RANDOMIZATION OF CERTAIN OPERATORS IN HARMONIC ANALYSIS

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Dedicated to my lovely wife

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ABSTRACT

RANDOMIZATION OF CERTAIN OPERATORS IN HARMONIC ANALYSIS

In this thesis, we study the Hardy-Littlewood majorant problem randomized via stochastic processes. Stationary processes, random walks and the Poisson processes are used for randomization, and we show the Hardy-Littlewood majorant property holds almost surely for deterministic sets perturbed by these processes. We also perturb a very large class of sparse sets, including the Green-Ruzsa set by Poisson processes and demonstrate that the Hardy-Littlewood majorant property remains valid up to a negligible probability. Additionally, we investigate how randomization affects the expected values of L^2-norm and L^4-norm of an exponential sum whose frequencies constitute an arithmetic progression of larger step size. Furthermore, we estimate the expected value of the L^n-norms, $n \in 2\mathbb{N}$ of exponential sums whose frequencies are randomized via Poisson processes, and these norms can be interpreted as lattice points in regions or solutions of diophantine equations in an average sense.

ÖZET

HARMONİK ANALİZDEKİ BAZI OPERATÖRLERİN RASTSALLAŞTIRILMASI

Bu tezde, stokastik süreçler aracılığıyla rastsallaştırılmış Hardy-Littlewood majorant problemi çalışılmıştır. Rastsallaştırma için durağan süreçler, rastgele yürüyüşler ve Poisson süreçleri kullanılmış ve bu süreçlerle pertürbe edilmiş deterministik kümeler için Hardy-Littlewood majorant özelliğinin neredeyse kesin olarak geçerli olduğu gösterilmiştir. Poisson süreçleri ile Green-Ruzsa kümesi de dahil olmak üzere çok büyük bir seyrek küme sınıfını pertürbe edilmiştir ve Hardy-Littlewood majorant özelliğinin ihmal edilebilir bir olasılıkla geçerliliğini sürdürdüğü gösterilmiştir. Ayrıca, frekansları daha büyük adım boyutuna sahip bir aritmetik ilerleme oluşturan bir üstel toplamın L^2-normu ve L^4-normunun beklenen değerlerinin rastsallaştırmadan nasıl etkilendiği incelenmiştir. Dahası, Poisson süreçleriyle rastsallaştırılmış frekanslara sahip üstel toplamların L^n-normlarının, $n \in 2\mathbb{N}$, beklenen değeri kestirilmiş ve bu normlar, ortalama anlamda, bölgeler üzerindeki tam sayı koordinatlı noktalar veya diyofant denklemlerinin çözümleri olarak yorumlanır.

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CHAPTER 1

INTRODUCTION

Hardy-Littlewood conjectured that for all $p \in [2, \infty)$, there exists a positive constant K_p such that for any set $\Gamma \subseteq \{1, 2, \ldots, M\}$, $M \in \mathbb{N}$ and any sequence $(a_n)_{n\in\Lambda}$ of complex numbers with the condition $\sup_{n\in\Lambda} |a_n| \leq 1$ we have

$$\Big\| \sum_{n\in\Gamma} a_n e^{2\pi i n x} \Big\|_{L^p(\mathbb{T})} \leq K_p \Big\| \sum_{n\in\Gamma} e^{2\pi i n x} \Big\|_{L^p(\mathbb{T})}, \tag{1.1}$$

see (Hardy and Littlewood, 1935, 304–308). This conjecture is referred to as the Hardy-Littlewood majorant problem in the literature, and the majorant property is the property that the inequality (1.1) holds. Many contributions have been made to this problem over the years. We now give details of these contributions in order.

First, Hardy and Littlewood pointed out that for even integer exponents p, one trivially has $K_p = 1$ due to Parseval's identity. They also observed that if K_3 exists, then it must be greater one . To see this, let $g(x) = 1 + se^{2\pi i x} - bs^3 e^{6\pi i x}$ where b is a positive real number and s is a sufficiently small positive real number. Let $Q(x) = g(x)^{3/2}$. By the Taylor series expansion of $Q(x)$ up to the third power, we have

$$\begin{aligned}
Q(x) =& (1 + se^{2\pi ix} - bs^3e^{6\pi ix})^{3/2}\\
=& 1 + \binom{3/2}{1}(se^{2\pi ix} - bs^3e^{6\pi ix}) + \binom{3/2}{2}(se^{2\pi ix} - bs^3e^{6\pi ix})^2\\
&+ \binom{3/2}{3}(se^{2\pi ix} - bs^3e^{6\pi ix})^3 + \cdots\\
=& 1 + \frac{3/2}{1!}(se^{2\pi ix} - bs^3e^{6\pi ix}) + \frac{3/2(3/2-1)}{2!}(se^{2\pi ix} - bs^3e^{6\pi ix})^2\\
&+ \frac{3/2(3/2-1)(3/2-2)}{3!}(se^{2\pi ix} - bs^3e^{6\pi ix})^3 + \cdots\\
=& 1 + 3/2se^{2\pi ix} + 3/8s^2e^{4\pi ix} - (1/16 + 3b/2)s^3e^{6\pi ix} + \cdots .
\end{aligned}$$

Then

$$
\begin{aligned}
\int_0^1 |g(x)|^3 dx &= \int_0^1 |Q(x)|^2 dx \\
&= \int_0^1 Q(x)\overline{Q}(x) dx \\
&= \int_0^1 \bigg[\Big(1 + 3/2se^{2\pi ix} + 3/8s^2e^{4\pi ix} - (1/16 + 3/2b)s^3e^{6\pi ix} + \cdots\Big) \qquad (1.2) \\
&\qquad \Big(1 + 3/2se^{-2\pi ix} + 3/8s^2e^{-4\pi ix} - (1/16 + 3/2b)s^3e^{-6\pi ix} + \cdots\Big)\bigg]dx \\
&= 1 + \frac{9}{4}s^2 + \frac{9}{64}s^4 + (1/16 + 3b/2)^2s^6 + \cdots .
\end{aligned}
$$

By choosing $b = 1$ and using (1.2) we see that $\|g(x)\|_3$ is greater than $\|G(x)\|_3$, where $G(x) = 1 + se^{2\pi ix} + bs^3e^{6\pi ix}$ and hence $\|g(x)\|_3/\|G(x)\|_3 > 1$.

Later on, (Boas, 1962, 255) proved that if K_p exists for every $p > 2$ that is not an even integer then it must be greater than one. Boas used a method similar to the one Hardy and Littlewood used to show $K_3 > 1$. Let $g_1(x) = 1 + se^{2\pi ix} - \gamma s^k e^{2k\pi ix}$ where $k \geq 3$ is an integer, $2k - 4 < p < 2k - 2$ and $s > 0$ is sufficiently small. By Taylor series expansion of $Q(x) = g_1(x)^{p/2}$ up to k-th power, we have

$$
\begin{aligned}
Q(x) = (1 + se^{2\pi ix} - \gamma s^k e^{2k\pi ix})^{p/2} =& 1 + d_1(se^{2\pi ix} - \gamma s^k e^{2k\pi ix}) + d_2(se^{2\pi ix} - \gamma s^k e^{2k\pi ix})^2 + \cdots \\
&+ d_k(se^{2\pi ix} - \gamma s^k e^{2k\pi ix})^k + O(s^{k+1}) \\
=& 1 + d_1 se^{2\pi ix} + d_2 s^2 e^{4\pi ix} + \cdots + (d_k - d_1\lambda)s^k e^{2k\pi ix} \\
&+ O(s^{k+1}).
\end{aligned}
$$

Then

$$
\begin{aligned}
\|g_1\|_p^p &= \int_0^1 |g_1(x)|^p dx \\
&= \int_0^1 |Q(x)|^2 dx \\
&= \int_0^1 Q(x)\overline{Q}(x)dx \\
&= \int_0^1 \Bigg[\Big(1 + d_1 s e^{2\pi i x} + d_2 s^2 e^{4\pi i x} + \cdots + (d_k - d_1\gamma) s^k e^{2k\pi i x} + O(s^{k+1})\Big) \\
&\qquad \Big(1 + d_1 s e^{-2\pi i x} + d_2 s^2 e^{-4\pi i x} + \cdots + (d_k - d_1\gamma) r^k e^{-2k\pi i x} + O(s^{k+1})\Big)\Bigg] dx \\
&= 1 + d_1^2 s^2 + d_2^2 s^4 + \cdots + (d_k - d_1\gamma)^2 s^{2k} + O(s^{2k+2}).
\end{aligned}
\tag{1.3}
$$

Since $k \geq 3$ is an integer and $2k - 4 < p < 2k - 2$, we have

$$
d_k = \binom{p/2}{k} = \frac{p/2(p/2-1)(p/2-2)\cdots(p/2-k+1)}{k!} < 0.
$$

If we take $\gamma = -d_k/d_1$, where $d_1 = p/2 > 0$ then from (1.3) we see that $\|g_1(x)\|_p$ is greater than $\|G_1(x)\|_p$, where $G_1(x) = 1 + se^{2\pi i x} + \gamma s^k e^{2k\pi i x}$ and hence $\|g_1(x)\|_p/\|G_1(x)\|_p > 1$.

1.1. Failure of the majorant property

Bachelis proved that the majorant property fails for each $p \notin \{2, 4, \dots\}$ by showing that the constant K_p in (1.1) grows without bound as $|\Gamma| \to \infty$ (Bachelis, 1973, 121). To do this, Bachelis applied a technique, which is introduced to him by Yitzhak Katznelson. First, observe that if $f_1(x)$ and $f_2(x)$ are integrable functions on $[0, 1]$, and if $f_2(x)$ is a periodic function with the period one then we have

$$
\lim_{t\to\infty} \int_0^1 f_1(x) f_2(tx) dx = \int_0^1 f_1(x) dx \int_0^1 f_2(x) dx. \tag{1.4}
$$

Let $\tilde{g}(x) = g_1(x)g_1(tx)$ and $\tilde{G}(x) = G_1(x)G_1(tx)$ where $g_1(x)$ and $G_1(x)$ are given as above. Since the absolute values of the coefficients of $g_1(x)$ are less than those of $G_1(x)$, the same holds for $\tilde{g}(x)$ and $\tilde{G}(x)$. Then using (1.4) and Boas's result above we obtain

$$\frac{\|\tilde{g}(x)\|_p}{\|\tilde{G}(x)\|_p} > C^2$$

for sufficiently large t, where $C > 1$. Continuing in this manner, we see that the constant K_p can not exist.

1.2. The quantitative behaviour of the constant K_p

Even though the Hardy-Littlewood majorant property does not hold for all $p \geq 2$, there has been interest in studying the behaviour of the constant in (1.1) to quantify this failure. For this purpose, let us define

$$K_p(M) := \sup_{\Gamma \subseteq \{1,2,\ldots,M\}} K_p(\Gamma),$$

where

$$K_p(\Gamma) = \frac{\sup_{|a_n| \leq 1} \left\| \sum_{n\in\Gamma} a_n e^{2\pi i n x} \right\|_{L^p(\mathbb{T})}}{\left\| \sum_{n\in\Gamma} e^{2\pi i n x} \right\|_{L^p(\mathbb{T})}}.$$

One of the main reasons to pay attention to the quantitative behaviour of $K_p(M)$ is its connection with restriction conjecture for the Fourier transform in harmonic analysis. Localized version of this conjecture can be stated as follows. Let f be a smooth function on the unit sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ with $\|f\|_\infty \leq 1$. Then for each $\varepsilon > 0$ and all $p > 2d/(d-1)$ there is a constant $B_{d,p}$ such that

$$\|\widehat{f d\sigma}\|_{L^p(B(R))} \leq B_{d,p} R^{\varepsilon},$$

where $B(R)$ is the closed ball centered at the origin on $\mathbb{R}^d$ and σ is the rotational invariant measure on the unit sphere. In light of (Mockenhaupt, 1996, 25–30), if we can show that $K_p(A) \leq_{p,\varepsilon} M^\varepsilon$ for certain sets $A \subseteq \{1, 2, \ldots, M\}$, then local restriction conjecture immediately follows. Mockenhaupt also proved that $K_p(M)$ has a lower bound $M^{c/\log\log M}$ for all $p \in (2, 4)$, where c is some positive constant (for details, see (Mockenhaupt, 1996, 1–56)).

Now it is natural to ask whether for any $p \geq 2$ and any $\varepsilon > 0$ we have

$$K_p(M) \leq_{p,\varepsilon} M^{\varepsilon} \tag{1.5}$$

or not. This question can be considered as a relatively less strong version of the Hardy-Littlewood majorant problem.

The inequality (1.5) was disproved by (Green and Ruzsa, 2004, 513) for $p = 3$, that is, there exists a constant $\alpha > 0$ such that $K_3(M) \gtrsim M^{\alpha}$. Furthermore, it was disproved by (Mockenhaupt and Schlag, 2009, 1194) for each $p > 2$ that is not an even integer. To be more precise, for a large enough M there is a positive constant γ_p, a sequence $\eta_j \in \{-1, 1\}$ and a frequency set $A = [0, M] \cap \mathbb{Z}$ such that we have

$$\Big\| \sum_{n \in A} \eta_j e^{2\pi i n x} \Big\|_p \geq M^{\gamma_p} \Big\| \sum_{n \in A} e^{2\pi i n x} \Big\|_p.$$

1.3. The majorant problem in combinatorial problems

Hardy-Littlewood majorant problem appears in some combinatorial problems as well. (Green, 2005, 1610) proved that for the set of the form $P_M = P \cap [1, M]$ and $p \geq 2$, where P denotes the set of prime numbers we have

$$\sup_{|a_n| \leqslant 1} \Big\| \sum_{n \in P_M} a_n e^{2\pi i n x} \Big\|_{L^p(\mathbb{T})} \leq K_p \Big\| \sum_{n \in P_M} e^{2\pi i n x} \Big\|_{L^p(\mathbb{T})},$$

for some constant $K_p > 0$ that depends only on p. Ben Green used a different form of this result for $p = 5/2$ to prove that every subset A of P with positive upper density, i.e. $\limsup_{M \to \infty} \frac{|A \cap P_M|}{|P_M|} > 0$ includes a nontrivial arithmetic progression of length three.

In (Krause et al., 2016, 168), the authors constructed a wide class of sparse deterministic sets $\Lambda \subset \mathbb{N}$ with zero upper density and showed that Hardy-Littlewood majorant property holds on these sets, that is, we have

$$\sup_{|a_n|\leqslant 1}\Big\|\sum_{n\in\Lambda\cap[1,M]} a_n e^{2\pi inx}\Big\|_{L^p(\mathbb{T})} \leq K_p\Big\|\sum_{n\in\Lambda\cap[1,M]} e^{2\pi inx}\Big\|_{L^p(\mathbb{T})},$$

where $p \geq p_\Lambda$. Here, the upper density of a set $\Lambda \subset \mathbb{N}$ is defined by

$$\limsup_{M\to\infty} \frac{|\Lambda \cap [1, M]|}{M}.$$

Piatetski-Shapiro primes of type $\gamma < 1$(γ is close enough to 1) are given by

$$P_\gamma = P \cap \{\lfloor n^{1/\gamma} \rfloor : n \in \mathbb{N}\}.$$

These sets are called thin subsets of primes. Leonidas Daskalakis introduced the sets of primes P_B, which can be seen as generalized Piatetski-Shapiro primes (for details, see (Daskalakis, 2024, 114)). On these sets, Leonidas Daskalakis showed that Hardy-Littlewood majorant property holds. More precisely, let $c_1 \in [1, 16/15)$, $c_2 \in [1, 17/6)$ and $p > 2 + \frac{62-62\gamma}{16\gamma_1+17\gamma_2-32}$. There exists a positive constant $C = C(p, h_1, h_2, \psi) > 0$ such that for any $M \in \mathbb{N}$ and any $(a_n)_{n\in\mathbb{N}}$ sequence of complex numbers such that $|a_n| \leq 1$ for all $n \in \mathbb{N}$, we have

$$\Big\|\sum_{n\in P_B\cap[1,M]} a_n e^{2\pi inx}\Big\|_{L^p(\mathbb{T})} \leq C\Big\|\sum_{n\in P_B\cap[1,M]} e^{2\pi inx}\Big\|_{L^p(\mathbb{T})}.$$

.

1.4. The majorant problem in d-dimesional space

A d-dimensional, slightly different version of the Hardy-Littlewood majorant property has been investigated in (Gressman et al., 2023, 146–147). The authors refer to this as the strict majorant property, which is defined as follows.

