A STUDY ON QUOTIENT QUANDLES OF KNOTS AND LINKS

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ABSTRACT

A STUDY ON QUOTIENT QUANDLES OF KNOTS AND LINKS

In this thesis, we study quandles, algebraic structures within knot theory. Following an overview of knots and links, we study knot invariants by focusing on n-coloring of knots and the knot group. Then, we examine the fundamental quandle of a knot, known also as the knot quandle. The knot quandle is stronger and almost complete invariant compared to the knot group. The fundamental quandle of a link is infinite except for the unlink and the Hopf link. However, the n-quandle quotient of the knot quandle is finite for some positive integers n. While studying quotients of the knot quandle, we expound the method of constructing the n-quandle quotient of the fundamental quandle of a knot, including computations of n-quandle quotients of the knot quandle of the trefoil knot for $n = 2, 3, 4, 5$.

ÖZET

DÜĞÜM VE LİNKLERİN BÖLÜM QUANDLELARI ÜZERİNE BİR ÇALIŞMA

Bu tez, düğüm teorisinde cebirsel bir yapı olan quandleları incelemektedir. Evvela düğüm ve link kavramlarını tanıtacak, ardından düğüm değişmezleri olan n-renklendirme ve düğüm gruplarını inceleyeceğiz. Daha sonra bir düğümün temel quandleını (düğüm quandleını) inceleyeceğiz. Düğüm quandleı düğüm grubuyla karşılaştırıldığında daha güçlü bir değişmezdir. Aşikar link ve Hopf link dışındaki bir linkin temel quandleı her zaman sonsuzdur. Ancak, bazı pozitif n tamsayı değerleri için bir düğümün n-quandle bölümü sonludur. Düğüm quandleının bölüm quandlelarını çalışırken, bir düğümün n-quandle bölümünü nasıl inşa edeceğimizi göstereceğiz ve bu yöntemi kullanarak trefoil düğümünün $n = 2, 3, 4, 5$ değerleri için n-quandle bölümünü hesaplayacağız.

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CHAPTER 1

INTRODUCTION

A mathematical knot is a simple closed curve embedded in $\mathbb{R}^3$, and a link is a disjoint union of knots. Each knot in a link is a component. In this case, every knot is a link. The main problem in knot theory is determining if two knots or links are equivalent. For this reason, invariants were established. Two knots are said to be equivalent if there exists an ambient isotopy of $\mathbb{R}^3$ mapping a knot to the second knot. In 1927, Kurt Reidemeister defined three moves as well as a planar isotopy move that change a link diagram locally and showed that two link diagrams represent the same link if and only if there is a finite sequence of Reidemeister moves between them (Reidemeister 1927, 24-32). In this thesis, we focus on one of the knot invariants, knot quandle and its n-quotients.

The structure as known *quandle* is a binary operation defined by David Joyce in 1982 (Joyce 1982, 37). On the other hand, it was defined by S.V. Matveev as *distributive grouppoids* in 1984 (Matveev 1984, 76), independently. The quandle corresponding a knot, the *knot quandle (or the fundamental quandle of a knot/link)* is an invariant of oriented links. Joyce proved that the knot quandle distinguishes all knots up to orientation (Joyce 1982, 60-61).

In this thesis, we will first give basic notions in knot theory in Section 2.1 and Section 2.2 and then study some knot invariants in Section 2.3. We will study colorability and the knot group along with the crossing number and linking number. While studying the knot group, we will give the proof of the theorem which states that the unknot is the only knot whose knot group is isomorphic to the set of integers. This theorem known as Dehn's Theorem (Manturov 2004, 53). Moreover, we will show that the knot group cannot distinguish all knots. Despite this, the knot group is a strong invariant which determines the knots up to mirror symmetry. In Chapter 3, we will introduce quandles and the fundamental quandle of a link. In Chapter 4, whilst we elaborate the knot quandle presentation and quotient quandles of knots/links, we will examine that how we can compute the n-quandle quotient of the knot quandle and give the proof of the thoerem which says that the fundamental quandle of a link L is finite if and only if L is either the unknot or the Hopf link (Crans et al. 2019, 4). Finally, in Section 4.1.1, we will compute the n-quandle quotient of the fundamental quandle of the trefoil, $Q_n(3_1)$ for $n = 2, 3, 4, 5$.

CHAPTER 2

PRELIMINARY NOTIONS OF KNOT THEORY

In this chapter, we recall the basic concepts of the knot theory that we will use in the following chapters.

2.1. Knots and Links in $\mathbb{R}^3$

Definition 2.1 A *knot* K is the image of a smooth map $K : S^1 \rightarrow \mathbb{R}^3$ satisfying the following:

(i) K is injective on S^1.

(ii) $\frac{dK}{dt} \neq 0$, for all $t \in S^1$.

Example 2.1

Figure 2.1.: The trefoil knot, denoted by (3_1) and the Figure-8 knot, denoted by (4_1).

Definition 2.2 If a knot bounds a disk in $\mathbb{R}^3$, then it is called the *unknot* or the *trivial knot*. See the illustration of the unknot in Figure 2.2.

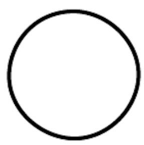


Figure 2.2.: The unknot.

Definition 2.3 A *link* L in $\mathbb{R}^3$ is a disjoint union of knots. Each knot forming the union is called a *component* of L. Every knot is a one-component link.

Example 2.2 In Figure 2.3 we see some of the most known links: the Hopf link with two components, eachbeing an unknot; the Borromean link with three components, each being an unknot; and the link with two components one being the trefoil knot and the other the Figure-8 knot.

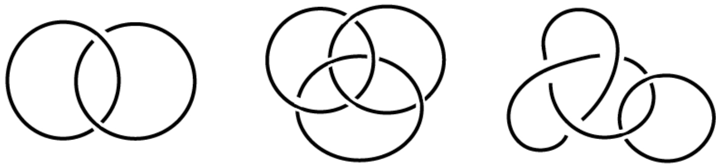


Figure 2.3.: The Hopf link, the Borromean link, and the link whose components are the trefoil knot and the Figure-8 knot, respectively.

Knots and links are studied up to an equivalence relation induced by *ambient isotopy*. We define the ambient isotopy as follows.

Definition 2.4 Two links K_1, K_2 in $\mathbb{R}^3$ are said to be *ambient isotopic* if there is a smooth map $\gamma : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that

(i) $\gamma(x,t) = \gamma_t(x)$ is a diffeomorphism for each $t \in [0,1]$,

(ii) $\gamma(x,0) = \gamma_0(x)$ is the identity map on $\mathbb{R}^3$, and

(iii) $\gamma(K_1, 1) = \gamma_1(K_1) = K_2$.

Example 2.3

Figure 2.4.: An example of a pair of ambient isotopic knots in $\mathbb{R}^3$.

Definition 2.5 We say that two links K and K' in $\mathbb{R}^3$ are *equivalent* or the *same link* if there is an ambient isotopy of $\mathbb{R}^3$ that takes K to K'. We denote two equivalent links K and K' with the notation $K \sim K'$.

2.2. Knots and Links in $\mathbb{R}^2$

Definition 2.6 Let L be a link in $\mathbb{R}^3$. A *diagram* of L is a projection of L to $\mathbb{R}^2$ with over and under information at transversal double points of the projection of L. We call these double points *crossings* in a diagram.

Note that not every link projection gives a link diagram.

Definition 2.7 We can assign an orientation to a link diagram by choosing a direction to travel around each component of a link. The resulting link is called an *oriented link*.

Example 2.4

Figure 2.5.: An oriented Hopf link and an oriented Figure-8 knot.

Definition 2.8 An *arc* in a link diagram is a piece of the diagram which starts at an under-passing crossing and ends at the next under-passing crossing.

Example 2.5

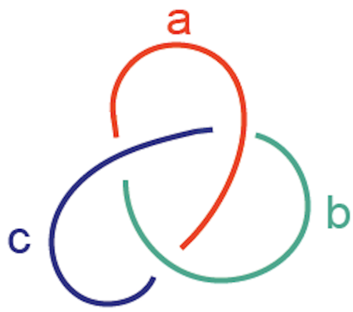


Figure 2.6.: The trefoil knot above has three arcs labeled with a, b, c.

Reidemeister moves consist of four moves **R0**, **R1**, **R2** and **R3** that change a link diagram locally as shown in Figure 2.7. **R0** is a planar isotopy move. The first Reidemeister move **R1** allows us to twist or untwist a strand. The second Reidemeister move **R2** allows us to delete or add a crossing, and the third Reidemeister move **R3** allows us to slide a strand of the knot or link from one side of a crossing to the other side of the crossing.

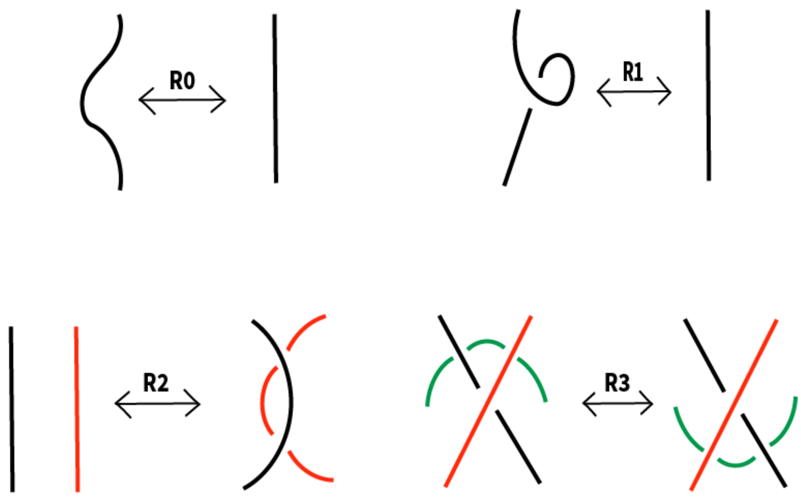


Figure 2.7.: Reidemeister moves **R0**, **R1**, **R2** and **R3**, respectively.

The Reidemeister Theorem says that two links are equivalent in $\mathbb{R}^3$ if there exists a finite sequence of Reidemeister moves in $\mathbb{R}^2$.

Theorem 2.1 (Reidemeister Theorem) Let K_1 and K_2 be two link diagrams. K_1 and K_2 represent the same link in $\mathbb{R}^3$ if and only if K_1 can be transformed into K_2 by using a finite sequence of Reidemeister moves (Reidemeister 1927, 24-32).

According to Kurt Reidemeister, ambient isotopy is generated by a single move type as called an *elementary isotopy* in $\mathbb{R}^3$, see Chapter 1 in *Knot Theory* (Reidemeister 1983, 3-10), and he proved his theorem (the Reidemeister Theorem) by using this correspondance. Therefore, before giving the proof of the Reidemeister Theorem, we will define polygonal knots, polygonal links, and elementary isotopy on polygonal links.

Every knot in $\mathbb{R}^3$ is homeomorphic to the unit circle S^1. In fact, the circle is a one-dimensional smooth manifold. Thus, every knot is a one-dimensional smooth manifold. According to the theorem of Cairns and Whitehead's, every smooth manifold with dimension ≤ 3 admits a compatible piecewise linear structure (Manolescu 2014, 22). Moreover, any piecewise linear manifold is a triangulation (Manolescu 2014, 22). Therefore, we can triangulate smooth knots and links.

Definition 2.9 A *polygonal knot* K in $\mathbb{R}^3$ is the union of a finite number of closed straight-line segments: $K = [A_0, A_1] \cup [A_1, A_2] \cup, ..., \cup [A_{N-2}, A_{N-1}] \cup [A_{N-1}, A_N]$, where $A_N = A_0$. Each $[A_i, A_{i+1}]$ is called an *edge*, and each A_i is called a *vertex* of K, where $i \in \{1, 2, ..., N-1\}$.

Example 2.6 In Figure 2.8, we have a polygonal trefoil knot with six straight edges and six vertices.

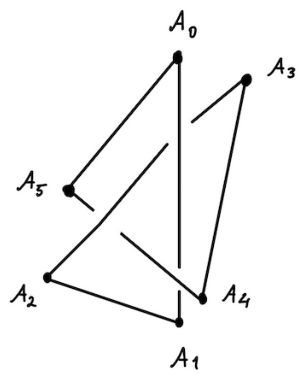


Figure 2.8.: An example of a polygonal knot.

Definition 2.10 A *polygonal link* in $\mathbb{R}^3$ is a disjoint union of a finite number of polygonal knots.

Example 2.7 In Figure 2.9, we see a polygonal Hopf link, which is the disjoint union of two polygonal unknots.

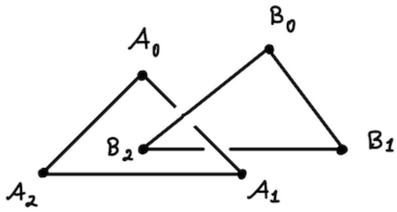


Figure 2.9.: An example of a polygonal link.

Definition 2.11 An *elementary isotopy* is a replacement of an edge of a link L in $\mathbb{R}^3$ with two edges of the triangle which does not intersect other edges of L.

An elementary isotopy in $\mathbb{R}^3$ is generetad by two moves shown in Figure 2.10. The first one is expansion (Δ-move) and the second one is the inverse of expansion move, contraction (Δ^{-1}-move) which are defined in the following.

Definition 2.12 Let L be a polygonal link in $\mathbb{R}^3$. Let us choose two vertices A_0 and A_2 on the link and put a new vertex A_1 in the complement of the link L. Then replace the edge $[A_0, A_2]$ with $[A_0, A_1] \cup [A_1, A_2]$. This move is called as *expansion*, and denoted by Δ. On the other hand, if we replace $[A_0, A_1] \cup [A_1, A_2]$ with $[A_0, A_2]$, we get *contraction* move and denote it by Δ^{-1}. These moves are illustrated in Figure 2.10.

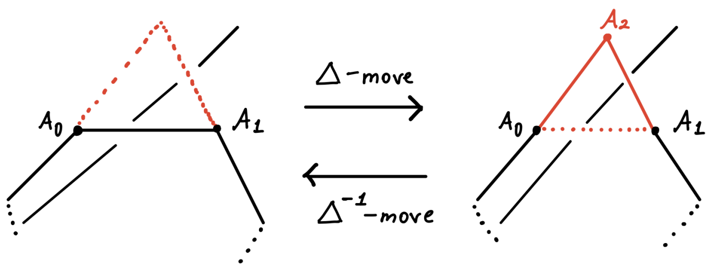


Figure 2.10.: Δ-moves on a polygonal link.

Definition 2.13 Two polygonal links L and L' in $\mathbb{R}^3$ are *isotopic* if there are a finite number of Δ-moves taking L to L'.

Now, we explain how Theorem 2.1 was proved by Kurt Reidemeister. He thought of smooth links as polygonal links. Thus, we consider polygonal links throughout the proof of the Reidemeister Theorem.

Proof (Proof of Theorem 2.1) If part is easy. Any sequence of Reidemeister moves do not change the equivalence class of a link diagram. Now, we prove that the "only if" part. Suppose that K_1 and K_2 are the same links in $\mathbb{R}^3$. To show that there is a finite sequence of Reidemeister moves that we need to illustrate that each elementary isotopy moves in $\mathbb{R}^3$ correspond to Reidemeister moves in $\mathbb{R}^2$. For this, we do induction on $n \in \mathbb{N}$, where n is the number of strands in triangular regions of Δ-moves.

For $n = 0$, we have just one move Δ or Δ^{-1} as shown in Figure 2.11. Let us denote this move as Δ_0.

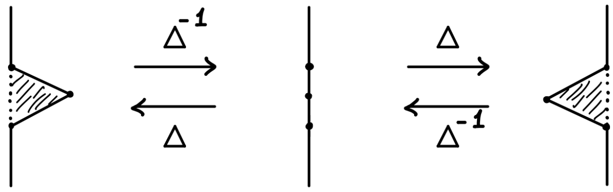


Figure 2.11.: The move Δ_0 in $\mathbb{R}^3$ corresponds to **R0** in $\mathbb{R}^2$.

For $n = 1$, we have three basic options. If the strand in the triangular region of a Δ-move comes from itself, the move will be called Δ_1 as shown in Figure 2.12:

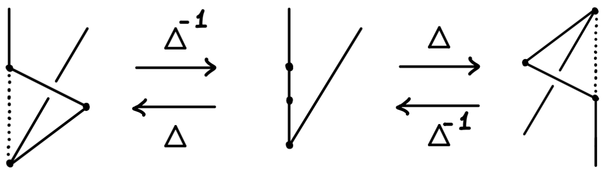


Figure 2.12.: The move Δ_1 in $\mathbb{R}^3$ corresponds to **R1** in $\mathbb{R}^2$.

The remaining two options are illustrated in Figure 2.13. We call these two moves as Δ_2.

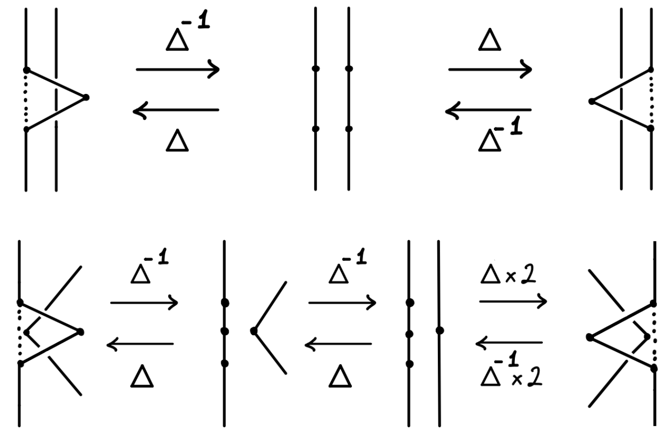


Figure 2.13.: The move Δ_2 in $\mathbb{R}^3$ corresponds to **R2** in $\mathbb{R}^2$.

For $n = 2$, we have two options shown in Figure 2.14. Let us call these moves as Δ_3.

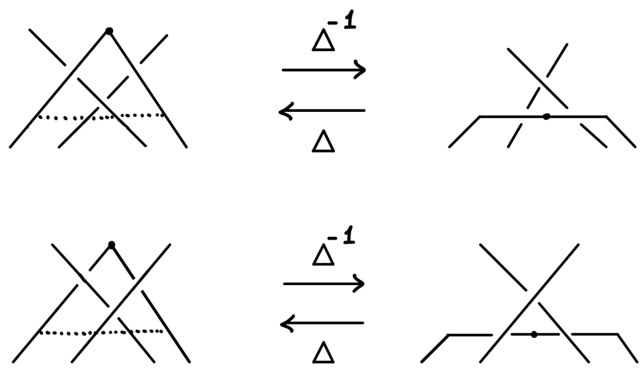


Figure 2.14.: The move Δ_3 in $\mathbb{R}^3$ corresponds to **R3** in $\mathbb{R}^2$.

