CX + D,$$

where X_1, X_2 are independent copies of i.i.d. random variable X.

Another expression of a stable random variable is given with its characteristics, as follows.

$$E(e^{i(u,X(t))}) = \exp\left(t\left\{i(b,u) - \frac{1}{2}(u,Au) + \int_{\mathbb{R}-\{0\}} (e^{i(u,x)} - 1 - i(u,x)1_{\|x\|<1})\nu(dx)\right\}\right).$$

Here, the vector $b \in \mathbb{R}^d$ determines the deterministic part of the motion, called the drift, and A is a positive definite $d \times d$ matrix. When $A \neq 0$ and $\nu = 0$, the process is equivalent to a Gaussian process with mean vector bt and covariance matrix At. The Lévy measure ν, which characterizes the behaviour of the discontinuous part in Lévy processes, gives the average number of jumps of the process; a Lévy process is determined by its "Lévy-Khintchine triplet" (a, A, ν).

Definition 1.16 (Stable Random Variable) *(equivalent to definition (1.15)Samoradnitsky and Taqqu (1994) A random variable X is called stable if its distribution has the following representation:*

$$\mathbb{E}(\exp(i\theta X)) = \begin{cases} -\sigma^\alpha|\theta|^\alpha(1 - i\beta sign(\theta)\tan(\frac{\pi\alpha}{2})) + i\mu\theta) & \text{if } \alpha \neq 1, \\ -\sigma|\theta|(1 + i\beta\frac{2}{\pi}sign(\theta)\ln|\theta|) + i\mu\theta) & \text{if } \alpha = 1. \end{cases}$$

Here, the index of the distribution, $0 < \alpha \leq 2$, specifies its asymptotic behavior. The scale parameter $\sigma \geq 0$, the skewness parameter, $-1 \leq \beta \leq 1$, is a measure of its asymmetry, and $\mu \in \mathbb{R}$ is the shift parameter. Together these fully characterize the random variable.

The distribution of a stable random variable is denoted $S_\alpha(\sigma,\beta,\mu)$. When $\alpha = 2$, the stable random variable defines a Gaussian random variable with mean μ and variance σ^2, denoted by $S_2(\sigma, 0, \mu)$. The Cauchy distribution is denoted $S_1(\sigma, 0, \mu)$, and the Lévy distribution $S_{1/2}(\sigma, 1, \mu)$.

Definition 1.17 (Stable Random Vector) *Samoradnitsky and Taqqu (1994) A random vector $X = (X_1, X_2, \cdots X_d)$ is called a stable random vector in $\mathbb{R}^d$ if, for any positive num-*

bers A, B, there exist a positive number C and a real vector $D \in \mathbb{R}^d$ satisfy the following distribution:

$$AX_1 + BX_2 = CX + D,$$

where X_1, X_2 are independent copies of i.i.d. random vector X.

Definition 1.18 (Stable Process) *Samoradnitsky and Taqqu (1994) A process $X(t)(\omega)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called stable if it follows finite dimensional distributions whose random vectors*

$$A_d(\omega) = [X(t_1)(\omega)\ X(t_2)(\omega)\ ...\ X(t_d)(\omega)]^T, \quad t_1, t_2, ..., t_d \in T, d \geq 1,$$

are stable where $A_d(\omega) \in \mathbb{R}^d$.

An α-stable process is a real-valued Lévy process. Therefore, its characteristic function is expressed in terms of the Lévy exponent, η, as follows:

$$\phi_X(u) = E(e^{i(u,X)}) = e^{t\eta}$$

where

$$\eta = \begin{cases} -\sigma^\alpha|\theta|^\alpha(1 - i\beta sign(\theta)\tan(\frac{\pi\alpha}{2})) + i\mu\theta) & \text{if } \alpha \neq 1, \\ \\ -\sigma|\theta|(1 + i\beta\frac{2}{\pi}sign(\theta)\ln|\theta|) + i\mu\theta) & \text{if } \alpha = 1. \end{cases}$$

Thus, the time-dependent variation of the parameters of the stable process is as follows:

$$X(t) \sim S_\alpha(\sigma t,\ \beta,\ \mu t).$$

The stable process is called α-stable Lévy motion if parameter μ equals to zero.

Definition 1.19 (Tempered Stable Processes) *Mantegna and Stanley (1994, 1995) A process is called a tempered stable process if its characteristic function with Lévy measure ν*

is defined as follows:

$$\varphi(z) = \exp\left(i(b,z) + \int_{\mathbb{R}^d - \{0\}} \left(e^{izx} - 1\right) \nu(dx)\right), \quad z \in \mathbb{R}^d$$

where

$$\nu(dx) = \left(\frac{r^+}{x^{1+\alpha^+}} e^{-\lambda^+ x} 1_{(0,\infty)}(x) + \frac{r^-}{|x|^{1+\alpha^-}} e^{-\lambda^-|x|} 1_{(-\infty,0)}(x)\right) dx.$$

Here, the parameters are $r^+, \lambda^+, r^-, \lambda^- \in (0,\infty)$; $\alpha^+, \alpha^- \in (0,2)$ and r^+, r^- are called tempering parameters. The process's Lévy-Ito decomposition is given as follows:

$$\mathcal{L}(t) = bt + \int_{\|x\| \geq 0} xN(t,dx). \tag{1.20}$$

In the literature, the following specific cases are given:

- If the parameters $r^- = r^+$ and $\alpha^- = \alpha^+$, the distribution is called a CGMY-distribution (here CGMY stands for the names Carr, Geman, Madan and Yor who worked on that type of process), or a classical tempered stable distribution.
- If $r^- = 0$ in CGMY-distribution, then it is called a totally positively skewed tempered α-stable distribution.

1.2.9. Lévy Type Stochastic Differential Equations

In order to expound the theory of stochastic analysis, some basic tools related to this subject are necessary. Calculation of Lévy type integrals is done over mappings, measurable with respect to sigma algebras formed by left-continuous mappings. This type of mappings is called predictable Applebaum (2009); Siakalli (2009). Furthermore, two spaces consisting of predictable mappings are needed to be defined under certain conditions. One of them is denoted $\mathcal{P}_2(T,E)$ for $E \in \mathcal{B}(\mathbb{R}^d - \{0\})$ and $0 < s < T$. In the space $\mathcal{P}_2(T,E)$, functions are defined as $H : [0,T] \times E \times \Omega \to \mathbb{R}^d$. The other is $\mathcal{P}_2(T)$, where the functions are defined as $F : [0,T] \times \Omega \to \mathbb{R}^d$. The conditions required for mappings to belong to these classes are given below.

Let predictable mappings $Y_2 \in \mathcal{P}_2(T,E)$ and $Y_1 \in \mathcal{P}_2(T)$ satisfy the following conditions by definition:

i) $P\left[\int_0^T \int_E |Y_2(s,x)|^2 \nu(dx)ds < \infty\right] = 1,$

ii) $P\left[\int_0^T |Y_1(s)|^2 ds < \infty\right] = 1$

A d-dimensional Lévy type stochastic differential equation is defined as follows:

$$\begin{aligned} dX(t) &= f(X(t))dt + g(X(t))dB(t) + \int_{||y||<c} H(X(t),y)\tilde{N}(dt,dy) \\ &+ \int_{||y||>c} K(X(t),y)N(dt,dy), \end{aligned} \tag{1.21}$$

where $f,\ g \in \mathcal{P}_2(T)$ and $H, K \in \mathcal{P}_2(T, E)$ which are called the drift and diffusion coefficients, respectively. Next, the necessary conditions will be seen for the existence and uniqueness of the solution of SDE (1.21).

1.2.9.1. Existence and Uniqueness Conditions

If the coefficient functions in equation (1.21) satisfy the following inequalities, which are called Lipschitz conditions, then the solution of the equation is unique Applebaum (2009).

1) Lipschitz Conditions: there exists a positive constant L, in relationship to the coefficient functions in equation (1.21), such that for all $x_1, x_2 \in \mathbb{R}^d$,

$$||f(x_1) - f(x_2)|| \leq L||x_1 - x_2||, \quad ||g(x_1) - g(x_2)|| \leq L||x_1 - x_2||$$

$$\int_{||y||<c} ||H(x_1,y) - H(x_2,y)||\nu(dy) \leq L||x_1 - x_2||.$$

If the coefficient functions in equation (1.21) satisfy the following inequalities, which are called growth conditions, then the solution of the equation exists.

2) Growth Conditions: For all x, there exists a positive constant K such that

$$||f(x)||^2 \leq K(1 + ||x||^2), \quad ||g(x)||^2 \leq K(1 + ||x||^2),$$

$$\int_{||y||<c} ||H(x,y)||^2 \nu(dy) \leq K(1 + ||x||^2).$$

3) Big Jumps Condition: $K(x, y) \in \mathbb{C}(D_c)$, where $D_c = \{(x, y)|x, y \in \mathbb{R}^d,\ ||y|| \geq c\}$.

Theorem 1.7 *Applebaum (2009) If coefficient functions in the stochastic equation (1.21) satisfy conditions (C1-C3), then the equation (1.21) has a unique, cadlag, adapted process solution.*

In the following section, definitions and theorems are given related to the stability analysis of stochastic differential equations in the form (1.21). Stochastic processes have no derivatives in the classic sense, so classical Lyapunov theory cannot be applied for the stability analysis of stochastic differential equations. Therefore, stability analysis for SDEs uses the infinitesimal operator, which exists for the semi-martingale class, instead of the classical derivative operator. Since the Lévy process is also a semi-martingale, the infinitesimal operator in the stability analysis of equation (1.21) is used. Now, definitions related to the infinitesimal operator are provided.

1.2.10. Ito Calculus

The Ito formula is a form of the change of variable formula belonging to classical calculus, extended to stochastic integrals with semi-martingale integrators and adapted, caglad integrands. The application of the Ito formula requires a semi-martingale process involving Lévy processes (see Applebaum (2009) for further information related to semi-martingale processes). Therefore, the system must be transformed into a semi-martingale by integration of the SDE (1.21), as follows Applebaum (2009); Siakalli (2009):

$$\begin{aligned} X(t) &= X_0 + \int_{t_0}^{t} f(X(t))dt + \int_{t_0}^{t} g(X(t))dB(t) + \int_{t_0}^{t}\int_{||y||<c} H(X(t), y)\tilde{N}(dt, dy) \\ &+ \int_{t_0}^{t}\int_{||y||>c} K(X(t), y)N(dt, dy) \end{aligned} \tag{1.22}$$

on $t_0 \leq t \leq T$, with initial value $X(t_0) = X_0,\ X_0 \in \mathbb{R}^d$ and $c \in \mathbb{R}^+$.

Before defining the Ito formula, note that the following condition on the jump function H must hold:

$$\sup_{0\leq s\leq t}\ \sup_{0<||y||<c} |H(s, y)| < \infty \quad \text{a.s.},\ t \geq 0. \tag{1.23}$$

Theorem 1.8 (Ito's Formula) *Applebaum (2009); Siakalli (2009) Let $f \in C^2(\mathbb{R}^d)$ and X be the solution of the Lévy type stochastic integral (1.22). Then the following equality is satisfied with probability 1:*

$$
\begin{aligned}
V(X(t)) - V(X(0)) \quad &= \int_0^t \partial_i V(X(s-)) \left[f^i(X(-s))ds + g^{ij}(X(-s))dB_j(s) \right] \\
&+ \frac{1}{2} \int_0^t \partial_i \partial_j V(X(s-)) d\left[g(X(-s)) g(X(-s))^T \right]^{ik} ds \\
&+ \int_0^t \int_{||y|| \geq c} [V(X(s-) + K(s,y)) - V(X(s-))] N(ds,dy) \\
&+ \int_0^t \int_{||y|| < c} [V(X(s-) + H(s,y)) - V(X(s-))] \tilde{N}(ds,dy) \\
&+ \int_0^t \int_{||y|| < c} [V(X(s-) + H(s,y)) - V(X(s-)) \\
&\quad - H^i(s,y) \partial_i V(X(s-))] \nu(dy) ds,
\end{aligned}
$$

where $V \in C(\mathbb{R}^d)$, $X \in \mathbb{R}^d$ and $c \in \mathbb{R}^+$.

1.2.11. Infinitesimal Generators

Definition 1.20 (C_0-Semigroup) *Gawarecki and Mandrekar (2011) A family $(T(t))_{t>0}$ of bounded linear operators on a Banach space V is called a strongly continuous C_0-semigroup, if the following conditions are satisfied:*

1) *T(0)=I,*
2) *$T(s+u) = T(u)T(s)$, for every $s, u > 0$; this equality is called the semigroup property,*
3) *$\lim_{t \to 0^+} T(t)x = x$ in norm for every $x \in V$; this equality is called the strong continuity property.*

Definition 1.21 (Infinitesimal Operator) *Gawarecki and Mandrekar (2011) Let X be a Banach space, $T(t)$ be a semigroup on it and A be a linear operator. Then*

$$
Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} \quad (in \;\; norm)
$$

is called the infinitesimal generator of the semigroup $T(t)$, when the domain of A is given by:

$$\mathbb{D}(A) = \left\{ x \in X : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \;\; \textit{exists} \right\}.$$

.

1.2.12. Stability Types for Stochastic Differential Equations

An equilibrium point or steady state $X = X_c,\ X_c \in \mathbb{R}^d$ of the SDE given by (1.21) is defined as a point which has the property $f(X_c) = 0, g(X_c) = 0,\ H(X_c, y) = \vec{0}$ for all $||y|| \leq c$, and $K(X_c, y) = \vec{0}$ for all $||y|| \geq c$. When $X_c = 0$, $X(t) = 0$ is obtained, it is defined as the trivial solution, and this corresponds to the initial value $X(t_0) = 0$.

In stability analysis of Lévy-type stochastic processes, the infinitesimal operator is used due to the lack of classical derivatives. Lévy processes belong to the class of Feller processes. The infinitesimal operator for Feller semigroups is defined as follows Siakalli (2009):

$$\begin{aligned} (\mathcal{L}V)(X) &= f^i(X)(\partial_i V)(X) + \frac{1}{2}[g(X)g(X)^T]^{ik}(\partial_i\partial_j V)(X) \\ &+ \int_{||y||<c} [V(X + H(X,y)) - V(x) - H^i(X,y)(\partial_i V)(x)]\nu(dy) \\ &+ \int_{||y||\geq c} [V(X + K(X,y)) - V(x)]\nu(dy), \end{aligned} \tag{1.24}$$

where $V \in C^2(\mathbb{R}^d), X \in \mathbb{R}^d$ and $c \in \mathbb{R}^+$.

For more details on Feller processes, see Applebaum (2009); Siakalli (2009). Just two types of stability are used, namely stable in probability and pth moment exponentially stable, defined below.

Definition 1.22 (Stable in Probability) *Siakalli (2009) The trivial solution of the solution of (1.21) is called stable in probability, if there exists $\delta = \delta(\epsilon, r, t_0)$ such that for any pair $\epsilon \in (0, 1)$ and $r > 0$,*

$$\mathbb{P}(\ ||X(t)|| < r \textit{ for all } t \geq t_0) \geq 1 - \epsilon$$

holds, whenever the initial solution $||X(t_0)|| < \delta$.

Definition 1.23 (pth Moment Exponentially Stable) *Siakalli (2009) The trivial solution of the solution (1.21) is called pth moment exponentially stable, if for a given constant* $p > 0$*, there exist two positive constants,* λ *and* C*, which satisfy the following inequality:*

$$\mathbb{E}(\|X(t)\|^p) \leq C\|X_0\|^p \exp(-\lambda(t-t_0)), \quad X_0 \in \mathbb{R}^d, \ t > t_0.$$

1.2.13. Stability Theorems for Lévy Type Stochastic Differential Equations

In this section, two theorems are presented which give conditions on the following Lévy type differential equation such that the trivial solution is stable in probability or pth moment exponentially stable Siakalli (2009)

$$dX(t) = f(X(t))dt + g(X(t))dB(t) + \int_{\|y\|<c} H(X(t), y)\tilde{N}(dt, dy) \tag{1.25}$$

with initial value $X(t_0) = X_0 \in \mathbb{R}^d$. Here the conditions on the functions $f, \ g \in \mathcal{P}_2(T)$ and $H, K \in \mathcal{P}_2(T, E)$ are presented in equations (1.21) and (1.23).

The integral form of the stochastic differential equation (1.25) is written as follows:

$$X(t) = X_0 + \int_{t_0}^{t} f(X(t))dt + \int_{t_0}^{t} g(X(t))dB(t) + \int_{t_0}^{t}\int_{\|y\|<c} H(X(t), y)\tilde{N}(dt, dy). \tag{1.26}$$

Here, the theorems are given which Siakalli proved for this model in her thesis Siakalli (2009) in chapter 3.

Theorem 1.9 *Siakalli (2009)(Chapter 3, Theorem 3.3.2, pp. 48) Assume that* $B_h = \{x \in \mathbb{R}^d, \|x\| \leq h\}$ *is a ball with radius h satisfying* $h > 2c$*, where c is the maximum jump size in (1.26). If the infinitesimal generator of* $\mathcal{L}$ *with respect to (1.26) satisfy the following* *condition:*

$$\mathcal{L}V < 0,$$

where $V : B_h \to \mathbb{R}^+$ is positive definite function, then the trivial solution of (1.26) is stable in probability.

Theorem 1.10 *Siakalli (2009)(Chapter 3, Theorem 3.5.1, pp. 64) If there is a function $V \in C^2(\mathbb{R}^d; \mathbb{R}_{\geq 0})$, which satisfy the following conditions with respect to (1.26):*

1) $a_1\|X\|^p \leq V(X) \leq a_2\|X\|^p, \;\; a_1, a_2 \in \mathbb{R}^+$

2) $\mathcal{L}V(X) \leq -a_3V(X), \;\; a_3 \in \mathbb{R}^+,$

then the following inequality is satisfied:

$$E(\|X\|^p) \leq \frac{a_2}{a_1}\|X_0\|^p \exp(-a_3(t - t_0)) \;\; for\ all \;\; t \geq t_0,$$

and the trivial solution of (1.26) is pth moment exponentially stable.

Theorem 1.11 *Applebaum (2009)Let X be a Lévy process with bounded jumps, then all moments of the process $\mathbb{E}(\|X(t)\|^m)$ are finite for all $m \in \mathbb{N}$.*

Theorem 1.12 *Applebaum (2009) Let X be a Lévy process. Then the moments of the process $\mathbb{E}(|X(t)|^n)$ are bounded for given each $n \in \mathbb{N}$ and all $t > 0$ if and only if the Lévy measure of the process satisfies the integral inequality $\int_{\|x\|>1} |x|^n \nu(dx) < \infty$.*

1.3. Stability Analysis of Nonlinear Dynamical Systems with Lévy Type Perturbations - An Application to Electrical Power Grids

Over the past decades, there has been a drastic increase in interest in Lévy processes. Having independent increments and stationary properties gives Lévy processes many applications, in areas such as macro-level engineering, economics and micro-level quantum physics. In fact, one of the essential property of Lévy processes, the stationary property, has been known since the introduction of infinite divisibility in the late 1920s by Lévy and KhincineLévy (1934); Khinchine (1937). Furthermore, Lévy processes were used to called as processes with stationary and independent increments in the 1960s and 1970s. Extension of stochastic calculus to Lévy processes was started with Khasminski, KushnerKhasminski (1969); Kushner (1967) and later on Mao Mao (2007). In 1980s, the current definition of a Lévy process was done by Finetti, Kolmogorov, Lévy and Ito Ito (1984).

Stable processes, also known as α-stable processes, are a subclass of Lévy processes with special importance. They were originally developed to explain anomalous diffusion, a relatively rare phenomenon that occurs in nature. Stable processes including pure jump processes are also non-Gaussian, due to the lack of a continuous process in their structure. α-stable processes have essential characteristic functions. These are very important and have applications in many areas of science. There exist two essential parameters in the characteristic function of an α-stable process: α and the skewness parameter. These parameters model the asymptotic behaviour and the skewness of the process distribution, respectively Samoradnitsky and Taqqu (1994). An α parameter takes its value between 0 and 2. The smaller α value models the slower decay, so results in the heavier tail in the distribution. Moreover, α-stable processes do not have moments larger than the α parameter. To overcome this, Mantegna and Stanley introduced tempered α-stable process Mantegna and Stanley (1994, 1995). Tempered α-stable process is produced by multiplying α-stable processes with a decreasing exponent on each half of the real axis. After this smoothing application, small jumps in the behaviour of the process retain their initial α-stable-like behaviour, while large jumps become much less violent. Hence Lévy processes find more application areas with tempered α-stable process.

One important application of Lévy processes is at modeling perturbations in power networks. In this thesis, the probabilistic synchronization and the stability control of an electricity power network under Lévy type stochastic perturbations are investigated. There are two essential types of stability relevant to power systems. These are the rotor angle stability and the frequency stability. Rotor angle stability refers to the ability of interconnected synchronous machines of a power system to maintain consistent generator rotor angles after perturbation. Frequency stability refers to the ability of a power system to maintain consistent frequency after perturbation. This depends on the balance between generation and consumption in the system.

The synchronization of the rotor angle is critical for the electrical connection of the power grids. As the rotor angle difference between the two regions increases, the voltage on the lines between them tends to drop. Larger rotor angle differences can result in more serious problems(blackouts). A lot of work has been done in order to minimize such problems. An efficient approach is to model the oscillations in the system with stochastic processes.

We investigate the rotor angle stability and frequency stability for two important equations used to model electrical power networks. These are the swing equation, which models the single machine infinite bus system (SMIBS), and the Kuramoto Model which models a large number of generators as coupled oscillators.

1.3.1. Model I: Swing Equation

The swing equation models the relationship between the mechanical power input and the electrical power output in a single machine connected to an infinite bus (SMIB) power system Kundur (1993). The deterministic swing equation is defined as follows:

$$\begin{aligned} \dot{\delta} &= \omega \\ M\dot{\omega} &= -D\omega + P_m - P_e. \end{aligned}$$

Here, variables δ and ω are respectively the relative rotor angle of the synchronous machine, and the rotor speed with respect to the synchronous reference. M, D are the moment of inertia and the damping constant, respectively. P_m is the mechanical input power and $P_e = P_{\max}\sin(\delta)$ is the electrical output power. $P_{\max}$ represents the maximum power output of the synchronous machine.

Much research on rotor angle stability has focused on the Single Machine Infinite Bus System (SMIBS). In stability study of the SMIBS, this was initially considered as a deterministic model. The stabilization of linearised models started with the first DeMello and Concordia deMello and Concordia (1969). Then Yang for the same structure H_∞ optimization method is used to develop control function Yang (1997). In Canizares (1995), the load was taken as bifurcation parameter. The chaotic behaviour of the system was analysed through system parameters by the Melnikov method. According to the research, it was observed that small perturbations in the load caused an unstable situation for the generator. Wang studied the dynamic properties of the SMIBS for periodic load disturbance X. Wang and Song (2015).