Definition 1.1 *Let $\Gamma \subseteq \mathbb{Z}^d$ and $p > 0$ be given. The* ***strict majorant property*** *on $L^p(\mathbb{T}^d)$ is* *the property that the inequality*

$$\Big\| \sum_{n\in\Gamma} a_n e^{2\pi i n\cdot x} \Big\|_{L^p(\mathbb{T}^d)} \le \Big\| \sum_{n\in\Gamma} A_n e^{2\pi i n\cdot x} \Big\|_{L^p(\mathbb{T}^d)}$$

holds for any real sequences $(a_n)_{n\in\Gamma}$ *and* $(A_n)_{n\in\Gamma}$ *with* $|a_n| \le A_n$.

This problem is considered for affinely independent sets.

Definition 1.2 *Let* $m \in \mathbb{Z}^d$ *and let* $(1, m)$ *be a vector in* $\mathbb{Z}^{d+1}$. *A set* $A \subseteq \mathbb{Z}^d$ *is said to be affinely independent if* $\tilde{A} = \{(1, m) \in \mathbb{Z}^{d+1} : m \in A\}$ *is a linearly independent set in* $\mathbb{Z}^{d+1}$.

Let us give an example of an affinely independent set to make it more understandable. Let

$$A = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0)\} \subseteq \mathbb{Z}^3,$$

and

$$\tilde{A} = \{(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (1, 1, 1, 0)\}.$$

Let M be a matrix whose rows consist of elements of $\tilde{A}$. Since $det(M) = 1 \neq 0$, A is affinely independent set.

The authors showed that the strict majorant property holds for affinely independent subsets of d-tuples of integers, that is, for an integer $d \ge 1$, a non-empty set $\Gamma \subseteq \mathbb{Z}^d$ satisfies the strict majorant property on $L^p(\mathbb{T}^d)$ for every $p > 0$ if and only if Γ is affinely independent. Moreover, if Γ is not affinely independent, there is a non-negative integer k, and real sequence $(a_n)_{n\in\Gamma}$ such that for every $p \in (2k, 2k + 2)$, we have

$$\Big\| \sum_{n\in\Gamma} |a_n| e^{2\pi i n\cdot x} \Big\|_{L^p(\mathbb{T}^d)} < \Big\| \sum_{n\in\Gamma} a_n e^{2\pi i n\cdot x} \Big\|_{L^p(\mathbb{T}^d)}.$$

Particularly, the strict majorant property holds for each set $\Gamma \subseteq \mathbb{Z}^d$ provided that the number of elements of Γ is at least $d + 2$.

1.5. A probabilistic approach to the majorant problem

Although (1.5) does not hold in general Mockenhaupt and Schlag approached this problem differently. This problem was considered for random sets of integers $\Gamma \subseteq \{1, 2, \ldots, M\}$ of size M^{γ}, $0 < \gamma < 1$ and it was shown that the majorant property is almost surely valid for these sets. Before we present how Schlag and Mockenhaupt did this randomization, we briefly summarize their work. We start with explaining where the size restriction on a set Γ comes from. By using Hausdorff-Young's inequality and the basic lower bound $\left\| \sum_{n\in\Gamma} e^{2\pi inx} \right\|_p \gtrsim |\Gamma| M^{-1/p}$ we have

$$\sup_{|a_n|\leqslant 1} \Big\| \sum_{n\in\Gamma} a_n e^{2\pi inx} \Big\|_p \lesssim \left(\frac{M}{|\Gamma|}\right)^{1/p} \Big\| \sum_{n\in\Gamma} e^{2\pi inx} \Big\|_p. \tag{1.6}$$

Therefore the desired size restriction follows from (1.6).

For any odd integer $p > 2$, it was shown in (Mockenhaupt and Schlag, 2009, 1191) that

$$\sup_{|a_n|\leqslant 1} \Big\| \sum_{n\in\Gamma} a_n e^{2\pi inx} \Big\|_p^p \leq \Big\| \sum_{n\in\Gamma} e^{2\pi inx} \Big\|_2 \Big\| \sum_{n\in\Gamma} e^{2\pi inx} \Big\|_{2(p-1)}^{p-1}. \tag{1.7}$$

If we suppose that

$$\Big\| \sum_{n\in\Gamma} e^{2\pi inx} \Big\|_2 \Big\| \sum_{n\in\Gamma} e^{2\pi inx} \Big\|_{2(p-1)}^{p-1} \leq C_\varepsilon M^\varepsilon \Big\| \sum_{n\in\Gamma} e^{2\pi inx} \Big\|_p^p \tag{1.8}$$

for any $\varepsilon > 0$ then clearly the inequality (1.5) holds.

For the set of squares if $p = 3m + 1$ for $m \in \mathbb{N}$ then it was observed in (Mockenhaupt and Schlag, 2009, 1192–1193) that (1.8) holds, and hence we have

$$\sup_{|a_n|\leqslant 1} \Big\| \sum_{n=1}^{M} a_n e^{2\pi in^2x} \Big\|_p \leq C_\varepsilon M^\varepsilon \Big\| \sum_{n=1}^{M} e^{2\pi in^2x} \Big\|_p.$$

The final argument is to plug random sets into the majorant problem. Let us consider the following random set defined by

$$\Gamma(\omega) = \{n \in \{1, 2, \ldots, M\} |\, \xi_n(\omega) = 1\}, \tag{1.9}$$

where ξ_n's are independent identically distributed selector variables satisfying $\mathbb{P}(\xi_n = 1) = \alpha = 1 - \mathbb{P}(\xi_n = 0)$ for $0 < \alpha < 1$. In other words, the set (1.9) is actually constructed by choosing every integer from the set $\{1, 2, \ldots, M\}$ with the same probability.

It was shown in (Mockenhaupt and Schlag, 2009, 1193) that

$$\mathbb{E}\Big\| \sum_{n\in\Gamma(\omega)} e^{2\pi inx} \Big\|_p^p \approx \alpha^p M^{p-1} + (\alpha M)^{p/2} \tag{1.10}$$

for $p \geq 2$. Observe that, if we take $\alpha = M^{\frac{2}{p}-1}$ then the right-hand side of (1.10) equals to $2M$. From this the inequality (1.8) holds and so does the inequality (1.5). But the inequality (1.8) does not hold except for $\alpha = M^{\frac{2}{p}-1}$.

Schlag and Mockenhaupt managed to show that the majorant property holds on the set $\Gamma(\omega)$ where $\alpha = M^{-\delta}$ and $0 < \delta < 1$, up to a negligible probability. To be more explicit we have for any $\varepsilon > 0$

$$\mathbb{P}\left[\sup_{|a_n|\leqslant 1} \Big\| \sum_{n\in\Gamma(\omega)} a_n e^{2\pi inx} \Big\|_{L^p(\mathbb{T})} \geq M^{\varepsilon} \Big\| \sum_{n\in\Gamma(\omega)} e^{2\pi inx} \Big\|_{L^p(\mathbb{T})}\right] \longrightarrow 0$$

as $M \longrightarrow \infty$. One can find more details in (Mockenhaupt and Schlag, 2009).

1.6. Main results of the first part

In the first part of this thesis, we perturb deterministic sets by stationary processes, Poisson processes and simple random walks and obtain the following theorems respectively.

Theorem 1.1 *Let $\{X_j\}_{j\in\mathbb{N}}$ be a stationary process taking only integer values. Let the probability mass function of the random variables in our process be denoted by $\mu : \mathbb{Z} \to \mathbb{R}$. Let $A \subset \mathbb{N}$ be any finite nonempty subset. Then for any $1 \leq p < \infty$ and any $\varepsilon > 0$ we have*

$$|A|^p \lesssim_{\mu,p} \mathbb{E}\Big\| \sum_{j\in A} e^{2\pi iyX_j} \Big\|_p^p \leq |A|^p, \tag{1.11}$$

$$\mathbb{E} \sup_{|a_j|\leq 1} \Big\| \sum_{j\in A} a_j e^{2\pi iyX_j} \Big\|_p^p \lesssim_{\mu,p} \mathbb{E}\Big\| \sum_{j\in A} e^{2\pi iyX_j} \Big\|_p^p, \tag{1.12}$$

$$\lim_{|A|\to\infty} \mathbb{P}\Big[\sup_{|a_j|\le 1} \Big\| \sum_{j\in A} a_j e^{2\pi i y X_j} \Big\|_p \ge |A|^{\varepsilon} \Big\| \sum_{j\in A} e^{2\pi i y X_j} \Big\|_p \Big] = 0. \tag{1.13}$$

Analogues of these results for $p = \infty$ also hold.

Let us consider the following heuristic argument. A stationary process repeats the same values with the same probabilities. Therefore the exponential sum in (1.11) behaves like a an $|A|$-fold sum of the same exponential $e^{2\pi i y X_j}$. The above result gives a confirmative answer to this heuristic. From this we obtain (1.12) and (1.13) immediately. (1.12) means that the Hardy-Littlewood majorant property holds on an average sense and (1.13) is indeed the Hardy-Littlewood majorant property for generic sets as in the contexts of Schlag-Mockenhaupt.

We need to consider the set A to be of interval form for Poisson processes. Since the analogue of (1.11) in this case does not hold, that is the left hand side depends not only on the cardinality but on the structure of the set. In spite of this technical difficulty we also prove the analogue of Theorem 1.2 for the set A, $2 \le p \le 4$.

Theorem 1.2 *Let $\{N(t)\}_{t\ge 0}$ be a Poisson process of intensity* 1. *Then for any* $2 \le p < \infty$ *we have*

$$\mathbb{E}\Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)} \Big\|_p^p \approx_p M^{p-1}, \tag{1.14}$$

$$\mathbb{E} \sup_{|a|\le 1} \Big\| \sum_{j=1}^{M} a_j e^{2\pi i y N(j)} \Big\|_p^p \lesssim_p \mathbb{E}\Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)} \Big\|_p^p, \tag{1.15}$$

$$\lim_{M\to\infty} \mathbb{P}\Big[\sup_{|a_j|\le 1} \Big\| \sum_{j=1}^{M} a_j e^{2\pi i y N(j)} \Big\|_p^p \ge M^{\varepsilon} \Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)} \Big\|_p^p \Big] = 0. \tag{1.16}$$

Analogues of these results for $p = \infty$ also hold.

Since $\mathbb{E}[N(j)] = \mathbb{V}[N(j)] = j$, heuristically we have $N(j) \sim j$ with high probability and using this we can guess (1.14).

Theorem 1.3 *Let $\{N(t)\}_{t\ge 0}$ be a Poisson process of intensity* 1. *Let $d > 1$ be an integer. Let $A \subset \mathbb{N}$ be a finite nonempty subset. Then for any* $2 \le p \le 4$ *we have*

$$|A|^{p/2} \le \mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_p^p \lesssim_p \begin{cases} |A|^{p/2} \log^{3p/8}(1+|A|) & \textit{for } d = 2 \\ |A|^{p/2} & \textit{for } d \ge 3, \end{cases} \tag{1.17}$$

$$\mathbb{E}\sup_{|a_j|\leq 1}\Big\|\sum_{j\in A}a_je^{2\pi iyN(j^d)}\Big\|_p^p\lesssim_p\begin{cases}C_\epsilon|A|^\varepsilon\mathbb{E}\Big\|\sum_{j\in A}e^{2\pi iyN(j^d)}\Big\|_p^p & \textit{for } d=2\\ \mathbb{E}\Big\|\sum_{j\in A}e^{2\pi iyN(j^d)}\Big\|_p^p & \textit{for } d\geq 3,\end{cases}\tag{1.18}$$

$$\lim_{|A|\to\infty}\mathbb{P}\Big[\sup_{|a_j|\leq 1}\Big\|\sum_{j\in A}a_je^{2\pi iyN(j^d)}\Big\|_p^p\geq |A|^\varepsilon\Big\|\sum_{j\in A}e^{2\pi iyN(j^d)}\Big\|_p^p\Big]=0.\tag{1.19}$$

Let us explain the relation between Chang's conjecture and the above theorem. Since the inequality (1.17) is obtained for an arbitrary $A\subseteq\{1,2,\ldots,M\}$, Chang's conjecture holds in an average sense. We refer to Section 1.12 for more details about this conjecture.

Theorem 1.4 *Let $\{R(j)\}_{j\in\mathbb{Z}_+}$ be a simple random walk. Then for any $2\leq p<\infty$ we have*

$$\mathbb{E}\Big\|\sum_{j=1}^{M}e^{2\pi iyR(j)}\Big\|_p^p\approx_p M^{p-1/2},\tag{1.20}$$

$$\mathbb{E}\sup_{|a_j|\leq 1}\Big\|\sum_{j=1}^{M}a_je^{2\pi iyR(j)}\Big\|_p^p\lesssim_p\mathbb{E}\Big\|\sum_{j=1}^{M}e^{2\pi iyR(j)}\Big\|_p^p,\tag{1.21}$$

$$\lim_{M\to\infty}\mathbb{P}\Big[\sup_{|a_j|\leq 1}\Big\|\sum_{j=1}^{M}a_je^{2\pi iyR(j)}\Big\|_p^p\geq M^\varepsilon\Big\|\sum_{j=1}^{M}e^{2\pi iyR(j)}\Big\|_p^p\Big]=0.\tag{1.22}$$

Analogues of these results for $p=\infty$ also hold.

Since $\mathbb{E}[|R(j)|]\approx\sqrt{j}$ and $\mathbb{V}[R(j)]=j$, heuristically we have $|R(j)|\approx\sqrt{j}$ with high probability and using this we can guess (1.20).

If for a process $X(j)$ we have $X(j)\approx j^{1-\alpha},\ 0\leq\alpha\leq 1$ with high probability, then we may see this as a value being repeated M^α times, and write

$$\mathbb{E}\Big\|\sum_{j=1}^{M}e^{2\pi iyX(j)}\Big\|_p^p\approx\mathbb{E}\Big\|M^\alpha\sum_{j=1}^{M^{1-\alpha}}e^{2\pi iyj}\Big\|_p^p\approx M^{p\alpha+(p-1)(1-\alpha)}=M^{p-1+\alpha},\tag{1.23}$$

and this is a very effective heuristic to guess what happens for other processes.

Next we consider the Green-Ruzsa set in (Green and Ruzsa, 2004) defined by

$$\Lambda_{D,k}=\Big\{\sum_{j=0}^{k-1}d_jD^j|\ \ d_j\in\{0,1,3\}\Big\},$$

for any integer $D \geq 5$ and $k \in \mathbb{N}$. We perturb this set via the Poisson process, and then investigate the Hardy-Littlewood majorant property on this set. Indeed, it stems solely from the sparsity of this set. Thus the following theorem is obtained for a very large class of sparse sets.

Theorem 1.5 *Let $\{N(t)\}_{t\geq 0}$ be a Poisson process of intensity 1. Let $A \subset \mathbb{N} \cup \{0\}$ be a finite nonempty subset such that for any $N \in \mathbb{N}, n \in \mathbb{Z}$*

$$\left|A \cap [n-M, n+M]\right| \leq C_A M^{\alpha},$$

with C_A is a constant that depends only on A. If $\alpha < 1/3$ for any $2 \leq p \leq 4$ we have

$$|A|^{p/2} \leq \mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y N(j)} \Big\|_p^p \lesssim_{\alpha, C_A} |A|^{p/2}, \tag{1.24}$$

$$\mathbb{E} \sup_{|a_j|\leq 1} \Big\| \sum_{j\in A} a_j e^{2\pi i y N(j)} \Big\|_p^p \lesssim_{\alpha, C_A} \mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y N(j)} \Big\|_p^p, \tag{1.25}$$

$$\lim_{|A|\to\infty} \mathbb{P}\Big[\sup_{|a_j|\leq 1} \Big\| \sum_{j\in A} a_j e^{2\pi i y N(j)} \Big\|_p^p \geq |A|^{\varepsilon} \Big\| \sum_{j\in A} e^{2\pi i y N(j)} \Big\|_p^p \Big] = 0, \tag{1.26}$$

where the limit is taken over A for which $\sup_A C_A < \infty$ and $\sup_A \alpha < 1/3$.