The moves in Figure 2.11, Figure 2.12, Figure 2.13 and Figure 2.14 are basically simple moves corresponding to **R0**, **R1**, **R2** and **R3**. When there are n strands in the Δ-moves, we can transform a link L to the link that is equivalent to L by using the combination of the finite number of moves. In this case, if it is needed we sometimes use a method called *subdivision*, illustrated in Figure 2.15. Moreover, the strands in the Δ-moves can cut themselves.

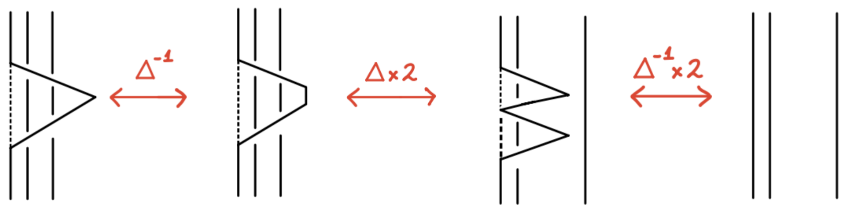


Figure 2.15.: A subdivision.

Now, let us take $n = 5$ strands in the triangular region of a Δ-move. In Figure 2.16, we see the Δ-moves for 5 strands with the subdivision method.

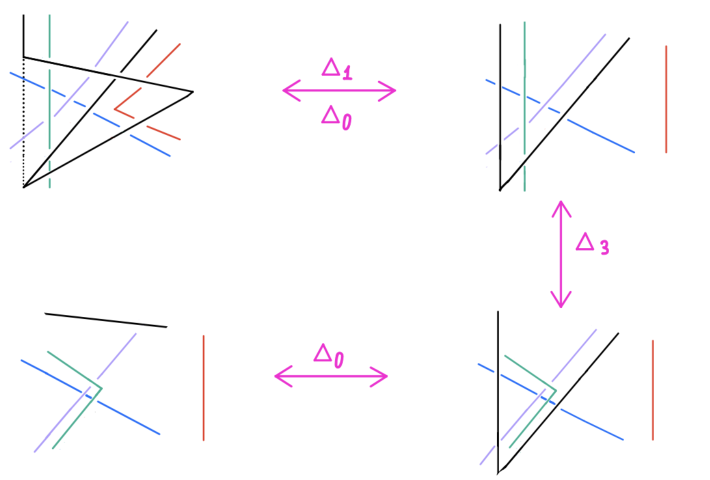


Figure 2.16.: Δ-moves for five strands with subdivision.

In the case above, to transform a link into its equivalent link, we use a finite sequence of the moves Δ_1, Δ_0, and Δ_3 in $\mathbb{R}^3$ that correspond to the Reidemeister moves

R1, **R0**, and **R3**, respectively, in $\mathbb{R}^2$. Similarly, we can extend this to n strands.

□

Note that Reidemeister moves give us an equivalence of links.

We can also apply Reidemeister moves for oriented knots and links. There are four different versions for each of the first and second Reidemeister moves, eight different versions for the third Reidemeister move as illustrated in Figure 2.17, Figure 2.18, and Figure 2.19:

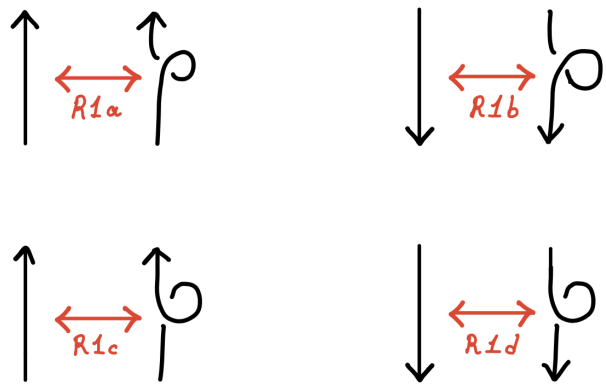


Figure 2.17.: The four versions of the first Reidemeister move for oriented links and knots.

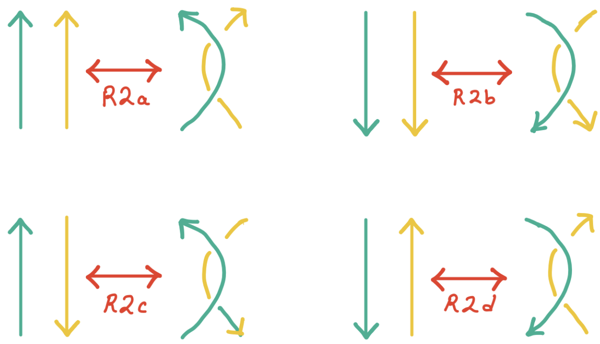


Figure 2.18.: The four versions of the second Reidemeister move for oriented links and knots.

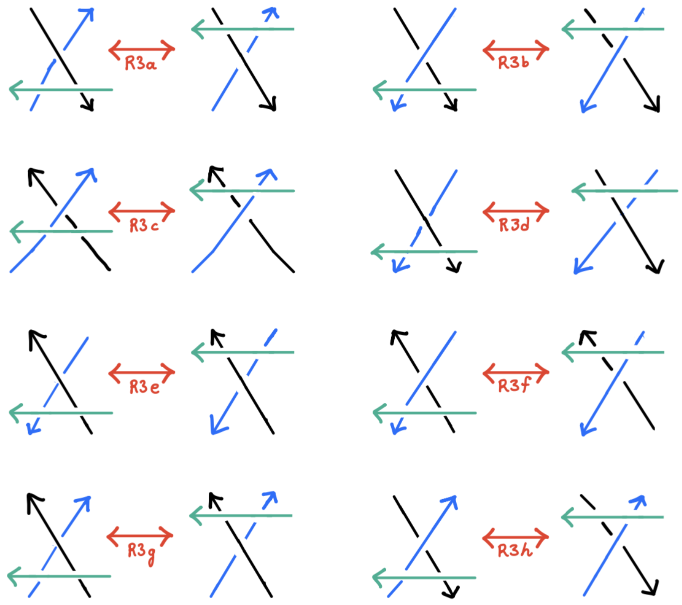


Figure 2.19.: The eight versions of the third Reidemeister move for oriented links and knots.

Theorem 2.2 Let L and L' be two oriented link diagrams that represent the same link in $\mathbb{R}^3$. Then L can be transformed into L' by using a finite sequence of the four oriented Reidemeister moves in Figure 2.20 (Polyak 2010, 399-411).

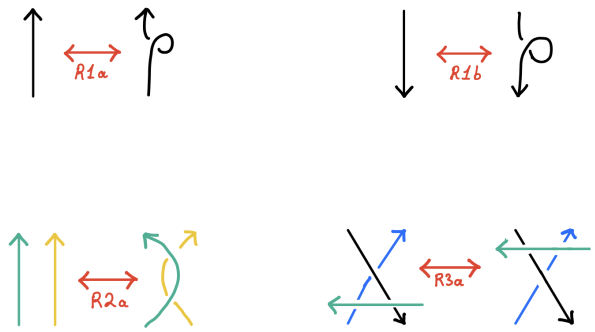


Figure 2.20.: **R1a**, **R1b**, **R2a** and **R3a** generate oriented Reidemeister moves.

2.3. Knot Invariants

In this chapter, we study some knot invariants such as crossing number, linking number, n-coloring, and the knot group.

Definition 2.14 Let K_1 and K_2 be two knot diagrams. Define

$$f : \{Knot\ Diagrams\} \to \mathbb{M}$$

be map, where $\mathbb{M}$ is a mathematical set. We say that f is a *knot invariant* if K_1 and K_2 represent the same knot in $\mathbb{R}^3$, then $f(K_1) = f(K_2)$ in $\mathbb{M}$.

Note that for two knot diagrams K_1 and K_2, it is not necessary for K_1 to be equivalent to K_2 when $f(K_1) = f(K_2)$.

Definition 2.15 Let K be a knot in $\mathbb{R}^3$. The crossing number of K is the minimum number of crossings in a diagram in the ambient isotopy class of K and is denoted by $c(K)$.

Proposition 2.1 Knot diagrams with maximum two crossings represent the unknot in $\mathbb{R}^3$.

Proof To prove this statement, we form all possible knot diagrams which have at most two crossings, as given in Figure 2.21. We observe that all these diagrams represent the unknot. □

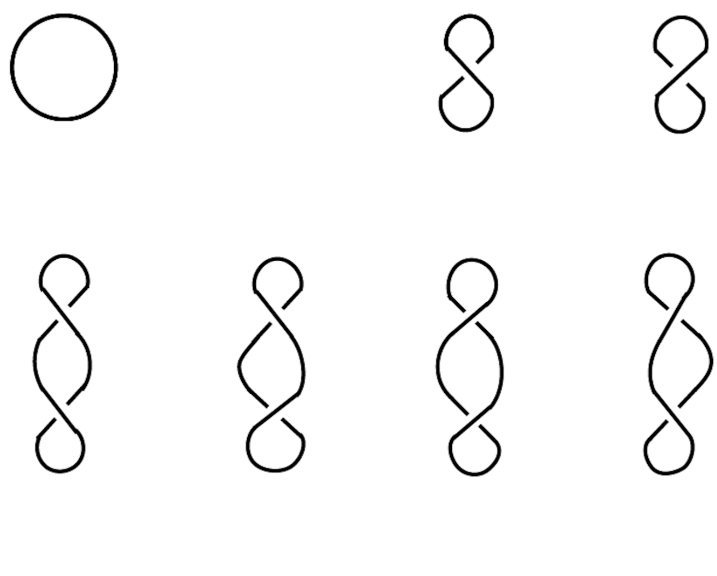


Figure 2.21.: The knots with maximum 2 crossings are equivalent to the unknot.

Definition 2.16 Let K_1 and K_2 be two oriented knots. A *connected sum (or composition)* of K_1 and K_2 is defined as an oriented knot obtained by cutting a piece from the portions of the arcs which do not contain crossings and attaching them while preserving the orientation of K_1 and K_2. We denote the connected sum of K_1 and K_2 by $K_1 \# K_2$. See the illustration in Figure 2.22.

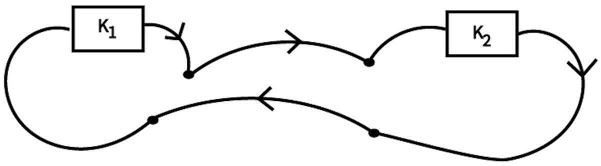


Figure 2.22.: A connected sum of two oriented knots K_1 and K_2.

Example 2.8 In Figure 2.23, we see the connected sum of an oriented trefoil and an oriented Figure-8 knot.

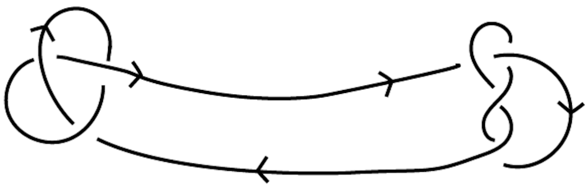


Figure 2.23.: A connected sum of the treofil and the Figure-8 knot.

Definition 2.17 If a knot K is not the connected sum of any two nontrivial knots, we say K is a *prime knot*.

The table in Figure 2.24 is taken from *Knot Theory and Its Applications* (Murasugi 1996, 326) which includes prime knots up to 9 crossings. The prime knots in this table are categorized by their crossing numbers: We observe that each knot has a number such as 3_1, 4_1, 5_2 which means that the only knot having the crossing number equals 3, the only knot having the crossing number 4, the second knot which has the crossing number equals 5, and so on.

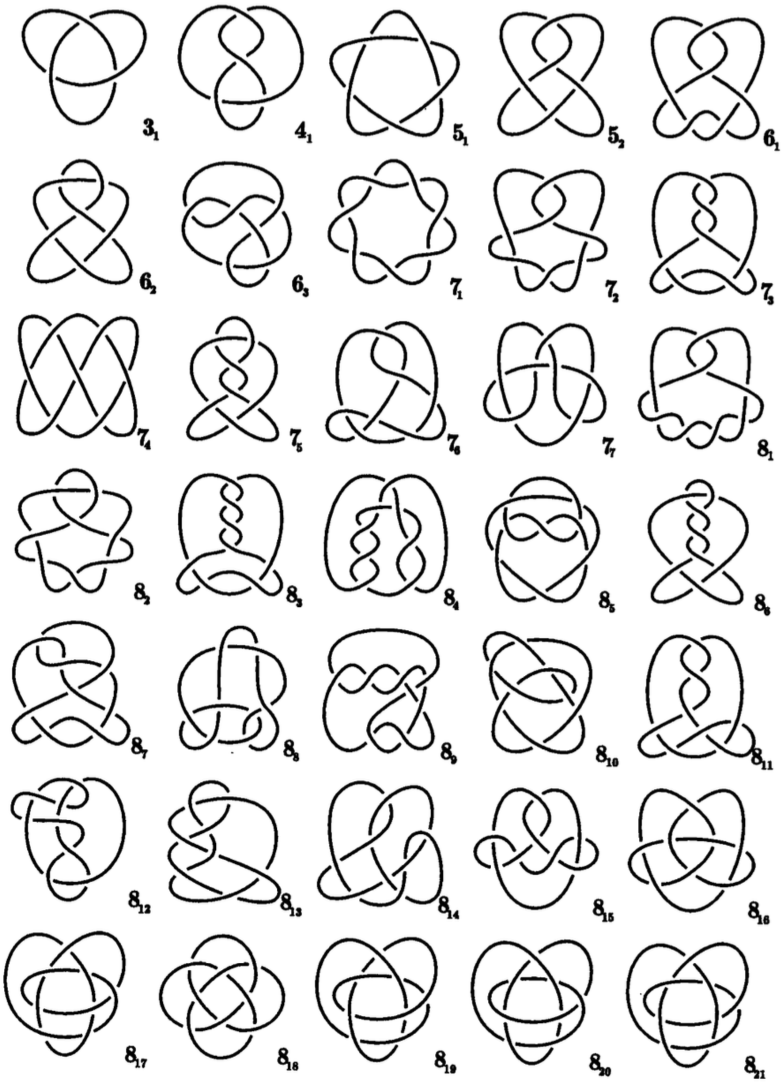


Figure 2.24.: A table of some prime knots.

Theorem 2.3 If two knots are the same, then their crossing numbers are equal.

Proof 2.1 *See Knot Theory and Its Applications (Murasugi 1996, 57).*

There are two types of crossings in an oriented knot diagram as indicated in Figure 2.25.

Definition 2.18 Let c be a crossing in a knot diagram K, and, at the crossing c, suppose the overpassing strand goes upward. Then, if the underpassing strand goes from the right to the left, we say that c is a *positive crossing*. If the underpassing strand goes from the left to the right, c is called a *negative crossing*.

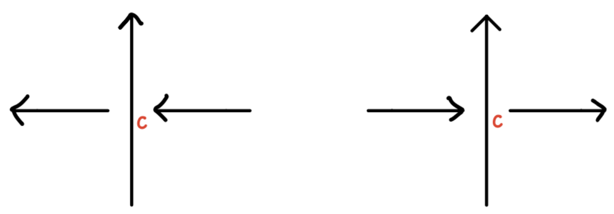


Figure 2.25.: Positive crossing and negative crossing, respectively.

Definition 2.19 Let L be an oriented link diagram and c a crossing in L. We define the *sign* of the crossing c as

$$sign(c) = \begin{cases} +1 & \text{if } c \text{ is a positive crossing,} \\ -1 & \text{if } c \text{ is a negative crossing.} \end{cases}$$

Definition 2.20 Let $\overrightarrow{L}$ be an oriented link diagram with two components. Let c be a crossing shared by these components. The *linking number* of $\overrightarrow{L}$ is given by

$$Lk(\overrightarrow{L}) = \frac{1}{2} \sum_{c} sign(c).$$

Example 2.9 The reader can examine that the linking number of the Hopf link is given -1, and the linking number of the Whitehead link is 0 as in Figure 2.26.

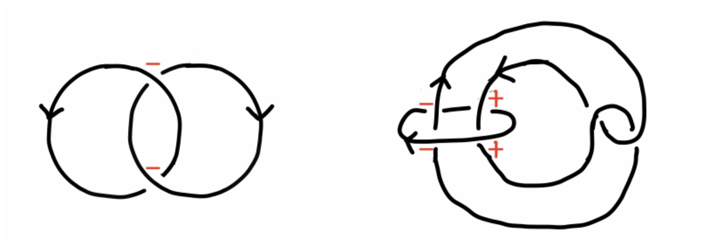


Figure 2.26.: An oriented Hopf link and an oriented Whitehead link, respectively with the shared crossings given with their signs.

Note: Although the linking number of the Whitehead link is 0, this link is not trivial.

Theorem 2.4 Let L be an oriented link. The linking number of L, $\overrightarrow{Lk}(L)$, is a link invariant.

Proof See *Knot Theory and Its Applications* (Murasugi 1996, 66-68). □

2.3.1. Coloring Knots and Links

Definition 2.21 Let K be a knot diagram. We say that K is *3-colorable* if each arc is drawn by using one of the colors red (R), yellow (Y), and blue (B) and the following conditions are satisfied.

(i) At least two different colors are used.

(ii) At each crossing, if two colors appear, then either three colors appear, or only one color is observed at that crossing.

Example 2.10

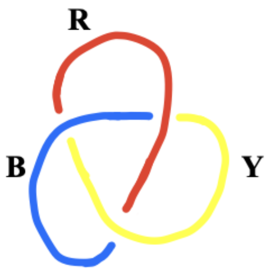


Figure 2.27.: The trefoil is 3-colorable.

Theorem 2.5 If a diagram of a knot or a link K is 3-colorable, then its all diagrams are 3-colorable.

Proof See Section 4.6 of *Knot Theory and Its Applications* (Murasugi 1996, 70-72). □

We can consider the colors of the arcs in a 3-colorable knot diagram K as the elements of $\mathbb{Z}_3$. Let x be the label of the overpassing arc, and let y and z be the labels of underpassing arcs at a crossing of K, where $x, y, z \in \mathbb{Z}_3$. Since K is 3-colorable, x, y, z are either distinct or all the same. Then, the second condition in Definition 2.21 becomes $2x - y - z \equiv 0 \pmod 3$ at each crossing. For generality, we give the following definition.