Other studies have used stochastic models. Demarco modelled stochastic noise using the Wiener process DeMarco and Bergen (1987). Wei and Luo established the necessary conditions on system parameters for a linearized SMIBS system to be stable under Gaussian noise Wei and Luo (2009). Stability analysis for small Gaussian oscillations of the system parameters was then performed for the linearised SMIBS system J. Y. Zhang and Wu (2012). Subsequently, the p-moment stability of linearized SMIBS under small Gaussian excitation was established Z. Lu and Li (2015). Another stochastic stability analysis on SMIBS was performed based on the Fokker-Plank equations. The effects of system parameters were observed for nonlinear stochastic systems with Gaussian type noise Wang and Crow (2013).

Later, Savacı and Yılmaz modelled the SMIBS system with noise from an alpha stable process Yılmaz and Savacı (2017). This idea was based on Weron's method of modeling electricity prices Weron (2009). In Yılmaz and Savacı (2017), changes in demand/supply cause sudden jumps in electricity prices. This suggests that load should be taken into account as an effective factor in electricity prices. Under α-stable process noise, changes in the basin of attraction and limit cycle of the stable equilibrium point(SEP), dependent on the parameters of the process, were observed. In another study, a control rule for the linearized SMIBS was created Savacı and Yılmaz (2020). In previous studies on stochastic models, the effect of noise on the system has just been observed that stabilization has been done for a limited range of system parameters, or stability control has been obtained on linearized models that is inefficient for modelling. In this thesis, the control function for the nonlinear model independently of the parameters have been constructed, and the control functions for the nonlinear model has three types of stability: stability in probability and pth moment exponential stability. Since the impact of big jumps is nearly zero is assumed, the modified tempered alpha stable process evaluated with an assuming finite jump condition on the tempered α-stable process for $\alpha < 1$ is considered to model the noise. According to our study with this model in the second chapter, using this control function, probability stability and 2-moment stability of the equilibrium points of the SMIBS were achieved. The validity of the theoretically obtained control function was obtained with the theoretical results.

1.3.2. Model II:Kuramoto Model

In complex power grids, a large number of generators are considered as coupled oscillators. First Winfree Winfree (1967), then Kuramoto also analyses the behaviour of systems whose state can be summarised by a single scalar variable θ, and in addition Kuramoto's formulation assumes conservation of the total phase. Kuramoto's systems can be modelled with a set of N coupled ordinary differential equations as follows:

$$\frac{\theta_i(t)}{\mathrm{d}t} = \left[\omega_i + \frac{K}{N}\sum_{j=1}^{N} a_{ij}\sin\Big(\theta_j(t) - \theta_i(t)\Big)\right], \quad i, j \in [1, \cdots N]. \tag{1.27}$$

Here, θ_i is the voltage phase angle, $\omega_i = 2\pi(f_i - f_{\mathbb{R}})$ is the angular velocity, f_i is the frequency of the oscillators $i = 1, \ldots, N$ and f_{R} is the reference frequency for $f_{\mathrm{R}} = 50$ Hz or $f_{\mathrm{R}} = 60$ Hz values and t represents time. Equation (1.27) describes a system describing the phase velocity of the ith oscillator. The quantity K is the coupling strength, representing the overall strength of connections between the oscillators. $\frac{1}{N}$ is a normalization constant. The a_{ij} are the weights of the adjacency matrix A which describes the (undirected) Kuramoto network. Specifically, $a_{ij} = a_{ji} = 1$ (for $i \neq j$) if there exists an edge between oscillator i and oscillator j, and $a_{ij} = 0$ otherwise.

The first stability studies related to the deterministic Kuramoto model were examined its system parameters. Kuramoto derived system parameters for synchronization of his model consisting of an infinite number of oscillators. Subsequently, Jadbabaie extended this work to models with a finite number of oscillators, a model more applicable to real life, and calculated the critical coupling value and order parameter for this structure Jadbabaie and Barahona (2004). Later Chopra and Spong also relaxed the constraints on the critical coupling for the synchronization of Jadbabe's system Chopra and Spong (2009). A similar study has been done for the multi rate Kuramoto model, including second and first order Kuramoto models, by Dofler and Bullo Dörfler and Bullo (2019). However, in these stability studies based on system parameters, initial conditions were assumed to be taken from the interval $(-\pi/2, \pi/2)$. This strict conditions reduce the applicability of the model to real life. It has been possible to relax conditions on the system with the introduction of pacemakers, which act as a kind of control function Li (Li). In the further studies carried out with control function construction, systems with multiple control layers to generate control functions, were modelled. These were able to reach stability in a finite and precise time Wu and Li (2019a). Guo and Rao then developed control functions to achieve this type of stability in systems with help of pacemaker dynamics. Rao (2022). In finite and fixed time stabilization studies, continuous control functions have been used because of the chattering problems caused by non-continuous control functions J. Wu (2021); X. Guo (2022).

Stochastic studies started in the 1980s with Sakaguchi, who defined the first stochastic system and analysed its behavior and stability using the Fokker-Plank function Sakaguchi (1988). Another study analysed stability conditions for phase oscillators in a complex Kuramoto model subject to Gaussian noise. Results obtained using the Fokker-Planck equation showed the existence of more than one stable cluster in the system for a stationary solution Park (1996). Later, stability research was extended to the two-dimensional Kuramoto Model. Acebron and Spigler's stationary solution of the Fokker-Planck equation was obtained analytically under certain conditions Acebrón and Spigler (1998). Some other stability studies have presented analytical results for the conditions required to obtain a stationary solution for the time periodic Fokker planck equation of the Kuramoto model with mean field structure P. Reimann (1999).

The importance of stochastically modeling power systems has recently been recognized and the idea of modeling the loss of system stability under a stochastic fault has become more popular due to the faults in the transition from power plants to renewable systems. Many important studies on stochastic models have already been done. Some of these are as follows: In D. and C. (2018), stability analysis was performed on the Fokker-Plank equation of the linearized Kuramoto model under the influence of tempered stable Lévy noise.

Initially, in power system studies, noise in the systems was modelled with Gaussian process for ease of implementation. However, a study published in Nature in 2018 Schafer et al. (2018), showing that the noise in the system contains sudden non-Gaussian changes, determined this continuous process model inefficient. In this article, second order nonlinear stochastic Kuramoto model which includes stable noise is reduced to aggregated swing equation under certain conditions to analyse the stability of the frequency of the power system. Also, the solution obtained has the same alpha and beta parameters as noise in the model but sigma parameter.

Thus, studies on non-gaussion models have started to gain importance. Roberts and Kalloniatis analysed the stability of the Kuramoto model subject to a tempered alpha stable process, which is a type of Lévy process, in different networks. With the help of the Fokker Planck equation obtained from the linearized Kuramoto model, they observed the variation in the distribution of the solution with respect to alpha D. and C. (2018).

However, in the studies related to non-Gaussian noise, the models were linearized, or any control function was not established on the model. In the third chapter, the strict conditions on system parameters are lost in previous studies, and model the fluctuations by Lévy processes in duplex power networks consisting of oscillator $\mathbb{K}$ and control layers $\mathbb{L}$.

$$\mathrm{d}\theta_i(t) = \left[\omega_i + K\sum_{j=1}^{N} a_{ij}\sin\Big(\theta_j(t) - \theta_i(t)\Big) + u_i(\theta(t))\right]\mathrm{d}t + \epsilon(\theta(t))dL(t), \quad i, j \in [1, \cdots N].$$

Here, the function ϵ models the Lévy processes noise, indicating the intensity of its effect on the system. If the oscillator layer $\mathbb{K}$ is affected, ϵ is defined as follows:

$$\epsilon(\theta(t)) = \varrho_1 \sum_{j=1}^{N} Ka_{ij}\sin(\theta_j(t) - \theta_i(t)).$$

If the noise affects the control layer $\mathbb{L}$, the noise intensity function $\epsilon(\theta(t))$ is defined as follows:

$$\epsilon(\theta(t)) = \varrho_2 \sum_{j=1}^{N} b_{ij}(\theta_j(t) - \theta_i(t),$$

where ϱ_1 and ϱ_2 are the intensity parameters of the noise in the Kuramoto layer $\mathbb{K}$ and control layer, $\mathbb{L}$ respectively. In the third chapter, the phase and frequency synchronization of Kuramoto oscillator systems in the form of a duplex network topology subject to Lévy process perturbation is investigated. The duplex network would lose phase and frequency synchronization under certain conditions is assumed, and in light of Lyapunov Theory, designed a control function to achieve system synchronization. The results obtained in this chapter are as follows. Comparing the experimental and theoretical results when the Kuramoto oscillator layer is subject to perturbation, the control function loses efficacy as noise intensity grows. When it is the control layer that is subject to peturbation, the numerical results indicate that noise intensity up to a certain point has minimal influence on the phase and frequency stability of the system, but that beyond that point it cannot be handled.

CHAPTER 2

STOCHASTIC STABILITY OF SINGLE MACHINE INFINITE BUS POWER SYSTEMS WITH MODIFIED TEMPERED α-STABLE LÉVY TYPE PROCESS

In electric power systems, synchronization can be described as the process of matching parameters such as voltage, frequency, phase angle, phase sequence, and waveform of an alternator (generator) or other sources working in rhythms. Any significant effect/noise in an electricity power network's supply or load can make the changes in synchronized movement of system elements. In such a case, the line voltage can drop and so the phase synchronism may be lost. This causes many problems in power systems, and can lead to blackouts. Therefore, in case of noise, the ability of a system to regain the state of equilibrium is crucial. To do this, we include control functions to the systems.

In this chapter, we introduce control functions in more generality, more precisely, control functions which provide trivial solutions for the probability and moment exponential stability of the general nonlinear stochastic swing equation modelled with modified totally positively skewed tempered α-stable (MTPSTS) process for $\alpha < 1$. Here, the MTPSTS distribution is evaluated by assuming the noise in the system can jump only up to a certain size and that these jumps come from the totally positively skewed tempered α-stable (TPSTS) distribution.

Roughly speaking, a stable equilibrium point (SEP) in the phase space of a system is the point to which all points nearby converge. If there is no disturbance in the system, equilibrium points are stable. However, if there is any disturbance, stability may be lost. In electrical power systems, disturbance can occur with load change, line tripping or loss of a generator. In such cases the balance between power input and electrical power output is ruined so the stability, i.e., the synchronization is lost.

In the following figure, we show an example of a deterministic swing equation. Here we take the damping parameter $D = 0.5$ and the critical value, $D_c \approx 0.39$ (as in Guckenheimer and Holmes (2013) and Yılmaz (2019)). Because $D > D_c$, all the equilibrium points are stable; trajectories of the system converge to a SEP under 10 initial states taken randomly from the phase space $[-\pi, \pi] \times [-20, 20]$.

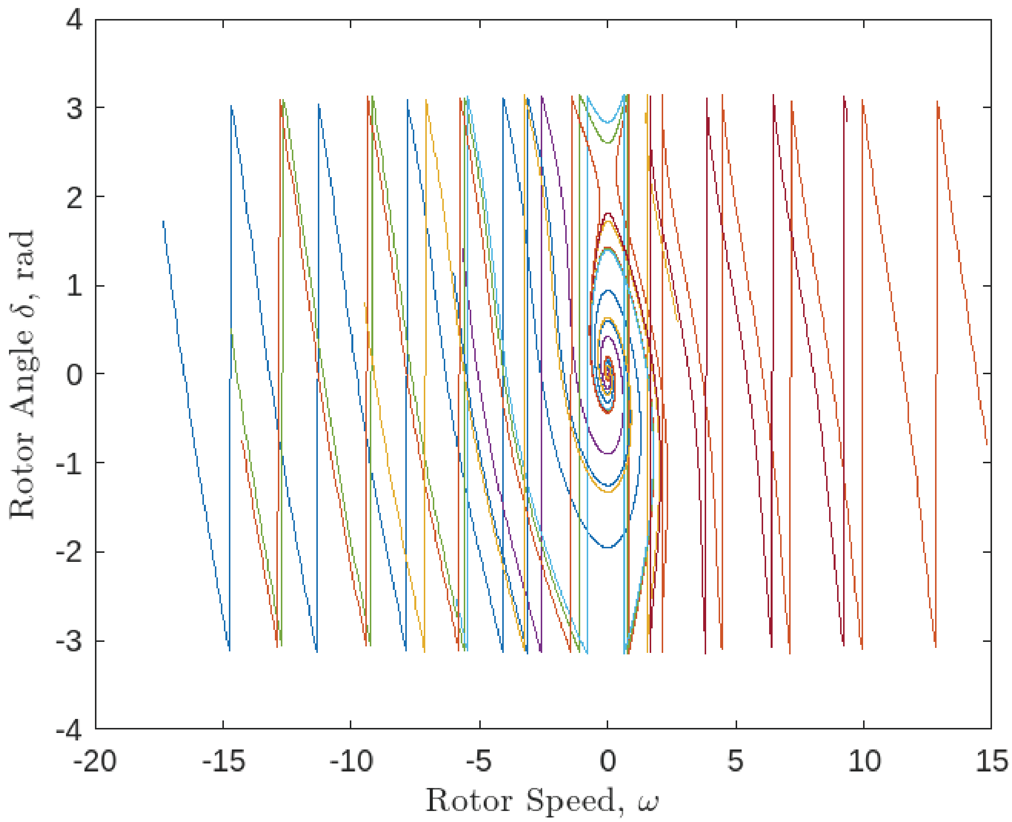

Figure 2.1. Phase portraits of deterministic SMIB system for P_m = 0.5 and D_c = 0.5

In Figure 2.2, the damping parameter D is set to 0.3, less than the critical value, D_c. In this case, the trajectories of the system converge to a stable rotating orbits (or 'limit cycles') and so the phase stability, that is, the synchronization of the power system networks is ruined.

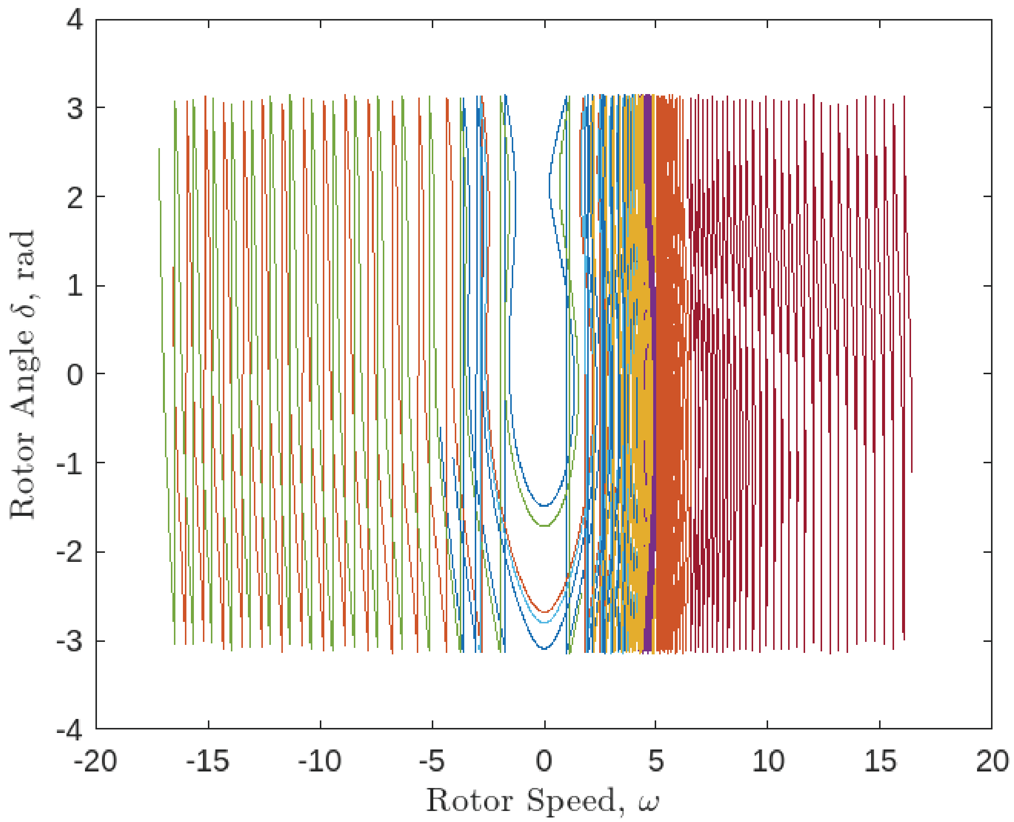

Figure 2.2. Phase portraits of deterministic SMIB system for $P_m = 0.5$ and $D_c = 0.3$

In the next section, we examine the probabilistic and exponential stability of the equilibrium for an SMIBS system with MTPSTS noise. We numerically test the effectiveness of the control function.

2.1. Modelling the Stochastic Swing Equation

A stochastic swing equation describes the relationship between mechanical power input and electrical power output in a single machine infinite bus (SMIB) power system subject to one dimensional MTPSTS noise $\mathrm{T}L_\alpha(t)$ for $\alpha < 1$. It is defined as follows:

$$\begin{bmatrix} d\delta(t) \\ Md\omega(t) \end{bmatrix} = \begin{bmatrix} \omega \\ -D\omega + P_m - P_{max}\sin\delta \end{bmatrix} dt + \begin{bmatrix} 0 \\ \epsilon([\delta\ \omega]^T) \end{bmatrix} dTL_\alpha(t). \tag{2.1}$$

Here variables δ and ω are respectively relative rotor angle of synchronous machine and rotor speed with respect to the synchronous reference. M, D are the polar moment of inertia and the damping constant respectively. P_m is the mechanical input power and $P_e = P_{max} sin(\delta)$ is the electrical output power where P_{max} represents the maximum power output of the synchronous machine. $\epsilon(x)$ is the intensity function of the Lévy noise. In this chapter, our aim is to find the control function that provides the stability types for the system (2.1). We take the inertia constant M and maximum power output P_{max} such that $M = P_{max} = 1$, the mechanical input power $0 \leq P_m < 0.7$, and a modified totally positively skewed tempered α-stable process as a noise. The system's equilibrium points are defined as $[\delta_1^*, \omega_2^*]^T = [\arcsin P_m, 0]$. In order to apply the Lyapunov Theory, it is necessary to shift the equilibrium points of the system to the origin with state variable $X = [x_1 \ x_2]^T$. Then (2.1) becomes as follows:

$$dX(t) = F(X(t))dt + \epsilon_1(X)dTL_\alpha(t) \tag{2.2}$$

$$F(X(t)) = \begin{bmatrix} x_2 \\ -Dx_2 + P_m - P_m \cos x_1 - \sqrt{1 - P_m^2} \sin x_1 \end{bmatrix}, \quad \epsilon_1(X) = \begin{bmatrix} 0 \\ \epsilon(X). \end{bmatrix} \tag{2.3}$$

Characteristics of the modified totally positively skewed tempered α-stable Lévy process $TL_\alpha(t)$ is defined with $(b, 0, \nu)$ and it has the following Lévy-Ito decomposition given in equation (1.20):

$$TL_\alpha(t) = bt + \int_{|y|<c} yN(t, dy), \quad \mathbb{E}(N(t, dy)) = t\nu(dy), \tag{2.4}$$

where the Lévy measure of the modified totally skewed tempered α-stable distribution is presented as follows:

$$\nu(dy) = \begin{cases} \frac{1}{y^{1+\alpha^+}} e^{-\lambda^+ y} 1_{(0,\infty)}(y) & \text{if } y \in (0, c], \\ 0 & \text{if } y \notin (0, c]. \end{cases} \tag{2.5}$$

where c is the size of the process's jumps limited due to the system's nature. In order to do stability analyses of the stochastic differential equation (2.2), the first step is to use the Lévy-Ito decomposition form of the tempered stable noise (2.4) in the system (2.2) to remodel the equation in a general Lévy type stochastic differential form (1.26) as follows:

$$X(t) = X_0 + \int_0^t F(X(s))ds + \epsilon_1(X)(bt + \int_{0\leq y\leq c} yN(t, dy)). \tag{2.6}$$

In the second step, the stochastic equation we have in (2.6) is transformed into the form (1.26) by applying the property (1.17) for the Poisson integral.

$$X(t) = X_0 + \int_0^t F(X(s))ds + \epsilon_1(X)(bt + t\int_{0\leq y\leq c} y\nu(dy) + \int_{0\leq y\leq c} y\tilde{N}(t, dy)), \tag{2.7}$$

where

$$\int_{0\leq y\leq c} yN(t, dy) = t\int_{0\leq y\leq c} y\nu(dy) + \int_{0\leq y\leq c} y\tilde{N}(t, dy). \tag{2.8}$$

Here, the equality given by (2.8) is valid only when $\int_{0\leq y\leq c} y\nu(dy)$ is finite. In the next section, under certain assumptions, the necessary conditions for the SDE (2.2) are given to provide stability in probability and 2-moment stability.

2.2. Stability Condition in SMIBS

We will now give the conditions required for SMIBS to achieve stability in probability and 2-moment exponential stability when perturbed with modified tempered alpha stable noise under the following assumptions:

Assumption 2.1 *We suppose that the function $\epsilon(x)$ satisfies following inequality for $\beta \in \mathbb{R}^+$ in the equation* (2.2)*:*

$$|\epsilon(x)| < \beta_2|x_2|.$$

Assumption 2.2 *Assuming that $\epsilon(x)$ satisfies the following inequality for $\beta \in \mathbb{R}^+$ in equation* (2.2)*, we have:*

$$|\epsilon(x)| < \beta||x||.$$

2.2.1. Stability in Probability

In this section, the theorem related to the control function to obtain the stability in probability of the SMIBS perturbed by a modified totally positively skewed tempered α-stable Lévy process is given, under certain assumptions.

Theorem 2.1 *Suppose the maximum jump amount of noise in the system is* $\pi/4$*, the maximum value of the mechanical input power* P_m *is* 0.7*, the maximum value of the stable parameter* α *is* 1 *and the Assumption 2.1 holds. Then the trivial solution of equation (2.2) is stable in probability with the following control function* $u(x) = [u_1(x) \;\; u_2(x)]^T$ *:*

$$u(x) \;=\; \begin{bmatrix} 0 \\ -Kx_2 \end{bmatrix},$$

where $K > M_1\beta_2^2 + \beta_2^2(|b| + N_1) - D$ *with*

$$M_1 = \int_{0\leq y\leq \pi/4} y^2\nu(dy) < \int_0^\infty y^2\frac{e^{-\lambda y}}{y^{1+\alpha}}dy = \lambda^{\alpha-2}\Gamma(2-\alpha) < \infty,$$

and

$$N_1 = \int_{0\leq y\leq \pi/4} y\nu(dy) < \int_0^\infty y\frac{e^{-\lambda y}}{y^{1+\alpha}}dy = \lambda^{\alpha-1}\Gamma(1-\alpha) < \infty, \;\; \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

Note : λ represents the tempering parameter and α signifies the stability parameter of the process.