1.7. Norms of random trigonometric polynomials

In d-dimensional space, a random trigonometric polynomial is defined by

$$f(x) := \sum_{A\subseteq \mathbb{Z}^n} X_n a_n e^{2\pi i n\cdot x}, \quad x \in [0,1]^d,$$

where A is finite subset, $a_n \in \mathbb{C}$, $n = (n_1, n_2, \ldots, n_d)$ and $\{X_n\}_{n\in A}$ is a sequence of independent identically distributed random variables

Definition 1.3 *A sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of random variables is called a **Rademacher sequence** if ε_n's are independent identically distributed random variables satisfying*

$$\mathbb{P}(\varepsilon_n = 1) = \frac{1}{2} \qquad \textit{and} \qquad \mathbb{P}(\varepsilon_n = -1) = \frac{1}{2}.$$

Especially, L^∞-norm of random trigonometric polynomials has been studied throughout the years. For the given random trigonometric polynomial

$$P(x) = \sum_{n=0}^{N} \varepsilon_n a_n \cos(nx + \phi_n),$$

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is Rademacher sequence and $\{\phi_n\}$ is a sequence of real numbers, (Salem and Zygmund, 1954, 245–248) showed that

$$\mathbb{P}\left(\|P\|_\infty < \lambda \left[\sum a_n^2 \log N\right]^{1/2}\right) \tag{1.27}$$

is approximately 1 when N or λ is big enough. One of the immediate result of (1.27) is significant. Let $a_1, a_2, \ldots, a_n$ be complex numbers and $\{\varepsilon_n\}_{n=1}^N$ be a Rademacher sequence. Then L^∞-norm of

$$f(t) = \sum_{n=1}^{N} \varepsilon_n a_n e^{int}$$

less than or equal to $C(\sum_{n=1}^N |a_n|^2 \log N)^{1/2}$, where $C > 0$. In other words,

$$\mathbb{P}\left(\|f\|_\infty \leq C \left[\sum_{n=1}^{N} |a_n|^2 \log N\right]^{1/2}\right)$$

is almost equal to 1 for a sufficiently large number N.

Salem and Zygmund's result was extended to a more general sequence of random variables by (Kahane, 1985, 69–72) , namely a sequence of independent random variables $\{\xi_n\}$, which satisfies the condition $\mathbb{E}(e^{\lambda \xi_n}) \leq e^{\lambda^2/2}$, and obtained the following result.

Theorem A *Let us consider a random trigonometric polynomial*

$$P(x) = \sum \xi_n f_n(x),$$

where f_n are real or complex trigonometric polynomials of degree less than or equal to

N, $\{\xi_n\}$ is subnormal sequence, and $\sum$ is finite sum. We have

$$\mathbb{P}\left(\|P\|_\infty \geq C\left[\sum \|f_n\|^2 \log N\right]^{1/2}\right) \leq \frac{1}{N^2},$$

for some absolute constant C.

Kahane also obtained an analogue of the previous theorem for random trigonometric polynomials in s-variables.

Theorem B *Let us consider a random trigonometric polynomial in s- variables*

$$P(t_1, t_2, \ldots, t_s) = \sum \xi_n f_n(t_1, t_2, \ldots, t_s),$$

where the f_n are complex trigonometric polynomials of degree less than or equal to N, ξ_n a subnormal sequence $\sum$ is a finite sum. Then we have

$$\mathbb{P}\left(\|P\|_\infty \geq C\left[s \sum \|f_n\|_\infty^2 \log N\right]^{1/2}\right) \leq N^{-2}e^{-s}$$

for some absolute constant C.

Littlewood polynomials are defined by

$$p(x) = \sum_{j=0}^{n-1} a_j x^j,$$

where $a_j \in \{-1, 1\}$. It was conjectured by (Littlewood, 1966, 367-370) that, we can find p_n such that

$$C_1\sqrt{n+1} \leq |p_n(z)| \leq C_2\sqrt{n+1} \tag{1.28}$$

for all complex numbers with property $|z| = 1$. The confirmative answer to this conjecture is given in (Balister et al., 2020, 980–985) by constructing an infinite family of Littlewood polynomials, which satisfies (1.28).

A notable effort was made to estimate the expected value of L^4-norm of a Littlewood polynomial of degree n in (Newman and Byrnes, 1990, 42–45) and it was shown that

$$\mathbb{E}(\|p\|_4^4) \leq 2(n+1)^2 - (n+1).$$

The following theorem was obtained by (Borwein and Lockhart, 2001, 1463-1467) to estimate the expected value of L^p- norm of a random polynomial

$$q_n(\theta) = \sum_{k=0}^{n-1} X_k e^{ik\theta},$$

where $\theta \in [0, 2\pi]$.

Theorem C *Fix $0 < p < \infty$. Assume that random variables X_k $k \geq 0$, are independent and identically distributed, have mean 0, variance equal to 1 and if $p > 2$, a finite p-th moment $\mathbb{E}(|X_k|^p)$. Then*

$$\frac{\mathbb{E}(\|q_n\|_p^p)}{n^{p/2}} \longrightarrow \Gamma(1 + p/2)$$

as $n \longrightarrow \infty$. If in addition, $\mathbb{E}(|X_k|^{2p}) < \infty$ then

$$\frac{\|q_n\|_p}{n^{1/2}} \longrightarrow \Gamma(1 + p/2)^{1/p}$$

in probability, and

$$\frac{\mathbb{E}(\|q_n\|_p)}{n^{1/2}} \longrightarrow (\Gamma(1 + p/2)^{1/p})^{1/p}.$$

1.8. On the convergence of random Fourier series

Over the years, random Fourier series has been studied. Most of the attention is on convergence of these series. A random fourier series is defined by

$$\sum_{n=0}^{\infty} a_n X_n e^{2\pi i n x}, \quad x \in [0, 1],$$

where $\{X_n\}_{n=0}^{\infty}$ is a sequence of independent random variables and $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers. (Paley and Zygmund, 1930, 337–345) studied a problem concerning the uniform convergence almost surely of the random Fourier series and obtained the following pioneering result. For the random Fourier series

$$\sum_{n=0}^{\infty} c_n \varepsilon_n e^{2\pi n x},$$

where $\{c_n\}_{n=0}^{\infty}$ is a sequence real numbers with $\sum_{n=0}^{\infty} c_n^2 = 1$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ is Rademacher sequence, it is uniformly almost surely convergent if

$$\sum_{n=2}^{\infty} c_n^2 (\log n)^{1+\varepsilon} < \infty, \quad \varepsilon > 0.$$

In (Salem and Zygmund, 1954, 250–260), the authors slightly changed the previous Fourier series and showed that for the random Fourier series

$$\sum_{n=0}^{\infty} c_n \varepsilon_n \cos(nx + \alpha_n),$$

where $0 \leq \alpha_n \leq 2\pi$ are real numbers and $\{\varepsilon_n\}_{n=0}^{\infty}$ is Rademacher sequence, it is uniformly almost surely convergent if

$$\sum_{j=2}^{\infty} \frac{\left(\sum_{n=k}^{\infty} c_n^2\right)^{1/2}}{k(\log k)^{1/2}} < \infty.$$

The results above were generalized by (Kahane, 1985, 50-60) replacing Rademacher sequence with symmetric complex valued random variable and obtained the following result. Let

$$X(x) = \sum_{n=0}^{\infty} c_n \eta_n \cos(nx + \phi_n), \quad x \in [0, 2\pi],$$

where $\{\eta_n e^{i\phi_n}\}$ is a sequence of independent symmetric complex valued random variable (η_n and ϕ_n real) with $\mathbb{E}|\eta_n|^2 = 1$ and $\sum_{n=0}^{\infty} c_n^2 = 1$. Then this series converges almost surely for each $x \in [0, 2\pi]$.

In (Talagrand, 1995, 777) the uniform convergence of the random Fourier series of the form

$$\sum_{n=1}^{\infty} \frac{X_n}{n} e^{2\pi n x}, \quad x \in [0, 1],$$

was investigated, where $\{X_n\}$ is a sequence of independent identically distributed random variables. The origin of this problem is to seek integrability conditions on a function f to ensure the convergence almost everywhere of its Fourier series. It was shown that for a given sequence $\{X_n\}$ of independent and identically distributed random variables with $\mathbb{E}(X_n) = 0$, the random Fourier series

$$\sum_{n=1}^{\infty} \frac{X_n}{n} e^{2\pi i n x}, \quad x \in [0, 1]$$

converges uniformly with negligible probability if and only if $\mathbb{E}(|X| \log\log(\max(e^e, |X|)))$ is finite.

1.9. Stochastic integrals

We dive into the world of stochastic integrals. For this purpose, we first give the definition of a Brownian motion and a simple stochastic process, respectively.

Definition 1.4 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Assume there exists a continuous function $W_t(\omega)$ with $t \geq 0$ and $W_0(\omega) = 0$ for every $\omega \in \Omega$. Then $W_t(\omega)$ is said to be a Brownian motion if for any partition $0 = t_0 < t_1 < \cdots < t_n$,*

$$W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$$

are all independent increments and every increment is normally distributed with mean 0 and variance $t_{i+1} - t_i$.

Definition 1.5 *Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of the interval $[0, T]$. A stochastic process $(S_t(\omega))_{t \geq 0}$ on $[0, T]$ is said to be simple if it is constant on each subinterval $[t_i, t_{i+1})$.*

1.9.1. Itô's integral

We first devise Itô's integral for simple integrands. Let $(S_t(\omega))_{t \geq 0}$ be a simple stochastic process adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ associated with a Brownian motion

$W_t(\omega)$. Then Itô integral of it is defined by

$$I(T) = \int_0^T S_t(\omega)dW_t(\omega) = \int_0^T S_t dW_t = \sum_{i=0}^{n-1} S_{t_i}[W_{t_{i+1}} - W_{t_i}].$$

It is possible to extend Itô's integral to non-simple integrands. Let X_t be a general stochastic process that is adapted to the filtration $\mathcal{F}_t$ and has the square-integrability condition

$$\mathbb{E}\int_0^T X_t^2 dt < \infty. \tag{1.29}$$

We can always construct a sequence $(S_t)_n$ of simple processes on $[0, T]$, which converges to X_t, that is

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |(S_t)_n - X_t|^2 dt.$$

Then Itô's integral of a general stochastic process X_t is defined by

$$I(T) := \int_0^T X_t dW_t = L^2\text{-}\lim_{n\to\infty}\int_0^T (S_t)_n dW_t.$$

Here the notation $L^2\text{-}\lim_{n\to\infty} Y_n = Y$ stands for $\lim_{n\to\infty}\mathbb{E}(|Y_n - Y|^2) = 0$, where $\{Y_n\}$ is a sequence of random variables.

Let X_t and Y_t be general stochastic processes that obey the integrability conditions. Then the Itô integral $I(t) = \int_0^t X_u dW_u$ for $0 \le t \le T$ enjoys with the following properties:

- $I(t)$ itself is a stochastic process.
- **(Continuity)** The function $t \mapsto I(t)$ is continuous.
- **(Adaptivity)** $I(t)$ is $\mathcal{F}_t$-measurable for all t.
- **(Linearity)** If $I(t) = \int_0^t X_u dW_u$ and $H(t) = \int_0^t Y_u dW_u$ then we have

  $$aI(t) \pm bH(t) = \int_0^t (aX_u \pm bY_u)dW_u$$

  for all $a, b \in \mathbb{R}$.
- **(Martingale)** $I(t)$ is a martingale.

- **(Itô Isometry)** $\mathbb{E}I^2(t) = \mathbb{E}\int_0^t X_u^2 du.$

- **(Quadratic variation)** $[I, I](t) = \int_0^t X_u^2 du.$

We refer to preliminaries for further information about the objects that we do not define explicitly in this subsection.

1.10. On the stochastic Fourier transformation

In (Ogawa, 2013, 286–294) the idea of the stochastic Fourier transformation was introduced. Let us first deploy preliminary concepts about this subject. We will consider the standard real Brownian function $W(x, \omega)$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_x\}, \mathbb{P})$, where $\{\mathcal{F}_x, x > 0\}$ stands for a natural filtration with respect to Brownian motion. Let $\mathbf{H}$ be the space of totality of measurable real random functions $f(x, \omega)$ on $\mathbb{B}_{[0,1]} \times \mathcal{F}$ where $\mathbb{B}_{[0,1]}$ is the Borel field. Let $\{\varphi_n(x) | n \in \mathbb{N}\}$ be an orthonormal basis in the real Hilbert space $L^2(0, 1)$ with the condition

$$\sup_x |\varphi_n(x)| < \infty \qquad \text{for all} \qquad n \in \mathbb{N}.$$

Definition 1.6 *Let $f(x, \omega) \in \mathbf{H}$ be a random function. The stochastic Fourier coefficient of $\hat{f}_n(\omega)$ of f with respect to a fixed orthonormal basis $\{\varphi_n\}$ is defined as follows.*

$$\hat{f}_n(\omega) := \int_0^1 f(x, \omega)\varphi_n(x) d_* W_x,$$

where $\int_0^1 d_* W_x$ denotes a stochastic integral of non-casual type. The function f is casual if f is adapted to the natural filtration $\{F_x\}$. Otherwise we call it noncasual.

In harmonic analysis, the following question holds significant importance: Under what conditions can we reconstruct a function from its Fourier coefficients? S. Ogawa studied this question for the following class of functions. Let $(\Omega, \mathcal{G}_{[0,1]}, \mathbb{P})$ be the Wiener space with $\mathcal{G}_{[0,1]} := \sigma\{W_s | s \in [0, 1]\}$. Let

$$\mathcal{M}^2 := \{f \in L^2([0, 1] \times \Omega, dx \times d\mathbb{P})|\}.$$

be adapted to the filtration $\{\mathcal{G}_{[0,x]}, x > 0\}$. The stochastic Fourier coefficient of $\hat{f}_n$ of the casual function $f \in \mathcal{M}^2$ is given by

$$\hat{f}_n(\omega) := \int_0^1 \varphi_n(x) f(x, \omega) d_0 W_x,$$

where $\int_0^1 d_0 W_x$ denotes the Itô integral. The following result is due to (Ogawa, 2013, 288).

Theorem D *Any casual random function $f \in \mathcal{M}^2$ can be reconstructed from its stochastic Fourier coefficients $\{\hat{f}_n(\omega)\}$.*

1.11. Stochastic oscillatory integrals

We start with the definition of a abstract Wiener space.

Definition 1.7 *Let X be a separable Banach space, μ be a Gaussian measure and H be the Cameron-Martin space. Any triple (X, μ, H) is called abstract Wiener space. Moreover, if we take the space of continuous paths $X = C[0, 1]$ as Banach space then it is called the classical Wiener space.*

A stochastic oscillatory integral is defined by

$$I(\lambda) := \int_X e^{i\lambda q(x)} \psi(x) d\mu(x),$$

where X is a real abstract Wiener space with Wiener measure μ and q, ψ are real valued Wiener functionals. On finite dimensional Euclidean spaces, we have asymptotic behaviour of oscillatory integrals as a consequence of method of principle of stationary phase (for details, see (Stein and Murphy, 1993, 60–90)). On the other hand, (Copson, 1965) showed that the principle stationary phase is still eligible in infinite dimensional space. This type of an asymptotic behaviour has been investigated for stochastic oscillatory integrals on infinite dimensional Wiener space X and some estimates are obtained in (Malliavin and Taniguchi, 1997, 470–475) and (Taniguchi, 1998, 424–428) . Note that, for general q and ψ, method of stationary phase has not been completely carried out yet.

Since we do not intend to bother readers with sophisticated arguments we do not give details of their study.

Let us consider the oscillatory integral

$$\int_0^1 e^{i\xi W(t)} dt,$$

where the phase function is the Brownian motion $W(t)$. More explicitly, the Brownian motion here is defined as

$$W(t) := X_0 t + \sqrt{2} \sum_{n=1}^{\infty} \frac{1}{2\pi n} \left(X_n sin(2\pi nt) + Y_n (1 - cos(2\pi nt)) \right),$$

where the sequences $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 1}$ consist of independent identically distributed random variables each of which has normalized Gaussian distribution.

The following result is due to (Kahane, 1985, 240–245).

Theorem E *Almost surely, there exists $C > 0$ such that for all $|\xi| \geq 1$, we have*

$$\left| \int_0^1 e^{i\xi W(t)} dt \right| \leq C |\xi|^{-1} \sqrt{\log |\xi|}.$$

This result is important since it shows that if we replace the phase function with the Brownian motion, we still observe a similar decay rate as in the following oscillatory integral, due to Van Der Corput, that is, assume that $\phi : [0, 1] \to \mathbb{R}$ is n times continuously differentiable phase function with $\phi^{(n)} \geq 1$. Then there exists an absolute constant c_n such that for each $|\xi| \geq 1$

$$\left| \int_0^1 e^{i\xi\phi(x)} dx \right| \leq c_n |\xi|^{-1/n},$$

where $n \geq 2$. Note that if the first derivative of the phase function above is monotone then this oscillatory integral decays as a rate of $1/|\xi|$.