Definition 2.22 Let K be a knot/link diagram, where each arc is labeled by one of the elements $x_0, x_1, \cdots, x_{n-1}$ from $\mathbb{Z}_n$. Then, if the following are satisfied, we say that K is *n-colorable*.

(i) At least two distinct elements from $\mathbb{Z}_n$ are observed on the arcs around each crossing.

(ii) At each crossing c, we have the relation $2x_i - x_j - x_k \equiv 0 \pmod n$, where x_i is the label of the overpassing arc, and x_j and x_k are the labels of underpassing arcs at c for $i, j, k \in \{0, 1, \cdots, n-1\}$.

Theorem 2.6 If a link diagram L is n-colorable, then any diagram of L is n-colorable.

Proof See Section 4.6 of *Knot Theory and Its Applications* (Murasugi 1996, 73-74). □

Example 2.11 The knot diagram of the Figure-8 knot, denoted as 4_1, is 5-colorable. If we assign the labels $0, 1, 2, 4 \in \mathbb{Z}_5$ to arcs of the diagram of 4_1 as in Figure 2.28, we will have the following relations at each crossing:

$$c_1 : 2.1 - 2 - 0 \equiv 0 \pmod 5$$
$$c_2 : 2.0 - 1 - 4 \equiv 0 \pmod 5$$
$$c_3 : 2.2 - 4 - 0 \equiv 0 \pmod 5$$
$$c_4 : 2.4 - 1 - 2 \equiv 0 \pmod 5.$$

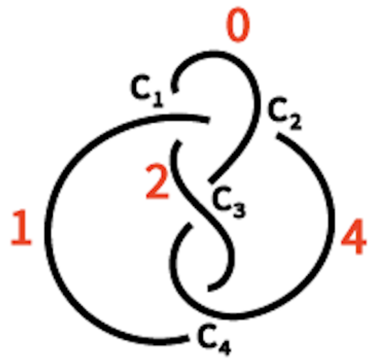


Figure 2.28.: The figure-8 knot is 5-colorable.

Definition 2.23 We can consider these relations as a system of linear equations from an n-colorable knot/link diagram K and present them with a matrix, which we will call a *coloring matrix* M_K. The relations at each crossing correspond to the rows, and the labels of arcs correspond to the columns of M_K.

Example 2.12

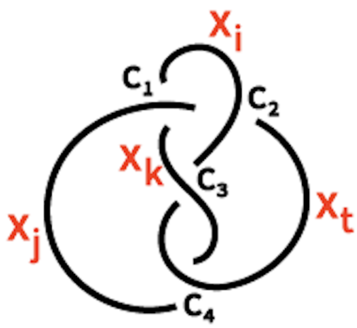


Figure 2.29.: The figure-8 knot with arcs labeled by some elements from $\mathbb{Z}_n$.

We have the following relations which come from the crossings of the figure-8 knot diagram in Figure 2.29:

$$c_1 : 2.x_j - x_k - x_i \equiv 0 \pmod 5$$
$$c_2 : 2.x_i - x_j - x_t \equiv 0 \pmod 5$$
$$c_3 : 2.x_k - x_t - x_i \equiv 0 \pmod 5$$
$$c_4 : 2.x_t - x_j - x_k \equiv 0 \pmod 5.$$

Therefore, the coloring matrix M_{4_1} of the diagram of the figure-8 knot is

$$\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{matrix} x_i & x_j & x_k & x_t \end{matrix} \\ \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & -1 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \end{array}.$$

Definition 2.24 Let L be a link in $\mathbb{R}^3$. If at least one component of L can be separated by a plane which is embedded in $\mathbb{R}^3$, we say that L is a *splittable link.*

Lemma 2.1 A knot/link diagram which is not splittable admits the same number of arcs and crossings, except the unknot and the unlink.

Proof By definition, an arc starts at a crossing and ends at the next crossing, which means, at each crossing, the underpassing strand is divided into two arcs, as illustrated in Figure 2.30). So, if we have n crossings in a knot/link diagram, then we see $2n$ arcs in total. In fact, each crossing is both the starting point of an arc and the end point of the other arc, which means each arc shares two crossings. Moreover, by definition, knots and links are closed curves. Therefore, we have $\frac{2n}{2} = n$ arcs in a knot or link diagram with n crossings. □

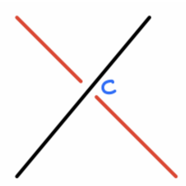


Figure 2.30.: At the crossing c, the underpassing arc is divided into two arcs drawn in red.

Note that the above lemma may not be true for splittable links. Consider a splittable link L consisting of a nontrivial link which has n arcs and n crossings, and m unknots. When we separate L by a plane, we have $n + m$ arcs yet n crossings in total since the unknot have no crossing. See in Figure 2.31.

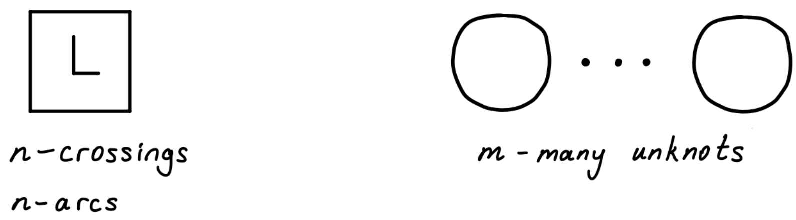


Figure 2.31.: A link diagram L has n-crossings and n-arcs, and m-many unknots which have contribution only m-arcs in total.

According to the above lemma, we have the following conclusion.

Corollary 2.1 If a link diagram L does not contain an unknot, then its coloring matrix M_L is a square matrix.

As we see in the coloring matrix in Example 2.12, the sum of the rows in each column is equal to zero vector, which means one of the relations is redundant. Indeed, in that example, we can observe that the negative of the third row is equal to the sum of the first, second, and fourth rows. Moreover, one of the relations in a coloring matrix is always redundant. Therefore, we delete the redundant row and a column from the coloring matrix to compute the determinant of a link.

Definition 2.25 Let L be a knot or a link diagram with n-crossings and n arcs, and let its $n \times n$ coloring matrix be M_L. When we delete each row can be chosen as redundant and an i^{th} column, the new $(n-1) \times (n-1)$ matrix is called the reduced coloring matrix *of* L. The absolute value of the determinant of the reduced coloring matrix is said to be the *determinant* of L.

Example 2.13 The reduced coloring matrix of M_{4_1} in Example 2.12, obtained by deleting the third row and the second column is the following matrix, say M'_{4_1},

$$M'_{4_1} = \begin{matrix} & x_i & x_k & x_t \\ c_1 & -1 & -1 & 0 \\ c_2 & 2 & 0 & -1 \\ c_4 & 0 & -1 & 2 \end{matrix}$$

$$M'_{4_1} = \begin{array}{c} c_1 \\ c_2 \\ c_4 \end{array} \begin{bmatrix} -1 & -1 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Then the determinant of the figure-8 knot is the following:

$$det(4_1) = \left|\det(M'_{4_1})\right| = 5.$$

Theorem 2.7 The determinant is a link invariant.

Proof The sketch of the proof can be found in *Knot Theory* (Livingston 1993, 46-47). □

Theorem 2.8 Let K be a knot diagram and p an odd prime number. Then K is p-colorable if and only if p divides $det(K)$.

Proof See Section 4.3 in *Knot Theory* (Livingston 1993, 45). □

Theorem 2.9 A link diagram L is n-colorable if and only if $gcd(n, det(L)) > 1$.

Proof The reader can find the proof in *Minimum Number of Fox Colors for Small Primes* (Lopes and Matias 2011, 6). □

In addition to the aforementioned invariants, we also have polynomial knot invariants such as the Alexander polynomial, the Alexander-Conway polynomial, the Kauffman bracket polynomial, and the Jones polynomial.

2.3.2. The Knot Group

In this section, we study one of the strong knot invariants, known as the knot group.

Definition 2.26 Let L be a knot (or link) in $\mathbb{R}^3$. The *knot group (or link group)* of L, shown as $\pi(L)$, is the fundamental group of the complement of L in $\mathbb{R}^3$, $\pi_1(\mathbb{R}^3 \setminus L)$.

Theorem 2.10 The knot group is a knot invariant.

Proof Suppose that K_1 and K_2 represent the same knot in $\mathbb{R}^3$. Then the ambient isotopy that takes K_1 to K_2 gives us a homeomorphism between $\mathbb{R}^3 \setminus K_1$ and $\mathbb{R}^3 \setminus K_2$. Therefore, $\pi_1(\mathbb{R}^3 \setminus K_1) \cong \pi_1(\mathbb{R}^3 \setminus K_2)$ since the fundamental group is a homotopy invariant (See Appendix A.1.3). □

Theorem 2.11 (Gordon–Luecke Theorem) Let K and K' be two knots which have homeomorphic complements, then K and K' are ambient isotopic (Gordon and Luecke 1989, 371).

Theorem 2.12 A knot is trivial if and only if its fundamental group is isomorphic to the set of integers (Manturov 2004, 53-54).

Proof Let K be the unknot and $N(K)$ its tubular neighborhood. Let us take a base-point x_0 in the boundary of $N(K)$, $\partial N(K) = T(K)$, which is the torus surrounding K (Scharlemann 1992, 68). Any closed loop c at x_0 can be isotoped to a loop that lies on the boundary of $N(K)$. On $\partial N(K) = T(K)$, this loop can be described as a combination of two loops that do not intersect each other: The meridian m wrapping around K once, and the longitude l, which is the parallel to K. The curves m and l generate the fundamental group of the torus, $\mathbb{Z} \times \mathbb{Z}$. Since the longitude l is contractible to a point in $\mathbb{R}^3 \setminus K$, only the meridian m generates the fundamental group of $\mathbb{R}^3 \setminus K$. Thus, the knot group of the unknot, $\pi_1(\mathbb{R}^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.

Conversely, assume that we have a knot K such that its knot group $\pi_1(\mathbb{R}^3 \setminus K)$ is isomorphic to $\mathbb{Z}$. Then $\pi_1(\mathbb{R}^3 \setminus K)$ contains $\{m\} = \mathbb{Z}$, where m is the meridian of $T(K)$ and $T(K)$ is the boundary of the tubular neighborhood of K, $N(K)$. However, the longitude l of $T(K)$ is not in a homotopy class generated by m. Hence, l can be shrunk to zero in the fundamental group of $\mathbb{R}^3 \setminus N(K)$, $\pi_1(\mathbb{R}^3 \setminus N(K)$. Therefore, there is an immersed 2-disk D whose boundary is l. Then, by Dehn's lemma (See in Appendix B.1), there exists an embedded disk D' in $\mathbb{R}^3 \setminus N(K)$ with the same boundary of D, l. When we contract $N(K)$ to K, we get a disk whose boundary is the knot K. It implies that K is the unknot by Definition 2.2. □

2.3.3. Wirtinger Presentation

We now study a presentation of the knot group, the *Wirtinger presentation*, which turns out to be the combinatorial definition for the knot group.

Definition 2.27 A *Wirtinger presentation*

$$\langle \alpha_1, \alpha_2, \cdots, \alpha_s : \ r_1, r_2, \cdots, r_t \rangle$$

is a finite presentation of a group G, where each relation r_i is written as $\alpha_j^{-1}\omega\alpha_k\omega^{-1}$ for some letter α_j, α_k and a word ω in $\alpha_1, \cdots, \alpha_s$. Here, G is a quotient group of a free group F with a basis $\alpha_1, \cdots, \alpha_s$.

Let K be a knot/link diagram with n arcs; each of them is labeled by $\alpha_1, \cdots, \alpha_n$. Let α_i, α_j, and α_k be arcs arounding a crossing of K, and let α_i denote the overpassing arc and α_j, α_k denote the underpassing arcs at that crossing, where $i, j, k \in \{1, 2, \cdots, n\}$. Then, depending on the type of crossings, we impose the relations in Figure 2.32.

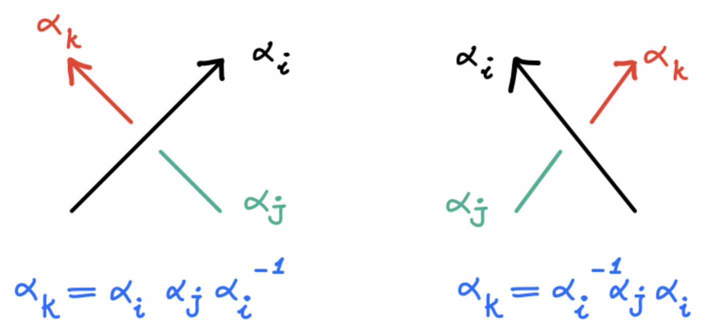


Figure 2.32.: The relations at crossings according to crossing types.

Definition 2.28 *The Wirtinger presentation of knot group of a knot K* is given as $\Gamma(K) = \langle \alpha_1, \alpha_2, \cdots, \alpha_n \mid r_1, r_2, \cdots, r_n \rangle$, where each α_i is a label of given arcs of K and each r_i is a relation at crossings for every $i \in \{1, 2, \cdots, n\}$.

Example 2.14

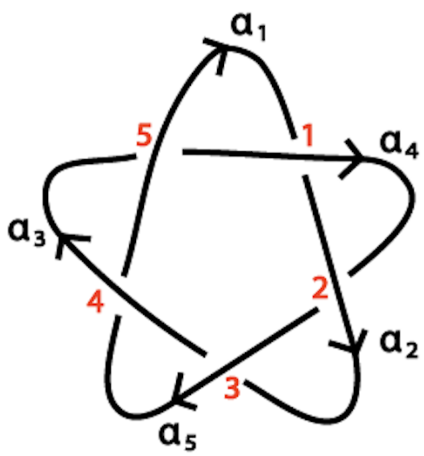


Figure 2.33.: An oriented Cinquefoil knot.

We show the Wirtinger presentation of the cinquefoil knot 5_1 illustrated in Figure 2.33.

At each crossing, we have the following relations.

at the crossing 1: $x_4^{-1}x_1x_4 = x_2$,
at the crossing 2: $x_2^{-1}x_4x_2 = x_5$,
at the crossing 3: $x_5^{-1}x_2x_5 = x_3$,
at the crossing 4: $x_3^{-1}x_5x_3 = x_1$,
at the crossing 5: $x_1^{-1}x_3x_1 = x_4$.

Then we have

$$\Gamma(5_1) = \langle x_1, x_2, x_3, x_4, x_5 \mid x_4^{-1}x_1x_4 = x_2,\ x_2^{-1}x_4x_2 = x_5,\ x_5^{-1}x_2x_5 = x_3,\ x_3^{-1}x_5x_3 = x_1,\ x_1^{-1}x_3x_1 = x_4\rangle.$$

Moreover, we have

$x_5 = x_4^{-1}x_1^{-1}x_4x_1x_4$, by the relations 1 and 2,
$x_5 = x_4^{-1}x_1^{-1}x_4x_1x_4$, by the relations from 3,
$x_3 = x_4^{-1}x_1^{-1}x_4^{-1}x_1x_4x_1x_4$, from the crossings 4 and 5.

Thus we obtain that $x_4x_1x_4x_1x_4 = x_1x_4x_1x_4x_1$.

Therefore, by setting as $x_1 = x$ and $x_4 = y$ we have

$$\Gamma(5_1) = \langle x, y \mid yxyxy = xyxyx \rangle.$$

Theorem 2.13 Let K be a knot diagram with n arcs in $\mathbb{R}^2$, and let α_i's be the labels of these arcs, where $i \in \{1, \cdots, n\}$. The knot group of K in $\mathbb{R}^3$, $\pi(K)$ is generated by $\alpha_1, \alpha_2, \cdots, \alpha_n$. In fact, $\pi_1(K)$ is equal to the Wirtinger presentation, $\Gamma(K)$.

Proof First, let us orient the given knot K in $\mathbb{R}^3$. Then, for some $n \in \mathbb{Z}^+$ we have a crossing type as shown in Figure 2.34, where $\alpha_{n+1} = \alpha_1$.

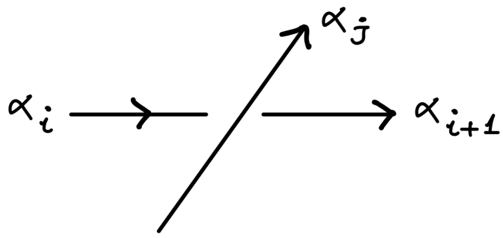


Figure 2.34.: A crossing type, where $i, j \in \{1, 2, \cdots, n\}$.

Secondly, we deform K in order to replace such crossing given above with a crossing like the following: While most of the knot K is inside the z=1 plane, at each crossing we drop a vertical line to the z=0 plane and connect with a line segment as labelled β_j in that place such that β_j starts from the foot of the dropped vertical line of α_i and ends at the foot of that starting α_{i+1}, as illustrated in Figure 2.35.

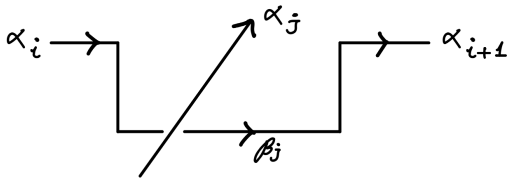


Figure 2.35.: The crossing after the deformation.

Thirdly, we remove a *tubular neighborhood* N of K from $\mathbb{R}^3$. We can imagine a tubular neighborhood in the following way: Suppose that we have a closed cube with a side length of ϵ, and it is fixed the knot K. Assume this cube can slide along the strands of K. When it meets with a crossing, as in Figure 23, it slides down the first vertical line with its faces parallel to the axes in $\mathbb{R}^3$, then moves along β_j, and finally return to the portion of α_{i+1} in the z=1 plane. See in Figure 2.36.

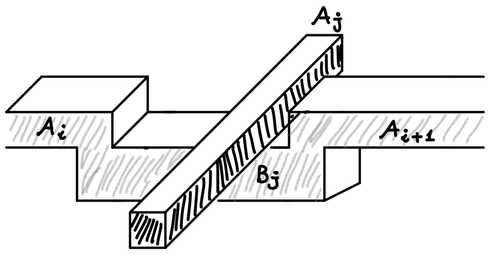


Figure 2.36.: We call the parts of the tubular neighborhood containing α_i, α_j, β_j and α_{i+1} as A_i, A_j, B_j, and A_{i+1}, respectively.

If we choose ϵ small enough, then we can ensure that $\mathbb{R}^3 \setminus N$ is a deformation retract of $\mathbb{R}^3 \setminus K$ (see *deformation retraction* in Appendix A.1.1). Thus the fundamental group of $\mathbb{R}^3 \setminus N$, $\pi_1(\mathbb{R}^3 \setminus N)$ is isomorphic to $\pi_1(\mathbb{R}^3 \setminus K)$, which is the knot group of K (See Appendix A.1.2).