Proof

We will use Theorem 1.9 to analyze the stability of the system. However, Theorem 1.9 can only be applied in the presence of a compensated Poisson integral in the system. Therefore, to ensure the applicability of the theorem, we apply the following decomposition to the Poisson integral (2.4):

$$TL_\alpha(t) = bt + t\int_{0\leq y\leq \pi/4} y\nu(dy) + \int_{0\leq y\leq \pi/4} y\tilde{N}(t,dy). \tag{2.9}$$

By decomposition of the tempered α stable process TL_α in (2.9), the functions $F(x)$ in (2.2) can be expressed as:

$$F(x) = \begin{bmatrix} x_2 \\ -Dx_2 - P_m(\cos x_1 - 1) - \sqrt{1 - P_m^2}\sin x_1 + u_2(x) + \epsilon(x)\left[b + \int_{0\le y\le \pi/4} y\nu(dy)\right] \end{bmatrix}.$$

Now, we construct the following positive definite Lyapunov function on a ball centered at the origin over conjugate gradient method (Khalil (2015),page 129). Suppose the system has the following model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -h_1(x) - h_2(x) \end{bmatrix}.$$

where $h_1(x)$ and $h_2(x)$ are locally Lipschitz, and satisfy

$$h_i(0) = 0, \;\; yh_i(y) > 0, \;\; \forall y \neq 0, \;\; y \in (-a,\, a), \;\; a \in \mathbb{R}^+, \;\; i \in [1, 2]. \tag{2.10}$$

Then the positive definite energy-like function V_1 can be defines as follows:

$$V_1(x) = \int_0^{x_1} h_1(y)dy + \frac{x_2^2(t)}{2}.$$

Here

$$\dot{V}_1 = h_1(x)x_2 + x_2[-h_1(x_1) - h_2(x_2)] = -x_2 h_2(x) \le 0$$

Now we apply this method to our model. We take the deterministic swing equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \; = \begin{bmatrix} x_2 \\ -h_2(x_2) - h_1(x_1) \end{bmatrix}.$$

$$
\begin{aligned}
h_1(x_1) &= P_m \cos x_1 + \sqrt{1-P_m^2}\sin x_1 - P_m,\\
h_2(x_2) &= Dx_2.
\end{aligned}
$$

Here h_1 and h_2 functions satisfy the conditions (2.10) over the domain $B_h = \{x \in \mathbb{R}^2 \mid \ \|x\| < \pi/2\}$ for $P_m < 0.7$. The positive definite function V_1 in B_h is:

$$
V_1(x) = \int_0^{x_1} [P_m(\cos x - 1) + \sqrt{1-P_m^2}\sin x]dx + \frac{x_2^2(t)}{2}.
$$

We now evaluate $\mathcal{L}V_1$: For this, first recall the formula of the infinitesimal operator (1.22) with respect to the d-dimensional SDE equation (1.25), given below:

$$
dx(t) = f(x(t))dt + \int_{\|y\|<c} H(x(t), y)\tilde{N}(dt, dy) \tag{2.11}
$$

Then the infinitesimal operator of the Lyapunov function with respect to the solution of the SDE (2.11) is:

$$
\begin{aligned}
(\mathcal{L}V)(x) &= f^i(x)(\partial_i V)(x) + \int_{\|y\|<c} [V(x + H(x,y)) - V(x) - H^i(x,y)(\partial_i V)(x)]\nu(dy),\\
& i \in [1, \cdots, d].
\end{aligned} \tag{2.12}
$$

Our model applies to (2.11) with

$$
\begin{aligned}
f(x) &= \begin{bmatrix} x_2 \\ -Dx_2 - P_m(\cos x_1 - 1) - \sqrt{1-P_m^2}\sin x_1 + u_2(x) + \epsilon(x)\Big(b + \int_{0\le y\le \pi/4} y\nu(dy)\Big) \end{bmatrix}\\
H(x,y) &= \begin{bmatrix} 0 \\ \epsilon(x)y \end{bmatrix},
\end{aligned}
$$

We can then evaluate $(\mathcal{L}V_1)(x)$ by

$$
\begin{aligned}
(\mathcal{L}V_1)(x) &= x_2\left[P_m(\cos x_1 - 1) + \sqrt{1-P_m^2}\sin x_1\right] \\
&+ x_2\Big[-Dx_2 - P_m(\cos x_1 - 1) - \sqrt{1-P_m^2}\sin x_1 + u_2(x) + \epsilon(x)(b + \int_{0\le y\le \pi/4} y\nu(dy))\Big] \\
&+ \int_{0\le y\le \pi/4}\Big[V_1(x + H(x,y) - V_1(x) - H^i(\partial_i V_1)(x)\Big]\nu(dy), \quad i \in [1,2]. \qquad (2.13)
\end{aligned}
$$

By applying the Mean Value Theorem to the integral part of $(\mathcal{L}V_1)$, we have:

$$V_1(x + H(x,y)) - V_1(x) = H^i(\partial_i V_1)(x) + \frac{1}{2}H_2^2(x,y)(\partial_2^2 V_1)(x_m),$$

where $x_m = \begin{bmatrix} x_1 \\ x_{2m} \end{bmatrix}$, $x_2 \le x_{2m} \le x_2 + H_2(X,y)$. Using Lyapunov function V_1, we evaluate the integral part of equation (2.13) as follows:

$$V_1(x + H(x,y)) - V_1(x) - H^i(\partial_i V_1)(x) = \frac{1}{2}\epsilon^2(x)y^2$$

Using the equality $\int_{0\le y\le\pi/4} y\nu(dy) = N_1$ and defining the control function $u_2 = -Kx_2$, we then have

$$
\begin{aligned}
(\mathcal{L}V_1)(x) &= x_2\Big[P_m(\cos x_1 - 1) + \sqrt{1-P_m^2}\sin x_1\Big] + x_2\Big[-Dx_2 - P_m(\cos x_1 - 1) \\
&- \sqrt{1-P_m^2}\sin x_1 + (b+N_1)\epsilon(x) - Kx_2\Big] + \frac{\epsilon^2(x)}{2}\int_{0\le y\le\pi/4} y^2\nu(dy) \\
&= -(D+K)x_2^2 + (b+N_1)\epsilon(x)x_2 + \frac{\epsilon^2(x)}{2}M_1.
\end{aligned}
$$

Under Assumption 2.1, we obtain the following inequality:

$$(\mathcal{L}V_1)(x) \le -(D + K - \beta_2(b+N_1))x_2^2 + \frac{M_1}{2}\beta_2^2 x_2^2.$$

Then the desired condition on the region B_h is satisfied if $K > \frac{M_1}{2}\beta_2^2 + \beta_2(|b| + N_1) - D$. This completes the proof. □

2.2.2. P-th Moment Exponential Stability in Probability

This section presents theory demonstrating that the control function provides 2-nd moment exponential stability in probability of the SMIBS disturbed by a modified totally positively skewed tempered α-stable Lévy process, given certain conditions.

Theorem 2.2 *Suppose that Assumption 2.2 holds and the maximum value of the stable parameter α is 1. Then the trivial solution of the equation (2.2) is 2nd moment exponentially stable for all initial values $X_0 \in \mathbb{R}^N$, given the following control function $u(x) = [u_1(x) \;\; u_2(x)]^T$:*

$$u(x) = \begin{bmatrix} -K_1 x_1 - x_2 \\ -2P_m \sin\frac{x_1^2}{2} - K_2 x_2 \end{bmatrix},$$

where

$$K_3 = \min(K_1,\ D + K_2) > (\frac{K_2}{2} + |b| + N_2)\,\beta + \frac{\beta^2}{2} M_2, \tag{2.14}$$

and in the light of the measure (2.5)

$$M_2 = \int_{0<y<c} y^2 \nu(dy) < \int_0^\infty y^2 \frac{e^{-\lambda} y}{y^{1+\alpha}} dy = \lambda^{\alpha-2}\Gamma(2-\alpha) < \infty \qquad .$$

and

$$N_2 = \int_{0<y<c} y\nu(dy) < \int_0^\infty y \frac{e^{-\lambda y}}{y^{1+\alpha}} dy = \lambda^{\alpha-1}\Gamma(1-\alpha) < \infty,\ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

Note : λ represents the tempering parameter and α signifies the stability parameter of the process.

Proof In this part, we check that the solution of (2.2) satisfies each of the conditions of Theorem 1.10:

1. If the Lyapunov function is taken as $V_2(x) = ||x||^2/2$, then the first condition is satisfied for $\alpha_1 = \alpha_2 = 1/2$ and $p = 2$.
2. The second condition $\mathcal{L}V_2 \leq -\alpha_3 V_2(X)$ can be satisfied by obtaining a positive, $\alpha_3 \in \mathbb{R}$ as follows.

In order to evaluate $\mathcal{L}V_2$, we must first write a Lévy decomposition of the MTPSTS noise $TL_\alpha(t)$ in the system, then reorganize the stochastic model, by considering the equality $\tilde{N}(t, dy) = N(t, dy) - \nu(dy)t$, as follows:

$$TL_\alpha(t) = bt + t\int_{0<y<c} y\nu(dy) + \int_{0<y<c} y\tilde{N}(t, dy), \tag{2.15}$$

Taking the process (2.15) and the control function (2.14), it is apparent that the function $F(x)$ in stochastic model (2.2) can be expressed as follows:

$$F(x) = \begin{bmatrix} x_2 + u_1(x) \\ -Dx_2 + P_m - P_m \cos x_1 - \sqrt{1 - P_m^2} \sin x_1 + u_2(x) + \epsilon(x)\Big(b + \int_{0<y<c} y\nu(dy)\Big) \end{bmatrix}.$$

Define the energy function as the Lyapunov function:

$$V_2(x) = xx^T/2.$$

We now can evaluate $\mathcal{L}V_2$. To do so, first recall the formula of the infinitesimal operator (1.22) with respect to the d-dimensional SDE equation (1.25):

$$dX(t) = f(X(t))dt + \int_{||y||<c} H(X(t), y)\tilde{N}(dt, dy) \tag{2.16}$$

In this case, the infinitesimal operator of the Lyapunov function with respect to the solution of the SDE is (2.16) as follows:

$$
\begin{aligned}
(\mathcal{L}V)(x) &= f^i(x)(\partial_i V)(x) \\
&+ \int_{\|y\|<c} [V(x + H(x,y)) - V(x) - H^i(x,y)(\partial_i V)(x)]\nu(dy), \quad i \in [1, \cdots, d].
\end{aligned}
$$

Taking into account the control function, $u(x)$, our system corresponds to the following:

$$
\begin{aligned}
f(x) &= \begin{bmatrix} x_2 + u_1(x) \\ -Dx_2 + P_m - P_m \cos x_1 - \sqrt{1 - P_m^2} \sin x_1 + \epsilon(x)\Big(b + \int_{|y|>0} y\nu(dy)\Big) + u_2(x) \end{bmatrix}, \\
H(x,y) &= \begin{bmatrix} 0 \\ \epsilon(x)y \end{bmatrix}.
\end{aligned} \tag{2.17}
$$

The image of the function V_2 under the infinitesimal operator $\mathcal{L}$, calculated similarly as in equation (1.24), is then as follows:

$$
\begin{aligned}
(\mathcal{L}V_2)(x) &= x_1x_2 + x_1u_1(x) + x_2\Big[-Dx_2 + P_m - P_m \cos x_1 - \sqrt{1 - P_m^2} \sin x_1 \\
&+ u_2(x) + \epsilon(x)(b + \int_{0 \le y \le c} y\nu(dy))\Big] \\
&+ \frac{1}{2} \int_{0 \le y \le c} \Big(x_1^2 + \Big(x_2 + \epsilon(x)y\Big)^2 - \Big[(x_1^2 + x_2^2) + 2\epsilon(x)yx_2\Big]\Big)\nu(dy).
\end{aligned}
$$

Considering the control function $u(x)$ given in (2.14) and given the equality $N_2 = \int_{0 \le y \le c} y\nu(dy)$, the integral component can be rearranged and the following expression evaluated:

$$
\begin{aligned}
(\mathcal{L}V_2)(x) &= -K_1x_1^2 - (D + K_2)x_2^2 + x_2(P_m - P_m \cos x_1 \\
&- \sqrt{1 - P_m^2} \sin x_1 - 2P_m \sin \frac{x_1^2}{2}) + (b + N_2)\epsilon(x)x_2 \\
&+ \frac{\epsilon^2(x)}{2} \int_{0<y<c} y^2 \nu(dy).
\end{aligned} \tag{2.18}
$$

Considering the half-angle identities for $(P_m - P_m \cos x_1) = 2P_m \sin \frac{x_1^2}{2}$ and the integral term $\int_{0\leq y<c} y^2\nu(dy)$ of the equality (2.18) equals to M_2, that equation can be rewritten as follows:

$$
\begin{aligned}
(\mathcal{L}V_2)(x) &= -K_1x_1^2 - (D + K_2)x_2^2 - K_2x_2 \sin(x_1) + \epsilon(x)x_2(b + N_2) \\
&+ \frac{\epsilon^2(x)}{2} M_2.
\end{aligned}
$$

By the inequality $|\sin x_1| \leq |x_1|$, it can be seen that:

$$
\begin{aligned}
(\mathcal{L}V_2)(x) &\leq -K_1x_1^2 - (D + K_2)x_2^2 + K_2|x_2x_1| + \epsilon(x)x_2(b + N_2) \\
&+ \frac{\epsilon^2(x)}{2} M_2.
\end{aligned}
$$

Then, by Assumption 2.2, the following inequality can be obtained:

$$
\begin{aligned}
(\mathcal{L}V_2)(x) &\leq -min(K_1, D + K_2)||x||^2 + \frac{K_2||x||^2}{2} + (|b| + N_2)\beta||x||^2 \\
&+ \frac{\beta^2}{2}||x||^2 M_2.
\end{aligned}
$$

Now, if the condition

$$
K_3 = \min(K_1,\ D + K_2) > \frac{K_2}{2} + (|b| + N_2)\beta + \frac{\beta^2}{2} M_2 \tag{2.19}
$$

is satisfied, then the desired stability condition is obtained:

$$(\mathcal{L}V_2)(x) \leq -\left[K_3 - (\frac{K_2}{2} + |b|\beta + \frac{\beta^2}{2}M_2)\right]||x||.$$

And the proof is completed. $\square$

2.3. Numerical Experiment

In this section, we will present the existing algorithm of modified positively skewed α stable process with maximum jumps size c and an example.

2.3.1. Algorithm

The modified totally skewed tempered α-stable random variable ($TL_\alpha^m(t)$) is simulated by the rejection method in three steps (algorithm is taken from Baeumer (2010)):

Step 1: We generate exponential random variable E with mean λ^{-1}, $\mathbb{P}(\mathrm{E} > x) = e^{-\lambda^{-1}x}$.

Step 2: We generate totally skewed $\alpha-$stable random variable $L_\alpha(\alpha < 1)$, by using the following formula Baeumer (2010); Assoc (1976):

$$L_\alpha(t) = (|c_1|t)^{\frac{1}{\alpha}} \frac{\sin\alpha\left(\gamma + \frac{\pi}{2}\right)}{(\cos\gamma)^{1/\alpha}} \left(\frac{\cos\left(\gamma - \alpha\left(\gamma + \frac{\pi}{2}\right)\right)}{W}\right)^{(1-\alpha)/\alpha}.$$

Here, the parameter $c_1 = \Gamma(2-\alpha)/(\alpha-1)$ is fixed where Γ is the known Gamma function, γ is uniformly distributed on $[-\pi/2, \pi/2]$, and W has exponential distribution with parameter 1,

Step 3: If $E > L_\alpha$ we replace $TL_\alpha^m(t) = L_\alpha(t) + c_1 t\alpha\lambda^{\alpha-1}$, otherwise we go back to step 1.

To simulate the entire path, follow these steps:

- Define $TL_\alpha^m(t)$ as the sum of increments over the specified time interval: $TL_\alpha^m(t) = \sum_{i=1}^n [TL_\alpha^m(k\Delta t) - TL_\alpha^m((k-1)\Delta t)]$

Here, t represents the total time ($t = n\Delta t$), and Δt is the time increment.

Next, rearrange the simulation process as follows:

- Start with $TL_\alpha^m((n-1)\Delta t)$. Calculate $TL_\alpha^m(n\Delta t) - TL_\alpha^m((n-1)\Delta t)$.
- If the calculated difference $(TL_\alpha^m(n\Delta t) - TL_\alpha^m((n-1)\Delta t))$ is greater than a specified threshold c, then set $TL_\alpha^m(n\Delta t)$ equal to $TL_\alpha^m((n-1)\Delta t)$. Otherwise, proceed to the next step

By using the Euler-Maruyama approximation presented in Janicki and Weron (1993), we obtain the numerical solutions of (2.2) using

$$X_{t_{i+1}} = X_{t_i} + f(X(t_{i-1}))\tau + \epsilon(X(t_{i-1}))\Delta TL_\alpha(\tau).$$

2.3.2. Example

In this section, we test the efficiency of the control function according to the change of α for given damping parameter $D = 0.5$, power input $P_M = 0.5$, intensity function $\epsilon(x) = \beta\|x\|$ and Lévy noise measure $\nu(dx)$ of the process.

- Take stability parameter as $\alpha = 0.2$ and tempering parameter $\lambda = 2$, then the Lévy measure of the noise is as follows:

$$\nu(dx) = \begin{cases} \frac{1}{x^{1+0.2}} e^{-2x} 1_{(0,\infty)}(x) & \text{if } x \in [0,3], \\ 0 & \text{if } x \notin [0,3]. \end{cases}$$

- Take the control function given in Theorem 2.2 accordance with condition (2.14).

The following Figure 2.3 shows the effected tempered α-stable Lévy noise for $\alpha = 0.2$ with intensity noise coefficient $\beta = 0.05$. (For each figure, simulations are repeated 10 times and time period is taken as $[0, 20]$.)

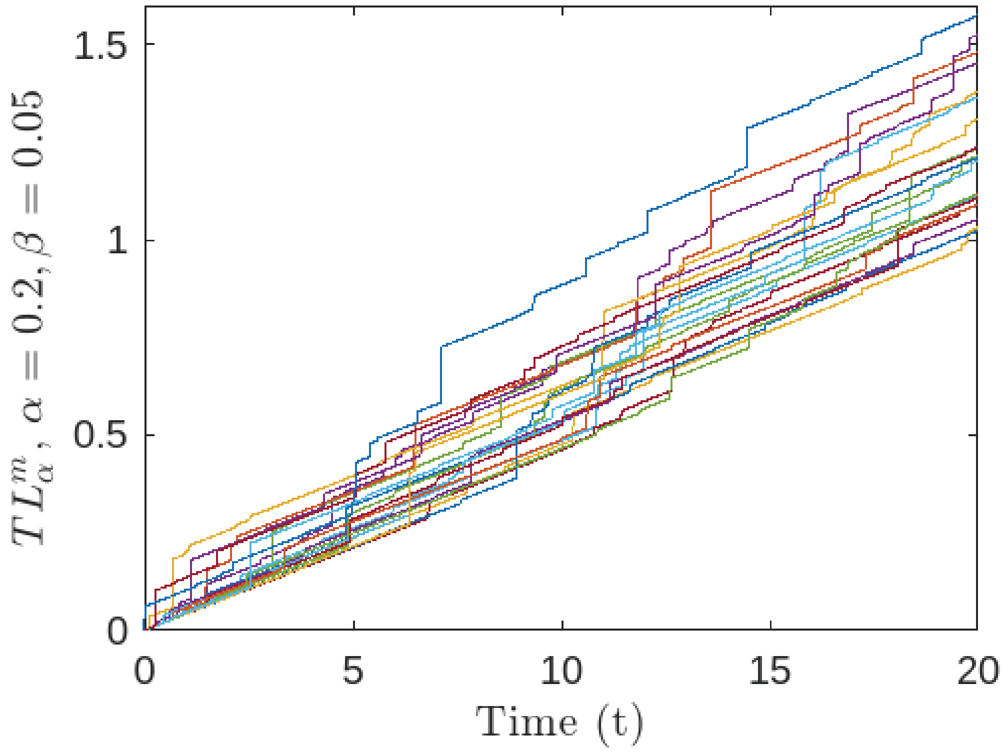

Figure 2.3. Modified totally positively skewed tempered α-stable Lévy Noise for $\alpha =$ 0.2 with intensity noise coefficient $\beta = 0.05$

The following Figure 2.4 shows how the system changes under the modified tempered α-stable Lévy Noise for $\alpha = 0.2$.

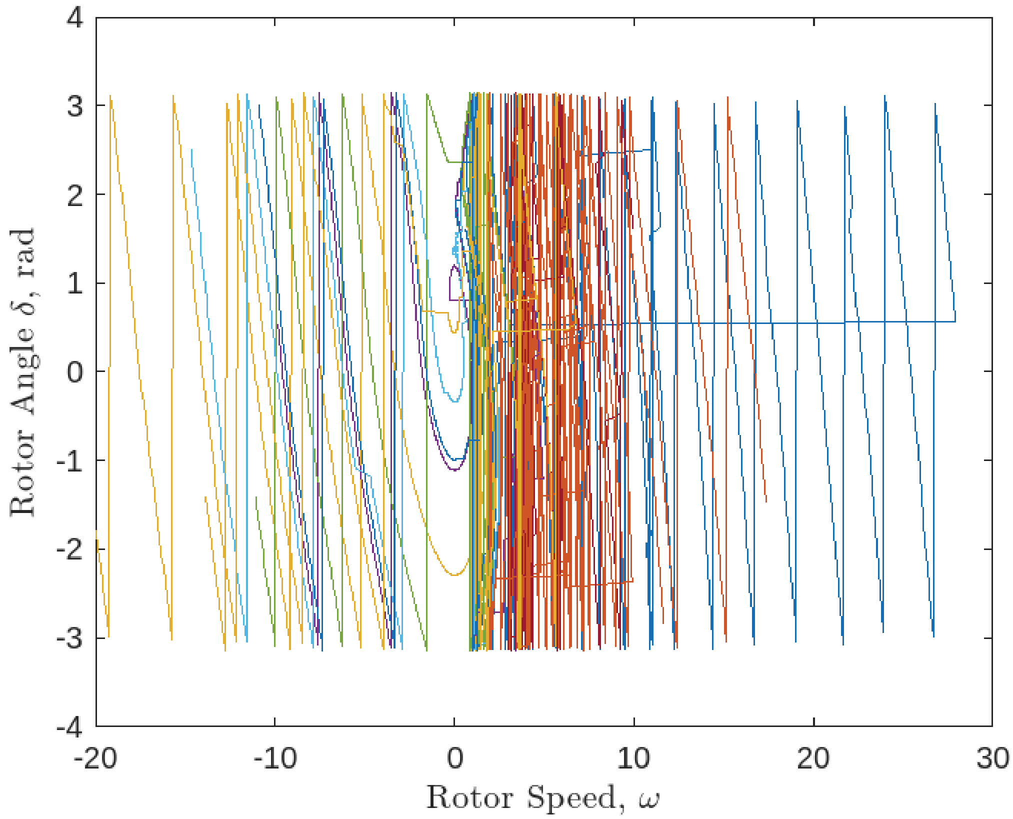

Figure 2.4. Phase portraits of perturbed SMIB system with totally positively skewed tempered α-stable Lévy noise for $\alpha = 0.2$ with intensity noise coefficient $\beta = 0.05$.