1.12. The number of solutions of diophantine equations via norms of exponential sums

L^{2n}-norms, where $n \in \mathbb{N}$ of exponential sums are related to solutions of symmetric homogeneous diophantine equations as follows:

$$\Big\| \sum_{j \in A} e^{2\pi i j y} \Big\|_{L^{2n}(\mathbb{T})}^{2n} = \left| \left\{ (j_1, j_2, \ldots, j_n, k_1, k_2, \ldots, k_n) \in A^{2n} : \sum_{i=1}^{n} j_i = \sum_{i=1}^{n} k_i \right\} \right|, \tag{1.30}$$

which also equals

$$\sum_{m \in \mathbb{Z}} \left| \{ (j_1, j_2, \ldots, j_n) \in A^n : j_1 + j_2 + \cdots + j_n = m \} \right|^2, \tag{1.31}$$

where $A \subset \mathbb{Z}$ is a finite set and $| \cdot |$ denotes cardinality. Estimating the number of solutions of (1.30),(1.31) in terms of cardinality of A has been a very popular topic among mathematicians. We have approximately $n!|A|^n$ trivial solutions for the sets in (1.30), and this can be obtained by choosing $(k_1, k_2 \ldots, k_n)$ to be a permutation of $(j_1, j_2, \ldots, j_n)$. Therefore we need to estimate the number of nontrivial solutions. Indeed there may be no nontrivial solutions for certain A, n. Observe that each element in the sets (1.31) is indeed representations of m by elements of A^n, and we will denote the cardinality of these sets by $R_{A^n}(m)$.

In general, A is taken as $d-$powers of the subset $\{1, 2, 3 \ldots, M\}$, and owing to this an estimation in terms of M is necessary. For these sets we aim to estimate the number of solutions of the diophantine equation given by a homogenous form:

$$\left| \left\{ (j_1, j_2, \ldots, j_n, k_1, k_2, \ldots, k_n) : 1 \leq j_i, k_i \leq M, \sum_{i=1}^{n} j_i^d = \sum_{i=1}^{n} k_i^d \right\} \right|. \tag{1.32}$$

In this particular case the elements of the sets (1.30) are representations of m by n d-powers and their cardinalities are denoted by

$$R_{n,d,M}(m) := \left| \{ (j_1, j_2, \ldots, j_n) : 1 \leq j_i \leq M, \ j_1^d + j_2^d + \cdots + j_n^d = m \} \right|. \tag{1.33}$$

These problems can be considered as finding lattice points on a variety in affine or projective plane as well.

We now utilize the following well known heuristic argument to have an opinion on the cardinality of the sets (1.32), (1.33). Since there are M choices for each j_i, where $i \in \{1, 2, \dots, n\}$ and any sum of the form $j_1^d + j_2^d + \cdots + j_n^d$ takes an integer value between 1 and nM^d there exist, on average, M^{n-d} representations for any integer $m \in [1, nM^d]$. More precisely, the asymptotic formula

$$R_{n,d}(m) = \frac{\Gamma(1 + 1/d)^n}{\Gamma(n/d)} \mathfrak{S}_{n,d}(m) m^{n/d-1} + o(m^{n/d-1})$$

was obtained in (Hardy and Littlewood, 1920, 36) for big enough s in terms of n, where

$$\mathfrak{S}_{n,d}(m) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(q^{-1} \sum_{r=1}^{q} e(ar^k/q) \right)^s e(-na/q).$$

Observe that for $d \geq n$ it is natural to expect that $R_{n,d,M}(m) \lesssim 1$, and therefore trivial solutions dominate in (1.33), which is called as paucity of nontrivial solutions. The heuristic argument above leads us to the Hardy-Littlewood Hypothesis K which can be stated as follows: The number of nonnegative solutions to the equation

$$j_1^n + \cdots + j_n^n = M$$

is $O(M^\varepsilon)$. This hypothesis was proved for $n = 2$, however in (Mahler, 1936, 138) it was disproved for $n = 3$, and nonetheless it may still hold for some of other values of n. We also have the conjecture, based on the same heuristic argument above that the cardinality of the set (1.32) is bounded by $C_\varepsilon M^\varepsilon \max\{M^n, M^{2n-d}\}$, see (Vaughan, 1997, 167–170). This was proved for $n = 2$, but it is still open for the other values of n. For $n = 3$ case, a significant effort (see, (Browning and Heath-Brown, 2004, 553–573),(Heath-Brown, 2002, 553–598) and (Heath-Brown, 2006, 51)) has been dedicated to merely obtaining the bounds $M^{7/2-\delta}$ for large enough d. the bound $M^{3+\varepsilon}$ was obtained by (Salberger, 2005, 93–115) for $d > 25$, and there has been no reduction in d so far. Even less information is available for larger values of n, see (Salberger and Wooley, 2010, 317–342) and (Marmon, 2011, 55–74). The only result for general n is due to (Salberger and Wooley, 2010, 317–

320). They showed the paucity of nontrivial solutions and from this the conjecture, for $d \geq (2n)^{4n}$.

If we consider d-powers of an arbitrary finite set $A \subseteq \mathbb{N}$ these problems becomes more challenging even for $n = 2$ case. The following conjecture, due to (Chang, 2004, 444) and addressing this case, is stated as follows: For any $\varepsilon > 0$ we have

$$\Big\| \sum_{j\in A} e^{2\pi i j^2 y} \Big\|_{L^4(\mathbb{T})}^4 \leq C_\varepsilon |A|^{2+\varepsilon}. \tag{1.34}$$

This conjecture is closely connected to numerous other problems in additive combinatorics, harmonic analysis, and number theory, see (Chang, 2004, 444–460),(Cilleruelo and Granville, 2006, 1–21) and (Sanders, 2012, 627–655). Trivial bound for (1.34) is $|A|^3$, and there have only been improvements that are not of a power type, see (Chang, 2004, 446) and (Sanders, 2012, 627–655).

1.13. Main results of the second part

In the second part of this thesis, we prove how randomization affects an exponential sum with frequencies forming an arithmetic progression when the step size is large and obtain the following result.

Theorem 1.6 *Let $\{N(t)\}_{t\geq 0}$ be a Poisson process of intensity 1. Let $r > 0$ be a real number. Then*

$$\mathbb{E}\Big\| \sum_{j=1}^{M} e^{2\pi i y N(jM^r)} \Big\|_2^2 \approx M, \qquad M^2 + M^{3-r} \lesssim \mathbb{E}\Big\| \sum_{j=1}^{M} e^{2\pi i y N(jM^r)} \Big\|_4^4 \lesssim_r M^2 \log M + M^{3-r}.$$

We do not have much of an effect when step size is one. One can see this by comparing (2.37) and (1.14). Observe that for $r < 1$ the term M^{3-r} dominates, and the logarithmic loss is not important. For $r > 1$, by using sharper methods it should be possible to remove the logarithmic loss. For $r = 1$ it might not be possible to remove it. One can also notice that randomization violates arithmetic progression structure in a way that we have M^{3-r} instead of having M^3 for the L^4 estimate.

We also estimate the average values of L^6-norm and L^n-norm, $n \in 2\mathbb{N}$ of exponential sums randomized by the Poisson processes and obtain the following results.

Theorem 1.7 *Let $\{N(t)\}_{t\geq 0}$ be a Poisson process of intensity* 1. *Let $d > 1$ be an integer. Let $A \subset \mathbb{N}$ be a finite nonempty subset. Then we have*

$$|A|^3 \lesssim \mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_6^6 \lesssim_{d,\varepsilon} \begin{cases} |A|^4 \log^2(1+|A|) \textit{ for } d = 2 \\ |A|^{7/2+\varepsilon} \textit{ for } d = 3 \\ |A|^{3+\varepsilon} \textit{ for } d = 4 \\ |A|^3 \textit{ for } d \geq 5. \end{cases} \tag{1.35}$$

Theorem 1.8 *Let $\{N(t)\}_{t\geq 0}$ be a Poisson process of intensity* 1. *Let $d \geq n$ be an integer and $n \in \mathbb{N}$. Let $A \subset \mathbb{N}$ be a finite nonempty subset. Then we have*

$$\mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_{2n}^{2n} \lesssim_{d,n} \begin{cases} \max\{|A|^n, |A|^{2n-\frac{d}{2}-1} \log^{1+\frac{1}{d}}(1+|A|)\} & \textit{if} \quad d \equiv 2 \pmod 4, \\ \max\{|A|^n, |A|^{2n-\frac{d}{2}-1} \log^{\frac{1}{d}}(1+|A|)\} & \textit{else}. \end{cases} \tag{1.36}$$

Let us now compare the consequences of Theorem 1.8 with deterministic results. (Salberger and Wooley, 2010, 317–320) achieve the bound $|A|^{n+\varepsilon}$ for $d \geq (2n)^{4n}$, while on an average sense we achieve the same bound for $d \geq 2n-2$. For the particular $n = 3$ case, in (Salberger, 2005, 93) the bound is $|A|^{3+\varepsilon}$ for $d > 25$, while we obtain it as soon as $d > 3$. The refinement over exponent $7/2$ in (Salberger, 2005, 94–95) happens only for $d > 8$. Notice also that our set A is arbitrary and not limited to just the natural numbers up to M.

CHAPTER 2

PRELIMINARIES

In this chapter, we deploy significant tools, definitions and results, which will be helpful for the rest of this thesis.

2.1. L^p space on the algebraic torus

We define the algebraic torus $\mathbb{T}$ as a factor group of the additive group of real numbers relative to the subgroup $\mathbb{Z}$ and it is given by

$$\mathbb{T} = \mathbb{R}/\mathbb{Z} := \{x + \mathbb{Z} | x \in \mathbb{R}\}.$$

As an illustration, we can consider the algebraic torus $\mathbb{T}$ as a circle, that is, we take the interval $[0, 1]$ bend it round and connect its end points. Note that, it is also possible to take another closed interval of length one as a model of $\mathbb{T}$. Observe that, there exists an identification between functions on $\mathbb{T}$ and 1-periodic functions on $\mathbb{R}$. This allow us to carry some concepts such as continuity, differentiability, integrability and so on. The Lebesgue measure on $\mathbb{T}$ is obtained by restricting the Lebesgue measure on $\mathbb{R}$ to the interval $[0, 1]$.

Let f be a complex-valued measurable function on $\mathbb{T}$. L^p-norm of f on $\mathbb{T}$ is defined by

$$||f||_{L^p(\mathbb{T})} = ||f||_p := \left(\int_{\mathbb{T}} |f|^p dt \right)^{1/p}$$

for $1 \leq p < \infty$. If $p = \infty$, then L^∞-norm is defined by

$$||f||_{L^\infty(\mathbb{T})} = ||f||_\infty := \operatorname{ess\,sup}_x |f(x)|.$$

The space $L^p(\mathbb{T})$ consists of complex-valued functions provided that $||f||_p < \infty$. Here, we need to pay attention to a technical point. If $||f||_p = ||g||_p$, then we do not have $f = g$.

Indeed we have $f = g$ almost everywhere instead. We define an equivalence relation

$$f \sim g \Longleftrightarrow f = g \quad \text{almost everywhere}$$

to get rid of this technical issue.

2.1.1. Some inequalities in $L^p(\mathbb{T})$

In this subsection we present inequalities, which appear in this thesis.

Theorem 2.1 (Minkowski's inequality) *Let $1 \leq p < \infty$ and $f, g \in L^p(\mathbb{T})$. Then we have $f + g \in L^p(\mathbb{T})$, and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Let us give the definition of dual exponents before we proceed. Let $1 \leq p, q \leq \infty$. Two exponents p and q are said to be **dual** if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Theorem 2.2 (Hölder's inequality) *Let $1 \leq p < \infty$ and q be the dual exponent of p. Then we have $fg \in L^1(\mathbb{T})$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proposition 2.1 *Let $0 < p < q \leq \infty$ then $L^q(\mathbb{T}) \subset L^p(\mathbb{T})$, and*

$$\|f\|_p \leq \|f\|_q. \tag{2.1}$$

Proof For $q = \infty$ we plainly have

$$\|f\|_p^p = \int_{\mathbb{T}} |f|^p dx \leq \|f\|_\infty^p \int_{\mathbb{T}} dx = \|f\|_\infty^p.$$

For $q < \infty$ we have by Hölder's inequality

$$\|f\|_p^p = \int_{\mathbb{T}} |f|^p \cdot 1 dx \leq \int_{\mathbb{T}} (|f|^p)^{\frac{q}{p}} dx \int_{\mathbb{T}} 1^{\frac{q}{q-p}} dx = \|f\|_q^p.$$

This concludes the proof.

□

Proposition 2.2 *Let $f, g \in L^2(\mathbb{T})$. Then we have*

- ***(Parseval's relation)***
$$\int_{\mathbb{T}} f(x)\overline{g(x)}dx = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}.$$
- ***(Parseval's identity)***
$$||f||_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2.$$

Remark 2.1 *The first two inequalities remain valid for general measure spaces, but Proposition 2.1 is valid only for finite measure spaces.*

We refer to (Loukas, 2014) for more knowledge.

2.2. Overview of probability theory

The sample space Ω consists of all possible outcomes of an experiment. An element of Ω is denoted by ω. Any subset of the sample space is called an event. A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space, where $\mathcal{F}$ is a σ-algebra on Ω.

Definition 2.1 *A function $\mathbb{P} : \mathcal{F} \to [0, 1]$ is called a probability measure if the following conditions hold:*

(i) $\mathbb{P}(\Omega) = 1$,

(ii) If the sets $B_1, B_2, \cdots \in \mathcal{F}$ are mutually disjoint then
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(B_i).$$

We have by a series of observations

- $\mathbb{P}(\emptyset) = 0$,
- if $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$ for any $A, B \in \mathcal{F}$,
- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ for any $A \in \mathcal{F}$,

- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ for any $A, B \in \mathcal{F}$.

Definition 2.2 *Two events A and B are said to be independent if*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Definition 2.3 (Conditional probability) *The conditional probability of A given B is defined by*

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cup B)}{\mathbb{P}(B)},$$

where $\mathbb{P}(B) > 0$.

One of the significant objects in the theory of probability is a random variable defined as follows.

Definition 2.4 *A random variable is a function that maps elements from a sample space* Ω *to the real numbers* $\mathbb{R}$, *and is denoted by* $X : \Omega \to \mathbb{R}$.

There exist two types of random variables based on their continuity, as described below.

- A random variable X is said to be **discrete** if it takes countable number of values. Its probability mass function is defined by $f_X(x) = \mathbb{P}(X = x)$.

- A random variable X is said to be **continuous** if there is a continuous nonnegative function f_X such that $\int_{-\infty}^{\infty} f_X(x)dx = 1$ and for any $a \leq b$ we have

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx.$$

  f_X is referred to as the probability density function of X.

Definition 2.5 *Two random variables* X_1 *and* X_2 *are said to be* ***independent*** *if*

$$\mathbb{P}(X_1 \in A_1, X_2 \in A_2) = \mathbb{P}(X_1 \in A_1)\mathbb{P}(X_2 \in A_2)$$

for all sets $A_1, A_2 \subseteq \mathbb{R}$.

We define the cumulative distribution function associated with a random variable X as follows.

Definition 2.6 *The cumulative distribution function $F_X : \mathbb{R} \to [0,1]$ is defined by*

$$F_X(x) = \mathbb{P}(X \leq x).$$

Theorem 2.3 *(Wasserman, 2013, Theorem 2.8) A function F is mapping the real line to $[0,1]$ is a CDF for some probability $\mathbb{P}$ if and only if F satisfies the following three conditions:*

(i) F is non-decreasing: $x_1 < x_2$ implies that $F(x_1) \leq F(x_2)$.

(ii) F is normalized:

$$\lim_{x \to -\infty} F(x) = 0,$$

and

$$\lim_{x \to \infty} F(x) = 1.$$

(iii) F is right-continuous: $F(x) = F(x^+)$ for all x, where

$$F(x^+) = \lim_{\substack{y \to x \\ y > x}} F(y).$$

Remark 2.2 *If two random variables X and Y have the same cumulative distribution function then they are called identically distributed.*

We now give fundamental distributions, which are closely related to our study.

Definition 2.7 (Poisson distribution) *A random variable X is said to have a Poisson distribution with intensity $\lambda > 0$ denoted by $X \sim Poisson(\lambda)$ if the probability density function of it is of the form*

$$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}, \; x \geq 0.$$

Definition 2.8 (Binomial distribution) *Let X be a random variable. The mass function of getting exactly x successes in n independent Bernoulli trials is given by*

$$f(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0, 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Then we say that X has a Binomial distribution, denoted as $X \sim Binomial(n, p)$.