Now, we will describe a decomposition of $\mathbb{R}^3 \setminus N$ by the union of two open sets in $\mathbb{R}^3$: Let us say $U := \{z > 0\} \setminus N$, and $V := \{z < \epsilon/2\} \setminus N$, where $\epsilon > 0$ is the length of a side of our closed cube. Here, U has a deformation retract onto n-leafed rose with loops $l_1, \cdots, l_n$, where each l_i is a loop passing under the part A_i of the tunnel neighborhood that contains α_i as in Figure 2.37. Thus, the fundamental group of U is $\pi_1(U) = (l_1, \cdots, l_n : \emptyset)$.

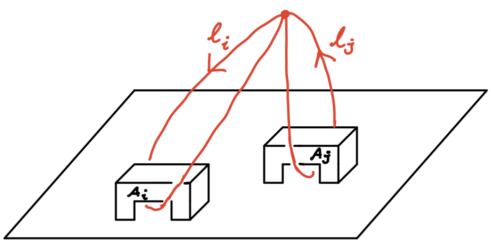


Figure 2.37.: l_i and l_i are loops that pass under the parts of the tubular neighborhood A_i and A_i, respectively, where $i, j \in \{1, 2, \cdots, n\}$.

Furthermore, V is an open half-space with trenches cut in it, as illustrated in Figure 2.38. Hence it is contractible and so $\pi_1(V) = \{1\}$, the trivial fundamental group (See Appendix A.1.3).

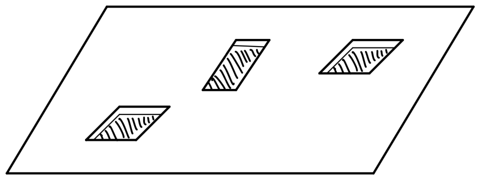


Figure 2.38.: The illustration of the open-half space V with trenches.

Now, consider that

$$U \cap V = \{0 < z < \epsilon/2\} \setminus N.$$

It is an infinite plane including m holes, where m is the number of crossings in $U \cap V$. See Figure 2.39.

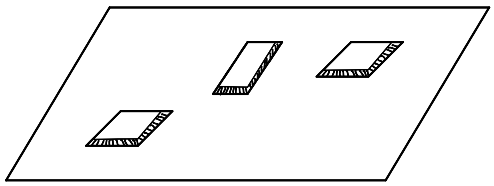


Figure 2.39.: The intersection of the spaces U and V is an infinite plane with m holes in it.

Therefore, the fundamental group of $U \cap V$ is a free group of rank m.

Now, a generator will correspond to a circuit around a hole, and so its image, a *relation*, in $\pi_1(U)$ has two forms according to crossing type. Hence, we have the number of m relations that are written as $r_1, \cdots, r_m$. They are shown in Figure 2.40 and Figure 2.41, respectively.

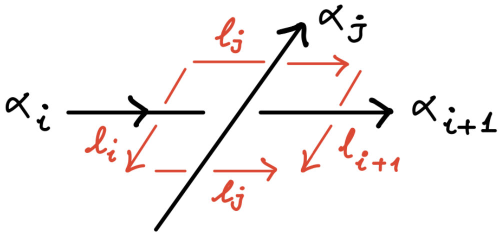


Figure 2.40.: The image of a generator has the form of $l_j l_{i+1}^{-1} l_j^{-1} l_i$ in $\pi_1(U)$. Thus we have a relation as $1 = l_j l_{i+1}^{-1} l_j^{-1} l_i$.

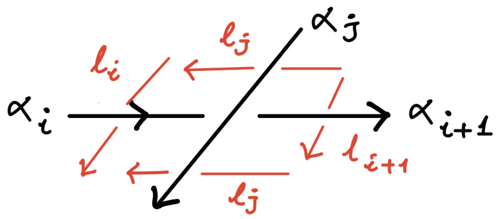


Figure 2.41.: The image of a generator has the form of $l_j^{-1} l_{i+1}^{-1} l_j l_i$ in $\pi_1(U)$. Thus we get a relation as $1 = l_j^{-1} l_{i+1}^{-1} l_j l_i$.

Finally, we can apply the van Kampen theorem (Hatcher 2001, 43):

$$\begin{aligned}\pi_1(\mathbb{R}^3 \setminus K) &\cong \pi_1(\mathbb{R}^3 \setminus N) \\ &= \frac{\pi_1(U) * \pi_1(V)}{\pi_1(U \cap V)} \\ &= \frac{\langle l_1, \cdots, l_n \rangle * \{1\}}{\langle r_1, \cdots, r_m \rangle} \\ &= \langle l_1, \cdots, l_n : r_1, \cdots, r_m \rangle,\end{aligned}$$

where $l_1, \cdots, l_n$ are loops that we have and $r_1, \cdots, r_m$ are the relations at each crossings. This is exactly how the knot group of a knot K, $\pi_1(\mathbb{R}^3 \setminus K)$, is presented as the Wirtinger presentation. □

Note that in the above proof, if we studied with the unknot or the unlink, or such a link as in Figure 2.42, by Lemma 1, we would have $m \neq n$. Otherwise, we have $n = m$ for knots and links.

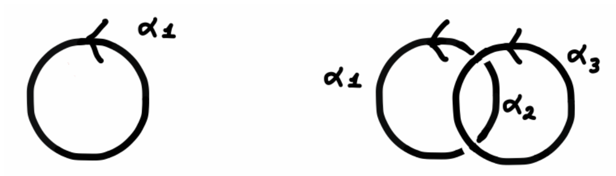


Figure 2.42.: The unknot and an the unlink with an orientation.

Theorem 2.12 indicates that the knot group is a comparatively strong knot invariant. Despite this, we cannot distinguish mirror symmetric knots and some further knots by calculating their knot groups. For instance, the knot group does not distinguish the mirror symmetric trefoils, and, also, it does not distinguish the granny knot and the square knot. We will examine them in the examples 2.15 and 2.16. Before that, we give the definition of *mirror symmetry*.

Definition 2.29 Let ϕ be a homeomorphism on $\mathbb{R}^3$ defined by $\phi(x, y, z) = (x, y, -z)$, where (x, y, z) is a point of a knot K in $\mathbb{R}^3$. Then, we say $\phi(K)$ is the *mirror image (or symmetry)* of the knot K.

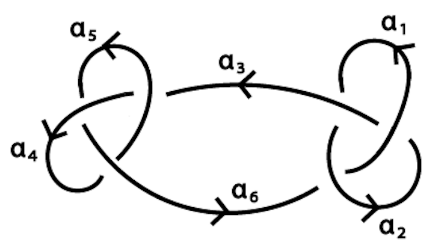
Example 2.15 In Figure 2.43, we see the right-handed trefoil, 3_1 whose all crossings are positive and its mirror image the left-handed trefoil, 3_1^* whose all crossings are negative. Now, we compute and examine the knot groups of 3_1 and 3_1^* by Wirtinger presentation.

Figure 2.43.: The right-handed trefoil and the left-handed trefoil, respectively.

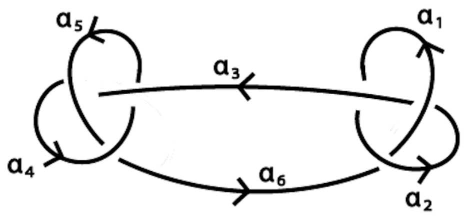
We have the following relations from the crossings of 3_1: $\alpha_2 = \alpha_3\alpha_1\alpha_3^{-1}$, $\alpha_1 = \alpha_2\alpha_3\alpha_2^{-1}$, and $\alpha_3 = \alpha_1\alpha_2\alpha_1^{-1}$. Thus we have the knot group
$\Gamma(3_1) = \langle \alpha_1, \alpha_2, \alpha_3 \mid \alpha_2 = \alpha_3\alpha_1\alpha_3^{-1}, \alpha_1 = \alpha_2\alpha_3\alpha_2^{-1}, \alpha_3 = \alpha_1\alpha_2\alpha_1^{-1} \rangle$. By using the relations, we obtain $\alpha_2\alpha_1\alpha_2 = \alpha_1\alpha_2\alpha_1$. Putting x for α_1 and y for α_2, the knot group $\Gamma(3_1)$ becomes $\Gamma(3_1) = \langle x, y \mid yxy = xyx \rangle$. On the other hand, we have the relations $\alpha_3 = \alpha_1^{-1}\alpha_2\alpha_1$, $\alpha_1 = \alpha_2^{-1}\alpha_3\alpha_2$, and $\alpha_2 = \alpha_3^{-1}\alpha_1\alpha_3$ from the crossings of 3_1^*. Hence, the knot group of 3_1^* is $\Gamma(3_1^*) = \langle \alpha_1, \alpha_2, \alpha_3 \mid \alpha_3 = \alpha_1^{-1}\alpha_2\alpha_1, \alpha_1 = \alpha_2^{-1}\alpha_3\alpha_2, \alpha_2 = \alpha_3^{-1}\alpha_1\alpha_3 \rangle$. Similarly, by the relations in $\Gamma(3_1^*)$, we obtain that $\alpha_2\alpha_1\alpha_2 = \alpha_1\alpha_2\alpha_1$ and so $\Gamma(3_1^*) = \langle a, b \mid bab = aba \rangle$ when we assign a and b to α_1 and α_2. Therefore, we have isomorphic knot groups $\Gamma(3_1)$ and $\Gamma(3_1^*)$. However, the right-handed trefoil 3_1 and the left-handed trefoil 3_1^* are not equivalent. Since their Jones polynomials are different (Manturov 2004, 83). Consequently, the knot group does not distinguish the knot 3_1 from its mirror symmetry 3_1^*.

The connected sum of two right-handed trefoil is called the *granny knot* and the connected sum of the right-handed trefoil and the left-handed trefoil is called the *square knot*.

Example 2.16 Let us compute the knot groups of the granny knot and the square knot demonstrated in Figure 2.44.



$3_1 \# 3_1$



$3_1 \# 3_1^*$

Figure 2.44.: The granny knot and the square knot, respectively.

We have the following relations from the crossings of the granny knot.

$$\alpha_2 = \alpha_3\alpha_1\alpha_3^{-1},$$
$$\alpha_1 = \alpha_2\alpha_6\alpha_2^{-1},$$
$$\alpha_3 = \alpha_1\alpha_2\alpha_1^{-1},$$
$$\alpha_4 = \alpha_5\alpha_3\alpha_5^{-1},$$
$$\alpha_6 = \alpha_4\alpha_5\alpha_4^{-1},$$
$$\alpha_5 = \alpha_6\alpha_4\alpha_6^{-1}.$$

Then, the knot group is

$$\begin{aligned}\Gamma(3_1\#3_1) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \mid {} & \alpha_2 = \alpha_3\alpha_1\alpha_3^{-1},\ \alpha_1 = \alpha_2\alpha_6\alpha_2^{-1}, \\ & \alpha_3 = \alpha_1\alpha_2\alpha_1^{-1}, \alpha_4 = \alpha_5\alpha_3\alpha_5^{-1}, \\ & \alpha_6 = \alpha_4\alpha_5\alpha_4^{-1}, \alpha_5 = \alpha_6\alpha_4\alpha_6^{-1}\rangle.\end{aligned}$$

From the crossings of the square knot, we have the following relations.

$$\alpha_2 = \alpha_3\alpha_1\alpha_3^{-1}$$
$$\alpha_1 = \alpha_2\alpha_6\alpha_2^{-1}$$
$$\alpha_3 = \alpha_1\alpha_2\alpha_1^{-1}$$
$$\alpha_5 = \alpha_3^{-1}\alpha_4\alpha_3$$
$$\alpha_4 = \alpha_5^{-1}\alpha_3\alpha_5$$
$$\alpha_6 = \alpha_4^{-1}\alpha_5\alpha_4.$$

Therefore, the knot group of the square knot is

$$\begin{aligned}\Gamma(3_1\#3_1^*) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \mid\, & \alpha_2 = \alpha_3\alpha_1\alpha_3^{-1},\ \alpha_1 = \alpha_2\alpha_6\alpha_2^{-1},\\ & \alpha_3 = \alpha_1\alpha_2\alpha_1^{-1}, \alpha_5 = \alpha_3^{-1}\alpha_4\alpha_3,\\ & \alpha_4 = \alpha_5^{-1}\alpha_3\alpha_5, \alpha_6 = \alpha_4^{-1}\alpha_5\alpha_4\rangle.\end{aligned}$$

By mapping $\Gamma(3_1\#3_1)$ to $\Gamma(3_1\#3_1^*)$ so that $\alpha_1 \mapsto \alpha_1$, $\alpha_2 \mapsto \alpha_2$, $\alpha_3 \mapsto \alpha_3$, $\alpha_4 \mapsto \alpha_5$, $\alpha_5 \mapsto \alpha_4$ and $\alpha_6 \mapsto \alpha_6$ we have isomorphic knot groups, $\Gamma(3_1\#3_1)$ and $\Gamma(3_1\#3_1^*)$. Despite the fact that, the granny knot and the square knot are different (Murasugi 1996, 223).

On the other hand, we have the following theorem for the knot group.

Theorem 2.14 The knot group distinguishes every knots up to mirror symmetry (Manturov 2004, 54).

Note that it is difficult to determine if given two knot groups are isomorphic. Although the knot group is almost a complete invariant (that is, it allows us to distinguish knots from each other up to mirror symmetry), knot theorists still require complete and computationally practical invariants. In the next chapter, we study an almost complete invariant (up to orientation), *the fundamental quandle of a knot (or the knot quandle)*, which is also a generalization of colorability.

CHAPTER 3

A REVIEW OF QUANDLES

3.1. Quandles

Definition 3.1 A *quandle* is a set X with a binary operation $\triangleright : X \times X \to X$ that satisfies the followings:

(i) For all $x \in X$, $x \triangleright x = x$,

(ii) For all $x, y \in X$, there exists a unique $z \in X$ such that $x = z \triangleright y$,

(iii) For all $x, y, z \in X$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

The second axiom is equivalent to say that the action of y on x defined by $f_y(x)$=$x \triangleright y$ is a bijection for all $x, y \in X$. Therefore, f_y is invertible and the inverse of f_y is given by $f_y^{-1}(x)$= $x \triangleright^{-1} y$. It means that there is an inverse (or dual) operation $\triangleright^{-1} : X \times X \to X$ of the operation $\triangleright$ such that for all $x, y \in X$, we have $(x \triangleright y) \triangleright^{-1} y$=x=$(x \triangleright^{-1} y) \triangleright y$. Here, $(X, \triangleright^{-1})$ is the *dual quandle* of $(X, \triangleright)$.

Quandle axioms are motivated by the Reidemeister moves on oriented knot and link diagrams. The elements of a quandle are labels of arcs of an oriented knot or a link diagram. If x and y are labels of two arcs, $x \triangleright y$ is read as x *crossing under* y *from right to left*. The crossing relations with quandle are as follows in Figure 3.1:

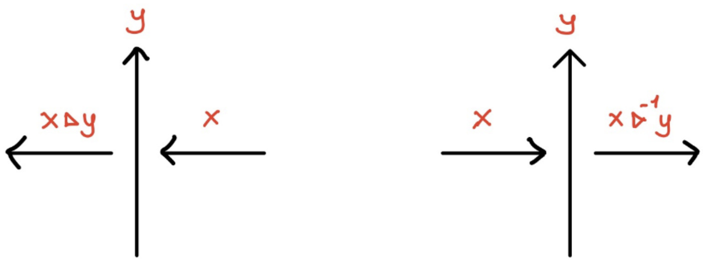


Figure 3.1.: The crossing relations come from the positive crossing and negative crossing, respectively.

We can understand the quandle axioms in terms of Reidemeister moves on diagrams of any knots/links. See in Figure 3.2, Figure 3.3 and Figure 3.4.

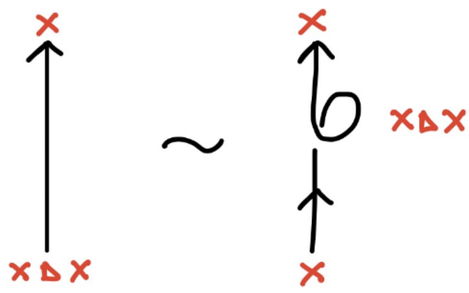


Figure 3.2.: The first Reidemeister move motivates the axiom (i).

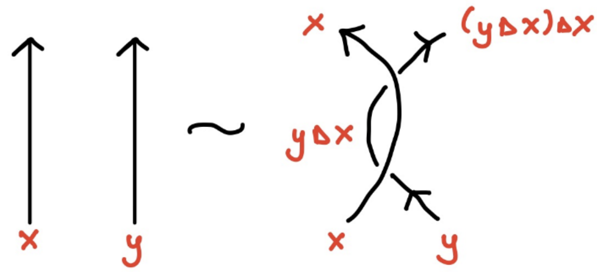


Figure 3.3.: The second Reidemeister move motivates the axiom (ii).

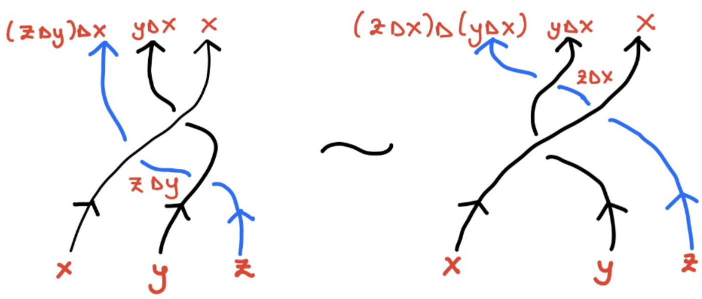


Figure 3.4.: The third Reidemeister move motivates the axiom (iii).

Example 3.1 Any $\mathbb{Z}$-module M is a quandle under the operation

$$a \triangleright b = 2b - a.$$

Example 3.2 Any module M over the ring $\mathbb{Z}\left[t^{\pm 1}\right]$ is a quandle under the operation

$$x \triangleright y = tx - ty + y.$$

Definition 3.2 We say that X is a *trivial quandle* if $x \triangleright y$=x, for any $x, y \in X$.

Definition 3.3 An n-*quandle* X is a quandle satisfying

$$x \triangleright^n y = (\cdots((x \triangleright \underbrace{y) \triangleright y) \triangleright \cdots) \triangleright y}_{n-times} = x$$

for all $x, y \in X$ and $n \in \mathbb{N} \setminus \{0\}$. Moreover, X is an *involutory quandle* if X is a *2-quandle.*

3.2. The Fundamental Quandle of a Knot/Link

Definition 3.4 Let L be an oriented knot/link, and let $D(L)$ be a diagram of L. The *fundamental quandle* (or *the knot/link quandle*) of L, denoted by $Q(L)$, is the quandle with generators corresponding to arcs in $D(L)$ and quandle relations at each crossing in the diagram.