According to the following Figure 2.5, trajectories tend to go into a rotating orbit. Then it shows phase angle by applying the control function $u(x)$ to the system.

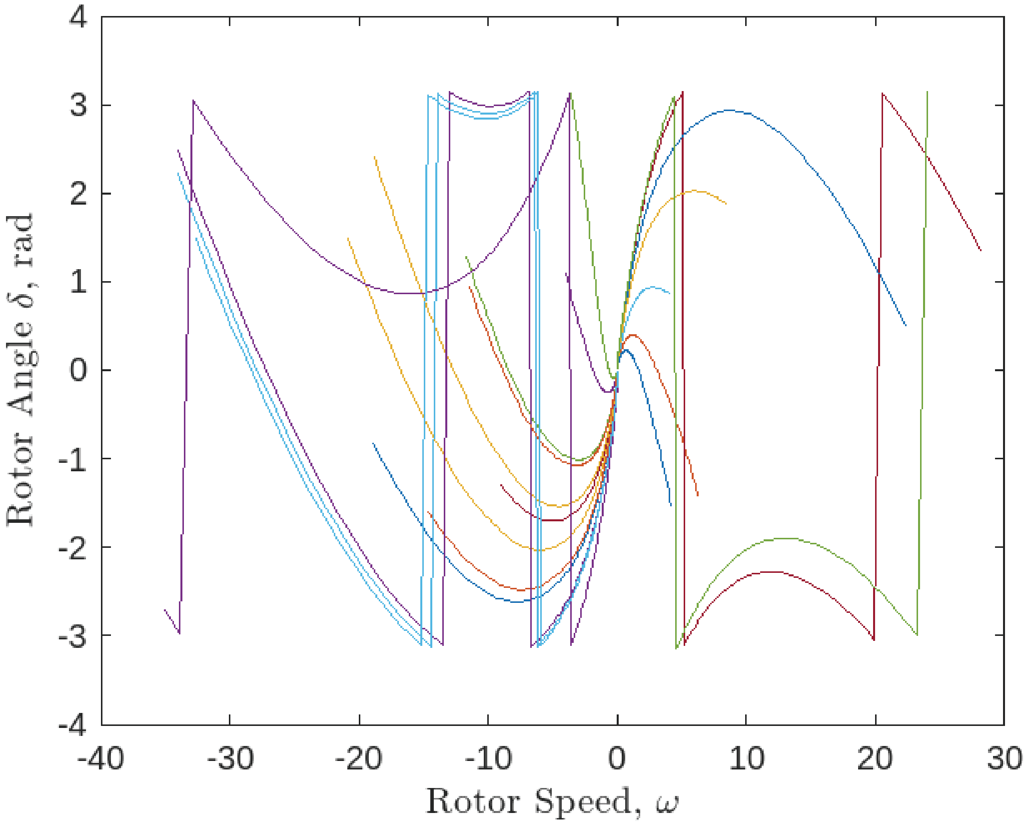

Figure 2.5. Controlled Phase portraits of perturbed SMIB system with totally positively skewed tempered α-stable Lévy noise for $\alpha = 0.2$ with intensity noise coefficient $\beta = 0.05$.

2.3.3. Summary

In this chapter, we obtained a control function for a perturbed SMIBS with modified totally positively skewed tempered α-stable noise by applying Lyapunov Theory. Using this control function, probability stability and 2nd moment exponential stability of the equilibrium points of the SMIBS was achieved. This validity of the control function was supported by results from numerical simulations.

CHAPTER 3

CONTROL OF KURAMOTO-OSCILLATOR NETWORKS DRIVEN BY LÉVY PROCESSES

In this chapter, the probabilistic frequency synchronization and phase agreement are investigated by control functions under Lévy-type stochastic perturbations of electricity power grid consisting of a large number of generators and/or loads (consumers) by using Kuromota Duplex network (coupled nonlinear oscillators). Stochastic perturbations in electric power systems include equipment failures, the effects of weather on wind, solar energy, etc. In this respect, it is assumed that perturbations can be modelled by Lévy Processes with jumps, and it is assumed that the cause of the frequent Blackouts observed in complex power networks in recent years is high variance. Although stochastic perturbations in both generators and loads in electrical power systems are modelled as Gaussian Noise (Wiener Processes) in the literature Wang et al. (2017), the hypothesis of the Lévy-type perturbations in electrical power systems Yılmaz and Savacı (2017); Yılmaz (2019) is supported by Weron (2009), which models oscillations in electricity prices with Lévy Processes and also supported in recent research Schafer et al. (2018). Now let's start to introduce the second problem by introducing the deterministic Kuramoto model.

.

3.1. Model

Consider a duplex network consisting of a Kuramoto-Oscillator layer $\mathbb{K}$ and a control layer $\mathbb{L}$ with the same topology.

$$\mathrm{d}\theta_i(t) = \left[\omega_i + \frac{K}{N}\sum_{j=1}^{N} a_{ij}\sin\Big(\theta_j(t) - \theta_i(t)\Big) + u_i(\theta(t))\right]\mathrm{d}t, \quad i, j \in [1, \cdots N], \tag{3.1}$$

where θ_i is the phase of the ith oscillator, ω_i is the natural frequency of the ith oscillator, and a_{ij} represents an element of the adjacency matrix A, and is indicative the strength of the interaction between oscillators i and j (which is assumed to be proportional to the sin of the difference in their phase angles). In this section, the structure of the duplex network is undirected graph $\mathcal{G}$ with N nodes and $a_{ij} = a_{ji} = 1 > 0$ for $i \neq j$. Otherwise, $a_{ij} = 0$. K, the coupling strength, characterizes the strength of interaction between the oscillators overall. The function u_i in the equation represents the control law.

$$\begin{aligned} u_i &= \frac{C}{N}\sum_{j=1}^{N} b_{ij}\left(\theta_j(t) - \theta_i(t)\right), \quad i, j \in [1, \cdots N], \\ u &= [u_1\ u_2 \cdots u_N]^T \end{aligned} \tag{3.2}$$

where $C > 0$ is the control strength and b_{ij} is the element of the adjacency matrix B that describes the structure of control layer $\mathbb{L}$. The ith element of the solution of the target system, which is desired to be reached in time, is denoted by the function $\tilde{\theta}_i(t)$. This is achieved with the help of the control function u. In the next section, a description of the Target system will be given in the next section. Now, the duplex power network model whose layers are perturbed by Lévy processes is given as follows:

$$\begin{aligned} \mathrm{d}\theta_i(t) &= \left[\omega_i + K\sum_{j=1}^{N} a_{ij}\sin\left(\theta_j(t) - \theta_i(t)\right) + u_i(\theta(t))\right] \\ &+ \mathrm{d}t + \epsilon(\theta(t))dL(t), \quad i, j \in [1, \cdots N]. \end{aligned} \tag{3.3}$$

Here, the function ϵ models the noise intensity of the Lévy processes. This function indicates the intensity of the effect of the Lévy process on the system. If the oscillator layer $\mathbb{K}$ is affected, ϵ is defined as follows:

$$\epsilon(\theta(t)) = \rho_1 \sum_{j=1}^{N} \frac{K}{N} a_{ij} \sin(\theta_j(t) - \theta_i(t))(t). \tag{3.4}$$

If the noise in the control layer $\mathbb{L}$, the noise intensity function $\epsilon(\theta(t))$ is defined as follows

$$\epsilon(\theta(t)) = \rho_2 \sum_{j=1}^{N} \frac{C}{N} b_{ij}(\theta_j(t) - \theta_i(t)), \tag{3.5}$$

where ρ_1 and ρ_2 are intensity parameters of the noise in the Kuramoto layer $\mathbb{K}$ and control layer $\mathbb{L}$ respectively. Regarding Lévy-type processes affecting the control and Kuramoto layers, the following assumption that the system contains finite jumps due to the nature of the system is considered.

Assumption 3.1 *We assume that the Lévy process $L(t)$ in the equation (3.11) is one dimensional and the size of its jumps is limited by the value $c > 1$ due to its nature. Thus, it has the following Lévy-Ito decomposition form.*

$$L(t) \;=\; bt + B_A(t) + \int_{|y|\leq 1} y\tilde{N}(t, dy) + \int_{1<|y|\leq c} yN(t, dy),$$

$$E(N(dt, dy)) = \nu(dy)dt. \tag{3.6}$$

Here $bt + B_A(t)$ is a Gaussian process with variance At, mean bt, $b \in \mathbb{R}$, and N is an independent Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R} - \{0\})$. ν represents the Lévy measure on $\mathbb{R}$ which is related to Poisson random measure N.

.

Assumption 3.2 *It is assumed that the Kuramoto oscillators have the same frequency $\omega_i = \omega$, where $i \in [1, \cdots N]$.*

The purpose of this section is to ensure that the network maintains phase agreement and frequency synchronization, defined as follows:

Definition 3.1 *The Kuramoto-oscillator network (3.1) achieves p-th moment exponential phase agreement if the following condition is satisfied:*
If there exist positive constants C and M such that

$$\mathbb{E}\left[\sum_{i=1}^{N}\sum_{j=1}^{N} |\theta_i(t) - \theta_j(t)|^p\right] \leq C \sum_{i=1}^{N}\sum_{j=1}^{N} |\theta_i(t_0) - \theta_j(t_0)|^p e^{-Mt}, \;\; t \geq t_0 \tag{3.7}$$

for all $\theta(t_0) \in \mathbb{R}^N$.

Definition 3.2 *The Kuramoto-oscillator network (3.1) achieves p-th moment exponential frequency synchronization, if there exist positive constants C and M such that*

$$\mathbb{E}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}|\dot{\theta}_i(t)-\dot{\theta}_j(t)|^p\right] \leq C\sum_{i=1}^{N}\sum_{j=1}^{N}|\dot{\theta}_i(t_0)-\dot{\theta}_j(t_0)|^p e^{-Mt}, \quad t \geq t_0 \tag{3.8}$$

for all $\theta(t_0),\ \omega(t_0) \in \mathbb{R}^N$.

Lemma 3.1 *Wu and Li (2019b) For an undirected graph* $\mathcal{G}$ *with N nodes, the following equality is satisfied*

$$x^T\mathcal{L}_A x = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left(x_i - x_j\right)^2, x = (x_1, x_2, \ldots, x_N)^T \tag{3.9}$$

Here, $\mathcal{L}_A = D^A - A$ *is the Laplacian matrix associated with the adjacency matrix A, where* $D^A \in R^{N\times N}$ *is a diagonal matrix with* $D^A_{ii} = \sum_{j=1}^{N} a_{ij}(\forall i \in [1, \cdots N])$. *The equality* (3.9) *implies that matrix* $\mathcal{L}_A$ *is positive semi-definite, with non-negative eigenvalues. With* $\mathcal{L}_A$ *represented using its eigenvalues* $0, \lambda_2, \cdots, \lambda_N$, *ordered to satisfy* $0 < \lambda_2 < \cdots < \lambda_N$, *and* 1_N *representing the N dimensional vector whose components are all equal to one, if* $1_N^T x = 0$, *then* $x^T L x \geq \lambda_2 x^T x$.

3.2. Control of Stochastic Phase Agreement and Frequency Synchronization of the Duplex Network

In this section, our aim is to ensure that the duplex system perturbed by Lévy processes reaches phase agreement and frequency synchronization with the help of the control function. To achieve this, the perturbed system is intended to reach the target system, where the oscillators exhibit a common behavior, by utilizing the control function $u(x)$ provided in equation (3.2).

Target System

$$\begin{aligned}\dot{\tilde{\theta}}_i(t) &= \omega, \;\; i \in [1, \cdots N], \\ \tilde{\theta}_i(t_0) &= \theta_0. \end{aligned} \tag{3.10}$$

Here the function $\tilde{\theta}_i(t)$ defined as $\tilde{\theta}_i(t) = \tilde{\theta}(t) = \frac{1}{N}\sum_{i=1}^{N}\theta_i(t)$ is the mean phase angle of oscillators on all N nodes on the graph, t_0 is the initial time and ω is the common natural frequency of the oscillators.

Now, the theorems about the formation of the control rule necessary for the system to have phase agreement and frequency synchronization when the oscillator and control layers are perturbed by the Lévy process given in the equation (3.6) are given.

3.2.1. Control of Stochastic Phase Agreement under Lévy Type Perturbations

In this section, assuming that the frequencies ω_i are the same, the system parameter-dependent control function required for phase synchronization when the Kuramoto model is disturbed by Lévy noise is constructed.

3.2.2. Lévy Type Perturbations on the Kuramoto-Oscillator Layer

This section concerns the synchronization of the system's phase angles when each node in the Kuramoto-oscillator layer $\mathbb{K}$ is perturbed by Lévy process and modeled with the following equation.

$$\begin{aligned}\mathrm{d}\theta_i(t) = \omega_i \;\; &+ \;\; \left[\frac{K}{N}\sum_{j=1}^{N} a_{ij}\sin\Big(\theta_j(t) - \theta_i(t)\Big) + u_i(\theta(t))\right]\mathrm{d}t \\ &+ \;\; \rho\frac{K}{N}\sum_{j=1}^{N} a_{ij}\sin\Big(\theta_j(t) - \theta_i(t)\Big)\,\mathrm{d}L(t), \;\; i, j \in [1, \cdots N]. \end{aligned} \tag{3.11}$$

Theorem 3.1 *Phase agreement of the Kuramoto-oscillator layer Lévy process-perturbed duplex network (3.11) is achieved under Assumption 3.1 and Assumption 3.2 and the following condition between the system parameters in (3.11):*

$$C > \left[(1+S)\frac{(\rho K)^2}{2} + K(1 + |\rho b| + |\rho R|)\right], \tag{3.12}$$

where

$$S = \int_{|y|\le c} y^2 \nu(dy), \quad R = \int_{1<|y|<c} y\nu(dy) \tag{3.13}$$

Note :These values are valid for all S and R values of this chapter.

Proof At this stage, the aim is to prove that all oscillators approach the $\tilde{\theta}(t)$ function asymptotically in time. In this respect, define the phase error/difference of each oscillator $e_i(t) = \theta_i(t) - \tilde{\theta}(t)$, should approach zero. Then, the phase error system evolves $\dot{e}_i(t) = \dot{\theta}_i(t) - \dot{\tilde{\theta}}(t) = \dot{\theta}_i(t) - \omega$ as and the stochastic error system can be calculated as follows:

$$\begin{aligned} \mathrm{d}e_i(t) = & \left[\frac{K}{N}\sum_{j=1}^{N} a_{ij}\sin\left(e_j - e_i\right) + \frac{C}{N}\sum_{j=1}^{N} b_{ij}\left(e_j - e_i\right)\right]\mathrm{d}t \\ & + \rho\sum_{j=1}^{N}\frac{K}{N}a_{ij} sin(e_j - e_i)\mathrm{d}L(t). \end{aligned} \tag{3.14}$$

The stability of the trivial solution of (3.14) can be examined with the help of Lyapunov Theory. For this purpose, the conditions given by Siakalli in her thesis Siakalli (2009), as stated in Theorem 1.10, are checked to see if the solution of (3.14) satisfies them.

1. If the Lyapunov function is taken as follows:

$$V_1 = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e} = \frac{1}{2}\sum_{i=1}^{N} e_i^2, \tag{3.15}$$

   where $\mathbf{e} = [e_1, \ldots, e_N]^{\mathrm{T}}$ is the phase error vector. Then first condition is satisfied for $\alpha_1 = \alpha_2 = 1/2$ and $p = 2$.

2. A $\alpha_3 \in \mathbb{R}^+$ can be found that will satisfy the second condition $\mathcal{L}V \leq -\alpha_3 V(X)$ as follows:

Before evaluating the infinitesimal operator $\mathcal{L}$ defined in (1.24) for the error dynamical system (3.14), the Poisson random measure is needed to be divided into compensated Poisson random measure and Lévy measure in the Lévy process as follows in order to apply the stability theorem given in (1.10):

$$L(t) = bt + B_A(t) + \int_{|y|\leq 1} y\tilde{N}(dt, dy) + t\int_{1<|y|\leq c} y\nu(dy) + \int_{1<|y|\leq c} y\tilde{N}(dt, dy). \quad (3.16)$$

By considering Theorem 1.11 and Theorem 1.12, the boundedness of the integral term is evaluated

$$R = \int_{1<|y|\leq c} y\nu(dy) \quad (3.17)$$

As the defined variable R is finite, it becomes possible to rearrange the L'evy-Ito decomposition into the following form.

$$\begin{aligned} L(t) &= bt + B_A(t) + t\int_{1<|y|\leq c} y\nu(dy) + \int_{|y|\leq c} y\tilde{N}(t, dy) \\ &= bt + B_A(t) + tR + \int_{|y|\leq c} y\tilde{N}(t, dy). \end{aligned} \quad (3.18)$$

Finally, let's evaluate the infinitesimal operator $\mathcal{L}$ defined in (1.24) for the error dynamical system (3.14) and Lévy-Ito decomposition given in (3.18) as follows:

$$\mathcal{L}V_1 = \sum_{i=1}^{N} e_i \left[[\sum_{j=1}^{N} \frac{K}{N} a_{ij}(1 + b\rho + R\rho)] \sin\left(e_j - e_i\right) + \frac{C}{N} \sum_{j=1}^{N} b_{ij} \left(e_j - e_i\right) \right]$$

$$+ \frac{\rho^2}{2} \sum_{i=1}^{N} \text{trace}\left\{ \Big[\sum_{j=1}^{N} \frac{K}{N} a_{ij} \sin(e_j - e_i) \Big] \times \Big[\sum_{j=1}^{N} \frac{K}{N} a_{ij} \sin(e_j - e_i) \Big]^T \right\}$$

$$+ \sum_{i=1}^{N} \int_{|y| \le c} \frac{1}{2} \Big[\Big(e_i + \rho \sum_{j=1}^{N} \frac{K}{N} a_{ij} \sin\left(e_j - e_i\right) y \Big)^2 - e_i^2$$

$$- e_i \rho \sum_{j=1}^{N} \frac{K}{N} a_{ij} \sin(e_j - e_i) y \Big] \nu(dy).$$

The above equality is transformed into the following expression, considering the Lyapunov function $V(e) = ee^T/2$ that has been chosen and the jump function in the form $H(e, y) = \rho \sum_{j=1}^{N} \frac{K}{N} a_{ij} \sin(e_j - e_i) y$

$$V_1(x + H(x, y)) - V_1(x) - H^i(x, y)\partial_i V_1(x) = \frac{1}{2} \left(\rho \sum_{j=1}^{N} \frac{K}{N} a_{ij} \sin\left(e_j - e_i\right) y \right)^2. \quad (3.19)$$

By considering the following inequality for

$$\frac{1}{2} \left(\rho \sum_{j=1}^{N} \frac{K}{N} a_{ij} \sin\left(e_j - e_i\right) y \right)^2 \le \frac{N}{2} y^2 \sum_{j=1}^{N} (\rho \frac{K}{N} a_{ij})^2 \sin^2\left(e_j - e_i\right).$$

Then the following inequality for the last term of the $\mathcal{L}V_1$ is obtained

$$\frac{1}{2} \int_{|y| \le c} \left(\rho \sum_{j=1}^{N} \frac{K}{N} a_{ij} \sin\left(e_j - e_i\right) y \right)^2 \nu(dy) \le \frac{N}{2} \sum_{j=1}^{N} (\rho \frac{K}{N} a_{ij})^2 \sin^2\left(e_j - e_i\right) \int_{|y| \le c} y^2 \nu(dy). \quad (3.20)$$

The integral term $S = \int_{|y|\le c} y^2\nu(dy)$ defined in (3.13) and established above is finite. It can be rewritten as follows,

$$\int_{|y|\le c} y^2\nu(dy) = \int_{0<|y|<1} y^2\nu(dy) + \int_{1<|y|<c} y^2\nu(dy). \tag{3.21}$$

First integral term in right side of the equation (3.21) is bounded by definition of Lévy noise, and second integral term is bounded result of Theorem 1.11 and Theorem 1.12 for the stochastic system includes bounded jumps. By taking into account the equality (3.13) and (3.20), $\mathcal{L}V_1$ can be reorganized in the following manner

$$\mathcal{L}V_1 \le \sum_{i=1}^{N} e_i \sum_{j=1}^{N}\Big[\frac{K}{N}a_{ij}(1+\rho b+\rho R)\Big]\sin\left(e_j - e_i\right)$$

$$+\frac{C}{N}\sum_{i=1}^{N} e_i \sum_{j=1}^{N} b_{ij}\left(e_j - e_i\right) + \frac{N\rho^2}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(\frac{K}{N}a_{ij})^2\sin^2(e_j - e_i))$$

$$+\frac{NS}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(\rho\frac{K}{N}a_{ij})^2\sin^2\left(e_j - e_i\right).$$

Now, the expression in parentheses $(e_j - e_i)^2$ is put to get the negative definiteness by editing the other terms.

$$\begin{aligned}
\mathcal{L}V_1 \quad &\le \quad -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\frac{K}{N}a_{ij}(1+\rho b+\rho R)\frac{\sin\left(e_j - e_i\right)}{e_j - e_i}\right](e_j - e_i)^2 \\
&- \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{C}{N}b_{ij}(e_j - e_i)^2 \\
&+ \quad \left[\frac{N\rho^2}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(\frac{K}{N}a_{ij})^2\frac{\sin^2\left(e_j - e_i\right)}{(e_j - e_i)^2}\right](e_j - e_i)^2 \\
&+ \quad \left[\frac{NS}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(\rho\frac{K}{N}a_{ij})^2\frac{\sin^2\left(e_j - e_i\right)}{(e_j - e_i)^2})\right](e_j - e_i)^2.
\end{aligned} \tag{3.22}$$

Let's sum all terms under the same sum expression

$$\mathcal{L}V_1 = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\Big[\frac{K}{N}a_{ij}(1+\rho b+\rho R)\frac{\sin\Big(e_j-e_i\Big)}{e_j-e_i}+\frac{C}{N}b_{ij} \\ -(1+S)\frac{(\rho K a_{ij})^2}{2N}\frac{\sin^2(e_j-e_i)}{(e_j-e_i)^2}\Big](e_j-e_i)^2.$$