The average value of a random variable X is called the expected value of X. We now give explicit definition of it.

Definition 2.9 *The expected value of a random variable X is given as follows:*

- *If X is discrete, then $\mathbb{E}[X] = \sum x\mathbb{P}(X = x) = \sum xf_X(x)$.*
- *If X is continuous, then $\mathbb{E}[X] = \int xf_X(x)dx$.*

Definition 2.10 (Conditional expectation) *Let X and Y be two random variables. The conditional expectation of X given $Y = y$ is as follows:*

- *If they are discrete, then $\mathbb{E}[X = x|Y = y] = \sum xf_{X|Y}(x|y)$.*
- *If they are continuous, then $\mathbb{E}[X|Y = y] = \int xf_{X,Y}(x, y)/f_Y(y)dx = \int xf_{X|Y}(x|y)dx$, where $f_{X,Y}(x, y)$ is the joint density function.*

Definition 2.11 *Let X be a random variable. The variance of X is defined by*

$$\sigma^2 = \mathbb{V}[X] = \mathbb{E}[X - \mathbb{E}(X)]^2.$$

*The square root of the variance is called **standard deviation**.*

Let X and Y be two random variables. These enjoy with the following properties:

- Linearity: $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$ for all $a, b \in \mathbb{R}$.
- Monotonicity: If $X \leq Y$, then $\mathbb{E}[X] \leq \mathbb{E}[Y]$.
- $|\mathbb{E}[X]| \leq \mathbb{E}[|X|]$.
- Let g be a function.
  - If X is discrete, then $\mathbb{E}[g(X)] = \sum g(x)f_X(x)$,
  - if X is continuous, then $\mathbb{E}[g(X)] = \int g(x)f_X(x)dx$.

Definition 2.12 (Filtration) *For a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration is a family of sub-σ-algebras of $\mathcal{F}$ given by $\mathbb{F} := (\mathcal{F}_i)_{i\in I}$ where I is an index set. Moreover, if $(X_n)_{n\in\mathbb{N}}$ is a sequence of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then $\mathcal{F}_n := \sigma(X_k|k \leq n)$ is a σ-algebra generated by $X_1, X_2, \ldots, X_n$ and $\mathbb{F} := (\mathcal{F}_i)_{n\in\mathbb{N}}$ is called a **natural filtration**.*

Definition 2.13 *A sequence of random variables* $(X_n)_{n\geq 0}$ *is said to be* ***adapted*** *if* $X_n \in \mathcal{F}_n$ *for all* n, *where* $(\mathcal{F}_n)_{n\geq 0}$ *is a filtration.*

Definition 2.14 *A sequence of adapted random variables* $\{X_n\}_{n\geq 0}$ *with* $\mathbb{E}(|X_n|) < \infty$ *for all* n *is said to be a* ***martingale*** *if*

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$$

for all n.

Definition 2.15 *A sequence of adapted random variables* $\{X_n\}_{n\geq 0}$ *with* $\mathbb{E}(|X_n|) < \infty$ *for all* n *is said to be a* ***submartingale*** *if*

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$$

for all n.

2.2.1. Some inequalities in probability theory

We introduce well known inequalities, which are necessary for our work. We start with the most fundamental inequality known as Markov's inequality.

Theorem 2.4 (Markov's inequality) *(Wasserman, 2013, Theorem 4.1) Let* X *be a non-negative random variable and suppose that* $\mathbb{E}(X)$ *exists. For any* $t > 0$,

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}(X)}{t}.$$

The following inequality is similar in spirit to Markov's inequality but it is a quite general one.

Theorem 2.5 *Let* h *be a nonnegative nondecreasing function with* $h(a) > 0$ *and* $\mathbb{E}(h(X))$ *exists. We have*

$$\mathbb{P}(X > a) \leq \frac{\mathbb{E}(h(X))}{h(a)}.$$

Proof We have by Markov's inequality

$$\mathbb{P}(X > a) = \mathbb{P}\big(h(X) > h(a)\big) \leq \frac{\mathbb{E}\big(h(X)\big)}{h(a)}.$$

$\square$

This result leads us to another important inequality.

Theorem 2.6 (Chebyshev's inequality) *(Wasserman, 2013, Theorem 4.2) Let X be a non-negative random variable and suppose that* $\mathbb{E}(X)$ *exists. For any* $t > 0$,

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}(X)}{t} \qquad \text{and} \qquad \mathbb{P}(|Z| \geq k) \leq \frac{1}{k^2}$$

where $Z(X - \mu)/\sigma$. *In particular,* $\mathbb{P}(|Z| \geq 2) \leq 1/4$ *and* $\mathbb{P}(|Z| \geq 3) \leq 1/9$.

Most of you are familiar with Cauchy-Schwarz inequality from analysis courses. Here, we give a version related to probability theory.

Theorem 2.7 (Cauchy-Schwarz inequality) *(Wasserman, 2013, Theorem 4.8) If X and Y have finite variances then*

$$\mathbb{E}|XY| \leq \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}.$$

Theorem 2.8 (Doob's martingale inequality) *(Revuz and Yor, 1999, Corollary 1.6) If X is a martingale or a positive submartingale indexed by finite set* $(0, 1, \ldots, N)$ *then for every* $p \geq 1$ *and* $\lambda > 0$

$$\lambda^p \mathbb{P}\left[\sup_n |X_n| \geq \lambda\right] \leq \mathbb{E}[|X_N|^p],$$

and for any $p > 1$,

$$\mathbb{E}[|X_N|^p] \leq \mathbb{E}\left[\sup_n |X_n|^p\right] \leq \left(\frac{p}{p-1}\right)\mathbb{E}[|X_N|^p].$$

2.3. Stochastic processes

A collection of random variables indexed by a subset of real numbers is called a stochastic process(or a random process). Possible values that the random variables take in a stochastic process is called the state space of the process. Index sets $\{n | n \in \mathbb{N}\}$ and $\{t | t \geq 0\}$ are called discrete and continuous, respectively. In general, the indices n and t are considered as "time". Now we deploy stochastic processes along with their properties, which are relevant to our study.

2.3.1. Random walks

Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of independent identically distributed random variables. A **random walk** $R(j)$ started at $R(0) = x_0 \in \mathbb{R}$ is a stochastic process defined by

$$R(j) := \sum_{i=1}^{j} X_i.$$

A random walk on $\mathbb{Z}$ is said to be simple if $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = -1) = q = 1 - p$ where $i \in \mathbb{N}$ and $0 < p < 1$, and the starting point of the walk is 0. If we take $p = 1/2$, then the simple random walk is called symmetric. It is called asymmetric otherwise. We can imagine a simple random walk as follows. Let us consider an object at the point 0 in the following figure.

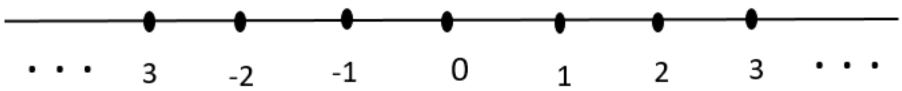

Figure 2.1.: An example of a simple random walk

This object is moved to the right by one unit with probability p and to the left by one unit with probability $q = 1 - p$. Location of an object after j moves is actually the simple random walk $R(j)$.

For a simple random walk, let an abject be moved j steps. Let j_r and j_l denote the number of movements to the right and to the left, respectively. Then we have

$$j_r + j_l = j. \tag{2.2}$$

If k is the final position of the object after j steps, then we also have

$$j_r - j_l = k. \tag{2.3}$$

Combining (2.2) and (2.3) gives $j_r = (j + k)/2$ and $j_l = (j - k)/2$. It shows that a simple random walk has binomial distribution, that is

$$\mathbb{P}(R(j) = k) = \binom{j}{(k+j)/2} p^{(k+j)/2} q^{(j-k)/2},$$

where j and k possess the same parity, $-j \leq k \leq j$. If j and k do not have the same parity, then $\mathbb{P}(R(j) = k) = 0$.

For a simple symmetric random walk if an object is moved $2j$ steps then it needs exactly j "+1" steps and j "−1" steps to come back to the origin. Then we have

$$\mathbb{P}(R(2j) = 0) = \binom{2j}{j} 2^{-2j}. \tag{2.4}$$

If the final location is $2k$, then

$$\mathbb{P}(R(2j) = 2k) = \binom{2j}{j+k} 2^{-2j}. \tag{2.5}$$

Remark 2.3 *A d-dimensional analogue of a random walk on $\mathbb{R}^d$ can be constructed as well.*

Let us now calculate the expected value and the variance of a simple symmetric random walk. Observe that we clearly have

$$\begin{aligned} \mathbb{E}[R(j)] = \mathbb{E}\left[\sum_{i=1}^{j} X_i\right] &= \sum_{i=1}^{j} \mathbb{E}[X_i] \\ &= \sum_{i=1}^{j} (\frac{1}{2} + \frac{1}{2} - 1) \\ &= 0. \end{aligned} \tag{2.6}$$

Owing to (2.6) it is reasonable to calculate average distance, i.e. $\mathbb{E}|R(j)|$. Since it is hard to deal with $\mathbb{E}|R(j)|$ we calculate $\mathbb{E}[R(j)]^2$ instead.

$$
\begin{aligned}
\mathbb{E}[R(j)]^2 &= \mathbb{E}\left[\left(\sum_{i=1}^{j} X_i\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^{j}\sum_{k=1}^{j} X_k X_i\right] \\
&= \sum_{i=1}^{j}\sum_{k=1}^{j} \mathbb{E}[X_k X_i] \\
&= \sum_{k\neq i} \underbrace{\mathbb{E}[X_k X_i]}_{=0} + j \\
&= j.
\end{aligned}
\tag{2.7}
$$

Using (2.6) and (2.7) we have $\mathbb{V}[R(j)] = \mathbb{E}[R(j)]^2 - (\mathbb{E}[R(j)])^2 = j$. We also infer from (2.7) that $\mathbb{E}[|R(j)|] \approx \sqrt{j}$, and yet it can be shown rigorously.

Lastly, we will present the following lemma in this subsection.

Lemma 2.1 *A simple symmetric random walk is a martingale.*

Proof It is sufficient to show that $\mathbb{E}[R(j+1)|\mathcal{F}_n] = R(j)$, where $\mathcal{F}_n = \sigma(X_k|k \leq n)$. We have by linearity of conditional expectation

$$
\mathbb{E}[R(j+1)|\mathcal{F}_n] = \mathbb{E}[R(j) + X_{n+1}|\mathcal{F}_n] = \mathbb{E}[R(j)|\mathcal{F}_n] + \mathbb{E}[X_{n+1}|\mathcal{F}_n]. \tag{2.8}
$$

Since X_n is known in F_n we have $\mathbb{E}[R(j)|\mathcal{F}_n] = R(j)$. Therefore (2.8) becomes

$$
\mathbb{E}[R(j+1)|\mathcal{F}_n] = R(j) + \mathbb{E}[X_{n+1}|\mathcal{F}_n]. \tag{2.9}
$$

Since X_{n+1} is independent of $\mathcal{F}_n$ we also have $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_{n+1}] = 0$. By plugging this into (2.9) we obtain $\mathbb{E}[R(j+1)|\mathcal{F}_n] = R(j)$. Hence we are done. □

2.3.2. Counting processes

A counting process is a type of stochastic process that counts number of events that have occurred up to time t. More precisely a stochastic process $\{N(t)\}_{t\geq 0}$ is said to be a **counting process** if it satisfies the following conditions:

i) $N(t)$ is nonnegative for all $t \geq 0$,

ii) $N(t)$ is integer valued for all $t \geq 0$,

iii) if $s \leq t$, then $N(s) \leq N(t)$.

2.3.3. Poisson processes

One of the most important counting processes is a Poisson process. This process is used in simulations to count the number of arrivals that occurs in some time interval at certain intensity. Nevertheless, exact time of arrivals remains unknown. Let us give the following illustration related to this process.

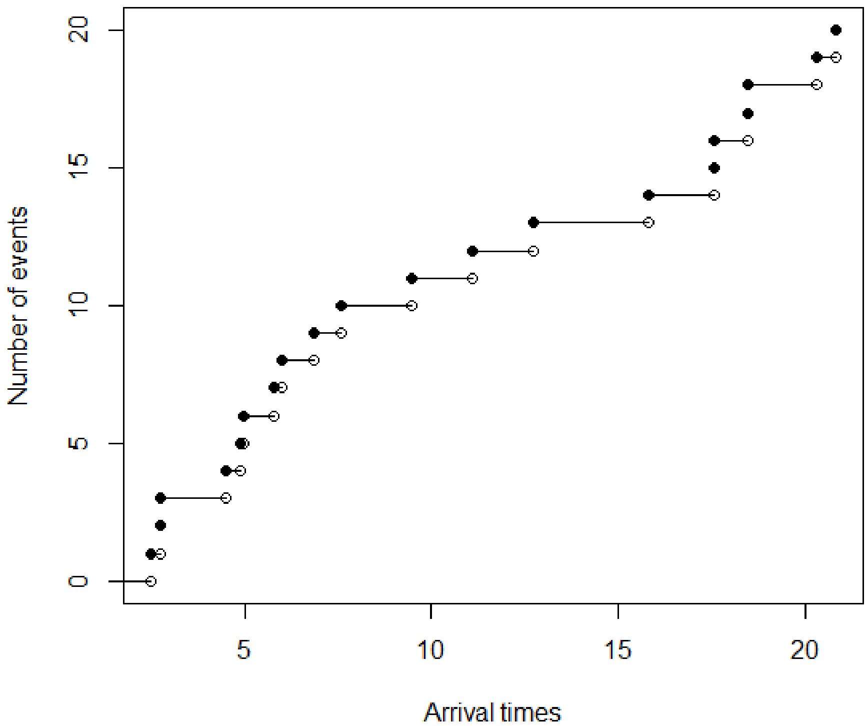

Figure 2.2.: An example of a Poisson process

Formally, it can be defined as follows. A counting process $\{N(t)\}_{t\geq 0}$ is said to be a **Poisson process** with intensity λ, $\lambda > 0$ if

i) $N(0) = 0$ almost surely.

ii) This process has independent increments, that is,

$$N(t_n) - N(t_{n-1}), N(t_{n-1}) - N(t_{n-2}), \ldots, N(t_1)$$

are independent random variables for $0 \le t_1 \le \cdots \le t_n$.

iii) The random variable $N(t) - N(s)$ is a Poisson random variable for every $0 \le s < t < \infty$.

Let us explicitly calculate the expected value and the variance of a Poisson process.

- The expected value of a Poisson process with intensity λ is

$$\begin{aligned}
\mathbb{E}[N(t)] = \sum_{n=0}^{\infty} n e^{-\lambda t} \frac{(\lambda t)^n}{n!} &= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{(n-1)!} \\
&= e^{-\lambda t} \lambda t \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} \\
&= e^{-\lambda t} \lambda t e^{\lambda t} \\
&= \lambda t.
\end{aligned}$$

- The variance of a Poisson process with intensity λ is

$$\begin{aligned}
\mathbb{V}[N(t)] = \mathbb{E}[N(t)^2] - (\mathbb{E}[N(t)])^2 &= \mathbb{E}[N(t)(N(t)-1)] + N(t)] - (\lambda t)^2 \\
&= \mathbb{E}[N(t)(N(t)-1)] + \mathbb{E}[N(t)] - (\lambda t)^2 \\
&= (\lambda t)^2 + \lambda t - (\lambda t)^2 \\
&= \lambda t.
\end{aligned}$$

2.3.4. Stationary processes

A stochastic process is said to be a stationary process if its statistical properties remain unchanged over time. We now provide a formal definition for both continuous-time and discrete-time versions.

(i) A continuous-time stochastic process $\{X(t)\}_{t \in I}$, where I is an index set, is called a stationary process if the cumulative distribution function remains the same under translation, that is

$$F_{X(t)}(x) = F_{X(t+t')}(x) \quad \text{for every} \quad t, t+t' \in I.$$

(ii) A discrete-time stochastic process $\{X(n)\}_{n \in I}$, where I is an index set consisting of integers, is called a stationary process if the cumulative distribution function remains the same under translation, that is

$$F_{X(n)}(x) = F_{X(n+s)}(x) \quad \text{for every} \quad n, n+s \in I.$$

Lastly, we give the following illustration related to this process.