There is also a geometric interpretation of the fundamental quandle of a link. Let L be an oriented link in $\mathbb{S}^3$ and $N(L)$ a regular neighborhood of L. We consider that $\mathbb{S}^3 \setminus N(L)$ with a fixed base point x_0 within it. The fundamental quandle of L is the set of homotopy class of the paths $x \triangleright y$ from x_0 to $N(L)$. The path $x \triangleright y$ is given by first following the path y, then travelling around the meridian μ linking L once, and then going backward along y, and finally following the path x as depicted in Figure 3.5.

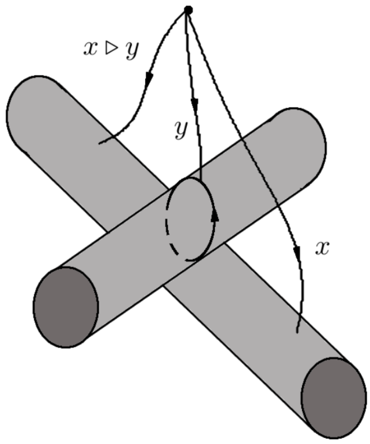


Figure 3.5.: The geometric description of the fundamental quandle of a link (Nelson and Tamagawa 2016, 253).

The knot group is derived from the knot quandle. By replacing $x \triangleright y$ with yxy^{-1} in above, we get a relation from the knot group of L. For more details, see *Knot Theory* (Manturov 2004, 63).

Example 3.3 We show the fundamental quandle of the Figure-8 knot 4_1, illustrated in Figure 3.6.

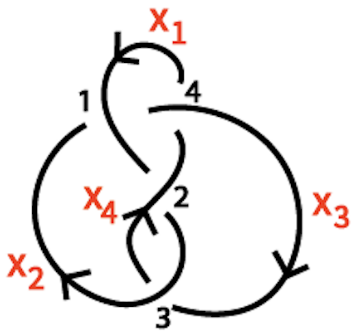


Figure 3.6.: The Figure-8 knot with an orientation. Its arcs are labeled as x_1, x_2, x_3, and x_4.

We have the quandle relations

$$x_2 \triangleright x_1 = x_3,$$
$$x_1 \triangleright^{-1} x_4 = x_2 \implies x_2 \triangleright x_4 = x_1, \qquad \text{by the second axiom of a quandle,}$$
$$x_3 \triangleright^{-1} x_2 = x_4 \implies x_4 \triangleright x_2 = x_3, \qquad \text{by the second axiom of a quandle,}$$
$$x_4 \triangleright x_3 = x_1, \qquad \text{at the crossings } 1, 2, 3 \text{ and } 4, \text{ respectively.}$$

Then the fundamental quandle presentation of the Figure-8 knot is

$$Q(4_1) = \langle x_1, x_2, x_3, x_4 \mid x_2 \triangleright x_1 = x_3,\ x_2 \triangleright x_4 = x_1,\ x_4 \triangleright x_2 = x_3,\ x_4 \triangleright x_3 = x_1 \rangle.$$

The fundamental quandle of the Figure-8 knot is infinite since elements like $x_1 \triangleright x_2$ or $x_3 \triangleright x_4$ cannot be expressed as one of x_1, x_2, x_3, or x_4. Any knot or link with a finite number of arcs in its diagram has a finitely generated fundamental quandle, as in this example.

Theorem 3.1 The knot quandle of a link is a link invariant (Joyce 1982, 58).

Proof Let L be a link in $\mathbb{R}^3$ and $Q(L)$ the fundamental quandle of L. To prove this theorem, we will show that $Q(L)$ is preserved under Reidemeister moves **R1**, **R2** and **R3**. Suppose the presentation of the fundamental quandle of L is given as

$$Q(L) = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_n \rangle,$$

where each x_i is a label of arcs and r_i's are the relations at crossings of L for all $i \in \{1, 2, \cdots, n\}$. Now, let x_j be the label of any arc of L, where $j \in \{1, 2, \cdots, n\}$. We show that the first Reidemeister move preserves the fundamental quandle of L illustrated in Figure 3.7.

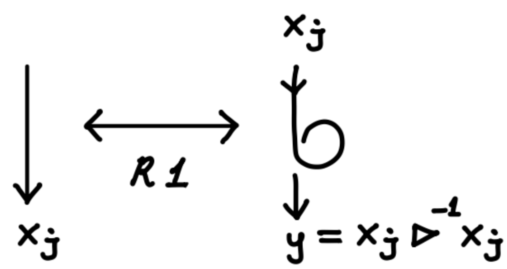


Figure 3.7.: The knot quandle is preserved under **R1**.

We get $x_j \triangleright^{-1} x_j = y$, and hence, by the second axiom of quandle, $y \triangleright x_j = x_j$. But, according to the first and second axioms of quandle, we obtain that $y = x_j$.

$$
\begin{aligned}
Q(L') &= \langle x_1, \cdots, x_n, y \mid r_1, \cdots, r_n, r_{new} : x_j = y \rangle \\
&= \langle x_1, \cdots, x_n \mid r_1, \cdots, r_n \rangle \\
&= Q(L).
\end{aligned}
$$

We see in Figure 3.7, the new generator y added after the first Reidemeister move, **R1**, is just x_j by the relation added by the crossing. Thus, $Q(L)$ remains the same under the first Reidemeister move.

Secondly, let us check whether $Q(L)$ will be preserved under the second Reidemeister move, **R2** or not. Assume that x_j, x_k are any two different generators in $Q(L)$.

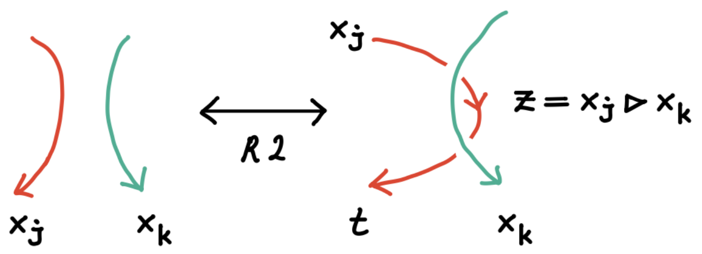


Figure 3.8.: The knot quandle is preserved under **R2**.

In Figure 3.8, we get two new relations such as $x_j \triangleright x_k = z$, and $z \triangleright^{-1} x_k = t$. Then, by the second new relation and the second axiom of quandle, $t \triangleright x_k = z$, and so t must be equal to x_j. Hence, we obtain

$$\begin{aligned} Q(L') &= \langle x_1, \cdots, x_n, z, t \mid r_1, \cdots, r_n, r_{new} : x_j \triangleright x_k = z, t \triangleright x_k = z, t = x_j \rangle \\ &= \langle x_1, \cdots, x_n, z \mid r_1, \cdots, r_n, r_{new} : x_j \triangleright x_k = z \rangle \\ &= \langle x_1, \cdots, x_n \mid r_1, \cdots, r_n \rangle \\ &= Q(L). \end{aligned}$$

Therefore, $Q(L)$ is preserved under the second Reidemeister move, **R2**.

Finally, we will show that the third Reidemeister move, **R3** preserves the fundamental quandle of L. Let x_i, x_j, x_k, x_p, x_q, and x_r denote the generators of $Q(L)$ that correspond to arc in the third Reidemeister move.

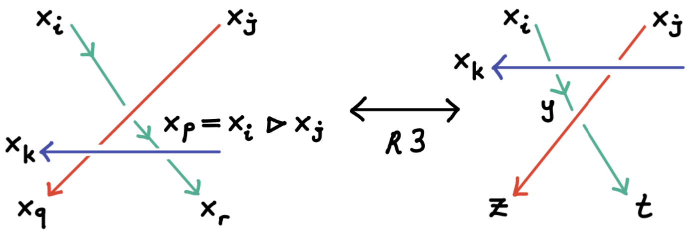


Figure 3.9.: The knot quandle is preserved under **R3**.

As shown on the left side of the Figure 3.9, we have the relations $x_i \triangleright x_j = x_p$, $x_j \triangleright x_k = x_q$, and $(x_i \triangleright x_j) \triangleright x_k = x_r$ in $Q(L)$, and on the right side of the Figure 3.9, we have these relations: $x_i \triangleright xk = y$, $x_j \triangleright xk = z$, and $(x_i \triangleright x_k) \triangleright (x_j \triangleright x_k) = t$. Thus, by the second and third axioms of quandle, we obtain that $z = x_q$ and $t = (x_i \triangleright x_k) \triangleright (x_j \triangleright x_k) =$

$(x_i \triangleright x_j) \triangleright x_k = x_r$. We now have

$$\begin{aligned} Q(L) &= \langle x_i, x_j, x_k, x_p, x_q, x_r, X \mid x_i \triangleright x_j = x_p, x_j \triangleright x_k = x_q, (x_i \triangleright x_j) \triangleright x_k = x_r, R\rangle \\ &= \langle x_i, x_j, x_k, X \mid r_1, \cdots, r_n\rangle \end{aligned}$$

and,

$$\begin{aligned} Q(L') &= \langle x_i, x_j, x_k, y, z, t, X'\} \mid R, r_{new} : x_i \triangleright x_k = y, x_j \triangleright x_k = z, (x_i \triangleright x_k) \triangleright (x_j \triangleright x_k) = t\rangle \\ &= \langle x_i, x_j, x_k, x_q, x_r, y \mid R, r_{new} : x_i \triangleright x_k = y,\ x_j \triangleright x_k = x_q, (x_i \triangleright x_j) \triangleright x_k = x_r\rangle \\ &= \langle x_i, x_j, x_k, X \mid r_1, \cdots, r_n\rangle \\ &= Q(L), \end{aligned}$$

where X is the set of the generators of $Q(L)$ except the generators $x_i, x_j, x_k, x_p, x_q, x_r$; R is the set of the relations in $Q(L)$ which are different from $x_i \triangleright x_j = x_p,\ x_j \triangleright x_k = x_q,\ (x_i \triangleright x_j) \triangleright x_k = x_r$, and X' is the set of the generators of $Q(L)$ except the generators x_i, x_j, x_k. Since we get $Q(L') = Q(L)$ after applying the third Reidemeister move, we conclude that $Q(L)$ is preserved under **R3**.

We have shown that the fundamental quandle $Q(L)$ of a link L is preserved under Reidemeister moves **R1**, **R2**, and **R3**. □

Theorem 3.2 The knot quandle distinguishes two knots up to orientation.

Proof The reader can find the proof in Section 5.3 in *Knot Theory* (Manturov 2004, 65-67). □

CHAPTER 4

QUOTIENTS OF THE FUNDAMENTAL QUANDLE

In this chapter, we focus on n-quandle quotients of the knot quandle. First, we give some definitions.

Let K be an oriented knot and $Q(K)$ the knot quandle of K.

Definition 4.1 We say that a relation in a quandle presentation is *short* if it is of the form $x_i \triangleright x_j = x_k$ for generators x_i, x_j, x_k.

Definition 4.2 If the generators are numbered $\{x_1, \ldots, x_n\}$, then we can express the quandle presentation as a short form presentation with a matrix whose ijth entry is k if $x_i \triangleright x_j = x_k$, otherwise 0 for all $i, j, k \in \{1, 2, \ldots, n\}$. Then we call this matrix as a *presentation matrix* for $Q(K)$.

Example 4.1 The Figure-8 knot in Example 3.3 has the presentation matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix}.$$

Definition 4.3 If a presentation matrix for $Q(K)$ has no zeros, we say that $Q(K)$ is a *finite quandle*.

The knot quandle is always infinite except for the Hopf link and the unknot. Despite this, the n-quandle quotient of the fundamental quandle of a link is finite for some natural numbers $n \in \mathbb{N} \setminus \{0\}$. Thus, it motivates us to look for n-quandle quotients of the knot quandle. Hoste and Shanahan classified links with finite n-quandles for some n (Hoste and Shanahan 2017, 14-15).

Definition 4.4 Let K be a knot and $Q(K)$ be the fundamental quandle of K. An *n-quandle quotient* of $Q(K)$ is $Q(K)/_{x \triangleright^n y}$, for any $x, y \in Q(K)$. This quotient is denoted by $Q_n(K)$.

4.1. Getting the n-quandle quotient of the knot quandle

We get the n-quandle quotient of the fundamental quandle of a knot from the following types of moves on the presentation matrix which do not change quandles:

(i) Fill in a zero in the presentation matrix with a value which is obtained by the quandle axioms and other relations.

(ii) Fill in a zero with a number which defines a new generator, and add a row and column of zeroes corresponding to this new generator.

(iii) If two generators are found to be equal, then delete a row and a column and replace all instances of larger generator with smaller generator.

Example 4.2 Let us get the 2-quandle quotient of the fundamental quandle of the Figure-8 knot (4_1) using the above moves. First, we consider the presentation matrix of $Q(4_1)$ from Example 4.1 and fill in the zeroes by quandle axioms and the 2-quandle condition:

By the first axiom of quandle, we have $x_i \triangleright x_i = x_i$ for all $i \in \{1, 2, 3, 4\}$ and we get the following matrix.

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 1 & 4 \end{bmatrix}.$$

By the 2-quandle condition, we obtain the followings.

$(x_2 \triangleright x_1) \triangleright x_1 = x_2$ and $x_2 \triangleright x_1 = x_3 \implies x_3 \triangleright x_1 = x_2$,

$(x_2 \triangleright x_4) \triangleright x_4 = x_2$ and $x_2 \triangleright x_4 = x_1 \implies x_1 \triangleright x_4 = x_2$,

$(x_4 \triangleright x_2) \triangleright x_2 = x_4$ and $x_4 \triangleright x_2 = x_3 \implies x_3 \triangleright x_2 = x_4$,

$(x_4 \triangleright x_3) \triangleright x_3 = x_4$ and $x_4 \triangleright x_3 = x_1 \implies x_1 \triangleright x_3 = x_4$.

By the third axiom of quandle, we have

$$\underbrace{(x_4 \triangleright x_4)}_{x_4} \triangleright \underbrace{(x_2 \triangleright x_4)}_{x_1} = \underbrace{(x_4 \triangleright x_2)}_{x_3} \triangleright x_4 \implies x_4 \triangleright x_1 = x_3 \triangleright x_4,$$

$$\underbrace{(x_2 \triangleright x_2)}_{x_2} \triangleright \underbrace{(x_4 \triangleright x_2)}_{x_3} = \underbrace{(x_2 \triangleright x_4)}_{x_1} \triangleright x_2 \implies x_2 \triangleright x_3 = x_1 \triangleright x_2,$$

$$\underbrace{(x_3 \triangleright x_3)}_{x_3} \triangleright \underbrace{(x_4 \triangleright x_3)}_{x_1} = (x_3 \triangleright x_4) \triangleright x_3 \implies \underbrace{x_3 \triangleright x_1}_{x_2} = (x_3 \triangleright x_4) \triangleright x_3 \implies x_2 = (x_3 \triangleright x_4) \triangleright x_3.$$

Since we have still zeros, we assign a new generator, say x_5. Let $x_3 \triangleright x_4$ be x_5. Then, $x_4 \triangleright x_1 = x_5$ and $x_5 \triangleright x_3 = x_2$ as $x_4 \triangleright x_1 = x_3 \triangleright x_4$, and $x_2 = (x_3 \triangleright x_4) \triangleright x_3$. As we assign a new generator x_5, we add a row and a column corresponding to x_5. Therefore, we get the following matrix by the previous relations and the first axiom of quandle.

$$\begin{bmatrix} 1 & 0 & 4 & 2 & 0 \\ 3 & 2 & 0 & 1 & 0 \\ 2 & 4 & 3 & 5 & 0 \\ 5 & 3 & 1 & 4 & 0 \\ 0 & 0 & 2 & 0 & 5 \end{bmatrix}.$$

Now, to fill the zeros we will use the relations in the present matrix and third axiom of quandle.

$$\underbrace{(x_1 \triangleright x_1)}_{x_1} \triangleright \underbrace{(x_3 \triangleright x_1)}_{x_2} = \underbrace{(x_1 \triangleright x_3)}_{x_4} \triangleright x_1 = x_5 \implies x_1 \triangleright x_2 = x_5,$$

and so $x_2 \triangleright x_3 = x_5$,

$$\underbrace{(x_4 \triangleright x_1)}_{x_5} \triangleright \underbrace{(x_3 \triangleright x_1)}_{x_2} = \underbrace{(x_4 \triangleright x_3)}_{x_1} \triangleright x_1 = x_1 \implies x_5 \triangleright x_2 = x_1,$$

$$\underbrace{(x_1 \triangleright x_1)}_{x_1} \triangleright \underbrace{(x_4 \triangleright x_1)}_{x_5} = \underbrace{(x_1 \triangleright x_4)}_{x_2} \triangleright x_1 = x_3 \implies x_1 \triangleright x_5 = x_3,$$

$$\underbrace{(x_2 \triangleright x_2)}_{x_2} \triangleright \underbrace{(x_1 \triangleright x_2)}_{x_5} = \underbrace{(x_2 \triangleright x_1)}_{x_3} \triangleright x_2 = x_4 \implies x_2 \triangleright x_5 = x_4,$$

$$\underbrace{(x_2 \triangleright x_1)}_{x_3} \triangleright \underbrace{(x_4 \triangleright x_1)}_{x_5} = \underbrace{(x_2 \triangleright x_4)}_{x_1} \triangleright x_1 = x_1 \implies x_3 \triangleright x_5 = x_1,$$

$$\underbrace{(x_4 \triangleright x_4)}_{x_4} \triangleright \underbrace{(x_3 \triangleright x_4)}_{x_5} = \underbrace{(x_4 \triangleright x_3)}_{x_1} \triangleright x_4 = x_2 \implies x_4 \triangleright x_5 = x_2,$$

$$\underbrace{(x_1 \triangleright x_2)}_{x_5} \triangleright \underbrace{(x_5 \triangleright x_2)}_{x_1} = \underbrace{(x_1 \triangleright x_5)}_{x_3} \triangleright x_2 = x_4 \implies x_5 \triangleright x_1 = x_4,$$

and

$$\underbrace{(x_2 \triangleright x_3)}_{x_5} \triangleright \underbrace{(x_1 \triangleright x_3)}_{x_4} = \underbrace{(x_2 \triangleright x_1)}_{x_3} \triangleright x_3 = x_3 \implies x_5 \triangleright x_4 = x_3.$$

Hence, we obtain that the final matrix in below. The 2-quandle quotient of $Q(4_1)$ is finite as the matrix has no zeros.

$$\begin{bmatrix} 1 & 5 & 4 & 2 & 3 \\ 3 & 2 & 5 & 1 & 4 \\ 2 & 4 & 3 & 5 & 1 \\ 5 & 3 & 1 & 4 & 2 \\ 4 & 1 & 2 & 3 & 5 \end{bmatrix}.$$

Theorem 4.1 The fundamental quandle of an oriented link L is finite if and only if L is either the unknot or the Hopf link with any orientation (Crans et al. 2019, 4).