Since $(\sin(r)/(r))_{r\in[-\pi,\pi]} \in [-1,1]$ and if $i \neq j$ then $a_{ij} = b_{ij} = 1$ else $a_{ij} = b_{ij} = 0$ for all $i, j \in [1, \cdots, N]$, this

$$\mathcal{L}V_1 \leq -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\Bigg[(-\frac{K}{N}(1+|\rho b|+|\rho R|)+\frac{C}{N}-(1+S)\frac{(\rho K)^2}{2N})a_{ij}\Bigg](e_j-e_i)^2.$$

By the Lemma 3.1, thus the following equality is satisfied

$$= -\Bigg[C-(1+S)\frac{(\rho K)^2}{2}-K(1+|\rho b|+|\rho R|)\Bigg]e\mathcal{L}_A e^T. \tag{3.23}$$

Taking into account the $1_N e = 0$, then the following inequality given in the Lemma can be used3.1

$$e\mathcal{L}_A e^T \geq \lambda_2(\mathcal{L}_A)ee^T. \tag{3.24}$$

If the following conditions are satisfied

$$C > \Bigg[(1+S)\frac{(\rho K)^2}{2}+K(1++|\rho b|+|\rho R|)\Bigg], \tag{3.25}$$

By considering (3.24) and (3.25), a bound for $\mathcal{L}V_1$ can be evaluated, as follows:

$$\mathcal{L}V_1 \leq -\Bigg[C-(1+S)\frac{(\rho K)^2}{2}-K(1+\rho b+\rho R)\Bigg]\lambda_2(\mathcal{L}_A)ee^T. \tag{3.26}$$

Finally, the negativeness condition of the Lyapunov operator $\mathcal{L}V_1 \leq -\alpha_3 V_1$ for Theorem1.10 is established, by setting α_3 as follows:

$$\alpha_3 = \left[C - (1+S)\frac{(\rho K)^2}{2} - K(1+\rho b + \rho R)\right]\lambda_2(\mathcal{L}_A)/2. \tag{3.27}$$

The trivial solution of the error stochastic system (3.14) is thus 2nd moment exponentially stable, and as a result the conditions for phase agreement are also fulfilled, completing the proof. □

3.2.3. Lévy Type Perturbations on the Control Layer

In this section, the situation where the control layer, which helps to provide synchronization to the system, is affected by the Lévy process is considered and determined the relationship between the system parameters necessary for phase agreement. First, let's examine the current Kuramoto model with distributed control u_i using equations (3.1) and (3.2).

$$\begin{aligned} d\theta_i(t) = \omega_i \quad &+ \quad \left[\frac{K}{N}\sum_{j=1}^{N} a_{ij}\sin\Big(\theta_j(t)) - \theta_i(t)\Big) + \frac{C}{N}\sum_{j=1}^{N} b_{ij}\Big(\theta_j(t) - \theta_i(t)\Big)\right]dt \\ &+ \quad \rho\frac{C}{N}\sum_{j=1}^{N} b_{ij}\Big(\theta_j(t) - \theta_i(t)\Big)dL(t), \;\; i, j \in [1, \cdots N]. \end{aligned} \tag{3.28}$$

Theorem 3.2 *If the following condition is satisfied between system parameters in (3.28), the duplex Kuramoto-oscillator network (3.28) achieves 2nd moment exponential stochastic phase agreement, even if the control layer environment is subject to the Lévy type process perturbations.*

$$-K + r^2 > 0, \quad C = r/z,$$

where

$$\begin{aligned} z^2 &= \sigma^2\rho^2 + S\rho^2 \\ r &= (1+\rho b + \rho R)/2z. \end{aligned}$$

Proof We will prove this theorem using a similar technique to that used in Theorem 3.1. To do this, first, the error function is constructed such that $e_i(t) = \theta_i(t) - \tilde{\theta}(t)$. Then The stochastic error system is evaluated as follows:

$$\begin{aligned} \mathrm{d}e_i(t) = \quad & \left[\frac{K}{N}\sum_{j=1}^{N} a_{ij}\sin\left(e_j - e_i\right) + \frac{C}{N}\sum_{j=1}^{N} b_{ij}\left(e_j - e_i\right)\right]\mathrm{d}t \\ & +\rho\sum_{j=1}^{N}\frac{C}{N}b_{ij}\left(e_j - e_i\right)\mathrm{d}L(t). \end{aligned} \tag{3.29}$$

As a result of Lyapunov Theory given in Siakalli (2009), the conditions of Theorem 1.10 are checked to determine if the trivial solution of (3.29) satisfies them.

1. If the Lyapunov function is defined as follows:

$$V_2 = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e} = \frac{1}{2}\sum_{i=1}^{N} e_i^2, \tag{3.30}$$

   where $\mathbf{e} = [e_1, \ldots, e_N]^{\mathrm{T}}$ is the phase error vector. So the first condition is met for $\alpha_1 = \alpha_2 = 1/2$ and $p = 2$.

2. As a result of Lyapunov Theory given in Siakalli (2009), the conditions of Theorem 1.10 are checked to determine if the trivial solution of (3.29) satisfies them.

$$\mathcal{L}V_2 \leq -\alpha_3 V_2(X). \tag{3.31}$$

In a similar way in Theorem 3.1, the Poisson random measure is divided into compensated Poisson random measure and Lévy measure in the Lévy process to evaluate the infinitesimal operator $\mathcal{L}$ defined in (1.24) for the error dynamical system (3.29) to apply the stability theorem given in (1.10). The representation of the L'evy process can be observed from the first theorem as follows:

As the variable R defined in (3.13) is finite, the L'evy-Ito decomposition can be rearranged to take the following form.

$$
\begin{aligned}
L(t) &= bt + B_A(t) + t\int_{1<|y|\leq c} y\nu(dy) + \int_{|y|\leq c} y\tilde{N}(t,dy) \\
&= bt + B_A(t) + tR + \int_{|y|\leq c} y\tilde{N}(t,dy). \qquad (3.32)
\end{aligned}
$$

where

$$
R = \int_{1<|y|\leq c} y\nu(dy) \qquad (3.33)
$$

Now, using the equality (3.32), now $\mathcal{L}V_2$ is calculated as follows:

$$
\begin{aligned}
\mathcal{L}V_2 &= \sum_{i=1}^{N} e_i \left[\frac{K}{N}\sum_{j=1}^{N} a_{ij}\sin\left(e_j - e_i\right) + \sum_{j=1}^{N}\Big[\frac{C}{N}b_{ij}(1 + b\rho + R\rho)\Big]\left(e_j - e_i\right)\right] \\
&+ \frac{\sigma^2}{2}\sum_{i=1}^{N}\text{trace}\left\{\left[\rho\sum_{j=1}^{N}\frac{C}{N}b_{ij}\left(e_j - e_i\right)\right]\times\left[\rho\sum_{j=1}^{N}\frac{C}{N}b_{ij}\left(e_j - e_i\right)\right]^T\right\} \\
&+ \sum_{i=1}^{N}\int_{|y|\leq c}\frac{1}{2}\Big[(e_i + \rho\sum_{j=1}^{N}\frac{C}{N}b_{ij}(e_j - e_i)y)^2 - e_i^2 \\
&- 2e_i\rho\sum_{j=1}^{N}\frac{C}{N}b_{ij}(e_j - e_i)y)\Big]\nu(dy).
\end{aligned}
$$

Unlike Theorem 3.1, the jump function is defined as $H(e,y) = \rho\sum_{j=1}^{N}\frac{C}{N}b_{ij}\left(e_j - e_i\right)y$ and same Lyapunov function $V_2(e) = ee^T/2$, the above equation leads to the following expression

$$
V_2(e + H(e,y)) - V_2(e) - H^i(e,y)\partial_i V_2(e) = \frac{1}{2}\left(\rho\sum_{j=1}^{N}\frac{C}{N}b_{ij}\left(e_j - e_i\right)y\right)^2 \qquad (3.34)
$$

By considering the inequality

$$\frac{1}{2}\left(\rho\sum_{j=1}^{N}\frac{C}{N}b_{ij}\left(e_j-e_i\right)y\right)^2 \leq \frac{N}{2}\sum_{j=1}^{N}(\rho\frac{C}{N}b_{ij})^2\left(e_j-e_i\right)^2 y^2,$$

Then, for the last term of $\mathcal{L}V_2$, the following inequality is obtained

$$\frac{1}{2}\int_{|y|\leq c}\left(\rho\sum_{j=1}^{N}\frac{C}{N}b_{ij}\left(e_j-e_i\right)y\right)^2 \nu(dy) \leq \frac{N}{2}\sum_{j=1}^{N}(\rho\frac{C}{N}b_{ij})^2\left(e_j-e_i\right)^2\int_{|y|\leq c}y^2\nu(dy). \tag{3.35}$$

The boundedness of the integral term $S = \int_{|y|\leq c} y^2\nu(dy)$ was showed in Theorem 3.1. Then $\mathcal{L}V_2$ is organized by considering equalities (3.13) and (3.35) Then the following inequality is evaluated

$$\begin{aligned}\mathcal{L}V_2 \leq \; & -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\frac{K}{N}a_{ij}\frac{\sin\left(e_j-e_i\right)}{e_j-e_i}\right](e_j-e_i)^2 \\ & -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{C}{N}b_{ij}(1+b\rho+R\rho)(e_j-e_i)^2 \\ & +\frac{N\sigma^2\rho^2}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{C^2}{N^2}b_{ij}^2(e_j-e_i)^2 \\ & +\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{NS}{2}(\rho\frac{C}{N}b_{ij})^2\left(e_j-e_i\right)^2.\end{aligned}$$

Now, the terms can be put in parentheses $(e_j-e_i)^2$ and alter the remaining terms to achieve negative definiteness.

$$\begin{aligned} = \; & -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\Big[\frac{K}{N}a_{ij}\frac{\sin(e_j-e_i)}{e_j-e_i}+\frac{C}{N}b_{ij}(1+b\rho+R\rho) \\ & -N\sigma^2\rho^2\frac{C^2}{N^2}b_{ij}^2-NS(\rho\frac{C}{N}b_{ij})^2\Big](e_j-e_i)^2.\end{aligned}$$

By defining the variables as follows:

$$z^2 = \sigma^2\rho^2 + S\rho^2$$
$$r = (1 + \rho b + \rho R)/2z$$

By using the quadratic property, $\max_{-\pi,\pi} |\frac{\sin(e_j - e_i)}{e_j - e_i}| = 1$ and $a_{ij} = b_{ij}$ (recall that control layer and Kuramoto layer have same topology)

$$\mathcal{L}V_2 = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}[-K + r^2 - [zC - r]^2]a_{ij}(e_j - e_i)^2. \tag{3.36}$$

Considering Lemma 3.1, the following equality is satisfied:

$$\mathcal{L}V_2 = -[-K + r^2 - [zC - r]^2]e\mathcal{L}_A e^T \tag{3.37}$$

Taking the following inequality into consideration given in the Lemma 3.1

$$e\mathcal{L}_A e^T \geq \lambda_2(\mathcal{L}_A)ee^T \tag{3.38}$$

, and assuming that the following conditions are satisfied

$$-K + r^2 > 0, \quad C = r/z, \tag{3.39}$$

, the following upper bound for $\mathcal{L}V_2$ can be calculated:

$$\mathcal{L}V_2 \leq -\left[-K + r^2)\right]\lambda_2(\mathcal{L}_A)ee^T. \tag{3.40}$$

As a result, 2nd moment exponentially stability of trivial solution of the error stochastic system (3.29) can be achieved, by setting α_3 in the second condition (3.31) to $[-K + r^2]\lambda_2(\mathcal{L}_A)$, completing the proof. □

3.3. Control of Stochastic Frequency Stability under Lévy Type Perturbations

This section, extends the results of Wu and Li (2021) for Lévy process noise. Thus, the equation obtained from the derivative of the deterministic Kuramoto model, which gives the frequency of the system, is assumed that is subjected to perturbation by Lévy-type noise, and then establish the relationship between the system parameters required for frequency synchronization of the duplex network under this assumption. No restrictions on the frequencies ω_i is required for the application of the methods of this section.

3.3.1. Lévy Type Perturbations on the Kuramoto-Oscillator Layer

The Kuramoto model of the perturbed oscillator network with distributed control function u_i is generated with composing derivative of (3.1) and (3.2) equations.

$$\begin{aligned} \mathrm{d}\dot{\theta}_i(t) &= \left[\frac{K}{N}\sum_{j=1}^{N} a_{ij}\cos\left(\theta_j-\theta_i\right)\left(\dot{\theta}_j-\dot{\theta}_i\right)+\frac{C}{N}\sum_{j=1}^{N} b_{ij}\left(\dot{\theta}_j-\dot{\theta}_i\right)\right]\mathrm{d}t \\ &+ \rho\sum_{j=1}^{N}\frac{K}{N}a_{ij}\cos\left(\theta_j-\theta_i\right)\left(\dot{\theta}_j-\dot{\theta}_i\right)\mathrm{d}L(t), \quad i,j\in[1,\cdots,N]. \end{aligned} \tag{3.41}$$

Theorem 3.3 *If the following criteria listed below is satisfied between system parameters in (3.41), the specified duplex Kuramoto-oscillator network in (3.41) has 2nd moment exponential stochastic frequency synchronization when the Kuramoto-oscillator is perturbed*

$$C > K(1+\rho|b|+\rho R)+\rho^2\sigma^2K^2+S\rho^2K^2. \tag{3.42}$$

Proof In this proof, a similar technique to the previous proofs is applied, but the parameter aimed to reach equilibrium is the frequency, which corresponds to the derivative of the oscillator variable.

From stochastic dynamical system (3.41) and Target system (3.10), it is known that the error function for frequency fluctuation $\dot{e}_i = \dot{\theta}_i - \dot{\bar{\theta}}$ and vector $\dot{e} = [\dot{e}_1, \dot{e}_2, ..., \dot{e}_d]^T$. Then the error model that its derivative of variable converges to zero is desired as follows:

$$\begin{aligned} \mathrm{d}\dot{e}_i(t) &= \left[\frac{K}{N}\sum_{j=1}^{N} a_{ij}\cos\left(e_j - e_i\right)\left(\dot{e}_j - \dot{e}_i\right) + \frac{C}{N}\sum_{j=1}^{N} b_{ij}\left(\dot{e}_j - \dot{e}_i\right)\right]\mathrm{d}t \\ &+ \rho\sum_{j=1}^{N}\frac{K}{N}a_{ij}\cos\left(e_j - e_i\right)\left(\dot{e}_j - \dot{e}_i\right)\mathrm{d}L(t). \end{aligned} \tag{3.43}$$

The stability of the trivial solution of (3.43) is examined with the help of Lyapunov Theory. For this aim, if the solution of (3.43) satisfies the conditions is checked in Theorem 1.10 given by Siakalli in her thesis Siakalli (2009), respectively.

1. If the Lyapunov function is taken as follows:

$$V_3 = \frac{1}{2}\dot{\mathbf{e}}^{\mathrm{T}}\dot{\mathbf{e}} = \frac{1}{2}\sum_{i=1}^{N}\dot{e}_i^2, \tag{3.44}$$

   where $\dot{\mathbf{e}} = [\dot{e}_1, \ldots, \dot{e}_N]^{\mathrm{T}}$ is the phase error vector. Then first condition is satisfied for $\alpha_1 = \alpha_2 = 1/2$ and $p = 2$.

2. It is possible to find an α_3 in $\mathbb{R}^+$ which satisfies the second condition $\mathcal{L}V \leq -\alpha_3 V(X)$ as follows:

In order to apply the stability theorem given in (1.10), the Poisson random measure must be divided into compensated Poisson random measure and Lévy measure in the Lévy process as follows before evaluating the infinitesimal operator $\mathcal{L}$ defined in (1.24) for the error dynamical system (3.43).

$$L(t) = bt + B_A(t) + \int_{|y|\leq 1} y\tilde{N}(dt, dy) + t\int_{1<|y|\leq c} y\nu(dy) + \int_{1<|y|\leq c} y\tilde{N}(dt, dy). \tag{3.45}$$

By considering Theorem 1.11 and Theorem 1.12, the boundedness of the integral term is evaluated

$$R = \int_{1<|y|\leq c} y\nu(dy). \tag{3.46}$$

Because of the finiteness of the defined variable R, the L'evy-Ito decomposition can be rearranged to take the following form.

$$
\begin{aligned}
L(t) &= bt + B_A(t) + t\int_{1<|y|\le c} y\nu(dy) + \int_{|y|\le c} y\tilde{N}(t,dy) \\
&= bt + B_A(t) + tR + \int_{|y|\le c} y\tilde{N}(t,dy) \qquad (3.47)
\end{aligned}
$$

Finally, let's evaluate the infinitesimal operator $\mathcal{L}$ defined in (1.24) for the error dynamical system (3.43) and Lévy-Ito decomposition given in (3.47) as follows: $\mathcal{L}$ defined in (1.9) for the above function V_3 along the dynamics (3.43) gives

$$
\begin{aligned}
&\mathcal{L}V_3 = \\
&\sum_{i=1}^{N} \dot{e}_i [\sum_{j=1}^{N} [\frac{K}{N} a_{ij}(1+\rho b+\rho R)] \cos\left(e_j - e_i\right)\left(\dot{e}_j - \dot{e}_i\right) + \frac{C}{N}\sum_{j=1}^{N} b_{ij}\left(\dot{e}_j - \dot{e}_i\right)] \\
&+\frac{\sigma^2}{2}\sum_{i=1}^{N} \text{trace}\left\{\left[\sum_{j=1}^{N}\frac{K}{N}a_{ij}\rho\cos\left(e_j-e_i\right)\left(\dot{e}_j-\dot{e}_i\right)\right]\times\left[\sum_{j=1}^{N}\frac{K}{N}a_{ij}\rho\cos\left(e_j-e_i\right)\left(\dot{e}_j-\dot{e}_i\right)\right]^T\right\} \\
&+\sum_{i=1}^{N}\int_{|y|\le c}\frac{1}{2}\Big[(\dot{e}_i+\rho\sum_{j=1}^{N}\frac{K}{N}a_{ij}\cos(e_j-e_i)(\dot{e}_j-\dot{e}_i)y)^2 \\
&-e_i^2 - 2e_i\rho\sum_{j=1}^{N}\frac{K}{N}a_{ij}\cos(e_j-e_i)(\dot{e}_j-\dot{e}_i)\Big]\nu(dy).
\end{aligned}
$$

According to the chosen Lyapunov function $V_3(e) = \dot{e}\dot{e}^T/2$ and the jump function in the form $H(e,y) = \rho\sum_{j=1}^{N}\frac{K}{N}a_{ij}\cos\left(e_j-e_i\right)\left(\dot{e}_j-\dot{e}_i\right)y$, the above equation returns to the following expression

$$
V_3(x+H(x,y)) - V_3(x) - H^i(x,y)\partial_i V_3(x) = \frac{1}{2}\left(\rho\sum_{j=1}^{N}\frac{K}{N}a_{ij}\cos\left(e_j-e_i\right)\left(\dot{e}_j-\dot{e}_i\right)y\right)^2 \qquad (3.48)
$$

By considering the following inequality for

$$\frac{1}{2}\left(\rho\sum_{j=1}^{N}\frac{K}{N}a_{ij}\cos\left(e_j-e_i\right)\left(\dot{e}_j-\dot{e}_i\right)y\right)^2\leq\frac{N}{2}y^2\sum_{j=1}^{N}(\rho\frac{K}{N}a_{ij})^2\left(\cos(e_j-e_i)(\dot{e}_j-\dot{e}_i)\right)^2.$$

Then the following inequality for the last term of the $\mathcal{L}V_3$ is obtained

$$\begin{aligned}&\frac{1}{2}\int_{|y|\leq c}\left(\rho\sum_{j=1}^{N}\frac{K}{N}a_{ij}(\cos\left(e_j-e_i\right)\left(\dot{e}_j-\dot{e}_i\right)y\right)^2\nu(dy)\\ \leq&\frac{N}{2}\sum_{j=1}^{N}(\rho\frac{K}{N}a_{ij})^2(\cos(e_j-e_i)(\dot{e}_j-\dot{e}_i))^2\int_{|y|\leq c}y^2\nu(dy).\end{aligned}\tag{3.49}$$

The defined integral term $S=\int_{|y|\leq c}y^2\nu(dy)$ have been shown that is finite in previous Theorem 3.1. If the bound of the $\mathcal{L}V_3$ (3.48) is rearranged by considering inequality (3.49)

$$\begin{aligned}\mathcal{L}V_3\leq\quad&\sum_{i=1}^{N}\dot{e}_i\left[\sum_{j=1}^{N}[\frac{K}{N}a_{ij}(1+\rho b+\rho R)]\cos\left(e_j-e_i\right)\left(\dot{e}_j-\dot{e}_i\right)+\frac{C}{N}\sum_{j=1}^{N}b_{ij}\left(\dot{e}_j-\dot{e}_i\right)\right]\\ +\quad&\frac{\sigma^2}{2}\sum_{i=1}^{N}\text{trace}\left\{\left[\sum_{j=1}^{N}\frac{K}{N}a_{ij}\rho\cos\left(e_j-e_i\right)\left(\dot{e}_j-\dot{e}_i\right)\right]\times\left[\sum_{j=1}^{N}\frac{K}{N}a_{ij}\rho\cos\left(e_j-e_i\right)\left(\dot{e}_j-\dot{e}_i\right)\right]^T\right\}\\ +\quad&\frac{N}{2}\sum_{i=1}\sum_{j=1}^{N}(\rho\frac{K}{N}a_{ij})^2\left(\cos\left(e_j-e_i\right)\left(\dot{e}_j-\dot{e}_i\right)\right)^2\int_{|y|\leq c}y^2\nu(dy).\end{aligned}$$

then the terms is written under the common sum

$$
\begin{aligned}
\mathcal{L}V_3 \leq \; & - \; \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{K}{N}a_{ij}(1+\rho b+\rho R)\cos(e_j-e_i)(\dot{e}_j-\dot{e}_i)^2 \\
& - \; \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{C}{N}b_{ij}(\dot{e}_j-\dot{e}_i)^2 \\
& + \; \frac{N\rho^2\sigma^2}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(\frac{K}{N}a_{ij})^2\cos^2(e_j-e_i)(\dot{e}_j-\dot{e}_i)^2 \\
& + \; \frac{NS}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho^2\frac{K^2}{N^2}a_{ij}^2\cos^2(e_j-e_i)(\dot{e}_j-\dot{e}_i)^2.
\end{aligned}
$$

Now, the expression in parenthesis $(\dot{e}_j - \dot{e}_i)^2$ is put to get the negative definiteness by editing the other terms.