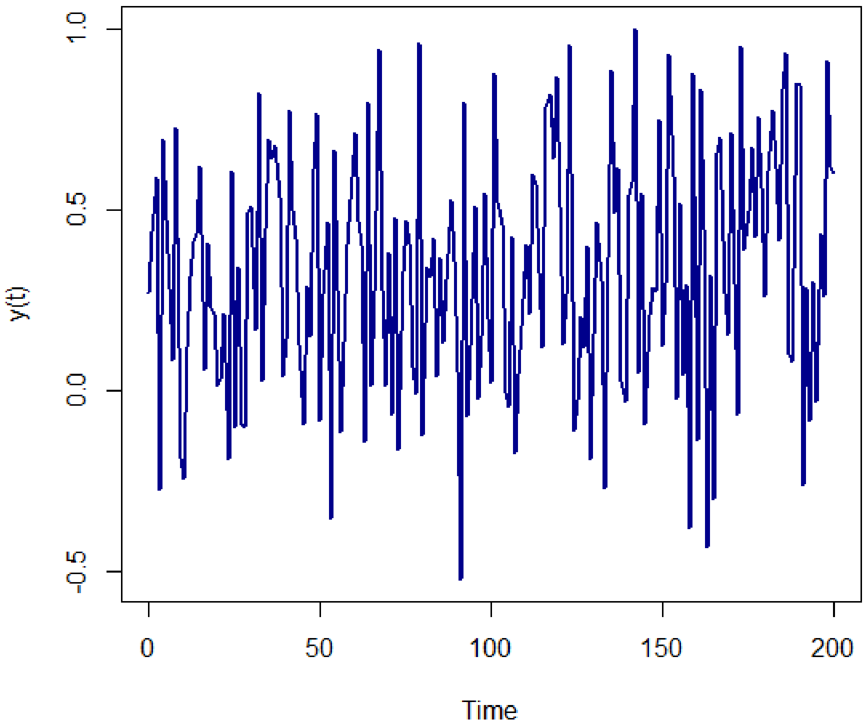

Figure 2.3.: An example of a stationary process

2.4. The sparsity of the Green-Ruzsa set

In this section, our goal is to measure how sparse the Green-Ruzsa set is.

Proposition 2.3 *The Green-Ruzsa set satisfies the sparsity condition*

$$\left|\Lambda_{D,k} \cap [n-M, n+M]\right| \leq 24M^{\frac{\log 3}{\log D}}$$

for any $M \in \mathbb{N}, n \in \mathbb{Z}$.

Proof Without loss of generality we can assume that $n \geq 0$. If the set in question has less than 2 elements, there is nothing to do. Therefore, we may assume that it has at least two elements. Let a be its least element and b be its largest element. Then we may write $m \leq k$

$$b - a = \sum_{j=0}^{m-1} c_j D^j \quad -3 \leq c_j \leq 3$$

with $c_{m-1} > 0$ as $b - a > 0$. But observe that

$$\begin{aligned}
(M+n) - (n-M) = 2M \geq b - a &\geq D^{m-1} + \sum_{j=0}^{m-2} c_j D^j \\
&\geq D^{m-1} - 3\sum_{j=0}^{m-2} D^j \\
&= D^{m-1} - 3\frac{D^{m-1}-1}{D-1} \\
&\geq D^{m-1} - \frac{3}{4}D^{m-1} \\
&= \frac{D^{m-1}}{4}.
\end{aligned}$$

By using this we obtain

$$8M \geq D^{m-1} \implies \log(8M) \geq (m-1)\log D \implies \frac{\log(8M)}{\log D} + 1 \geq m.$$

By changing the first k digits of a when written in base D we might obtain elements of Λ not exceeding b, but we cannot change higher digits even if they exist, for numbers thus obtained will certainly lie outside $[a,b]$. Therefore there are at most 3^m elements of Λ in $[a,b]$, which means

$$\begin{aligned}\left|\Lambda_{D,k} \cap [n-M, n+M]\right| = \left|\Lambda_{D,k} \cap [a,b]\right| \leq 3^m &\leq 3^{1+\frac{\log 8M}{\log D}} \\ &= 3 \cdot 3^{\log_D 8M} \\ &= 3(8M)^{\frac{\log 3}{\log D}} \\ &\leq 24M^{\frac{\log 3}{\log D}}.\end{aligned}$$

□

2.5. Arithmetic Problem

In this section, we aim to estimate the cardinality of the following expression

$$\sup_{D \leq C \leq D^2} \left|\{(j,k) \in A^2 : j > k,\ |j^d - k^d - C| < D\}\right|, \tag{2.10}$$

for large C, D and suitable A. This set appears in the proof of Theorem 1.3, Theorem 1.7 and Theorem 1.8, respectively. We can see that (2.10) is clearly dominated by

$$\sup_{D \leq C \leq D^2} \left|\{(j,k) \in \mathbb{N}^2 : j > k,\ |j^d - k^d - C| < D\}\right|. \tag{2.11}$$

We can estimate (2.11) either directly, or by viewing it as a margin of the set

$$\{(j,k) \in \mathbb{N}^2 :\ 0 < j^d - k^d < D\}. \tag{2.12}$$

Fortunately, we do have estimates on sets of this type in number theory literature. Actually they are some of the oldest and most famous problems in number theory. We first give a historical background on these problems, and at the end state the result, which will be utilized to find an estimate on (2.11) .

We begin with the Dirichlet divisor problem. The divisor summatory function $D(x)$ is defined as

$$D(x) := \sum_{n \leq x} d(n),$$

where $d(n)$ is the number of positive divisors of n. (Dirichlet, 1849, 69–72) showed that

$$D(x) = x \log x + (2\gamma - 1)x + \Delta(x), \qquad \Delta(x) = O(\sqrt{x}),$$

where γ is the Euler-Mascheroni constant

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.57721\ldots.$$

The next step in the Dirichlet divisor problem is to make improvements on the bound of the error term $\Delta(x)$. The Dirichlet summatory function can be written in the following form:

$$D(x) = \sum_{n\le x} d(n) = \sum_{0<ab\le x} 1 = \{(a, b) \in \mathbb{N}^2 : ab \le x\}. \tag{2.13}$$

In this form it turns into a lattice point problem. This problem has been studied intensely for two centuries. Dirichlet's result can be obtained by carefully rearranging the lattice point sum. (Voronoi, 1903, 243) obtained $O(x^{1/3} \log x)$. This exponent $1/3$ represents an important milestone, in that it can still be obtained by relatively easy methods, and yet is very difficult to significantly improve. It was shown in (Hardy, 1915, 263–265) and (Hardy, 1917, 1–5) that $\Delta(x) = O(x^{\theta})$ is not possible for $\theta < 1/4$. Using exponential sum estimates of Weyl, the exponent θ was decreased to $27/82$ in (Van der Corput, 1928, 699–700). This method in time was enhanced and sharpened to obtain the method of exponent pairs. As the method developed, its application to the Dirichlet divisor problem yielded better exponents, but by 50's this method had reached its natural limits, and progress on Dirichlet divisor problem stuck. Then the idea of double exponential sums in (Srinivasan, 1963, 153–172) and (Srinivasan, 1965, 280–311) made appearance and led to another round of improvements for the problem. After this method also ran its course, Bombieri and Iwaniec combined the estimates on exponential sums of Weyl and Van der Corput with the large sieve and Vinogradov mean value theorem to initiate another wave of improvements, which peaked in the work of (Huxley, 2003, 592) proving that $\Delta(x) = O(x^{\theta} \log^{\eta} x)$ with

$$\theta = \frac{131}{416} = 0.3149\ldots, \qquad \eta = 1 + \frac{18627}{8320} = 3.2513\ldots.$$

Progress on the Dirichlet divisor problem once again is stuck after this result, and yet it is expected that we should be able to reduce it all the way to 0.25.

One related problem for which progress almost entirely parallels that of the Dirichlet divisor problem is the Gauss circle problem. This problem searches for estimating the error term for the number of lattice points inside a circle. Let

$$r(n) := |\{(j,k) \in \mathbb{Z}^2 : j^2 + k^2 = n\}|,$$

be the number of lattice points on a circle of radius $\sqrt{n}$. Summing these over

$$R(x) := \sum_{n \leq x} r(n),$$

we obtain the number of lattice points in and on a circle of radius $\sqrt{x}$. Then it is easily seen that

$$R(x) = \pi x + \Upsilon(x), \qquad \Upsilon(x) = O(x^{1/2}).$$

The number of lattice points on a circle is related to the divisors of the radius of the circle. Due to this, this problem is very closely connected to the Dirichlet divisor problem, and the progress on the problem entirely reflects that on the Dirichlet divisor problem.

The Gauss circle problem has been generalized by taking other closed curves instead of circles. One obvious family of such curves is the Lamé curves:

$$r_d^+(n) := |\{(j,k) \in \mathbb{N}^2 : |j|^d + |k|^d = n\}|,$$

where $d \geq 2$.

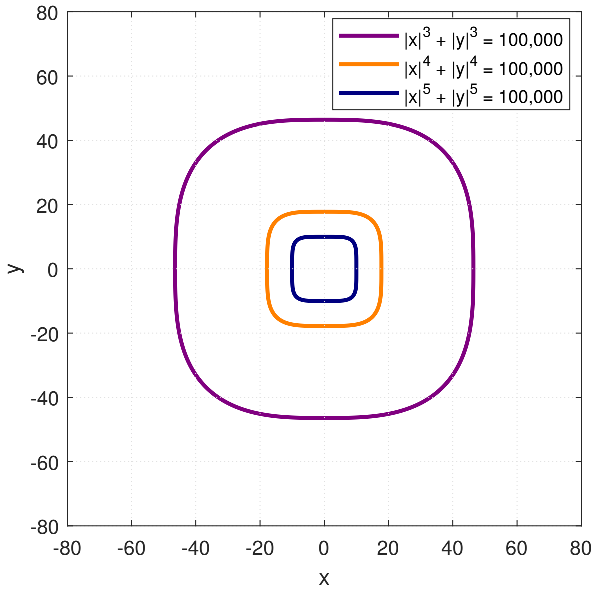

Figure 2.4.: The graph of Lamé curve for $d = 3, 4, 5$ and $n = 100,000$.

Summing these over

$$R_d^+(x) := \sum_{n \leq x} r_d^+(n),$$

we obtain the number of lattice points in and on a Lamé curve. When $d \geq 3$ the behaviour of $R_d^+(x)$ changes drastically owing to existence of points of zero curvature at points where the curve intersects the axes. To understand this issue we turn our look at broader generalizations of the Gauss circle problem.

A very general result of (Steinhaus, 1947, 1–5) is that for a closed continuous curve J with area F and length $l \geq 1$, the number of lattice points in and on this curve G satisfies $|F - G| < l$. Let us pick a fixed such curve and dilate it with a large real number x we observe that the number of lattice points $G(x)$ will satisfy $|G(x) - Fx^2| < lx$. So at this extreme generality the error term is of the order of length of the curve, and the error term satisfies $O(x)$. Let us consider the following example. Let J be a square centered at the origin with the side length of 2. If we multiply J with a large positive integer x, then the number of lattice points on the curve xJ will exactly be $8x$. This example shows that improvement is not possible at this generality.

But if we focus on curves with nonzero curvature then improvement is possible. The reason behind this is that the number of lattice points on a line segment can be proportional to its length. But for a strictly convex curve of length l (Jarník, 1926, 500–

503) showed that the number of lattice points on this curve can not exceed $3(4\pi)^{-1/3}l^{2/3} + O(l^{1/3})$. For algebraic curves and transcendental functions even better bounds are possible. (Bombieri and Pila, 1989, 337–338) obtained the following results:

- For a subset K of a algebraic curve of degree n lying within a square of side length M the number of lattice points on this set is bounded by

$$C(n, \varepsilon)M^{\frac{1}{n}+\varepsilon}$$

  for each $\varepsilon > 0$.

- Let g be a transcendental function on $[0, 1]$ and M be the graph of g. If M is dilated by a factor of x, $x \geq 1$ then the number of lattice points on xM is bounded by

$$C(g, \varepsilon)x^{\varepsilon}$$

  for each $\varepsilon > 0$.

Let S be a compact convex set in $\mathbb{R}^2$ with the origin as an interior point, for which the radius of curvature of the boundary curve is continuous in the direction of the tangent vector. Also assume that the radius curvature has a maximum r_{max} and a minimum r_{min} satisfying $0 < r_{min} \leq r_{max} < \infty$. (Van der Corput, 1920, 1–5) showed that the number of lattice points $S(x)$ in the set xS satisfies

$$S(x) = x^2|S| + O_S(x^{2/3}) = x^2|S| + O_S((|S|x^2)^{1/3}).$$

It showed by (Jarník, 1926, 510–517) that this estimate is best possible for plane convex sets. Indeed, there are plane convex sets whose error terms are equal to $\Omega(x^{2/3})$.

So returning to the Lamé curves, these curves' points with zero curvature lead to very different behaviour for lattice points in and on them. In this case for $d \geq 3$ we have, mostly using the formalism of (Nowak, 1998, 421–422),

$$R_d^+(x) = A_d^+ x^{2/d} + D_d^+ F_d^+(x^{1/d})x^{1/d-1/d^2} + \Delta_d^+(x),$$

where the first term comes from the area, and the coefficient A_d^+ is the area inside the Lamé curve $|x|^d + |y|^d = 1$ given by

$$A_d^+ = \frac{2\Gamma^2(1/d)}{d\Gamma(2/d)},$$

the $x^{1/d-1/d^2}$ term comes out of the neighborhood of the points $(\pm x^{1/d}, 0)$ and $(0, \pm x^{1/d})$, where the curvature of the boundary curve disappears and its coefficients are

$$D_d^+ = 2^{3-1/d}\pi^{-1-1/d}d^{1/d}\Gamma\Big(1+\frac{1}{d}\Big), \quad F_d^+(x) = \sum_{n=1}^{\infty} n^{-1-1/d}\sin\Big(2\pi nx - \frac{\pi}{2d}\Big).$$

So

$$|F_d^+(x)| \leq \sum_{n=1}^{\infty} n^{-1-1/d},$$

is a bounded function of x, and therefore

$$D_d^+F_d^+(x^{1/d})x^{1/d-1/d^2} = O(x^{\frac{1}{d}-\frac{1}{d^2}}), \quad D_d^+F_d^+(x^{1/d})x^{1/d-1/d^2} = \Omega(x^{\frac{1}{d}-\frac{1}{d^2}}).$$

But we also see that we cannot write

$$D_d^+F_d^+(x^{1/d})x^{1/d-1/d^2} = K_d^+x^{\frac{1}{d}-\frac{1}{d^2}} + O(x^{\theta})$$

for a fixed constant K_d^+ depending on d. So an asymptotic expansion for $R_d^+(x)$ cannot be developed beyond this exponent. But at least we know explicitly the terms preventing this. The error term $\Delta_d^+(x)$ is analogous to the error term of the Gauss circle problem, i.e. the case $d = 2$, and applying results of (Huxley, 1996, 100–120) and (Kuba, 1993, 87–95) obtained

$$\Delta_d^+(x) = O(x^{\frac{46}{73}\frac{1}{d}}\log^{\frac{315}{146}} x).$$

Also it is known that, see (Krätzel, 1988, 53–74),

$$\Delta_d^+(x) = \Omega_{\pm}(x^{\frac{1}{2d}}).$$

There are also logarithmic improvements to this lower bound, see (Ivic et al., 2004) for a summary.

Turning back to our set (2.12), estimating the cardinality of this set is a 'hyperbolic 'analogue of the 'elliptic' Lamé curves problem. On the other hand, for the case $d = 2$ the linear transformation $(a, b) = T(k, j) = (j - k, j + k)$ makes this problem essentially equivalent to the Dirichlet divisor problem, see (Kühleitner, 1992, 117–123). Hence, the cardinality of (2.12) for $d = 2$ can be estimated using the bounds obtained for the Dirichlet divisor problem. But for $d \geq 3$ this transformation does not provide any simplification or reduction, and the study of this problem largely proceeds along the analogies with the Lamé curves problem. Indeed again using (Nowak, 1998, 421–422),

$$r_d^-(n) := |\{(j, k) \in \mathbb{Z}^2 : |j|^d - |k|^d = n\}|,$$

where $d \geq 3,\ n \in \mathbb{N}$.