Proof Let L_1 be the oriented unknot and L_2 the oriented Hopf link as shown in Figure 4.1 and Figure 4.2, respectively. We will first show that the fundamental quandles of L_1 and L_2 are finite.

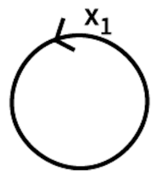


Figure 4.1.: An oriented unknot, L_1.

The fundamental quandle of the unknot is $Q(L_1) = \langle x_1 : x_1 = x_1 \triangleright x_1 \rangle$ by quandle axiom (i) and the presentation matrix is $\begin{bmatrix} 1 \end{bmatrix}$. Since we have no zeros,it is a finite quandle.

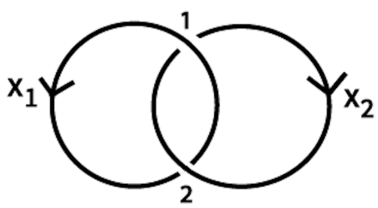


Figure 4.2.: An oriented Hopf link, L_2.

The quandle relations at each crossings of L_2 are $x_1 \triangleright x_2 = x_1$, $x_2 \triangleright x_1 = x_2$. Thus the fundamental quandle of the Hopf link is $Q\left(L_2\right) = \langle x_1, x_2 : x_1 = x_1 \triangleright x_2, x_2 = x_2 \triangleright x_1 \rangle$. Then the presentation matrix for the fundamental quandle is by the quandle relations on the diagram of the Hopf link and quandle axiom (i):

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

Thus, we find that $x_1 = x_2$ by the second axiom of a quandle. Hence the final presentation matrix is $\begin{bmatrix} 1 \end{bmatrix}$, and so $Q(L_2)$ is finite.
Now, suppose that the fundamental quandle $Q\left(L\right) = \langle x_1, x_2, ..., x_n : x_i \triangleright x_j = x_k \rangle$ of the link L is finite, where $x_1, x_2, ..., x_n$ are labels of arcs of the diagram of L and for $1 \leq i, j, k \leq n$ with $i \neq j \neq k$ and $x_k = x_i \triangleright x_j$ are the quandle relations at each crossing of the diagram of L. Thus, any generator x_i can be expressed by some combination the other generators, where $i \in \{1, 2, ..., n\}$. So every quotient of $Q(L)$ is finite, and hence the n-quandle quotient of the fundamental quandle of L, $Q_n(L)$, is finite for all $n > 1$. Let $\widetilde{M}_n(L)$ be the n-fold cyclic branched cover of S^3, over L. Then, we know that by Theorem 4 in *Links with finite n-quandles* (Hoste and Shanahan 2017, 9), $Q_n(L)$ is finite if and only if the fundamental group of the n-fold cyclic branched cover of S^3 over L, $\pi_1\left(\widetilde{M}_n(L)\right)$ is finite. Define $\mathcal{O}(L, n)$ to be the 3-orbifold with underlying space $\mathbb{S}^3$ and singular locus L, where each component of L is labelled n. Now, we have a manifold covering of the orbifold, $p : \widetilde{M}_n(L) \to \mathcal{O}(L, n)$, and the covering map p inducing a homomorphism $p_* : \pi_1\left(\widetilde{M}_n(L)\right) \to \pi_1^{orb}(\mathcal{O}(L, n))$ for which the index of $p_*\left(\pi_1\left(\widetilde{M}_n(L)\right)\right)$ in $\pi_1^{orb}(\mathcal{O}(L, n))$ is the branch index n. Since $\pi_1\left(\widetilde{M}_n(L)\right)$ is finite, $\pi_1^{\text{orb}}(\mathcal{O}(L, n))$ is finite. Moreover, the universal orbifold cover of $\mathcal{O}(L, n)$ is equal to the universal cover of $\widetilde{M}_n(L)$ and a simply-connected manifold. Moreover, since $\pi_1^{orb}(\mathcal{O}(L, n))$ is finite, the universal cover is also compact. Now, Thurston's geometrization theorem establishes that the only compact, simply connected 3-manifold is $\mathbb{S}^3$ (Hoste and Shanahan 2017, 14). Thus, $\mathcal{O}(L, n)$ is a spherical 3-orbifold. In 1988, Dunbar classified all geometric, non-hyperbolic 3-orbifolds (Dunbar 1988, 75-93). From Dunbar's list, we see that the only link whose n-quandle quotient is finite for $n > 1$ is the Hopf link. □

The n-quandle quotient of the knot quandle, $Q_n(L)$ is also a link invariant (Hoste and Shanahan 2017, 3-4). In his PhD thesis, Winker showed that the 4-quandle quotients of the granny knot and the square knot are not isomorphic (Winker 1984, 98-102).

4.1.1. Computation of some quotient quandles of the trefoil

In 2017, J. Hoste and P. D. Shanahan showed that the n-quandle quotient of the knot quandle of the trefoil is finite for $n = 3$, $n = 4$ and $n = 5$ in $\mathbb{S}^3$ (Hoste and Shanahan 2017, 15) by using Dunbar's classification (Dunbar 1988, 75-93). In the present section, we compute the n-quandle quotient of the fundamental quandle of the trefoil knot in Figure 4.3 for $n = 2, 3, 4, 5$ applying the process given in Section 4.1.

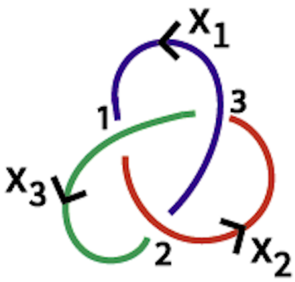


Figure 4.3.: An oriented trefoil knot whose arcs are labeled by x_1, x_2 and x_3.

We have the fundamental quandle of the trefoil as
$Q(3_1) = \langle x_1, x_2, x_3 \mid x_1 \triangleright x_3 = x_2,\ x_3 \triangleright x_2 = x_1,\ x_2 \triangleright x_1 = x_3 \rangle$, and so the presentation matrix for $Q(3_1)$ is

$$\begin{bmatrix} 0 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

For $n = 2$:
We fill the zeroes on the diagonal with the first quandle axiom, and we get

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}.$$

The involutory quandle condition says $(x_i \triangleright x_j) \triangleright x_j = x_i$ for all $i, j \in \{1, 2, 3\}$. Thus, since $(x_3 \triangleright x_2) \triangleright x_2 = x_3$ and $x_3 \triangleright x_2 = x_1$, we have $x_1 \triangleright x_2 = x_3$. Similarly, we get

$x_2 \triangleright x_3 = x_1$ and $x_3 \triangleright x_1 = x_2$. Then the final presentation matrix is

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

For $n = 3$:

With the first quandle axiom, we get

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}.$$

We cannot fill the other zeroes with the 3-quandle condition, so we choose the zero $x_2 \triangleright x_3$ and say it x_4. Then, we have $x_4 \triangleright x_3 = x_1$ since $((x_1 \triangleright x_3) \triangleright x_3) \triangleright x_3 = x_1$ by the 3-quandle condition, and so $x_1 \triangleright x_3 = x_2$ by the relations. Moreover, we always have $x_4 \triangleright x_4 = x_4$ by the first quandle axiom. Thus, we get

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Now, we use the relations and the right distributive law of quandle to fill the remaining zeroes:

$$\underbrace{(x_3 \triangleright x_3)}_{x_3} \triangleright \underbrace{(x_2 \triangleright x_3)}_{x_4} = \underbrace{(x_3 \triangleright x_2)}_{x_1} \triangleright x_3 = x_2 \implies x_3 \triangleright x_4 = x_2,$$

$$\underbrace{(x_2 \triangleright x_3)}_{x_4} \triangleright \underbrace{(x_1 \triangleright x_3)}_{x_2} = \underbrace{(x_2 \triangleright x_1)}_{x_3} \triangleright x_3 = x_3 \implies x_4 \triangleright x_2 = x_3,$$

$$\underbrace{\underbrace{(x_4 \triangleright x_4)}_{x_4} \triangleright \underbrace{(x_3 \triangleright x_4)}_{x_2}}_{x_3} = \underbrace{(x_4 \triangleright x_3)}_{x_1} \triangleright x_4 \implies x_1 \triangleright x_4 = x_3,$$

$$\underbrace{(x_3 \triangleright x_3)}_{x_3} \triangleright \underbrace{(x_4 \triangleright x_3)}_{x_1} = \underbrace{(x_3 \triangleright x_4)}_{x_2} \triangleright x_3 = x_4 \implies x_3 \triangleright x_1 = x_4,$$

$$\underbrace{\underbrace{(x_4 \triangleright x_2)}_{x_3} \triangleright \underbrace{(x_3 \triangleright x_2)}_{x_1}}_{x_4} = \underbrace{(x_4 \triangleright x_3)}_{x_1} \triangleright x_2 \implies x_1 \triangleright x_2 = x_4,$$

$$\underbrace{(x_1 \triangleright x_3)}_{x_2} \triangleright \underbrace{(x_2 \triangleright x_3)}_{x_4} = \underbrace{(x_1 \triangleright x_2)}_{x_4} \triangleright x_3 = x_1 \implies x_2 \triangleright x_4 = x_1,$$

$$\underbrace{(x_2 \triangleright x_3)}_{x_4} \triangleright \underbrace{(x_4 \triangleright x_3)}_{x_1} = \underbrace{(x_2 \triangleright x_4)}_{x_1} \triangleright x_3 = x_2 \implies x_4 \triangleright x_1 = x_2.$$

Then we have the matrix

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \end{bmatrix}.$$

For $n = 4$:

Let us say $x_4 := x_1 \triangleright x_2$. Then, by the 4-quandle condition, we get $((x_4 \triangleright x_2) \triangleright x_2) \triangleright x_2 = x_1$, and the matrix

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

With the relations and the right distributive law of quandle, we get these relations:

$$\underbrace{\underbrace{(x_1 \triangleright x_1)}_{x_1} \triangleright \underbrace{(x_2 \triangleright x_1)}_{x_3}}_{x_2} = \underbrace{(x_1 \triangleright x_2)}_{x_4} \triangleright x_1 \implies x_4 \triangleright x_1 = x_2,$$

$$\underbrace{(x_2 \triangleright x_2)}_{x_2} \triangleright \underbrace{(x_1 \triangleright x_2)}_{x_4} = \underbrace{(x_2 \triangleright x_1)}_{x_3} \triangleright x_2 = x_1 \implies x_2 \triangleright x_4 = x_1.$$

Then the matrix is

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 3 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 2 & 0 & 0 & 4 \end{bmatrix}.$$

Now, let $x_5 := x_4 \triangleright x_2$. Then, we have $(x_5 \triangleright x_2) \triangleright x_2 = x_1$ and the relations:

$$\underbrace{(x_4 \triangleright x_2)}_{x_5} \triangleright \underbrace{(x_1 \triangleright x_2)}_{x_4} = \underbrace{(x_4 \triangleright x_1)}_{x_2} \triangleright x_2 = x_2 \implies x_5 \triangleright x_4 = x_2,$$

$$\underbrace{(x_2 \triangleright x_2)}_{x_2} \triangleright \underbrace{(x_4 \triangleright x_2)}_{x_5} = \underbrace{(x_2 \triangleright x_4)}_{x_1} \triangleright x_2 = x_4 \implies x_2 \triangleright x_5 = x_4.$$

As $(x_5 \triangleright x_2) \triangleright x_2 = x_1$, let us say $x_5 \triangleright x_2 =: x_j$. Then, $x_j \triangleright x_2 = x_1$, and hence $x_j = x_3$ by the previous matrix (by the second axiom of quandle). Thus, $x_5 \triangleright x_2 = x_3$. Then, we have the following relations:

$$\underbrace{\underbrace{(x_5 \triangleright x_5)}_{x_5} \triangleright \underbrace{(x_2 \triangleright x_5)}_{x_4}}_{x_2} = \underbrace{(x_5 \triangleright x_2)}_{x_3} \triangleright x_5 \implies x_3 \triangleright x_5 = x_2,$$

$$\underbrace{(x_2 \triangleright x_2)}_{x_2} \triangleright \underbrace{(x_5 \triangleright x_2)}_{x_3} = \underbrace{(x_2 \triangleright x_5)}_{x_4} \triangleright x_2 = x_5 \implies x_2 \triangleright x_3 = x_5,$$

$$\underbrace{\underbrace{(x_1 \triangleright x_3)}_{x_2} \triangleright \underbrace{(x_2 \triangleright x_3)}_{x_5}}_{x_4} = \underbrace{(x_1 \triangleright x_2)}_{x_4} \triangleright x_3 \implies x_4 \triangleright x_3 = x_4,$$

$$\underbrace{(x_4 \triangleright x_2)}_{x_5} \triangleright \underbrace{(x_3 \triangleright x_2)}_{x_1} = \underbrace{(x_4 \triangleright x_3)}_{x_4} \triangleright x_2 = x_5 \implies x_5 \triangleright x_1 = x_5,$$

$$\underbrace{(x_5 \triangleright x_2)}_{x_3} \triangleright \underbrace{(x_1 \triangleright x_2)}_{x_4} = \underbrace{(x_5 \triangleright x_1)}_{x_5} \triangleright x_2 = x_3 \implies x_3 \triangleright x_4 = x_3,$$

$$\underbrace{(x_3 \triangleright x_2)}_{x_1} \triangleright \underbrace{(x_4 \triangleright x_2)}_{x_5} = \underbrace{(x_3 \triangleright x_4)}_{x_3} \triangleright x_2 = x_1 \implies x_1 \triangleright x_5 = x_1,$$

$$\underbrace{(x_3 \triangleright x_4)}_{x_3} \triangleright \underbrace{(x_2 \triangleright x_4)}_{x_1} = \underbrace{(x_3 \triangleright x_2)}_{x_1} \triangleright x_4 = x_1 \triangleright x_4 \implies x_3 \triangleright x_1 = x_1 \triangleright x_4,$$

and

$$\underbrace{(x_3 \triangleright x_2)}_{x_1} \triangleright \underbrace{(x_1 \triangleright x_2)}_{x_4} = (x_3 \triangleright x_1) \triangleright x_2.$$

Then we have $x_3 \triangleright x_1 = (x_3 \triangleright x_1) \triangleright x_2$ and so $x_3 \triangleright x_1 = x_2$ by the first axiom. Thus, $x_1 \triangleright x_4 = x_2$.

$$\underbrace{(x_4 \triangleright x_3)}_{x_4} \triangleright \underbrace{(x_2 \triangleright x_3)}_{x_5} = \underbrace{(x_4 \triangleright x_2)}_{x_5} \triangleright x_3 \implies x_4 \triangleright x_5 = x_5 \triangleright x_3,$$

and

$$\underbrace{(x_1 \triangleright x_2)}_{x_4} \triangleright \underbrace{(x_4 \triangleright x_2)}_{x_5} = \underbrace{(x_1 \triangleright x_4)}_{x_2} \triangleright x_2 = x_2 \implies x_4 \triangleright x_5 = x_2.$$

Then, $x_5 \triangleright x_3 = x_2$.

Thus, the matrix is

$$\begin{bmatrix} 1 & 4 & 2 & 2 & 1 \\ 3 & 2 & 5 & 1 & 4 \\ 2 & 1 & 3 & 3 & 2 \\ 2 & 5 & 4 & 4 & 2 \\ 5 & 3 & 2 & 2 & 5 \end{bmatrix}.$$

Now, we found that $x_1 = x_5$ and $x_3 = x_4$. Therefore, we will replace all instances of x_5 with x_1 and x_4 with x_3. Then the final matrix is

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

For $n = 5$:

Let $x_4 := x_1 \triangleright x_2$. Then by the 5-quandle condition we get $(((x_4 \triangleright x_2) \triangleright x_2) \triangleright x_2) \triangleright x_2 = x_1$.