$$
\begin{aligned}
= \; & -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\bigg[\frac{K}{N}a_{ij}(1+\rho b+\rho R)\cos(e_j-e_i)+\frac{C}{N}b_{ij} \\
& -N\rho^2\sigma^2\frac{K^2}{N^2}a_{ij}^2\cos^2(e_j-e_i)-NS\rho^2\frac{K^2}{N^2}a_{ij}^2\cos^2(e_j-e_i)\bigg](\dot{e}_j-\dot{e}_i)^2.
\end{aligned}
$$

By using the property that $\max_{x\in[-\pi,\pi]}\cos x = 1$ and if $i \neq j$ then $a_{ij} = b_{ij} = 1$ otherwise $a_{ij} = b_{ij} = 0$ for all $i, j \in [1, \cdots, N]$,

$$
\mathcal{L}V_3 \leq -\frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\Big[C-\Big(K(1+\rho|b|+\rho R)+\rho^2\sigma^2K^2+S\rho^2K^2\Big)\Big]a_{ij}(\dot{e}_j-\dot{e}_i)^2.
$$

Considering the inequality given in the Lemma 3.1

$$
\dot{e}\mathcal{L}_A\dot{e}^T \geq \lambda_2(\mathcal{L}_A)\dot{e}\dot{e}^T, \tag{3.50}
$$

and assuming that the following condition is fulfilled

$$
C > K(1+\rho|b|+\rho R)+\rho^2\sigma^2K^2+S\rho^2K^2, \tag{3.51}
$$

the following upper bound for $\mathcal{L}V_3$ can be determined:

$$\mathcal{L}V_3 \leq -\left[C - K(1+\rho|b| + \rho R) + \rho^2\sigma^2K^2 + S\rho^2K^2\right]\lambda_2(\mathcal{L}_A)\dot{e}\dot{e}^T. \tag{3.52}$$

This, satisfies the conditions of Theorem 1.9, thus establishing the 2nd moment exponentially stability of the trivial solution of error stochastic system (3.41). The perturbed system's frequency exponentially asymptotically converges to the same mean square values, also establishing stability in frequency as a consequence. □

3.3.2. Lévy Type Perturbations on the Control Layer

The Kuramoto model of the perturbed oscillator network with distributed control function u_i is generated by composing the derivatives of (3.1) and (3.2), giving the following equation:

$$\begin{aligned} \mathrm{d}\dot{\theta}_i(t) &= \left[\frac{K}{N}\sum_{j=1}^{N} a_{ij}\cos\left(\theta_j - \theta_i\right)\left(\dot{\theta}_j - \dot{\theta}_i\right) + \frac{C}{N}\sum_{j=1}^{N} b_{ij}\left(\dot{\theta}_j - \dot{\theta}_i\right)\right]\mathrm{d}t \\ &\quad +\rho\sum_{j=1}^{N}\frac{C}{N}b_{ij}\left(\dot{\theta}_j - \dot{\theta}_i\right)\mathrm{d}L(t),\ i,j \in [1,\cdots N]. \end{aligned} \tag{3.53}$$

Theorem 3.4 *If the following requirement upon the system parameters in (3.53) is fulfilled, the frequency 2nd moment exponential stability of the system can be restored if the control layer of the duplex Kuramoto network in (3.53) is perturbed.*

$$-\frac{K}{N}a_{ij} + r^2 > 0,\ \ \frac{C}{N} = -r/z, \tag{3.54}$$

where

$$\begin{aligned} z^2 &= \sigma^2\rho^2 + S\rho^2 \\ r &= (1+\rho b + \rho R)/2z, \end{aligned}$$

Proof Our purpose while proving the approximation of each frequency ω_i to the aver-

age frequency $(1/N)\sum_{i=1}^{N}\omega_i = \bar{\omega}$. The convergence of their difference $\dot{e}_i = \dot{\theta}_i - \bar{\omega}$, which is equivalent to this convergence, to go to zero is aimed. So the Lyapunov theory can be used.
Now let's define the stochastic system related to the error function

$$\begin{aligned} \mathrm{d}\dot{e}_i(t) &= \left[\frac{K}{N}\sum_{j=1}^{N} a_{ij}\cos\left(e_j - e_i\right)\left(\dot{e}_j - \dot{e}_i\right) + \frac{C}{N}\sum_{j=1}^{N} b_{ij}\left(\dot{e}_j - \dot{e}_i\right)\right]\mathrm{d}t \\ &\quad +\rho\sum_{j=1}^{N}\frac{C}{N}b_{ij}\left(\dot{e}_j - \dot{e}_i\right)\mathrm{d}L(t). \end{aligned} \tag{3.55}$$

The stability of the trivial solution of (3.55) can be examined with the help of Lyapunov Theory. For this purpose, it is checked that if the solution of (3.55) satisfies the conditions in Theorem 1.10 given by Siakalli in her thesis Siakalli (2009), respectively

1. If the Lyapunov function is taken as follows:

$$V_4 = \frac{1}{2}\dot{\mathbf{e}}^{\mathrm{T}}\dot{\mathbf{e}} = \frac{1}{2}\sum_{i=1}^{N}\dot{e}_i^2.$$

   where $\mathbf{e} = [e_1, \ldots, e_N]^{\mathrm{T}}$ is the phase error vector. Then first condition is satisfied for $\alpha_1 = \alpha_2 = 1/2$ and $p = 2$.

2. A $\alpha_3 \in \mathbb{R}^+$ can be found that will satisfy the second condition $\mathcal{L}V_4(X) \leq -\alpha_3 V_4(X)$ as follows:

As in the previous Theorem 3.1, the Poisson random measure is divided into compensated Poisson random measure and Lévy measure in the Lévy process as follows in order to apply the stability theorem given in (1.10).

$$L(t) = bt + B_A(t) + \int_{|y|\leq 1} y\tilde{N}(dt, dy) + t\int_{1<|y|\leq c} y\nu(dy) + \int_{1<|y|\leq c} y\tilde{N}(dt, dy). \tag{3.56}$$

By considering Theorem 1.11 and Theorem 1.12, the boundedness of the integral term is evaluated

$$R = \int_{1<|y|\leq c} y\nu(dy). \tag{3.57}$$

The defined variable R in is finite thus, the Lévy-Ito decomposition can be rearranged., so it will take the form as follows:

$$\begin{aligned} L(t) &= bt + B_A(t) + t\int_{1<|y|\leq c} y\nu(dy) + \int_{|y|\leq c} y\tilde{N}(t, dy) \\ &= bt + B_A(t) + tR + \int_{|y|\leq c} y\tilde{N}(t, dy). \end{aligned} \tag{3.58}$$

Finally, let's evaluate the infinitesimal operator $\mathcal{L}$ defined in (1.24) for the error dynamical system (3.55) and Lévy-Ito decomposition given in (3.58) as follows:

$$\begin{aligned} \mathcal{L}V_4 &= \sum_{i=1}^{N} \dot{e}_i \left[\sum_{j=1}^{N} \frac{K}{N} a_{ij} \cos\left(e_j - e_i\right)\left(\dot{e}_j - \dot{e}_i\right) + \frac{C}{N} \sum_{j=1}^{N} b_{ij}(1+\rho b)\left(\dot{e}_j - \dot{e}_i\right) \right] \\ &+ \frac{\sigma^2}{2} \sum_{i=1}^{N} \text{trace}\{\left[\sum_{j=1}^{N} \frac{C}{N} b_{ij}\rho\left(\dot{e}_j - \dot{e}_i\right)\right] \times \left[\sum_{j=1}^{N} \frac{C}{N} b_{ij}\rho\left(\dot{e}_j - \dot{e}_i\right)\right]^T \} \\ &+ \sum_{i=1}^{N} \int_{|y|<1} \frac{1}{2}\Big[(\dot{e}_i + \rho \sum_{j=1}^{N} \frac{C}{N} b_{ij}(\dot{e}_j - \dot{e}_i) y)^2 - e_i^2 \\ &- 2e_i\rho \sum_{j=1}^{N} \frac{C}{N} b_{ij}(\dot{e}_j - \dot{e}_i) \Big] \nu(dy). \end{aligned} \tag{3.59}$$

According to the chosen Lyapunov function $V_4(e) = ee^T/2$ and the jump function in the form $H(e, y) = \rho \sum_{j=1}^{N} \frac{K}{N} a_{ij} \sin(e_j - e_i)y$, the above equation returns to the following expression

$$V_4(x + H(x, y)) - V_4(x) - H^i(x, y)\partial_i V_4(x) = \frac{1}{2}\left(\rho \sum_{j=1}^{N} \frac{C}{N} b_{ij}\left(\dot{e}_j - \dot{e}_i\right) y \right)^2. \tag{3.60}$$

By considering the inequality

$$\frac{1}{2}\left(\rho\sum_{j=1}^{N}\frac{C}{N}b_{ij}\left(\dot{e}_j-\dot{e}_i\right)\right)^2 \leq \frac{N}{2}\sum_{j=1}^{N}(\rho\frac{C}{N}b_{ij})^2\left(\dot{e}_j-\dot{e}_i\right)^2.$$

Then the following inequality is obtained for the last term of the $\mathcal{L}V_4$

$$\frac{1}{2}\int_{|y|\leq c}\left(\rho\sum_{j=1}^{N}\frac{C}{N}b_{ij}\left(\dot{e}_j-\dot{e}_i\right)y\right)^2\nu(dy) \leq \frac{N}{2}\sum_{j=1}^{N}(\rho\frac{C}{N}b_{ij})^2\left(\dot{e}_j-\dot{e}_i\right)^2\int_{|y|\leq c}y^2\nu(dy). \tag{3.61}$$

In Theorem 3.1, bound of the defined integral term $S = \int_{|y|\leq c}y^2\nu(dy)$ is shown. If the upper bound of $\mathcal{L}V_4$ (3.59) is reorganized by considering equality (3.13) and (3.61) then

$$\begin{aligned}
\mathcal{L}V_4 \quad \leq \quad & -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{K}{N}a_{ij}\cos(e_j-e_i)(\dot{e}_j-\dot{e}_i)^2 \\
& -\frac{1}{2}\frac{C}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}b_{ij}(1+\rho b+\rho R)\left(\dot{e}_j-\dot{e}_i\right)^2 \\
& +\frac{N\sigma^2\rho^2}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{C^2}{N^2}b_{ij}^2(\dot{e}_j-\dot{e}_i)^2 \\
& +\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{NS}{2}(\rho\frac{C}{N}b_{ij})^2(\dot{e}_j-\dot{e}_i)^2.
\end{aligned}$$

We now put the terms in parentheses $(\dot{e}_j - \dot{e}_i)^2$ and alter the remaining terms to achieve negative definiteness.

$$\begin{aligned}
= \quad & -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\Bigg[\frac{K}{N}a_{ij}\cos(e_j-e_i)+\frac{C}{N}b_{ij}(1+b\rho+R\rho) \\
& -N\sigma^2\rho^2\frac{C^2}{N^2}b_{ij}^2-NS\,(\rho\frac{C}{N}b_{ij})^2\Bigg](\dot{e}_j-\dot{e}_i)^2.
\end{aligned}$$

By defining the following variables,

$$\begin{aligned} z^2 &= \sigma^2\rho^2 + S\rho^2 \\ r &= (1 + \rho b + \rho R)/2z, \end{aligned}$$

and applying the quadratic property, $\max_{-\pi,\pi} |\cos(e_j - e_i)| = 1$ and the property that $i \neq j$ then $a_{ij} = b_{ij} = 1$ otherwise $a_{ij} = b_{ij} = 0$ for all $i, j \in [1, \cdots, N]$, the following inequality can be established:

$$\mathcal{L}V_4 \leq - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}[-K + r^2 - [zC - r]^2](\dot{e}_j - \dot{e}_i)^2. \tag{3.62}$$

Taking into account the following inequality given in the Lemma 3.1

$$\dot{e}\mathcal{L}_A\dot{e}^T \geq \lambda_2(\mathcal{L}_A)\dot{e}\dot{e}^T, \tag{3.63}$$

and assuming that the following conditions are satisfied,

$$-K + r^2 > 0, \quad C = r/z, \tag{3.64}$$

By considering the equations in (3.63) and (3.64), the following estimate for the upper limit for $\mathcal{L}V_2$ given in inequality (3.62) can be derived:

$$\mathcal{L}V_4 \leq -\left[-K + r^2)\right]\lambda_2(\mathcal{L}_A)\dot{e}\dot{e}^T. \tag{3.65}$$

Thus, the trivial solution of the error stochastic system (3.55) is 2nd moment exponentially stable. Since the perturbed system's frequency converges exponentially asymptotically to the same mean square values, stability in frequency is also established as a consequence.

□

3.4. Numeric Simulation

In this section, the accuracy of the control rules obtained is numerically tested for the phase and frequency synchronization of the system. To obtain the numerical results, the Euler-Maruyama scheme as follows is employed Janicki and Weron (1993):

$$
\begin{aligned}
&\theta_i^{n+1}(t) = \\
&\theta_i^n(t) + \left[\omega_i + K\sum_{j=1}^{N} a_{ij}\sin\left(\theta_j^n(t) - \theta_i^n(t)\right)\right. \\
&\left. +C\sum_{j=1}^{N} b_{ij}\left(\theta_j^n(t) - \theta_i^n(t)) + H(\theta^n)(b + dB(t_n) - \int_{|y|<1} s\nu(ds)\right) dt\right. \\
&+H(\theta^n)\int_{t_n}^{t^{n+1}}\int_{|y|<c} sN(dt, ds), \quad i \in [1, \cdots N].
\end{aligned}
$$

The H function indicates which layer is affected by noise

$$
H(\theta) = \begin{cases} \rho_1 \sum_{j=1}^{N} Ka_{ij}\sin(\theta_j(t) - \theta_i(t) \\ \\ \rho_2 \sum_{j=1}^{N} Cb_{ij}\left(\theta_j(t) - \theta_i(t)\right) \end{cases}. \tag{3.66}
$$

where ρ_1 and ρ_2 are intensity parameters of the noise in the Kuramoto layer $\mathbb{K}$ and control layer $\mathbb{L}$ respectively.

Natural frequencies are selected as $\omega_i = 0$ for computation simplicity, and all oscillators are interconnected.

3.4.1. Algorithm:

In the Matlab experiment, a detailed representation of each discretized trajectory taken from the article H. Yu (2012) is provided in the following:

Before moving on to the steps of the algorithm, define $f(\nu)$ density function and λ parameter from the equality $\nu(ds)dt = \lambda f(s)dsdt$

1 Evaluate a path by simulating

$$\begin{aligned}\tilde{\theta}_i^n(t_{n+1}) &= \theta_i^n(t_n) + \Bigg[\omega_i + K \sum_{j=1}^{N} a_{ij} \sin\left(\theta_j^n(t_n) - \theta_i^n(t)\right) \\ &+ C \sum_{j=1}^{N} b_{ij}(\theta_j^n(t_n) - \theta_i^n(t_n)) + H(\theta^n(t_n))b\Bigg]\mathrm{d}t + H(\theta^n(t_n))B(t_n)\end{aligned}$$

2 Simulate $N(t_{n+1}) - N(t_n)$ samples from a random variable $N(t_n)$ with parameter λt_n

3 Evaluate $N(t_{n+1}) - N(t_n)$ samples $\tilde{t}_i$ from a uniformly distributed random variable defined on $[-N(t_n),\ N(t_{n+1})]$.

4 Evaluate $N(t_{n+1}) - N(t_n)$ samples ξ_i from a random variable with a density function $f(s)$

5 Calculate $\theta_i^{n+1}(t_{n+1}) = \tilde{\theta}_i^n(t_n) + H(\tilde{\theta}^n) \sum_{i=N(t_n)+1}^{N(t_{n+1})} \chi_{(t_{n+1} \leq \tilde{t}_i < t_n)} \xi_i$

Now let's give simulations of the deterministic basic version of the phase angle and frequency of the model for certain parameters in which our theoretical inference will be tested. In all simulations, the same values are taken for $K = 0.2$, and the initial values are chosen from the normal distribution from the range $[-\pi,\ \pi]$.

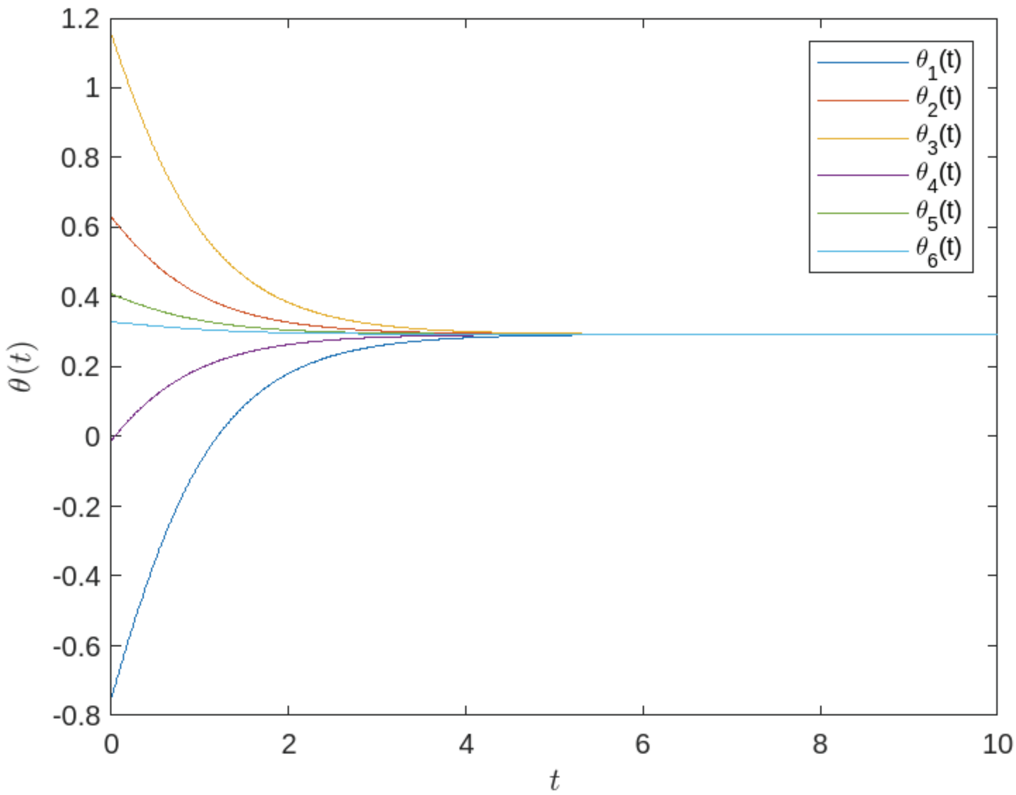

Figure 3.1. Evolution of phases θ_i in time for 6 identical oscillator

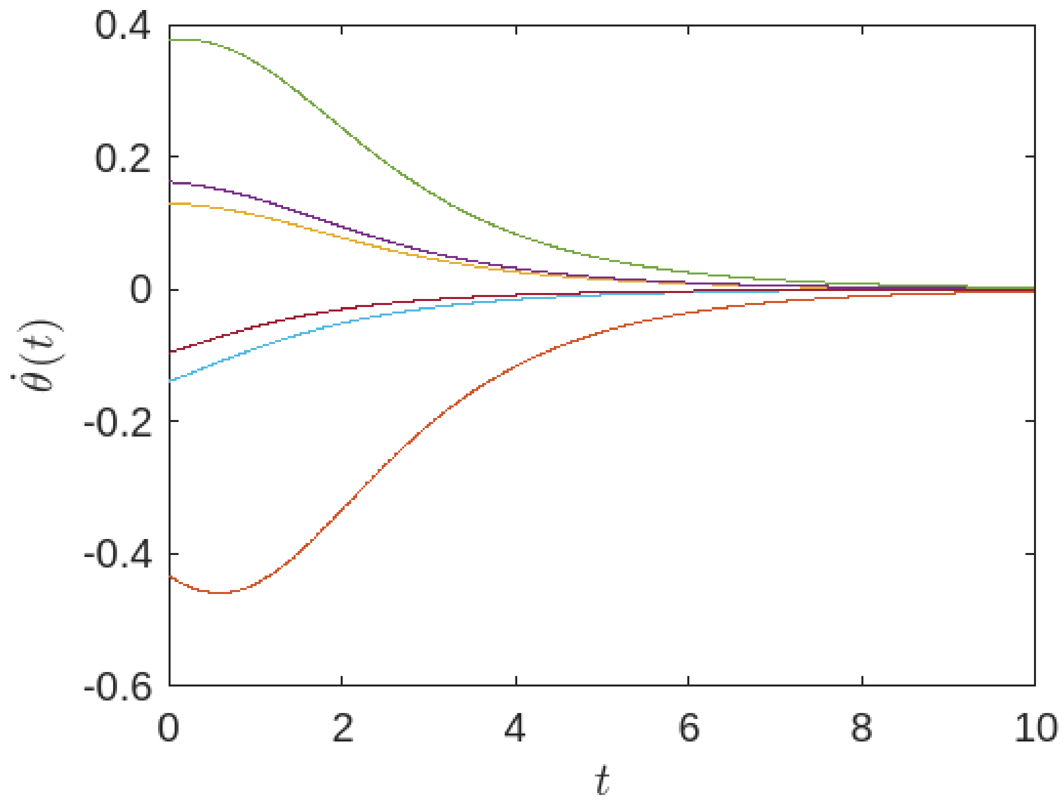

Figure 3.2. Evolution of frequencies $\dot{\theta}_i$ in time for 6 identical oscillator

The Lévy process placed in the examples is taken with a triplet as

$$(0.5,\ 1\ , \nu(dy)) \tag{3.67}$$

where $\nu(dy) = \exp(-x^2/2)/(\sqrt{2\pi})$.

3.4.2. Case 1: Kuramoto-oscillator layer is perturbed by the Lévy process

In the model (3.66), ω_s are taken zero for simplicity and $H(\theta)$ is taken as follows:

$$\rho_1 \sum_{j=1}^{N} K a_{ij} \sin\left(\theta_j(t) - \theta_i(t)\right) \tag{3.68}$$

I : **Numerical Control of Phase agreement** :

In this section, the control of phase agreement of the oscillators is examined when Lévy processes affect the oscillator layer.

Example1 :In this example, the intensity coefficient $\rho = 2$ is considered for Lévy process (3.67) in the system (3.41).