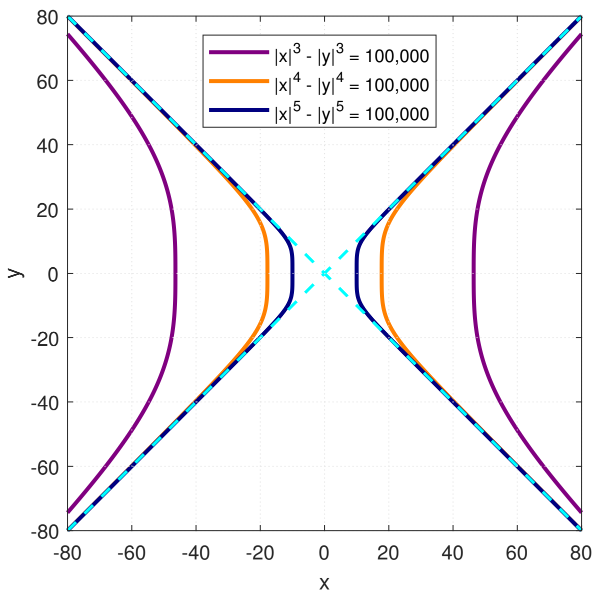

Figure 2.5.: The graph of $|x|^d - |y|^d = n$ for $d = 3, 4, 5$ and $n = 100, 000$.

Summing these over

$$R_d^-(x) := \sum_{n \leq x} r_d^-(n) = |\{(j, k) \in \mathbb{Z}^2 : 0 < |j|^d - |k|^d \leq x\}|,$$

and we have by the work of (Krätzel, 1969, 111–115), (Nowak, 1995, 335–339) and (Nowak, 1998, 421–422)

$$R_d^-(x) = A_d^- x^{2/d} + B_d^- x^{1/(d-1)} + D_d^- F_d^-(x^{1/d}) x^{1/d-1/d^2} + \Delta_d^-(x), \tag{2.14}$$

where the first term comes from the area, the second from the length of the boundary and their coefficients are given by

$$A_d^- = \frac{\Gamma^2(1/d)}{d\cos(\pi/d)\Gamma(2/d)}, \quad B_d^- = 4\zeta(1/(d-1)) d^{-1/(d-1)},$$

the $D_d^- F_d^-(x^{1/d}) x^{1/d-1/d^2}$ term emerges from the neighborhood of the points $(-x^{1/d}, 0)$ and $(x^{1/d}, 0)$ where the curvature of the boundary curve disappears and its coefficients are

$$D_d^- = 2^{2-1/d} \pi^{-1-1/d} d^{1/d} \Gamma\Big(1 + \frac{1}{d}\Big), \quad F_d^-(x) = \sum_{n=1}^{\infty} n^{-1-1/d} \sin\Big(2\pi n x + \frac{\pi}{2d}\Big)$$

So

$$|F_d^-(x)| \leq \sum_{n=1}^{\infty} n^{-1-1/d}$$

is a bounded function of x, and therefore

$$D_d^- F_d^-(x^{1/d}) x^{1/d-1/d^2} = O(x^{\frac{1}{d}-\frac{1}{d^2}}), \quad D_d^- F_d^-(x^{1/d}) x^{1/d-1/d^2} = \Omega(x^{\frac{1}{d}-\frac{1}{d^2}}).$$

But we also see that we cannot write

$$D_k^- F_d^-(x^{1/d}) x^{1/d-1/d^2} = K_d^+ x^{\frac{1}{d}-\frac{1}{d^2}} + O(x^{\theta})$$

for a fixed constant K_d^- depending on d. So an asymptotic expansion for $R_d^-(x)$ cannot be developed beyond this exponent. The error term $\Delta_d^-(x)$ is analogous to the error terms of other problems above. Applying the results in (Huxley, 1993, 279–285), (Huxley, 2003, 47–53), (Nowak, 1995, 335–339) and (Nowak, 1998, 421–422) obtained

$$\Delta_d^-(x) = O(x^{\frac{46}{73}\frac{1}{d}+\varepsilon}), \quad \Delta_d^-(x) = \Omega(x^{\frac{1}{2d}}). \tag{2.15}$$

Plainly, the cardinality of our set (2.12) can be obtained from the equations

$$\begin{aligned}|\{(j,k) \in \mathbb{Z}^2 : 0 < |j|^d - |k|^d \le x\}| = |\{(j,k) \in \mathbb{Z}^2 : 0 < |j|^d - |k|^d < x\}| \\ + |\{(j,k) \in \mathbb{Z}^2 : |j|^d - |k|^d = x\}|\end{aligned}$$

$$\begin{aligned}|\{(j,k) \in \mathbb{Z}^2 : 0 < |j|^d - |k|^d < x\}| = 4|\{(j,k) \in \mathbb{N}^2 : 0 < j^d - k^d < x\}| \\ + 2|\{j \in \mathbb{N} : 0 < j^d < x\}|.\end{aligned}$$

Plainly

$$|\{j \in \mathbb{N} : 0 < j^d < x\}| = x^{1/d} + O(1). \tag{2.16}$$

We also have by considering signs of k and j

$$|\{(j,k) \in \mathbb{Z}^2 : |j|^d - |k|^d = x\}| = 4|\{(j,k) \in \mathbb{N}^2 : j^d - k^d = x\}| + O(1).$$

The set on the right hand side is nonempty, provided that x must be an integer. Since

$$x = j^d - k^d = (j-k)(j^{d-1} + j^{d-2}k + \cdots + k^{d-1}),$$

$j - k$ is a divisor of x. This shows that the number of values of it can not exceed $2d(x)$ different values. Let $a = j - k$ be fixed . We then have

$$x = j^d - k^d = (k+a)^d - k^d,$$

where $a \neq 0$. By the binomial expansion the equation above can be written as

$$\begin{aligned} x = j^d - k^d &= (k+a)^d - k^d \\ &= \sum_{n=0}^{d} \binom{d}{n} k^n a^{d-n} - k^d \\ &= \sum_{n=0}^{d-1} \binom{d}{n} k^n a^{d-n}. \end{aligned}$$

This means k is the root of a degree $d-1$ polynomial. If k is fixed, then so is j. Therefore there exist $2(d-1)d(x)$ different pairs (j,k) on this set. We know that the number of divisors of an integer x is quite small, that is

$$d(x) \leq C_\varepsilon x^\varepsilon$$

for any $\varepsilon > 0$. This gives $2(d-1)d(x) \leq C_{d,\varepsilon} x^\varepsilon$. Combining all of these we obtain

$$|\{(j,k) \in \mathbb{Z}^2 : |j|^d - |k|^d = x\}| = O_{d,\varepsilon}(x^\varepsilon). \tag{2.17}$$

Using (2.16), and (2.17) we finally have

$$|\{(j,k) \in \mathbb{N}^2 : 0 < j^d - k^d < x\}| = \frac{1}{4}\Big|\{(j,k) \in \mathbb{Z}^2 : 0 < |j|^d - |k|^d \leq x\}\Big| - \frac{1}{2}x^{1/d} + O_{d,\varepsilon}(x^\varepsilon),$$

which equals by (2.14), and (2.15)

$$\begin{aligned} &\frac{1}{4}\Big[A_d^- x^{2/d} + B_d^- x^{1/(d-1)} + D_d^- F_d^-(x^{1/d}) x^{1/d-1/d^2} + \Delta_d^-(x)\Big] - \frac{1}{2}x^{1/d} + O_{d,\varepsilon}(x^\varepsilon) \\ &= \frac{1}{4}\Big[A_d^- x^{2/d} + B_d^- x^{1/(d-1)} + D_d^- F_d^-(x^{1/d}) x^{1/d-1/d^2}\Big] - \frac{1}{2}x^{1/d} + O_{d,\varepsilon}(x^{\frac{46}{73}\frac{1}{d}+\varepsilon}). \end{aligned} \tag{2.18}$$

Setting $\varepsilon = 1/100d$ in both (2.15), and (2.17) we obtain the following estimation, whose constants depend only on d:

$$= \frac{1}{4}\Big[A_d^- x^{2/d} + B_d^- x^{1/(d-1)} + D_d^- F_d^-(x^{1/d}) x^{1/d-1/d^2}\Big] - \frac{1}{2} x^{1/d} + O_d(x^{\frac{2}{3d}}). \tag{2.19}$$

As we explained the term $D_d^- F_d^-(x^{1/d}) x^{1/d-1/d^2}$ is considered as an error term for our application, therefore

$$= \frac{1}{4}\Big[A_d^- x^{2/d} + B_d^- x^{1/(d-1)}\Big] - \frac{1}{2} x^{1/d} + O_d(x^{1/d-1/d^2}). \tag{2.20}$$

By utilizing this we will deduce a theorem on the set (2.11).

Theorem 2.9 *Let $C, D > 0$ be real numbers, and let $d \geq 3$ be an integer. Then*

$$\sup_{D \leq C \leq D^2} \Big|\{(j,k) \in \mathbb{N}^2 : j > k,\ |j^d - k^d - C| < D\}\Big| \leq C_d D^{2/d}.$$

Proof We separate this estimate into two:

$$\begin{aligned} &\sup_{D \leq C \leq 4D} \Big|\{(j,k) \in \mathbb{N}^2 : j > k,\ |j^d - k^d - C| < D\}\Big| \\ &\sup_{4D \leq C \leq D^2} \Big|\{(j,k) \in \mathbb{N}^2 : j > k,\ |j^d - k^d - C| < D\}\Big|. \end{aligned} \tag{2.21}$$

Since the first estimate is easy to handle we start with this first estimate. It equals

$$\begin{aligned} &\sup_{D \leq C \leq 4D} \Big|\{(j,k) \in \mathbb{N}^2 : j > k,\ C - D < j^d - k^d < C + D\}\Big| \\ &\leq \Big|\{(j,k) \in \mathbb{N}^2 :\ 0 < j^d - k^d < 5D\}\Big|. \end{aligned}$$

By applying (2.18) we immediately bound this by $C_d D^{2/d}$. As for the second estimate in (2.21) we fix a C on the interval $[4D, D^2]$, and then we can write

$$\begin{aligned} &\Big|\{(j,k) \in \mathbb{N}^2 : j > k,\ C - D \leq j^d - k^d < C + D\}\Big| \\ =&\Big|\{(j,k) \in \mathbb{N}^2 : j > k,\ 0 < j^d - k^d < C + D\}\Big| - \Big|\{(j,k) \in \mathbb{N}^2 : j > k,\ 0 < j^d - k^d < C - D\}\Big|. \end{aligned}$$

By applying (2.20) for each of these two expressions

$$\begin{aligned}&\leq \frac{1}{4}\Big[A_d^-[(C+D)^{2/d}-(C-D)^{2/d}]+B_d^-[(C+D)^{1/(d-1)}-(C-D)^{1/(d-1)}]\Big]\\&-\frac{1}{2}[(C+D)^{1/d}-(C-D)^{1/d}]+O_d(C^{\frac{1}{d}-\frac{1}{d^2}}).\end{aligned} \tag{2.22}$$

Using mean value theorem for the functions $f_1(x) = x^{2/d}$, $f_2(x) = x^{1/(d-1)}$ on the interval $[C-D, C+D]$ respectively we have the estimates

$$(C+D)^{2/d}-(C-D)^{2/d} = \frac{4}{d}DC_1^{2/d-1} \lesssim_d DC^{2/d-1},$$

and

$$(C+D)^{1/(d-1)}-(C-D)^{1/(d-1)} = \frac{2}{d-1}DC_2^{1/(d-1)-1} \lesssim_d DC^{1/(d-1)-1}.$$

where $C_1, C_2 \in (C-D, C+D)$. Since exponents of C are negative the right hand sides of these estimates decrease as C increases for $4D \leq C \leq D^2$. We have

$$(C+D)^{2/d}-(C-D)^{2/d} \lesssim_d DC^{2/d-1} \leq DD^{2/d-1} = D^{2/d}$$

$$(C+D)^{1/(d-1)}-(C-D)^{1/(d-1)} \lesssim_d DC^{1/(d-1)-1} \leq DD^{1/(d-1)-1} = D^{1/(d-1)}.$$

Plugging these into (2.22) we obtain

$$\begin{aligned}&\leq \frac{1}{4}[A_d^-D^{2/d}+B_d^-D^{1/(d-1)}]-\frac{1}{2}D^{2/d-1}\\&\leq C_dD^{2/d}.\end{aligned}$$

□

We now turn our look to a special case of this. In this case we obtain some gain in the exponent by fixing $C = D^s$, $1 < s \leq 2$.

Theorem 2.10 *Let $D > 0$ be a real number, and $1 < s \leq 2$. Let $d \geq 3$ be an integer. Then*

$$\left|\{(j,k)\in\mathbb{N}^2 : j>k,\ |j^d-k^d-D^s|<D\}\right| \le \begin{cases} C_d D^{1+s(\frac{2}{d}-1)} & \textit{if } 1<s\le \frac{d^2}{d^2-d-1}, \\ C_d D^{\frac{s}{d}(1-\frac{1}{d})} & \textit{if } \frac{d^2}{d^2-d-1}\le s\le 2. \end{cases}$$

Proof

Let $D^s \ge 2D$ and write

$$\begin{aligned} &\left|\{(j,k)\in\mathbb{N}^2 : j>k,\ D^s-D\le j^d-k^d<D^s+D\}\right| \\ &= \left|\{(j,k)\in\mathbb{N}^2 : j>k,\ 0<j^d-k^d<D^s+D\}\right| - \left|\{(j,k)\in\mathbb{N}^2 : j>k,\ 0<j^d-k^d<D^s-D\}\right|. \end{aligned}$$

By applying (2.20) for each of these two expressions

$$\begin{aligned} &\le \frac{1}{4}\Big[A_d^-[(D^s+D)^{2/d}-(D^s-D)^{2/d}] + B_d^-[(D^s+D)^{1/(d-1)}-(D^s-D)^{1/(d-1)}]\Big] \\ &- \frac{1}{2}[(D^s+D)^{1/d}-(D^s-D)^{1/d}] + O_d(D^{\frac{s}{d}[1-\frac{1}{d}]}). \end{aligned} \tag{2.23}$$

Using mean value theorem for the functions $f_3(x)=x^{2/d}$, $f_4(x)=x^{1/(d-1)}$ on the interval $[D^s-D, D^s+D]$ respectively we have the estimates

$$(D^s+D)^{2/d}-(D^s-D)^{2/d} = \frac{4}{d}DC_4^{2/d-1} \lesssim_d D^{1+s(2/d-1)},$$

and

$$(D^s+D)^{1/(d-1)}-(D^s-D)^{1/(d-1)} = \frac{2}{d-1}DC_5^{1/(d-1)-1} \lesssim_d D^{1+s(1/(d-1)-1)}$$

where $C_4, C_5 \in (D^s-D, D^s+D)$. Plugging these into (2.23) we obtain

$$\begin{aligned} &\le \frac{1}{4}[A_d^- D^{1+s(2/d-1)} + B_d^- D^{1+s(1/(d-1)-1)}] - \frac{1}{2}D^{1+s(1/d-1)} \\ &\le C_d D^{1+s(2/d-1)} + O_d(D^{\frac{s}{d}[1-\frac{1}{d}]}). \end{aligned}$$

We now compare the $D^{1+s(2/d-1)}$ term with the $D^{\frac{s}{d}[1-\frac{1}{d}]}$ error term. Let

$$D^{1+s(2/d-1)} = D^{\frac{s}{d}[1-\frac{1}{d}]}. \tag{2.24}$$

Arranging (2.24) we obtain

$$s = \frac{d^2}{d^2 - d - 1}.$$

If $1 < s \leq \frac{d^2}{d^2-d-1}$, then the $D^{1+s(2/d-1)}$ term dominates, and otherwise the error term.

If $D^s < 2D$ then we apply (2.20) with $x = 3D$ to obtain the set above has cardinality at most $C_d D^{2/d}$. But $D^s < 2D$ leads us to

$$D^{\frac{2}{d}} = DD^{\frac{2}{d}-1} \leq D(D^s/2)^{\frac{2}{d}-1} \leq 2D^{1+s(\frac{2}{d}-1)}.$$

This completes the proof.