Moreover we have the following equations:

$$\underbrace{(x_2 \triangleright x_2)}_{x_2} \triangleright \underbrace{(x_1 \triangleright x_2)}_{x_4} = \underbrace{(x_2 \triangleright x_1)}_{x_3} \triangleright x_2 = x_1 \implies x_2 \triangleright x_4 = x_1,$$

$$\underbrace{(x_1 \triangleright x_2)}_{x_4} \triangleright \underbrace{(x_3 \triangleright x_2)}_{x_1} = \underbrace{(x_1 \triangleright x_3)}_{x_2} \triangleright x_2 = x_2 \implies x_4 \triangleright x_1 = x_2.$$

Then the matrix is

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 3 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 2 & 0 & 0 & 4 \end{bmatrix}.$$

Now, let us say $x_5 := x_4 \triangleright x_2$. Then we have $((x_5 \triangleright x_2) \triangleright x_2) \triangleright x_2 = x_1$, and the following equations:

$$\underbrace{(x_2 \triangleright x_2)}_{x_2} \triangleright \underbrace{(x_4 \triangleright x_2)}_{x_5} = \underbrace{(x_2 \triangleright x_4)}_{x_1} \triangleright x_2 = x_4 \implies x_2 \triangleright x_5 = x_4,$$

$$\underbrace{(x_4 \triangleright x_2)}_{x_5} \triangleright \underbrace{(x_1 \triangleright x_2)}_{x_4} = \underbrace{(x_4 \triangleright x_1)}_{x_2} \triangleright x_2 = x_2 \implies x_5 \triangleright x_4 = x_2,$$

and then the matrix is

$$\begin{bmatrix} 1 & 4 & 2 & 0 & 0 \\ 3 & 2 & 0 & 1 & 4 \\ 0 & 1 & 3 & 0 & 0 \\ 2 & 5 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 5 \end{bmatrix}.$$

Now, let $x_6 := x_5 \triangleright x_2$. Then, we have the relation $(x_6 \triangleright x_2) \triangleright x_2 = x_1$. Then, by the second axiom of quandle, we get $x_6 \triangleright x_2 = x_3$. Moreover, we have these relations:

$$\underbrace{(x_2 \triangleright x_2)}_{x_2} \triangleright \underbrace{(x_5 \triangleright x_2)}_{x_6} = \underbrace{(x_2 \triangleright x_5)}_{x_4} \triangleright x_2 = x_5 \implies x_2 \triangleright x_6 = x_5,$$

$$\underbrace{(x_5 \triangleright x_2)}_{x_6} \triangleright \underbrace{(x_4 \triangleright x_2)}_{x_5} = \underbrace{(x_5 \triangleright x_4)}_{x_2} \triangleright x_2 = x_2 \implies x_6 \triangleright x_5 = x_2,$$

$$\underbrace{(x_2 \triangleright x_2)}_{x_2} \triangleright \underbrace{(x_6 \triangleright x_2)}_{x_3} = \underbrace{(x_2 \triangleright x_6)}_{x_5} \triangleright x_2 = x_6 \implies x_2 \triangleright x_3 = x_6,$$

$$\underbrace{(x_3 \rhd x_3)}_{x_3} \rhd \underbrace{(x_2 \rhd x_3)}_{x_6} = \underbrace{(x_3 \rhd x_2)}_{x_1} \rhd x_3 = x_2 \implies x_3 \rhd x_6 = x_2,$$

$$\underbrace{\underbrace{(x_5 \rhd x_4)}_{x_2} \rhd \underbrace{(x_2 \rhd x_4)}_{x_1}}_{x_3} = \underbrace{(x_5 \rhd x_2)}_{x_6} \rhd x_4 \implies x_6 \rhd x_4 = x_3,$$

$$\underbrace{\underbrace{(x_3 \rhd x_6)}_{x_2} \rhd \underbrace{(x_2 \rhd x_6)}_{x_5}}_{x_4} = \underbrace{(x_3 \rhd x_2)}_{x_1} \rhd x_6 \implies x_1 \rhd x_6 = x_4,$$

$$\underbrace{\underbrace{(x_1 \rhd x_3)}_{x_2} \rhd \underbrace{(x_2 \rhd x_3)}_{x_6}}_{x_5} = \underbrace{(x_1 \rhd x_2)}_{x_4} \rhd x_3 \implies x_4 \rhd x_3 = x_5,$$

$$\underbrace{\underbrace{(x_6 \rhd x_5)}_{x_2} \rhd \underbrace{(x_2 \rhd x_5)}_{x_4}}_{x_1} = \underbrace{(x_6 \rhd x_2)}_{x_3} \rhd x_5 \implies x_3 \rhd x_5 = x_1,$$

and

$$\underbrace{(x_4 \rhd x_2)}_{x_5} \rhd \underbrace{(x_3 \rhd x_2)}_{x_1} = \underbrace{(x_4 \rhd x_3)}_{x_5} \rhd x_2 = x_6 \implies x_5 \rhd x_1 = x_6.$$

Therefore, we get the matrix

$$\begin{bmatrix} 1 & 4 & 2 & 0 & 0 & 4 \\ 3 & 2 & 6 & 1 & 4 & 5 \\ 0 & 1 & 3 & 0 & 1 & 2 \\ 2 & 5 & 5 & 4 & 0 & 0 \\ 6 & 6 & 0 & 2 & 5 & 0 \\ 0 & 3 & 0 & 3 & 2 & 6 \end{bmatrix}.$$

Now, we assign a new generator x_7 as $x_7 = x_1 \rhd x_4$. Then, we have these relations:

$$\underbrace{(x_4 \rhd x_4)}_{x_4} \rhd \underbrace{(x_1 \rhd x_4)}_{x_7} = \underbrace{(x_4 \rhd x_1)}_{x_2} \rhd x_4 = x_1 \implies x_4 \rhd x_7 = x_1,$$

$$\underbrace{(x_1 \rhd x_4)}_{x_7} \rhd \underbrace{(x_2 \rhd x_4)}_{x_1} = \underbrace{(x_1 \rhd x_2)}_{x_4} \rhd x_4 = x_4 \implies x_7 \rhd x_1 = x_4,$$

$$\underbrace{\underbrace{(x_3 \rhd x_5)}_{x_1} \rhd \underbrace{(x_2 \rhd x_5)}_{x_4}}_{x_7} = \underbrace{(x_3 \rhd x_2)}_{x_1} \rhd x_5 \implies x_1 \rhd x_5 = x_7,$$

$$\underbrace{(x_5 \triangleright x_4)}_{x_2} \triangleright \underbrace{(x_1 \triangleright x_4)}_{x_7} = \underbrace{(x_5 \triangleright x_1)}_{x_6} \triangleright x_4 = x_3 \implies x_2 \triangleright x_7 = x_3,$$

$$\underbrace{(x_5 \triangleright x_5)}_{x_5} \triangleright \underbrace{(x_1 \triangleright x_5)}_{x_7} = \underbrace{(x_5 \triangleright x_1)}_{x_6} \triangleright x_5 = x_2 \implies x_5 \triangleright x_7 = x_2,$$

$$\underbrace{(x_1 \triangleright x_4)}_{x_7} \triangleright \underbrace{(x_6 \triangleright x_4)}_{x_3} = \underbrace{(x_1 \triangleright x_6)}_{x_4} \triangleright x_4 = x_4 \implies x_7 \triangleright x_3 = x_4.$$

Thus, we get the matrix

$$\begin{bmatrix} 1 & 4 & 2 & 7 & 7 & 4 & 0 \\ 3 & 2 & 6 & 1 & 4 & 5 & 3 \\ 0 & 1 & 3 & 0 & 1 & 2 & 0 \\ 2 & 5 & 5 & 4 & 0 & 0 & 1 \\ 6 & 6 & 0 & 2 & 5 & 0 & 2 \\ 0 & 3 & 0 & 3 & 2 & 6 & 0 \\ 4 & 0 & 4 & 0 & 0 & 0 & 7 \end{bmatrix}.$$

Along with these, we have

$$\underbrace{(x_6 \triangleright x_4)}_{x_3} \triangleright \underbrace{(x_2 \triangleright x_4)}_{x_1} = \underbrace{(x_6 \triangleright x_2)}_{x_3} \triangleright x_4 \implies x_3 \triangleright x_1 = x_3 \triangleright x_4,$$

$$\underbrace{(x_2 \triangleright x_7)}_{x_3} \triangleright \underbrace{(x_4 \triangleright x_7)}_{x_1} = \underbrace{(x_2 \triangleright x_4)}_{x_1} \triangleright x_7 \implies x_3 \triangleright x_1 = x_1 \triangleright x_7,$$

$$\underbrace{(x_1 \triangleright x_2)}_{x_4} \triangleright \underbrace{(x_4 \triangleright x_2)}_{x_5} = \underbrace{(x_1 \triangleright x_4)}_{x_7} \triangleright x_2 \implies x_4 \triangleright x_5 = x_7 \triangleright x_2,$$

$$\underbrace{(x_1 \triangleright x_6)}_{x_4} \triangleright \underbrace{(x_2 \triangleright x_6)}_{x_5} = \underbrace{(x_1 \triangleright x_2)}_{x_4} \triangleright x_6 \implies x_4 \triangleright x_5 = x_4 \triangleright x_6,$$

$$\underbrace{(x_1 \triangleright x_4)}_{x_7} \triangleright \underbrace{(x_5 \triangleright x_4)}_{x_2} = \underbrace{(x_1 \triangleright x_5)}_{x_7} \triangleright x_4 \implies x_7 \triangleright x_2 = x_7 \triangleright x_4,$$

$$\underbrace{(x_4 \triangleright x_3)}_{x_5} \triangleright \underbrace{(x_2 \triangleright x_3)}_{x_6} = \underbrace{(x_4 \triangleright x_2)}_{x_5} \triangleright x_3 \implies x_5 \triangleright x_6 = x_5 \triangleright x_3,$$

$$\underbrace{(x_5 \triangleright x_1)}_{x_6} \triangleright \underbrace{(x_2 \triangleright x_1)}_{x_3} = \underbrace{(x_5 \triangleright x_2)}_{x_6} \triangleright x_1 \implies x_6 \triangleright x_3 = x_6 \triangleright x_1.$$

Now let us say $x_8 := x_3 \triangleright x_1$. Therefore, since $x_3 \triangleright x_1 = x_3 \triangleright x_4 = x_1 \triangleright x_7$, $x_3 \triangleright x_4 = x_8$ and $x_1 \triangleright x_7 = x_8$. In addition to these, we have

$$\underbrace{\underbrace{(x_3 \triangleright x_5)}_{x_1} \triangleright \underbrace{(x_1 \triangleright x_5)}_{x_7}}_{x_8} = \underbrace{(x_3 \triangleright x_1)}_{x_8} \triangleright x_5 \implies x_8 \triangleright x_5 = x_8,$$

$$\underbrace{(x_1 \triangleright x_1)}_{x_1} \triangleright \underbrace{(x_3 \triangleright x_1)}_{x_8} = \underbrace{(x_1 \triangleright x_3)}_{x_2} \triangleright x_1 = x_3 \implies x_1 \triangleright x_8 = x_3,$$

$$\underbrace{(x_4 \triangleright x_1)}_{x_2} \triangleright \underbrace{(x_3 \triangleright x_1)}_{x_8} = \underbrace{(x_4 \triangleright x_3)}_{x_5} \triangleright x_1 = x_6 \implies x_2 \triangleright x_8 = x_6,$$

$$\underbrace{(x_3 \triangleright x_1)}_{x_8} \triangleright \underbrace{(x_2 \triangleright x_1)}_{x_3} = \underbrace{(x_3 \triangleright x_2)}_{x_1} \triangleright x_1 = x_1 \implies x_8 \triangleright x_3 = x_1,$$

$$\underbrace{(x_3 \triangleright x_4)}_{x_8} \triangleright \underbrace{(x_2 \triangleright x_4)}_{x_1} = \underbrace{(x_3 \triangleright x_2)}_{x_1} \triangleright x_4 = x_7 \implies x_8 \triangleright x_1 = x_7,$$

$$\underbrace{(x_3 \triangleright x_1)}_{x_8} \triangleright \underbrace{(x_5 \triangleright x_1)}_{x_6} = \underbrace{(x_3 \triangleright x_5)}_{x_1} \triangleright x_1 = x_1 \implies x_8 \triangleright x_6 = x_1,$$

$$\underbrace{(x_7 \triangleright x_7)}_{x_7} \triangleright \underbrace{(x_1 \triangleright x_7)}_{x_8} = \underbrace{(x_7 \triangleright x_1)}_{x_4} \triangleright x_7 = x_1 \implies x_7 \triangleright x_8 = x_1,$$

$$\underbrace{(x_2 \triangleright x_1)}_{x_3} \triangleright \underbrace{(x_8 \triangleright x_1)}_{x_7} = \underbrace{(x_2 \triangleright x_8)}_{x_6} \implies x_3 \triangleright x_7 = x_6 \triangleright x_1,$$

$$\underbrace{(x_6 \triangleright x_4)}_{x_3} \triangleright \underbrace{(x_1 \triangleright x_4)}_{x_7} = \underbrace{(x_6 \triangleright x_1)} \triangleright x_4 \implies x_3 \triangleright x_7 = (x_6 \triangleright x_1) \triangleright x_4,$$

then we get $(x_6 \triangleright x_1) \triangleright x_4 = x_6 \triangleright x_1$ and so $x_6 \triangleright x_1 = x_4$ by the second axiom of quandle. Since $x_6 \triangleright x_3 = x_6 \triangleright x_1$, $x_6 \triangleright x_3 = x_4$. In addition we have

$$\underbrace{(x_2 \triangleright x_1)}_{x_3} \triangleright \underbrace{(x_3 \triangleright x_1)}_{x_8} = \underbrace{(x_2 \triangleright x_3)}_{x_6} \triangleright x_1 = x_4 \implies x_3 \triangleright x_8 = x_4,$$

$$\underbrace{(x_6 \triangleright x_1)}_{x_4} \triangleright \underbrace{(x_5 \triangleright x_1)}_{x_6} = \underbrace{(x_6 \triangleright x_5)}_{x_2} \triangleright x_1 = x_3 \implies x_4 \triangleright x_6 = x_3,$$

and then $x_4 \triangleright x_5 = x_3$, $x_7 \triangleright x_2 = x_3$, $x_7 \triangleright x_4 = x_3$ as we found $x_4 \triangleright x_5 = x_4 \triangleright x_6 = x_7 \triangleright x_2 = x_7 \triangleright x_4$,

$$\underbrace{(x_4 \triangleright x_4)}_{x_4} \triangleright \underbrace{(x_3 \triangleright x_4)}_{x_8} = \underbrace{(x_4 \triangleright x_3)}_{x_5} \triangleright x_4 = x_2 \implies x_4 \triangleright x_8 = x_2,$$

$$\underbrace{(x_4 \triangleright x_2)}_{x_5} \triangleright \underbrace{(x_6 \triangleright x_2)}_{x_3} = \underbrace{(x_4 \triangleright x_6)}_{x_3} \triangleright x_2 = x_1 \implies x_5 \triangleright x_3 = x_1,$$

then $x_5 \triangleright x_6 = x_1$, as $x_5 \triangleright x_6 = x_5 \triangleright x_3$,

$$\underbrace{(x_5 \triangleright x_1)}_{x_6} \triangleright \underbrace{(x_3 \triangleright x_1)}_{x_8} = \underbrace{(x_5 \triangleright x_3)}_{x_1} \triangleright x_1 = x_1 \implies x_6 \triangleright x_8 = x_1,$$

$$\underbrace{(x_8 \triangleright x_1)}_{x_7} \triangleright \underbrace{(x_5 \triangleright x_1)}_{x_6} = \underbrace{(x_8 \triangleright x_5)}_{x_8} \triangleright x_1 = x_7 \implies x_7 \triangleright x_6 = x_7,$$

$$\underbrace{(x_7 \triangleright x_6)}_{x_7} \triangleright \underbrace{(x_2 \triangleright x_6)}_{x_5} = \underbrace{(x_7 \triangleright x_2)}_{x_3} \triangleright x_6 = x_2 \implies x_7 \triangleright x_5 = x_2,$$

$$\underbrace{(x_3 \triangleright x_1)}_{x_8} \triangleright \underbrace{(x_4 \triangleright x_1)}_{x_2} = \underbrace{(x_3 \triangleright x_4)}_{x_8} \triangleright x_1 = x_7 \implies x_8 \triangleright x_2 = x_7,$$

$$\underbrace{(x_3 \triangleright x_1)}_{x_8} \triangleright \underbrace{(x_6 \triangleright x_1)}_{x_4} = \underbrace{(x_3 \triangleright x_6)}_{x_2} \triangleright x_1 = x_3 \implies x_8 \triangleright x_4 = x_3,$$

$$\underbrace{(x_3 \triangleright x_1)}_{x_8} \triangleright \underbrace{(x_8 \triangleright x_1)}_{x_7} = \underbrace{(x_3 \triangleright x_8)}_{x_4} \triangleright x_1 = x_2 \implies x_8 \triangleright x_7 = x_2,$$

$$\underbrace{(x_5 \triangleright x_5)}_{x_5} \triangleright \underbrace{(x_8 \triangleright x_5)}_{x_8} = (x_5 \triangleright x_8) \triangleright x_5 \implies (x_5 \triangleright x_8) \triangleright x_5 = x_5 \triangleright x_8 \implies x_5 \triangleright x_8 = x_5,$$

$$\underbrace{(x_4 \triangleright x_1)}_{x_6} \triangleright \underbrace{(x_8 \triangleright x_1)}_{x_7} = \underbrace{(x_4 \triangleright x_8)}_{x_2} \triangleright x_1 = x_3 \implies x_6 \triangleright x_7 = x_3.$$

Thus we have the matrix

$$\begin{bmatrix} 1 & 4 & 2 & 7 & 7 & 4 & 8 & 3 \\ 3 & 2 & 6 & 1 & 4 & 5 & 3 & 6 \\ 8 & 1 & 3 & 8 & 1 & 2 & 4 & 4 \\ 2 & 5 & 5 & 4 & 3 & 3 & 1 & 2 \\ 6 & 6 & 1 & 2 & 5 & 1 & 2 & 5 \\ 4 & 3 & 4 & 3 & 2 & 6 & 3 & 1 \\ 4 & 3 & 4 & 3 & 2 & 7 & 7 & 1 \\ 7 & 7 & 1 & 3 & 8 & 1 & 2 & 8 \end{bmatrix}.$$

As we see, $x_6 \triangleright x_1 = x_7 \triangleright x_1$ in the above matrix, by the second axiom of quandle, $x_6 = x_7$. Similarly $x_5 = x_8$, $x_7 = x_8$ and $x_2 = x_6$. Then,

$$\begin{bmatrix} 1 & 4 & 2 & 2 \\ 3 & 2 & 2 & 1 \\ 2 & 1 & 3 & 2 \\ 2 & 2 & 2 & 4 \end{bmatrix}.$$

We still have equal generators: Since $x_3 \triangleright x_1 = x_4 \triangleright x_1$, $x_1 \triangleright x_3 = x_2 \triangleright x_3$ and $x_1 \triangleright x_4 = x_3 \triangleright x_4$, we have $x_3 = x_4$, $x_1 = x_2$ and $x_1 = x_3$, respectively. Thence, the final matrix is $\begin{bmatrix} 1 \end{bmatrix}$.

CHAPTER 5

CONCLUSION

In this thesis, we studied quandles and the n-quandle quotients of the knot quandle. We defined the knot quandle and indicated its geometric description. By examining this structure, we observed that the knot group is derived from the knot quandle and the knot quandle is stronger invariant than the knot group. Indeed, whilst the knot group cannot distinguish the granny knot and the square knot, the 4-quandle quotient of their knot quandles determine these two knots. Through out this thesis, we have studied one of the strongest link invariants in knot theory.

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APPENDIX A

ALGEBRAIC TOPOLOGY

In this chapter, we recall some fundamental definitions and theorems in algebraic topology.

A.1. Some Basic Notions of Topology

Definition A.1 Let X be a topological space and $\mathcal{T}$ a collection of subsets of X. If for every $x \in X$ there is a neighborhood U of x that intersects only a finite number of sets in $\mathcal{T}$, we say that $\mathcal{T}$ is *locally finite*.

Definition A.2 Let X be a topological space and $\mathcal{T}, \mathcal{S} \subseteq \mathcal{P}(X)$ be two collections. We say that $\mathcal{S}$ is a *refinement* of $\mathcal{T}$ if each set $S \in \mathcal{S}$ is contained in some set $T \in \mathcal{T}$.

Definition A.3 A topological space X is said to be *paracompact* if for every open cover U of X there exists an open cover $\tilde{U}$ of X that is a refinement of U and locally finite.

A.1.1. Homotopy

Definition A.4 Let X be a topological space, and $\gamma : [0, 1] \to X$ be a continuous map. Then we say γ is *a path* in X.