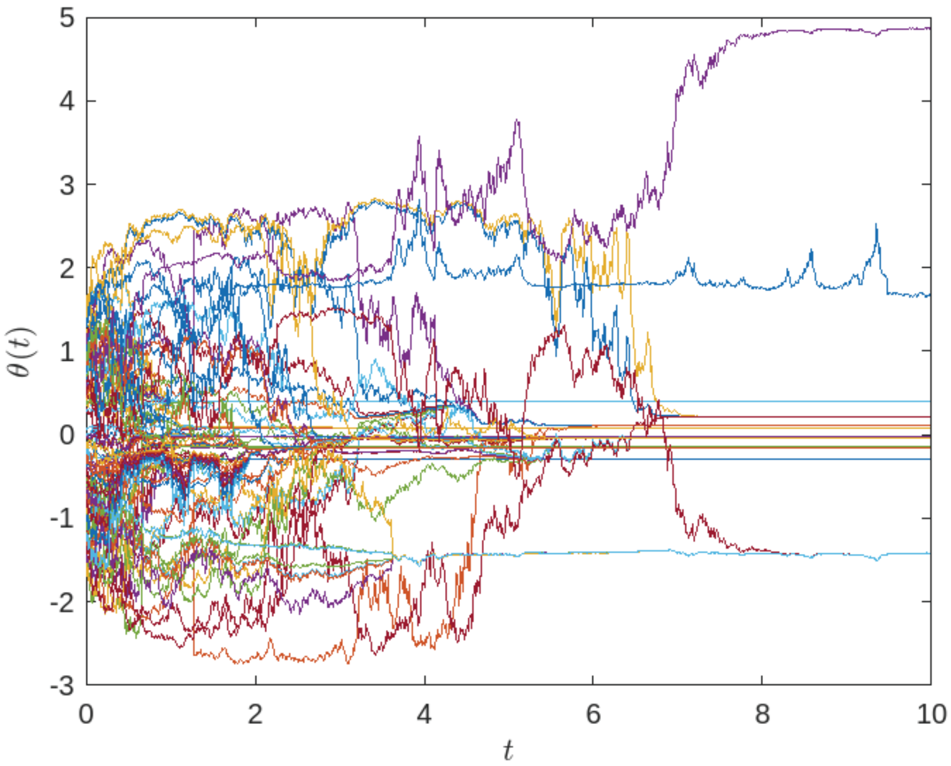

Figure 3.3. Evolution of ocillator layer perturbated phases θ_i with intensity coefficient $\rho = 2$

By applying the control function $u(x)$ to the system, trajectories tend to go into stable equilibrium points

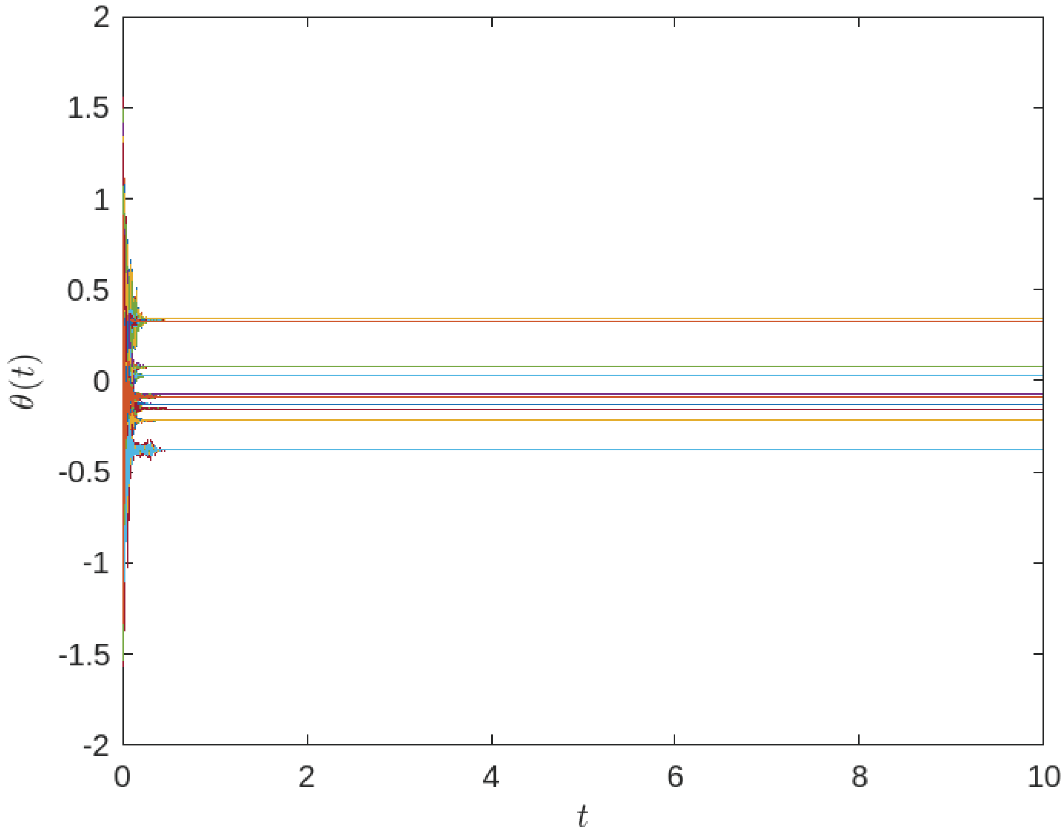

Figure 3.4. Controlled phase θ_i evolution of oscillator layer perturbated system with Lévy process with intensity coefficient $\rho = 2$

Example2 :In this example, it is considered different intensity coefficient $\rho = 16$ for the system (3.41) with the Lévy process (3.67)

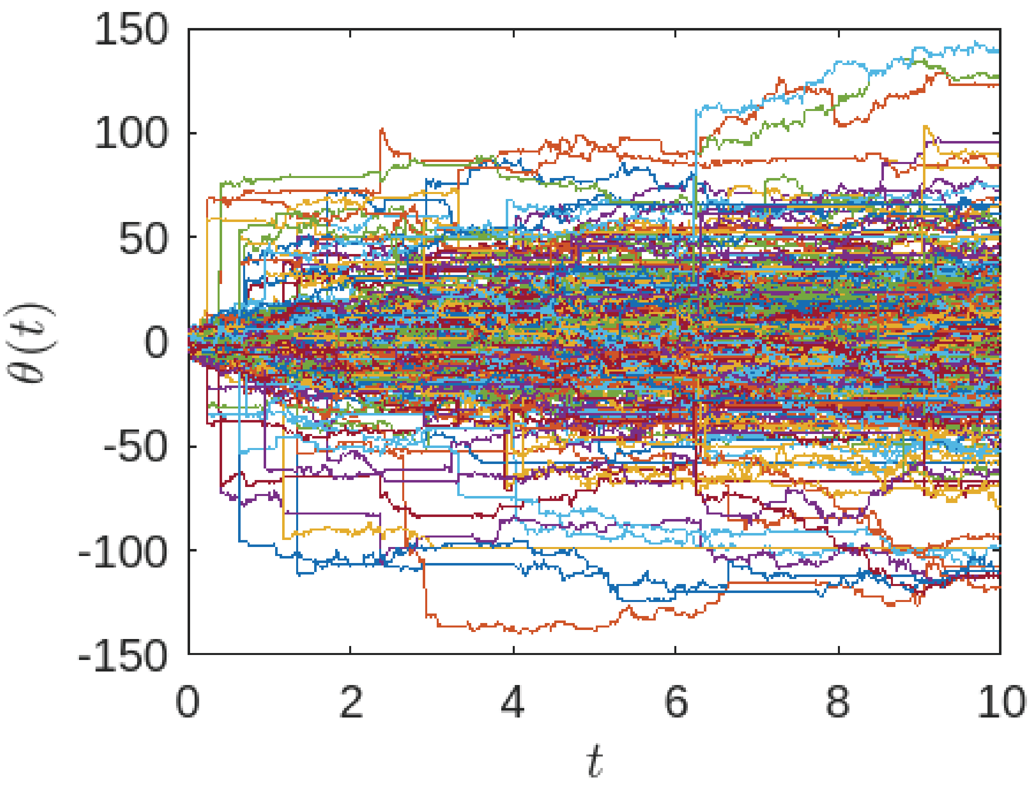

Figure 3.5. Evolution of oscillator layer perturbated phases θ_i with Lévy process with intensity coefficient $\rho = 16$

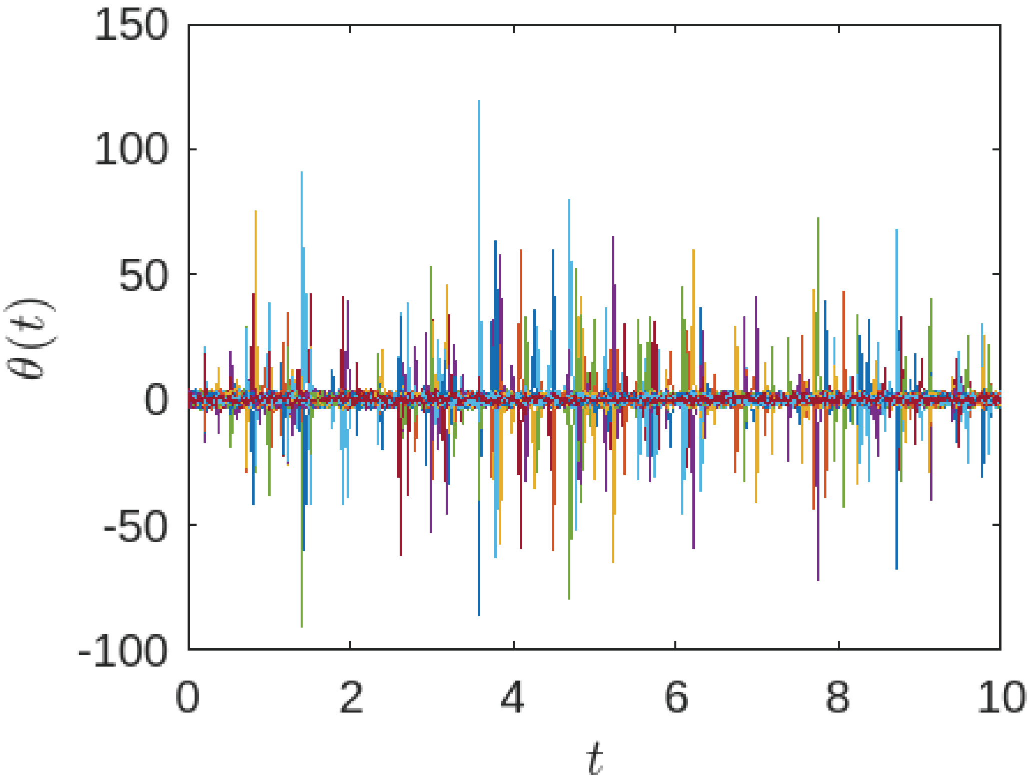

Figure 3.6. Controlled phase θ_i evolution of ocillator layer perturbated system with Lévy process with intensity coefficient $\rho = 16$

When the noise in intensity coefficient in the system is increased, the control function is insufficient to provide stability is observed.

II : **Numerical Control of Frequency Sychronization agreement** :

The figure 3.7 shows that frequency parameters evaluation in time under the (3.67) with intensity coefficient.

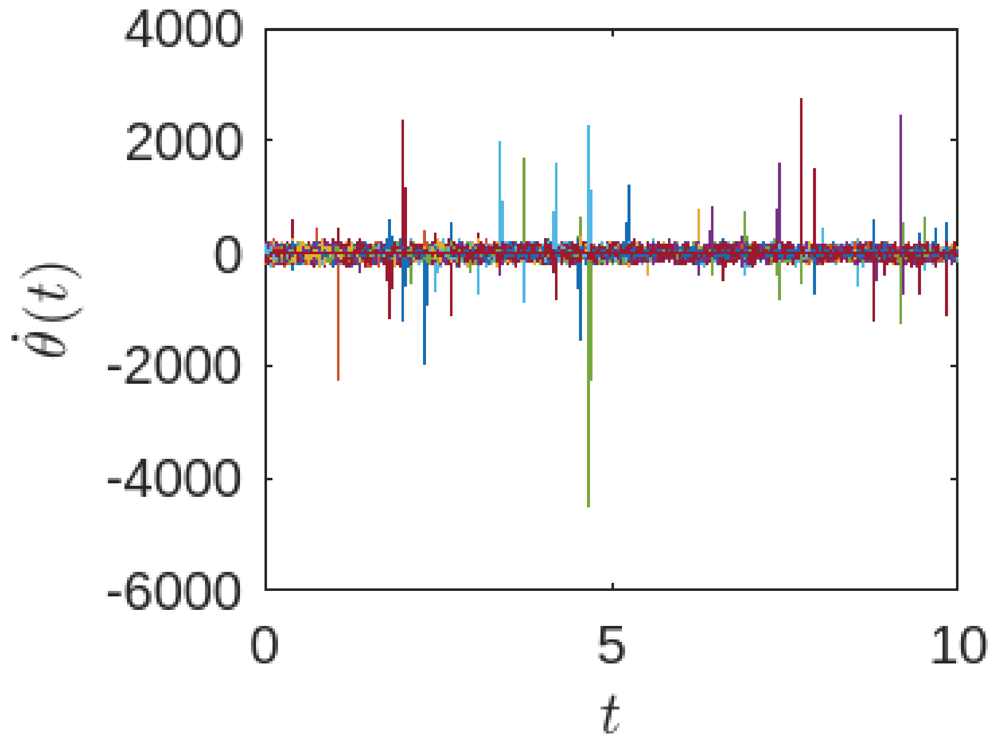

Figure 3.7. Evolution of ocillator layer perturbated frequencies $\dot{\theta}_i$ with Lévy process with intensity coefficient $\rho = 2$

The Figure 3.8 shows that frequency parameters evaluation in perturbed system (3.41) goes into stability by control function $u(x)$

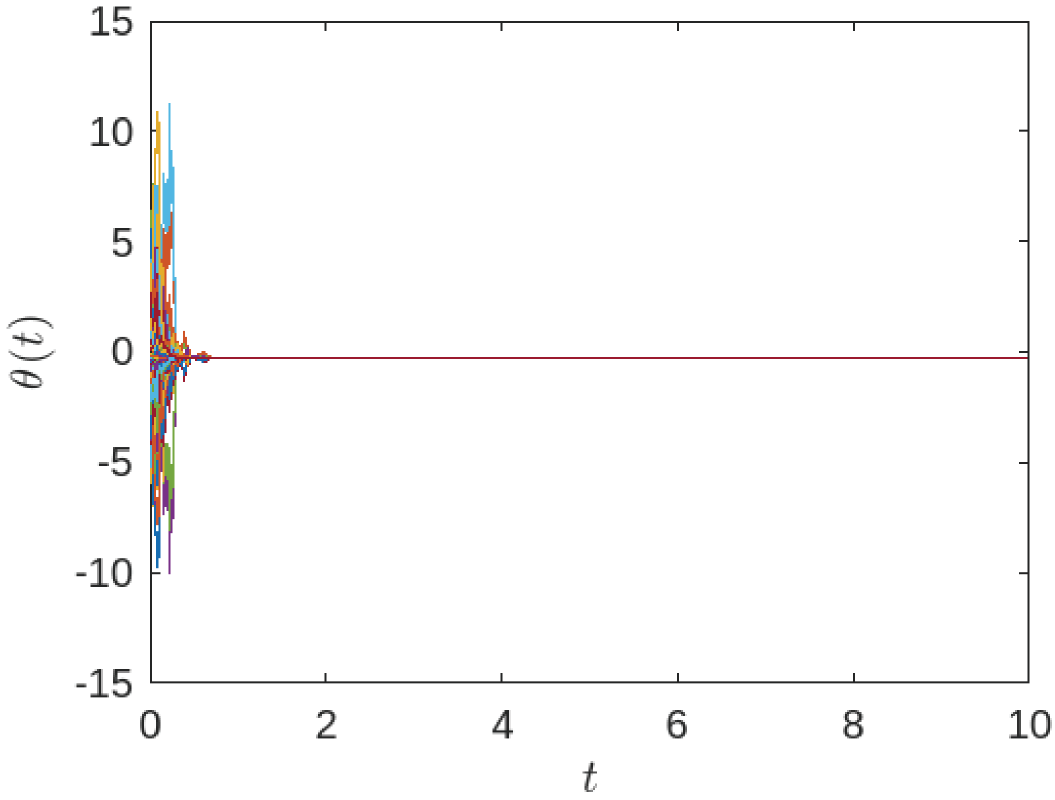

Figure 3.8. Controlled frequencies $\dot{\theta}_i$ evolution of ocillator layer perturbated system with Lévy process with intensity coefficient $\rho = 2$

The following Figure 3.9 shows that frequency evolution in time when the intensity coefficient $\rho = 16$ for the system.

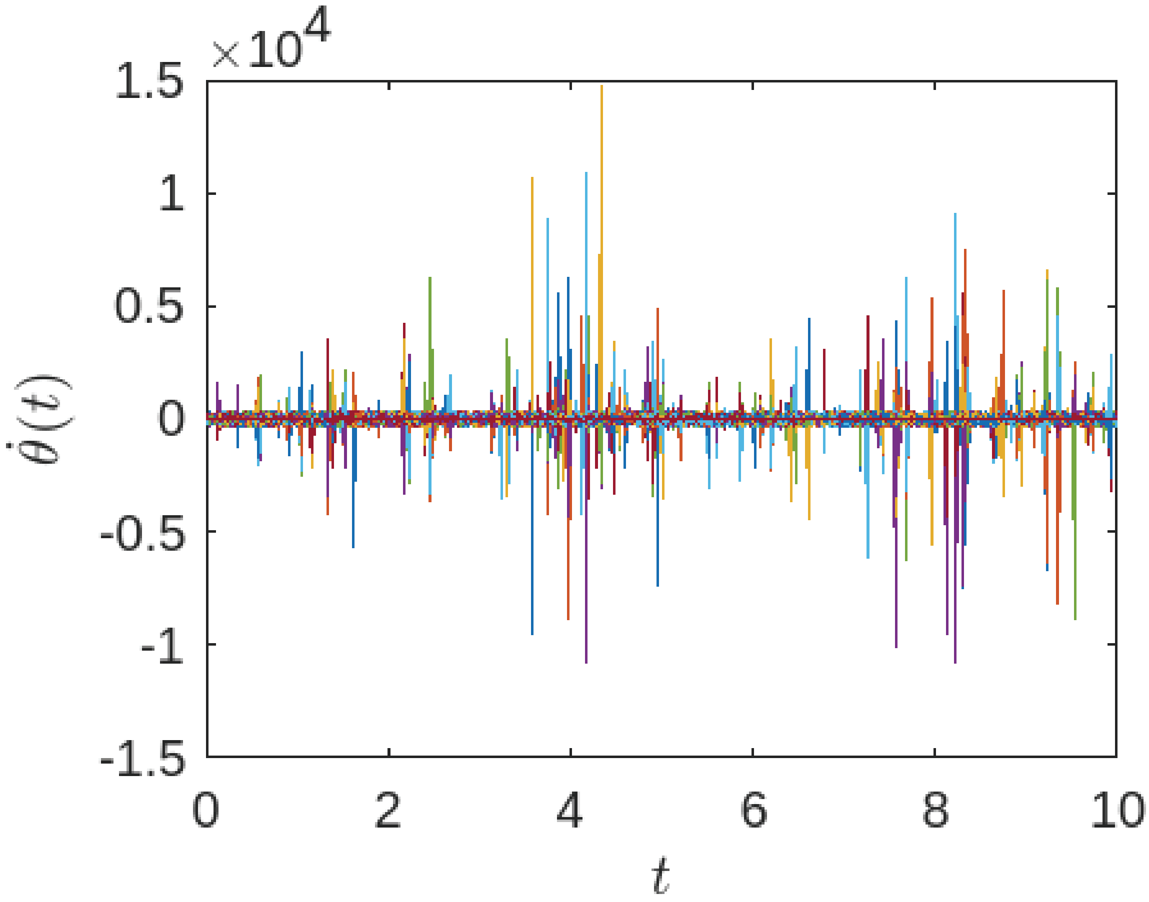

Figure 3.9. Evolution of oscillator layer perturbated frequencies $\dot{\theta}_i$ with Lévy process with intensity coefficient $\rho = 16$

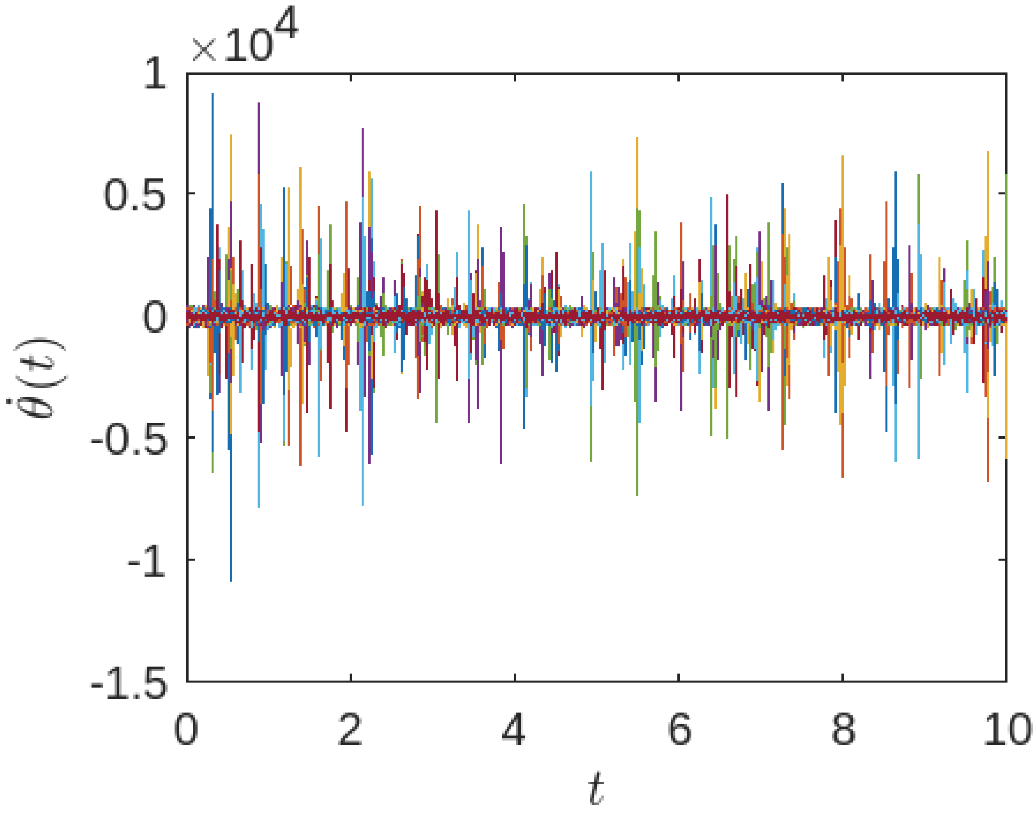

Figure 3.10. Controlled frequencies $\dot{\theta}_i$ evolution of oscillator layer perturbated system with Lévy process with intensity coefficient $\rho = 16$

3.4.3. Case 2: Control layer is perturbed by the Lévy process

I : **Numerical Control of Phase agreement** : In this section, the control of phase agreement of the oscillators is examined when Lévy processes affect the control layer.

The following Figure 3.11 shows that, the intensity coefficient $\rho = 2$ for Lévy process (3.67) is considered in the system (3.41).

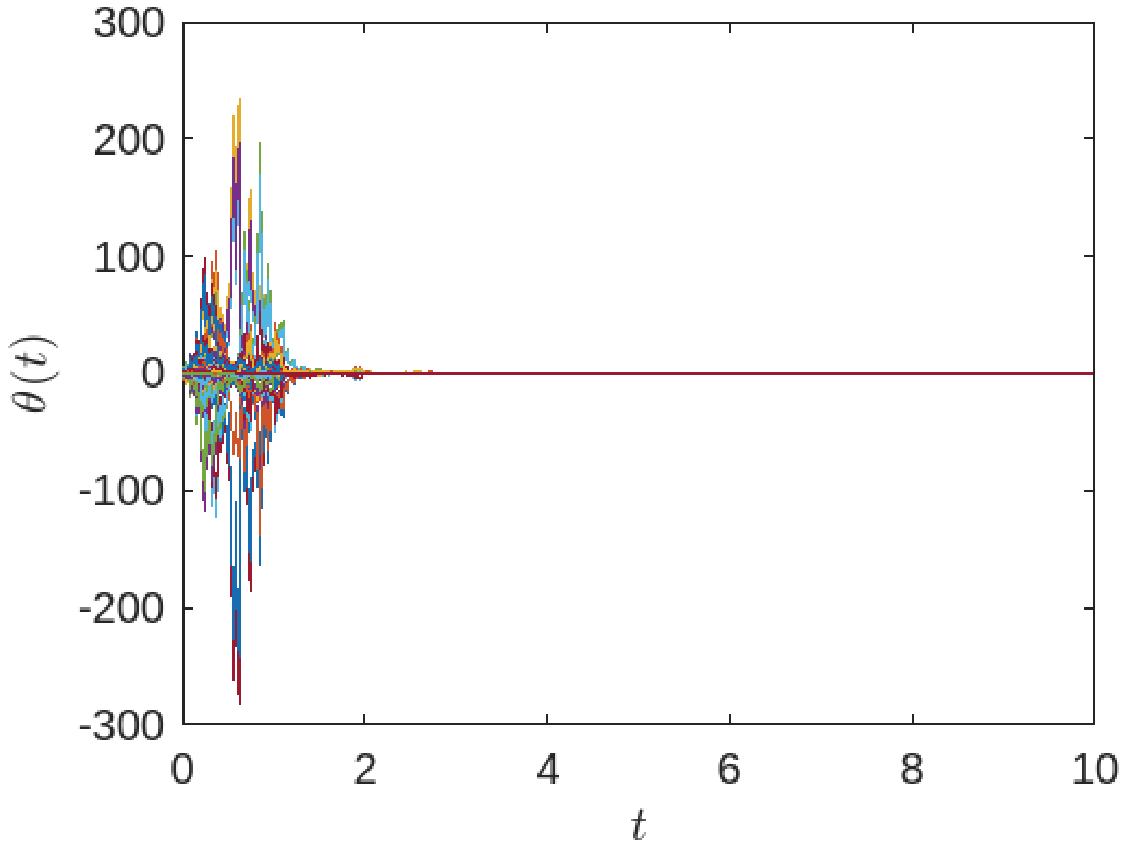

Figure 3.11. Evolution of control layer perturbated phases θ_i with intensity coefficient $\rho = 2$

By applying the control function $u(x)$ to the system, trajectories tend to go into stable equilibrium points in shorter time given with following Figure 3.12

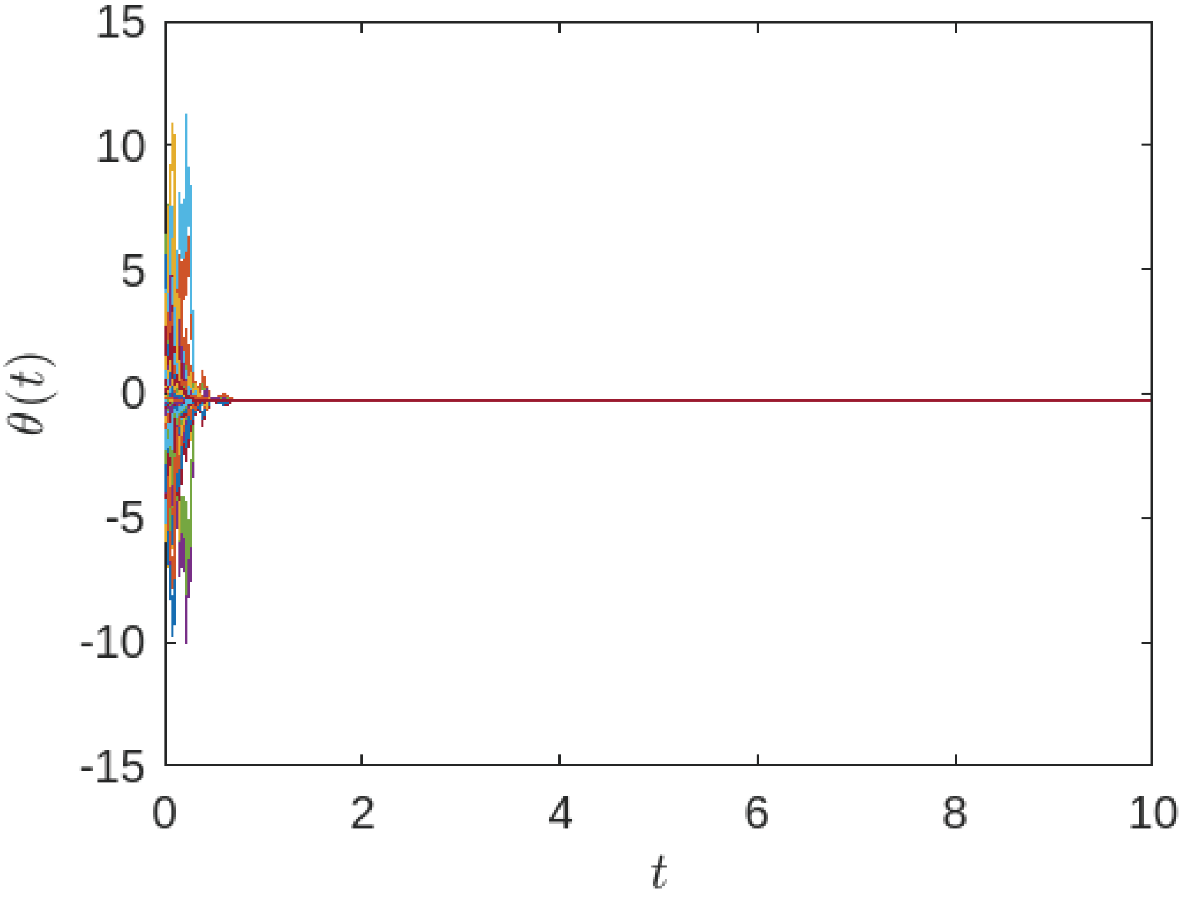

Figure 3.12. Controlled phase θ_i evolution of control layer perturbated system with Lévy process with intensity coefficient $\rho = 2$

The Figure 3.13 shows that, intensity coefficient $\rho = 16$ is considered for Lévy process (3.67) in the system (3.41).

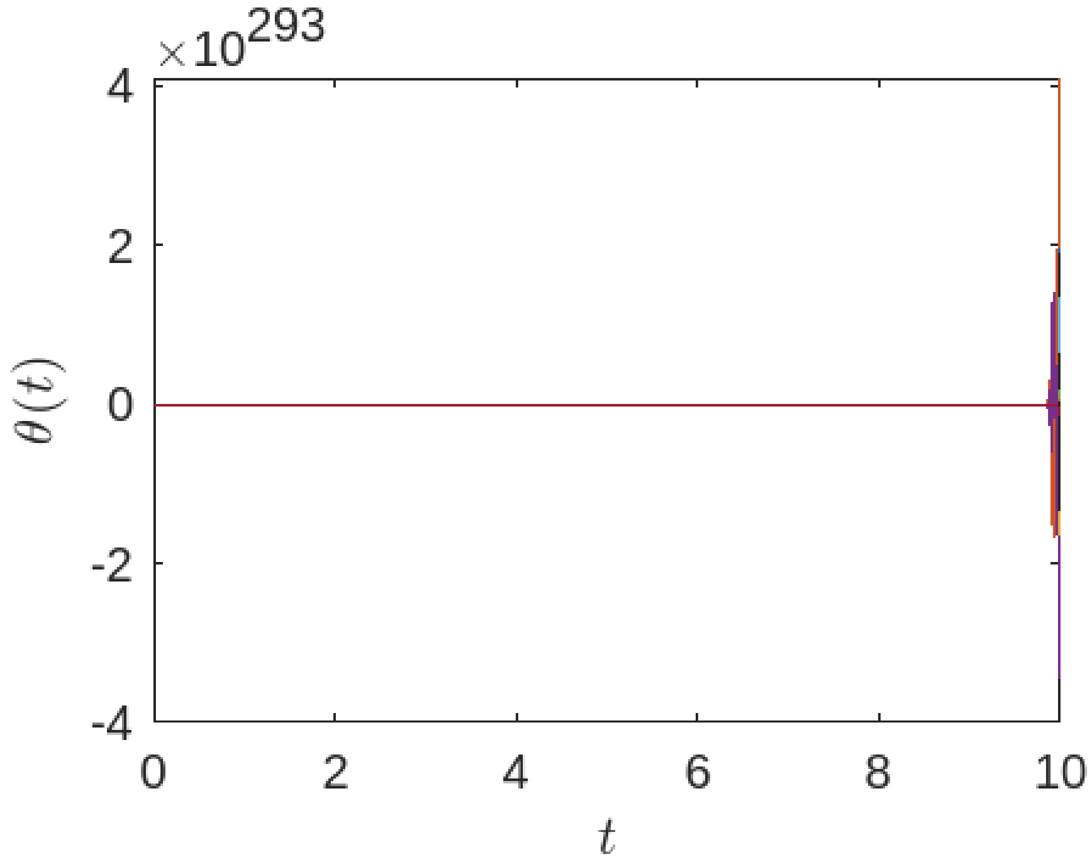

Figure 3.13. Evolution of control layer perturbated phases θ_i with intensity coefficient $\rho = 16$

II : **Numerical Control of Freqency Sychronization** :

The Figure 3.14 shows trajectories of the frequencies of the system in time, when intensity coefficient $\rho = 2$ is considered for Lévy process (3.67) in the system (3.41)

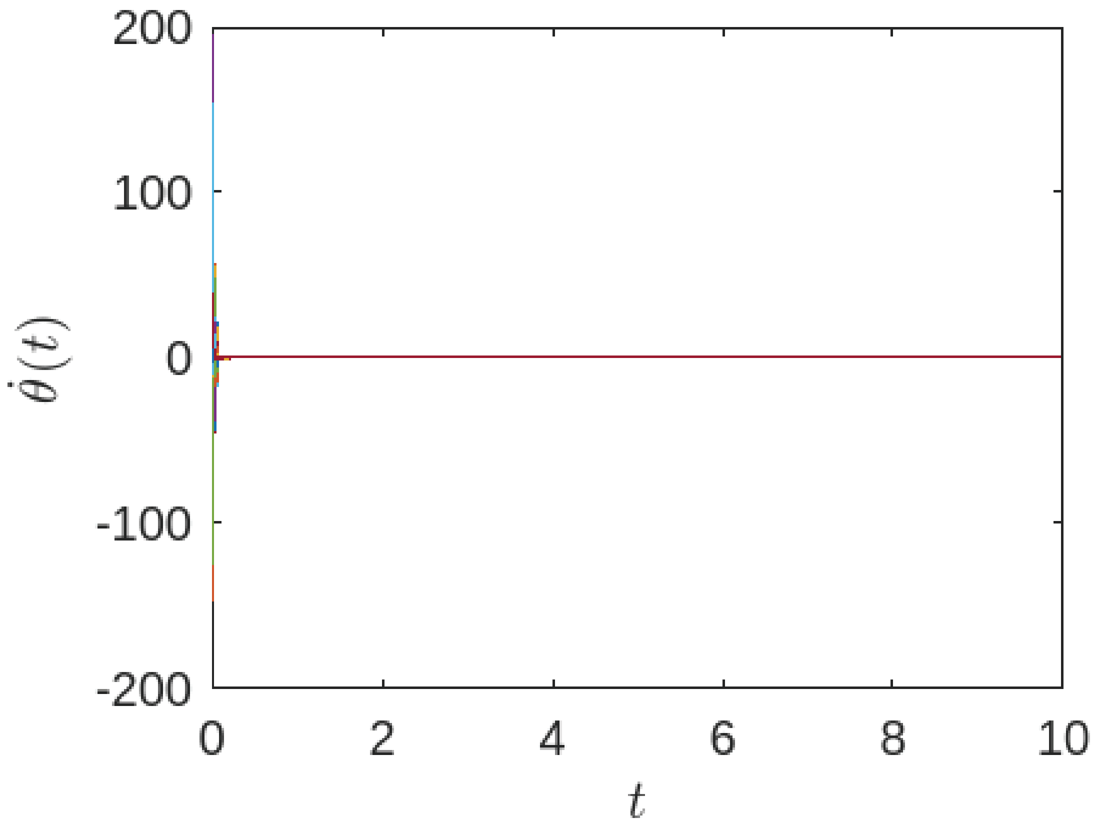

Figure 3.14. Evolution of control layer perturbed frequencies $\dot{\theta}_i$ with intensity coefficient $\rho = 2$

After adding control function into system, the Figure 3.15 shows that frequency evolution in time

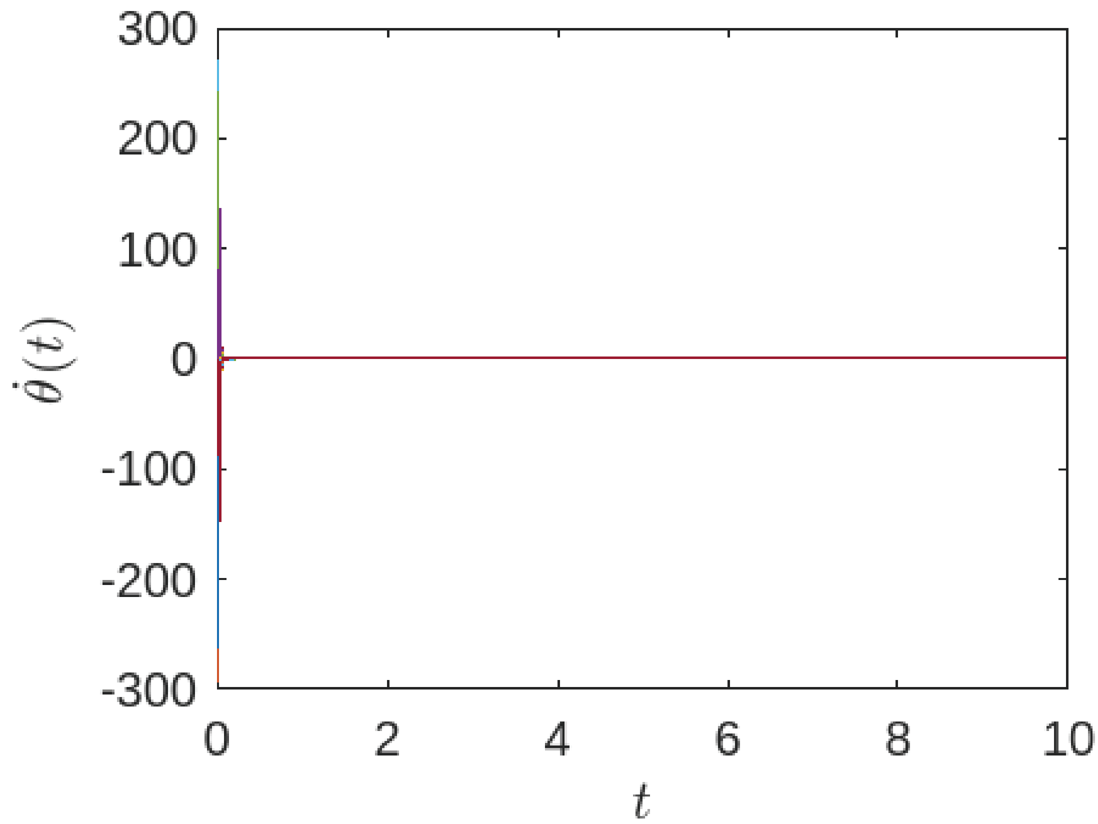

Figure 3.15. Controlled phase $\dot{\theta}_i$ evolution of control layer perturbated system with Lévy process with intensity coefficient $\rho = 2$

The Figure 3.16 shows trajectories of the frequencies of the system in time, when intensity coefficient $\rho = 16$ is considered for Lévy process (3.67) in the system (3.41)

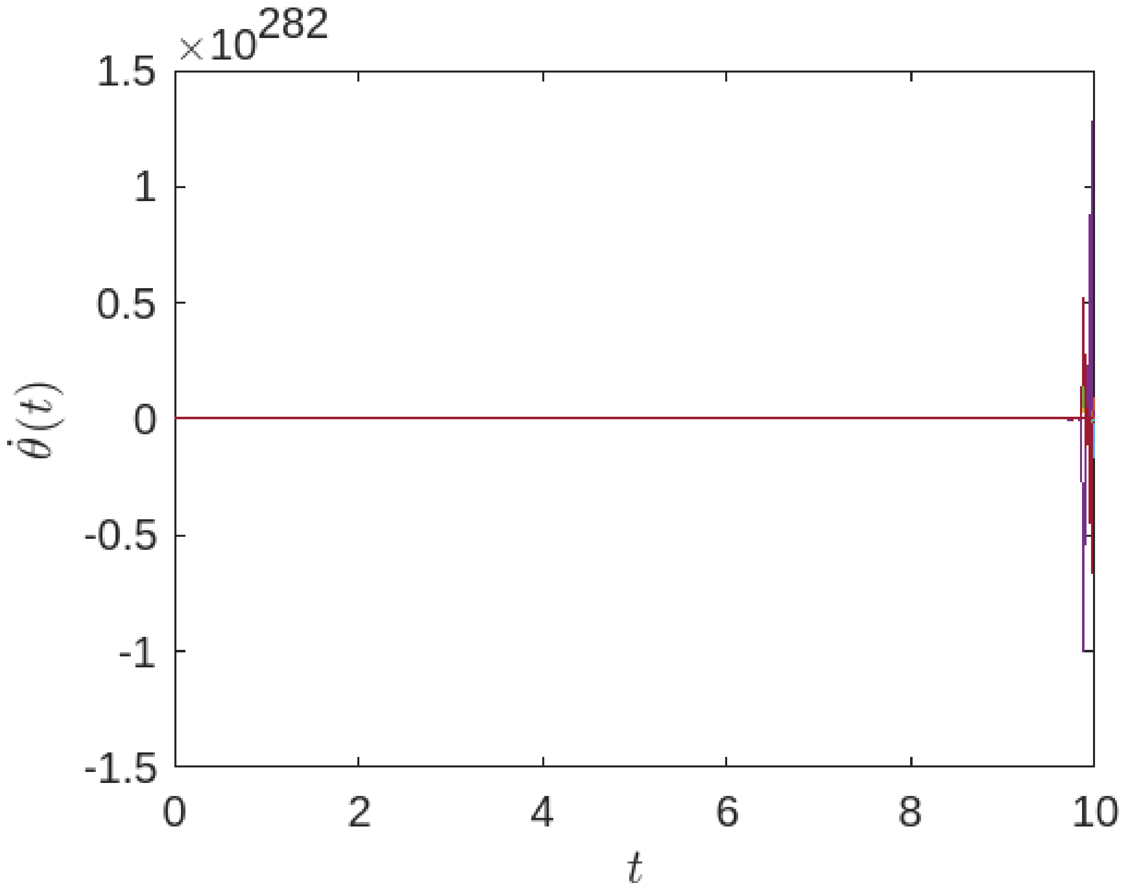

Figure 3.16. Evolution of control layer perturbed frequencies $\dot{\theta}_i$ with intensity coefficient $\rho = 16$

According to the obtained results, compared to the theoretical results, when the Kuramoto oscillator layer is affected, the control function loses power according to the increase in noise intensity. Where the control layer is affected, the numerical results indicate that noise up to a certain intensity does not affect the phase and frequency stability of the control system, but that after this point it cannot be controlled.

CHAPTER 4

CONCLUSIONS

In conclusion, this thesis has demonstrated that the normal distribution is not always suitable for modeling noise in power networks, particularly when sudden peaks occur. Instead, a combination of a continuous process and a jump process is a more suitable approach for modeling such noise. The study focused on analyzing the stability and control of two fundamental equations used in the modeling of power grids, the swing equation, and the Kuramoto Model. For the SMIBS, it was concluded that a modified (tempered) α-stable process is a better model for the noise, and control functions are designed to make the system stable in probability and 2nd moment exponential stability stable at its equilibrium point.

In the case of the Kuramoto model, the researchers studied the synchronization of phase and frequency in Kuramoto oscillator systems. These systems were organized in a duplex network topology and subjected to perturbations from a L'evy process. Our investigation aimed to determine the conditions under which the duplex network would experience a loss of phase and frequency synchronization. To address this, we utilized Lyapunov Theory and devised a control function to facilitate system synchronization. It was concluded that control of the system frequency and phase synchronization can be evaluated up to a certain noise intensity level however not beyond that level. In sum, the thesis provides valuable insights into the stability and control of power networks under Lévy noise, and the results have practical implications for the design and operation of power systems.

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X. Guo, P Rao, M. L. (2022). Synchronization of kuramoto-oscillator networks based on cyber-physical system. *Journal of the Korean Physical Society*.

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Z. Lu, W. Zhao, D. X. and G. Li (2015). P-moment stability of power system under small gauss type random excitation. *Chaos Solutions Fractals 69*, 30–37.

VITA

EDUCATION

2015 - 2023 Doctor of Philosophy in Mathematics

Graduate School of Engineering and Sciences, Izmir Institute of Technology,
Izmir -Turkey

Thesis Title: Stability Analysis of Nonlinear Dynamical Systems with Lévy Type Perturbations

Supervisor: Asst. Prof. Dr. Ahmet BATAL Cosupervisor: Prof. Dr. Ferit Acar Savacı

2011 - 2014 Master of Science in Mathematics and Computer

Department of Mathematics and Computer, Democracy University
Izmir-Turkey

2002 - 2006 Bachelor of Mathematics

Department of Mathematics, Faculty of Science, Ege University,
Izmir -Turkey

PROFESSIONAL EXPERIENCE

2021 - present,

Project Assistant | TUBITAK, Ankara, Turkiye

2013 - 2016,

Research and Teaching Assistant | Democracy University, Izmir, Turkiye

PUBLICATIONS

- B. Pektaş and E. Tamcı, "The Heat Flux Identification Problem for a Nonlinear Parabolic Equation in 2d", Journal of Computational and Applied Mathematics, 2016.