Definition A.5 A path γ is called *a loop* if $\gamma(0) = \gamma(1)$. If $\gamma(0) = \gamma(1) = x \in X$, we say that γ is *a loop based at x.*

Definition A.6 Let X be a topological space. A *homotopy* of paths in X is a family $f_t : [0, 1] \to X$ for $0 \leq t \leq 1$ that satisfies the following:

i) The endpoints $f_t(0) = x \in X$ and $f_t(1) = y \in X$ are independent of t.

ii) The map $H : [0, 1] \times [0, 1] \to X$ defined by $H(s, t) := f_t(s)$ is continuous.

If two paths f_0 and f_1 are connected by a homotopy, we say f_0 and f_1 are *homotopic*, and denote by $f_0 \simeq f_1$. See Figure A.1.

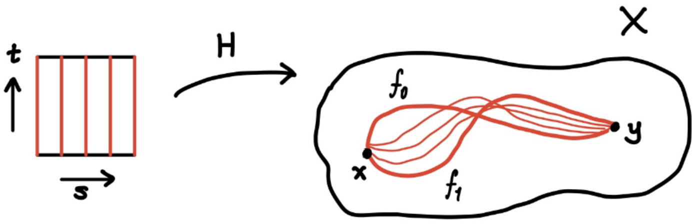

Figure A.1.: Homotopy of paths.

Proposition A.1 The homotopy relation given above is an equivalence relation.

Definition A.7 Let f be a path in a topological space X. The equivalence class of f under the equivalence relation of homotopy is denoted by $[f]$ and called as *the homotopy class* of f.

Definition A.8 Let X be a topological space and A be a subspace of X. A *deformation retraction* of X onto A is a family of continuous maps $f_t : X \to X$, where $t \in [0,1]$ such that $f_0 = 1_X$ the identity map, $f_1(X) = A$, and $f_t|_A = 1_X$ for all $t \in [0,1]$. In this case, we say that A is a *deformation retract* of X.

Note that a deformation retraction is a special case of a homotopy.

Definition A.9 Let X be a topological space and $H : [0,1] \times [0,1] \to X$ be a continuous map. We say H is *a free homotopy of loops* if $\gamma_s(t) =: H(s,t)$ is a loop for every fixed point $s \in [0,1]$, that is $H(s,0) = H(s,1), \forall s \in [0,1]$ (See Figure A.2]. If $x \in X$ is a base point, we have $H(s,0) = H(s,1) = x$ for every $s \in [0,1]$ and H is called *a homotopy based at* x (See Figure A.3).

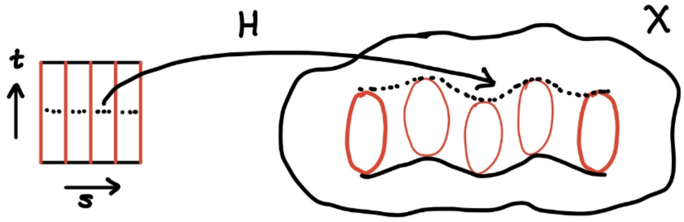

Figure A.2.: A free homotopy of loops.

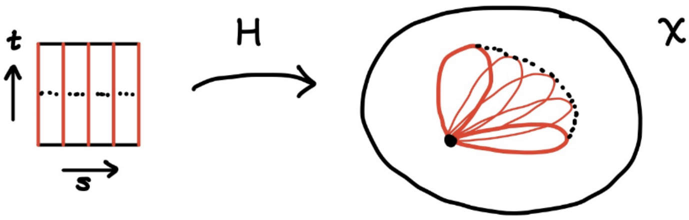

Figure A.3.: A homotopy based at the base point x.

Definition A.10 If we have a based homotopy γ_s, we say γ_0 is *homotopic* to γ_1, and denote by $\gamma_0 \simeq \gamma_1$.

Lemma A.1 If $\gamma : [0,1] \to \mathbb{R}^n$ is a loop based at the origin, then γ is based-homotopic to the constant loop $c : [0,1] \to \mathbb{R}^n$ such that $c(t) = 0$ for all $t \in [0,1]$.

Definition A.11 Let γ be a loop, and c be the constant loop. A homotopy from γ to c is called *a nullhomotopy.* Then we say that γ is *contractible.*

Definition A.12 Let X be a topological space and $p, t : [0,1] \to X$ be two paths with $p(0) = t(0) = x \in X$ and $p(1) = t(1) = y \in X$. If there is a map $H : [0,1] \times [0,1] \to X$ such that $H(s,0) = p(s)$ and $H(s,1) = t(s)$, $\forall s \in [0,1]$; $H(0,r) = x$ and $H(1,r) = y$, $\forall\, r \in [0,1]$, we say p is *path-homotopic* to t.

Definition A.13 Let X be a topological space and $\alpha, \beta : [0,1] \to X$ be paths with $\alpha(1) = \beta(0)$. The *composition (product) path* $\alpha \cdot \beta$ is defined by

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t) & \text{if } t \in [0, 1/2] \\ \beta(2t-1) & \text{if } t \in [1/2, 1] \end{cases} \tag{A.1}$$

A.1.2. The Fundamental Group

Definition A.14 Let X be a topological space. The set of homotopy classes of loops in X at a fixed basepoint $x_0 \in X$ is denoted by $\pi_1(X, x_0)$.

Proposition A.2 Let f and g be two paths in X. Then $\pi_1(X, x_0)$ is a group with respect to the product $[f \cdot g] = [f] \cdot [g]$.

Definition A.15 The group $\pi_1(X, x_0)$ above is called *the fundamental group* of X at the basepoint x_0.

Definition A.16 If $\pi_1(X, x_0) = \{1_X\} = \{1\}$, we say that X has the *trivial fundamental group*.

Definition A.17 If a topological space X is path-connected and has trivial fundamental group then, it is called as *simply-connected*.

Proposition A.3 If A is a deformation retract of X, then the map

$$g : \pi_1(A, x_0) \to \pi_1(X, x_0) \tag{A.2}$$

is an isomorphism.

A.1.3. Homotopy of Maps

Definition A.18 Let X_1, X_2 be two topological spaced and $F, G : X_1 \to X_2$ continuous maps. If there is a continuous homotopy $H : X_1 \times [0, 1] \to X_2$ with $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$ for all $X \in X$, we say that F and G are *homotopic*. We write it as $F \simeq G$.

Definition A.19 Let X be a topological space. If $id_X \simeq c$ where id_X is the identity map on X and c is a constant map, we say X is *contractible*.

Definition A.20 Let X, Y be topological spaces and F be a map from X to Y. We say that F is *a homotopy equivalence* if there exists a map G from Y to X with $F \circ G \simeq id_Y$ and $G \circ F \simeq id_X$, and G is called as *a homotopy inverse* of F. Moreover, we say that X and Y are *homotopy equivalent* and denoted by $X \simeq Y$.

Theorem A.1 The fundamental group is a homotopy invariant. It means that is two topological spaces X and Y are homeomorphic, then $\pi_1(X) \cong \pi_1(Y)$.

Lemma A.2 If X is a contractible space, then X is homotopy equivalent to $\{p\}$, where p is any point in X.

Corollary A.1 Let X be a contractible space, and $x_0 \in X$. Then X has the trivial fundamental group; in fact, $\pi_1(X, x_0) = \{1\}$.

Corollary A.2 If a space X is contractible to a point $x \in X$, then X is simply-connected.

A.1.4. Covering Spaces

Definition A.21 Let X_1 and X_2 be topological space, I be any index set, and $p : X_2 \to X_1$ be a map that satisfies the following condition: There exists an open cover U_α of X_1 such that $\forall \alpha \in I$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in X_2, each of these is mapped by p homeomorphically onto U_α. *We say that* p *is a covering map and* X_2 *is a covering space of* X_1 *together with the map* p. If for every $x \in X_1$, the preimage $p^{-1}(x)$ includes exactly n points in X_2 we say that p is *an n-fold covering map*.

Example A.1 $\mathbb{R}$ is a covering space of the unit circle S^1 with the covering map $p : \mathbb{R} \to S^1$ with $p(x) = (cos(2\pi x), sin(2\pi x))$. Examine it in Figure A.4.

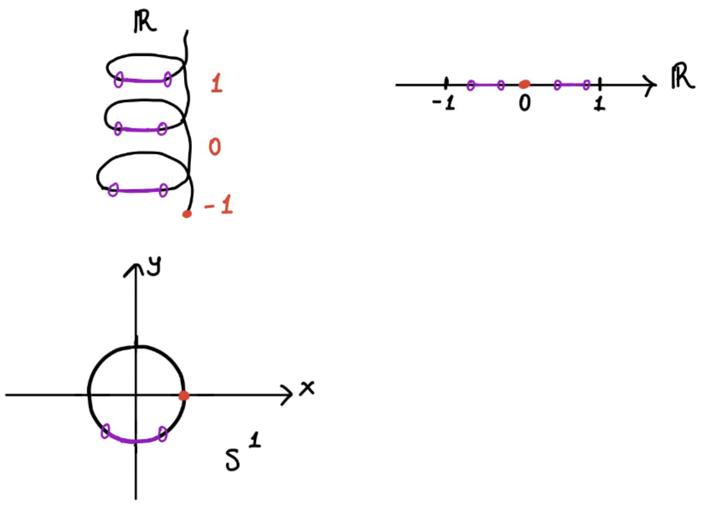

Figure A.4.: A covering space of the unit circle is $\mathbb{R}$.

APPENDIX B

MANIFOLDS AND ORBIFOLDS

In this section, we recall the fundamental definitions/notions of manifolds and orbifolds.

B.1. Manifolds

Definition B.1 Let M be a topological space. We say that M is *locally Euclidean of dimension n* if for any point $m \in M$ there is an open set $U \subseteq M$ that contains m and it is homeomorphic to an open set V of $\mathbb{R}^n$.

Definition B.2 We say a topological space M is a *(topological) manifold of dimension n*, or an *n-manifold* if M is Hausdorff, second-countable and locally Euclidean of dimension n. Abbreviately, we denote as M^n.

Definition B.3 A two-dimensional submanifold of a 3-manifold is said to be a *surface*.

Definition B.4 A *simply-connected manifold* is a simply-connected topological space.

Lemma B.1 (Dehn's lemma) Let M be a 3-manifold with the boundary ∂M, and let γ be a closed curve on ∂M. Then, if there exists an immersed 2-disk $D \to M$ such that $\partial D = \gamma$, there exists an embedded disk $D' \subset M$ with $\partial D' = \gamma$.

Definition B.5 Let M^n be a manifold, $U \subseteq M$ open and $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ a homeomorphism. A *coordinate chart* on M is the pair (U, ϕ).

Definition B.6 Let (U, ϕ) and (V, ψ) be two coordinate charts on M with $U \cap V \neq \emptyset$. The composition map $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is called the *transition map from ψ to ϕ*.

Definition B.7 We say that two charts (U, ϕ) and (V, ψ) on M are *compatible* if the transition map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.

Definition B.8 Let M^n be a manifold. An *atlas* is a collection of coordinate charts $\{(U_i, \phi_i)\}_{i \in I}$ that satisfies

$$\bigcup_{i \in I} U_i = M.$$

Definition B.9 Let A be an atlas on a manifold M^n. If each transition map between charts in A is a diffeomorphism, then we say A is a *smooth atlas*.

Definition B.10 A smooth atlas A on M^n is said to be *maximal* if there is no strictly larger smooth atlas on M^n that contains A. Moreover, a *smooth structure* on a topological manifold M is a maximal smooth atlas.

Definition B.11 Let M be a manifold and A is a smooth structure on M. Then the pair (M, A) is called a *smooth manifold*.

Definition B.12 Let M_1, M_2 be two smooth manifolds and $\phi : M_1 \to M_2$ be a smooth map. If the derivative of ϕ is injective for all $m \in M_1$, we say ϕ is an *immersion*. If the immersion ϕ is a homeomorphism onto its image $\phi(M_1)$, it is called as a *smooth embedding*.

Definition B.13 Let M be an n-dimensional manifold, and S be a subset of M. Then S is called as a *submanifold of dimension* $k \leq n$, if for any point $s \in S$ there exists a coordinate chart (U, ψ) around s such that $\psi(U \cap S) = \psi(U) \cap \mathbb{R}^k$.

Definition B.14 Let M be an n-dimensional manifold and N be a k-dimensional submanifold of M. We say that N is a *codimension-2 submanifold* of M if $dim(M) - dim(N) = n - k = 2$.

B.1.1. Coverings of Manifolds (or Covering Spaces)

Definition B.15 Let M_1 and M_2 be two manifolds. A function f is *locally injective at* $m \in M_1$ if there is a neighborhood $U \subseteq M_1$ of m such that $f|_U$ is injective. If f is locally injective at all points of M_1, we say that f is *locally injective*.

Definition B.16 Let M_1 and M_2 be two manifolds and $g : M_1 \to M_2$ be a continuous map. If for any two charts (U, ϕ) of M_1 and (V, ψ) *of* M_2*, the map* $\psi \circ g \circ \phi^{-1} : \phi(U \cap g^{-1}(V)) \to \psi(V)$ *is smooth, then we say that* g *is smooth.*

Definition B.17 Let $f : M_1 \to M_2$ be a smooth map that fails to be locally injective at some points $m_i \in M_1$, where $i \in I$ and I is any index set. Then we say the set $\{m_i \in M_1\}$ is a *branch set*.

Definition B.18 Let $f : M_1 \to M_2$ be a smooth map, where M_1 and M_2 are manifolds. The *branch index* is the number of times f fails to be locally injective at a point $m \in M_1$.

Definition B.19 Let M_1 and M_2 be compact manifolds of dimension n with proper submanifolds $A_1^{n-2} \subset M_1$, $A_2^{n-2} \subset M_2$. The continuous function $f : M_1 \to M_2$ is called as a *branched covering* with branch sets A_1 and A_2 if

i) components of preimages of open sets of M_2 are a basis for the topology on M_1,

ii) $f(A_1) = A_2$, $f(M_1 \setminus A_1) = M_2 \setminus A_2$, and $M_2 \setminus A_2$ is the set of points in M_2 that are evenly covered, that means, the points in M_2 have neighborhoods V such that f sends each component of $f^{-1}(V)$ onto V, homeomorphically.

We say that M_2 is *the base* of branched the covering, and M_1 is *a branched covering* of M_2.

Definition B.20 A diffeomorphism $d : M_1 \to M_1$ is called *an automorphism* of the branched covering f above, if $f \circ d = f$. If the map $p : M_1/Aut(f) \to M_2$ is a homeomorphism, where $Aut(f)$ is the automorphism group of the branched covering f, then we say that f is *regular.*

Definition B.21 If the automorphism group of a regular branched covering is cyclic, then the regular branched covering is called as *cyclic.*

B.2. Orbifolds

We first recall group actions and then give some fundamental mydefinition*s and notions about orbifolds.

Definition B.22 An *action of a group G on a set X* is a map $f : G \times X \to X$ such that $f(g, x) = gx$ with $f(e, x) = x$ and $f(g, f(h, x)) = f(gh, x)$ for all x in X and g, h in G, where e is the identitiy element of G. When G acts on X, we say that X is a ***G**-set*.

Definition B.23 Let a group G act on a set X. The *isotropy subgroup* of $x \in X$ is $G_x := \{g \in G \mid gx = x\}$.

Definition B.24 Let $f : G \times X \to X$ be a group action and G_x is the isotropy subgroup of any $x \in X$. If

$$\bigcap_{x \in X} G_x = \{e\},$$

where e is the identity element of G, we say the action f is *effective*. If $G_x = \{e\}$ for every $x \in X$, f is called as *free action*.

Definition B.25 Let $f : G \times X \to X$ be a group action. If f is smooth, we say that G *acts on* X *smoothly*.

Definition B.26 Let G be a group that acts on a set X with an action map f, and Y be a subset of X. If $\forall g \in G$ we have $gy \in Y$ whenever $y \in Y$, Y is called as *a G-invariant*.

Definition B.27 Let X be a topological space. An *orbifold chart* $\{\tilde{U}, H, f\}$ *of dimension* $n \in \mathbb{N}$ for an open set $U \subseteq X$ includes a connected open set $\tilde{U} \subseteq \mathbb{R}^n$, a finite group H that acts smoothly and effectively on $\tilde{U}$ and a continuous H-invariant map $f : \tilde{U} \to X$ that induces a homeomorphism between $\tilde{U}/H$ and U. Here an H*-invariant map* is a map $f : \tilde{U} \to X$ that satisfies for all u in $\tilde{U}$ and for all h in H, $f(hu) = f(u)$.

Definition B.28 Let $\{\tilde{U}_1, H_1, f_1\}$ and $\{\tilde{U}_2, H_2, f_2\}$ be two orbifold charts. An *embedding* between two orbifold charts is a smooth embedding $g : \tilde{U}_1 \to \tilde{U}_2$ satisfying $f_2 \circ g = f_1$.

Definition B.29 Let $\{\tilde{U}_1, H_1, f_1\}$ and $\{\tilde{U}_2, H_2, f_2\}$ be two orbifold charts. Let $\tilde{U}_1, \tilde{U}_2 \subseteq \mathbb{R}^n$ are included in $U_1, U_2 \subseteq X$, respectively. We say that the orbifold charts are *locally compatible* if for any $x \in U_1 \cap U_2$ there exists an open neighborhood $U_3 \subseteq U_1 \cap U_2$ and an orbifold chart $\{\tilde{U}_3, H_3, f_3\}$ for U_3 that admits embeddings in $\{\tilde{U}_1, H_1, f_1\}$ and $\{\tilde{U}_2, H_2, f_2\}$.

Definition B.30 Let X be a topological space and I be an index set. An *orbifold atlas* for X is a collection $\mathcal{A} = \{(\tilde{U}_i, H_i, f_i)\}_{i \in I}$ of locally compatible orbifold charts that covers X.

Definition B.31 Let $\mathcal{A}_1, \mathcal{A}_2$ be two orbifold atlases. If every chart in $\mathcal{A}_1$ admits an embedding in some chart in $\mathcal{A}_2$, we say $\mathcal{A}_1$ *refines* $\mathcal{A}_2$. When two orbifold atlases have a common refinement, we say these atlases are *equivalent*.

Definition B.32 An *n-dimensional smooth orbifold* $\mathcal{O}$ is a pair $(X_{\mathcal{O}}, [\mathcal{A}])$ where $X_{\mathcal{O}}$ is a Hausdorff and paracompact topological space that is called the *underlying space of* $\mathcal{O}$, and $[\mathcal{A}]$ is an equivalance class of atlases for $X_{\mathcal{O}}$.