□

Theorem 2.11 *Let $C, D > 0$ be real numbers, and let $d \geq 3$ be an integer. Then we have*

$$\sup_{D \leq C \leq D^2} \left|\{(j,k) \in \mathbb{N}^2 : j > k,\ |j^d - k^d - C| < D\}\right| \leq C_d D^{2/d}. \tag{2.25}$$

Proof We divide this estimate into two:

$$\begin{aligned} &\sup_{D \leq C \leq 4D} \left|\{(j,k) \in \mathbb{N}^2 : j > k,\ |j^d - k^d - C| < D\}\right| \\ &\sup_{4D \leq C \leq D^2} \left|\{(j,k) \in \mathbb{N}^2 : j > k,\ |j^d - k^d - C| < D\}\right|. \end{aligned} \tag{2.26}$$

Observe that (2.25) is the maximum of these two estimates. The first estimate is easier to handle. So we begin with the first estimate. It equals

$$\begin{aligned} &\sup_{D \leq C \leq 4D} \left|\{(j,k) \in \mathbb{N}^2 : j > k,\ C - D < j^d - k^d < C + D\}\right| \\ &\leq \left|\{(j,k) \in \mathbb{N}^2 :\ 0 < j^d - k^d < 5D\}\right|. \end{aligned}$$

To count the pairs (j,k), our strategy is to count the number of possible (j,k) for each fixed k. We observe that

$$(j-k)dk^{d-1} < (j-k)(j^{d-1} + j^{d-2}k + \ldots + jk^{d-2} + k^{d-1}) = j^d - k^d < 5D.$$

From this we have

$$k^{d-1} \leq dk^{d-1} < (k-j)dk^{d-1} < 5D. \tag{2.27}$$

(2.27) implies

$$k < 5D^{\frac{1}{d-1}}.$$

For each such k

$$0 < j - k < 5Dk^{1-d}.$$

Therefore for a fixed k there are at most $5D/k^{d-1}$ pairs (j,k). We will utilize this estimate for $k > D^{1/d}$. On the other hand, for $k \leq D^{1/d}$ we have

$$j^d - k^d < 5D \implies j < (k^d + 5D)^{1/d} \implies j < (D + 5D)^{1/d} = 6D^{1/d}.$$

So there are at most $6D^{1/d} + 1$ pairs (j,k) for these k. Combining all of these our set can be bounded by

$$\begin{aligned}
&\left(\sum_{1<k\leq D^{\frac{1}{d}}} + \sum_{D^{\frac{1}{d}}<k<5D^{\frac{1}{d-1}}} \right) \left| \{ j \in \mathbb{N} : \ 0 < j^d - k^d < 5D \} \right| \\
\leq &\sum_{1<k\leq D^{\frac{1}{d}}} 6D^{1/d} + \sum_{D^{\frac{1}{d}}<k<5D^{\frac{1}{d-1}}} 5Dk^{1-d} \\
\leq &6D^{\frac{1}{d}}D^{\frac{1}{d}} + 5D \cdot \sum_{D^{\frac{1}{d}}<k<5D^{\frac{1}{d-1}}} k^{1-d} \\
\leq &6D^{\frac{2}{d}} + 5D \cdot \sum_{D^{\frac{1}{d}}<k<5D^{\frac{1}{d-1}}} k^{1-d}.
\end{aligned}$$

We can estimate this last sum by using the integral

$$\begin{aligned}
D^{\frac{1-d}{d}} + \int_{D^{\frac{1}{d}}}^{\infty} x^{1-d} dx = D^{\frac{1-d}{d}} + \left. \frac{x^{2-d}}{2-d} \right|_{D^{\frac{1}{d}}}^{\infty} &\leq D^{\frac{1-d}{d}} + 0 + \frac{1}{d-2} D^{-1+\frac{2}{d}} \\
&\leq 2D^{-1+\frac{2}{d}}.
\end{aligned}$$

Thus the final bound is

$$6D^{2/d} + 10D^{2/d} \leq 16D^{2/d}.$$

Now we focus on the second estimate in (2.26)

$$\sup_{4D\le C\le D^2} \left|\{(j,k)\in\mathbb{N}^2 : j>k,\ |j^d-k^d-C|<D\}\right|. \tag{2.28}$$

We will count this estimate via differences $b = j-k$ as done in $p = 4$ case. Since $b^d = (j-k)^d < j^d-k^d < C+D$ we must have $1 \le b < (C+D)^{1/d}$. Then for a fixed $C \in [4D, D^2]$ we can decompose (2.28) into two:

$$\begin{aligned}
&\left|\{(j,k)\in\mathbb{N}^2 : C-D < j^d-k^d < C+D\}\right| \\
&= \left(\sum_{1\le b<(C-2D)^{\frac{1}{d}}} + \sum_{(C-2D)^{\frac{1}{d}}\le b<(C+D)^{\frac{1}{d}}}\right)\left|\{k\in\mathbb{N} : C-D<(k+b)^d-k^d<C+D\}\right| \\
&= I + II.
\end{aligned}$$

In order to handle these sets let us introduce the function

$$g(x) := (x+b)^d - x^d - b^d = \sum_{j=1}^{d-1}\binom{d}{j}x^{d-j}b^j, \quad x\in\mathbb{R}.$$

Observe that this function is strictly increasing on $x \ge 0$ with $g(0) = 0$, and thus its restriction to $[0,\infty)$ is a bijection onto $[0,\infty)$. Consequently it has a strictly increasing inverse $g^{-1} : [0,\infty) \to [0,\infty)$. We observe that

$$g(x) \le xg'(x) = \sum_{j=1}^{d-1}\binom{d-1}{j}(d-j)x^{d-j}b^j \le dg(x), \quad x \ge 0.$$

For $x \ge 0$ we have

$$g(x) = \sum_{j=1}^{d-1}\binom{d}{j}x^{d-j}b^j \underset{\text{for } j=1,d-1}{\ge} dx^{d-1}b + dxb^{d-1}, \tag{2.29}$$

and

$$g'(x) \geq g'(0) = db^{d-1}. \tag{2.30}$$

Furthermore from (2.29) we obtain

$$\begin{aligned} g(x) \geq dbx^{d-1} &\implies (g^{-1} \circ g)(x) \geq g^{-1} \circ (dbx^{d-1}) \\ &\implies x \geq g^{-1}(dbx^{d-1}) \\ &\implies \Big(\frac{x}{db}\Big)^{\frac{1}{d-1}} \geq g^{-1}(x), \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} g(x) \geq dxb^{d-1} &\implies (g^{-1} \circ g)(x) \geq g^{-1} \circ (dxb^{d-1}) \\ &\implies x \geq g^{-1}(dxb^{d-1}) \\ &\implies \frac{x}{db^{d-1}} \geq g^{-1}(x). \end{aligned} \tag{2.32}$$

Depending on the sizes x, b usefulness of (2.31) and (2.32) changes. We first estimate the second sum. For fix b. Then

$$\begin{aligned} &\Big|\{k \in A : C - D < (k+b)^d - k^d < C + D\}\Big| \\ =&\Big|\{k \in A : C - D - b^d < g(k) < C + D - b^d\}\Big| \\ \leq&\Big|\{k \in A : 0 < g(k) < C + D - b^d\}\Big|. \end{aligned} \tag{2.33}$$

From (2.32) we obtain

$$g(k) < C + D - b^d \implies k < \frac{C + D - b^d}{db^{d-1}}.$$

Since $k > 0$, and $b^d \geq C - 2D \geq 2D$ the last term of (2.33) is bounded by

$$\begin{aligned} \Big|\{k \in A : 1 \leq k < \frac{C + D - b^d}{db^{d-1}}\}\Big| \leq \frac{C + D - b^d}{db^{d-1}} &\leq \frac{C + D - 2D}{db^{d-1}} \\ &\leq \frac{3D}{db^{d-1}} \\ &\leq \frac{3D}{dD^{\frac{d-1}{d}}} \\ &\leq \frac{3D^{1/d}}{d}. \end{aligned}$$

By our assumption the cardinality of integers b is bounded by

$$\begin{aligned}
3D &= (C+D) - (C-2D) \\
&= \big[(C+D)^{\frac{1}{d}} - (C-2D)^{\frac{1}{d}}\big]\Big[\sum_{j=1}^{d}(C+D)^{\frac{d-j}{d}}(C-2D)^{\frac{j-1}{d}}\Big].
\end{aligned}$$

This implies that

$$\begin{aligned}
3D &= \big[(C+D)^{\frac{1}{d}} - (C-2D)^{\frac{1}{d}}\big]\Big[\sum_{j=1}^{d}(C+D)^{\frac{d-j}{d}}(C-2D)^{\frac{j-1}{d}}\Big] \\
&\underset{\text{for} j=1}{>} \big[(C+D)^{\frac{1}{d}} - (C-2D)^{\frac{1}{d}}\big](C+D)^{\frac{d-1}{d}}.
\end{aligned} \tag{2.34}$$

By (2.34) we have

$$(C+D)^{\frac{1}{d}} - (C-2D)^{\frac{1}{d}} < 3D(C+D)^{\frac{1}{d}-1} < 3D^{\frac{1}{d}}.$$

Thus II is bounded by

$$\begin{aligned}
II &\le (3D^{\frac{1}{d}}+1)\Big(\frac{3D^{1/d}}{d}\Big) \\
&\le \frac{9D^{2/d}}{d} + \frac{3D^{1/d}}{d} \\
&\le 3D^{2/d} + D^{1/d} \\
&\le 4D^{2/d}.
\end{aligned}$$

We turn back to I. Fix b.

$$\begin{aligned}
&\Big|\{k \in \mathbb{N} : C - D < (k+b)^d - k^d < C + D\}\Big| \\
=&\Big|\{k \in \mathbb{N} : C - D - b^d < g(k) < C + D - b^d\}\Big| \\
=&\Big|\{k \in \mathbb{N} : g^{-1}(C - D - b^d) < k < g^{-1}(C + D - b^d)\}\Big|.
\end{aligned}$$

By Mean Value Theorem the cardinality of this set is bounded by

$$\begin{aligned}\left|g^{-1}(C+D-b^d)-g^{-1}(C-D-b^d)\right|&\\ &=(C+D-b^d-C+D+b^d)(g^{-1})'(x)=2D(g^{-1})'(x),\quad (2.35)\end{aligned}$$

for some $C-D-b^d<x<C+D-b^d$. Observe that since both g', g^{-1} are strictly increasing on the positive real axis, $(g^{-1})'$ is strictly decreasing there. Thus (2.35) can be bounded by $2D(g^{-1})'(C-D-b^d)$. By properties of g for $y>0$

$$\begin{aligned}\frac{1}{g(g^{-1}(y))}\geq\frac{1}{g^{-1}(y)g'(g^{-1}(y))}&\implies\frac{1}{y}\geq\frac{1}{g^{-1}(y)g'(g^{-1}(y))}\\ &\implies(g^{-1})(y)=\frac{g^{-1}(y)}{y}.\end{aligned}$$

Combining this with (2.31) and (2.32) we have

$$\begin{aligned}(g^{-1})'(C-D-b^d)&\leq\frac{g^{-1}(C-D-b^d)}{C-D-b^d}\\ &\leq\frac{1}{C-D-b^d}\min\Big[\Big(\frac{C-D-b^d}{db}\Big)^{\frac{1}{d-1}},\frac{C-D-b^d}{db^{d-1}}\Big]\\ &\leq\min\Big[\frac{1}{(C-D-b^d)^{\frac{d-2}{d-1}}(db)^{\frac{1}{d-1}}},\frac{1}{db^{d-1}}\Big].\end{aligned}\qquad(2.36)$$

By using (2.35) we bound I in the following way

$$\begin{aligned}I&\leq\sum_{1\leq b<(C-2D)^{\frac{1}{d}}}1+2D(g^{-1})'(C-D-b^d)\\ &\leq(C-2D)^{\frac{1}{d}}+2D\Big[\sum_{1\leq b<(C-2D)^{\frac{1}{d}}}\frac{g^{-1}(C-D-b^d)}{(C-D-b^d)}\Big].\end{aligned}$$

Since $C\leq D^2$ we have $(C-2D)^{1/d}\leq D^{2/d}$. By using (2.36) we split up the sum in the parenthesis into two:

$$\sum_{1\le b<D^{\frac{1}{d}}} \frac{1}{(C-D-b^d)^{\frac{d-2}{d-1}} b^{\frac{1}{d-1}}} + \sum_{D^{\frac{1}{d}}\le b<(C-2D)^{\frac{1}{d}}} \frac{1}{db^{d-1}}$$
$$\le \frac{2}{(C-D)^{\frac{d-2}{d-1}}}\Big[\sum_{1\le b<D^{\frac{1}{d}}} b^{-\frac{1}{d-1}}\Big] + \frac{1}{d}\Big[\sum_{D^{\frac{1}{d}}\le b<(C-2D)^{\frac{1}{d}}} b^{1-d}\Big].$$

These sums can be estimated by the integrals

$$\begin{aligned}\sum_{1\le b<D^{\frac{1}{d}}} b^{-\frac{1}{d-1}} \le 1 + \int_1^{D^{\frac{1}{d}}} b^{-\frac{1}{d-1}}db &= 1 + \frac{d-1}{d-2}\Big[b^{\frac{d-2}{d-1}}\Big|_1^{D^{\frac{1}{d}}}\Big]\\ &= 1 + \frac{d-1}{d-2}\Big[D^{\frac{d-2}{d(d-1)}} - 1\Big]\\ &\le 2D^{\frac{d-2}{d(d-1)}},\end{aligned}$$

and

$$\begin{aligned}\sum_{D^{\frac{1}{d}}\le b<(C-2D)^{\frac{1}{d}}} b^{1-d} \le D^{\frac{1-d}{d}} + \int_{D^{\frac{1}{d}}}^{\infty} &= D^{\frac{1-d}{d}} + \frac{b^{2-d}}{2-d}\Big|_{D^{\frac{1}{d}}}^{\infty}\\ &D^{\frac{1-d}{d}}\frac{1}{2-d}\Big[0-(D^{\frac{1}{d}}-1)^{2-d}\Big]\\ &\le 2D^{-1+\frac{2}{d}}.\end{aligned}$$

Combining all of these we obtain

$$\begin{aligned}I &\le D^{\frac{2}{d}} + 2D\Big[\frac{4D^{\frac{d-2}{d(d-1)}}}{(C-D)^{\frac{d-2}{d-1}}} + \frac{2}{d}D^{-1+\frac{2}{d}}\Big]\\ &\le D^{\frac{2}{d}} + 8D^{1+\frac{d-2}{d(d-1)}-\frac{d-2}{d-1}} + \frac{2}{d}D^{\frac{2}{d}}\\ &\le D^{\frac{2}{d}} + 8D^{\frac{2}{d}} + 2D^{\frac{2}{d}}\\ &\le 11D^{\frac{2}{d}}.\end{aligned}$$

Taking the maximum of these two estimates we finally obtain

$$\sup_{D\le C\le D^2} \Big|\{(j,k)\in\mathbb{N}^2 : j>k,\ |j^d-k^d-C|<D\}\Big| \le 16D^{2/d}.$$

This concludes the proof.

□

2.6. Auxiliary results

In this section, we give preparatory results most of which are very crucial to prove our main theorems.

Lemma 2.2 *Let* $\sum_{n=1}^{M} e^{2\pi iny}$ *be the Dirichlet kernel . Then we have*

$$\left\| \sum_{n=1}^{M} e^{2\pi iny} \right\|_{L^p(\mathbb{T})} \approx M^{\frac{p-1}{p}}, \tag{2.37}$$

where $1 < p < \infty$.

Proof We first try to obtain an upper bound. For this purpose we divide our integral into two parts:

$$\begin{aligned} \left\| \sum_{n=1}^{M} e^{2\pi iny} \right\|_{L^p(\mathbb{T})}^p &= \int_0^1 \left| \sum_{n=1}^{M} e^{2\pi iny} \right|^p dy \\ &= \int_{|y|\le \frac{1}{M}} \left| \sum_{n=1}^{M} e^{2\pi iny} \right|^p dy + \int_{|y|> \frac{1}{M}} \left| \sum_{n=1}^{M} e^{2\pi iny} \right|^p dy. \end{aligned} \tag{2.38}$$

By using the fact $\left|\sum_{n=1}^{M} e^{2\pi iny}\right| \lesssim \min(M, 1/|y|)$, (2.38) is bounded by

$$\begin{aligned} &\le \int_{|y|\le \frac{1}{M}} M^p dy + \int_{|y|> \frac{1}{M}} \frac{1}{|y|^p} dy. \\ &= 2M^{p-1} + 2\left[\frac{1}{2^{1-p}(1-p)} + \frac{M^{p-1}}{p-1} \right] \\ &= C_p M^{p-1}. \end{aligned}$$

Now, we will obtain a lower bound as follows:

$$\begin{aligned}
\left\|\sum_{n=1}^{M} e^{2\pi i n y}\right\|_{L^p(\mathbb{T})}^{p} = \int_0^1 \left|\sum_{n=1}^{M} e^{2\pi i n y}\right|^p dy \geq \int_0^{1/100M} \left|\sum_{n=1}^{M} e^{2\pi i n y}\right|^p dy \\
\geq \int_0^{1/100M} \left|\sum_{n=1}^{M} \cos(2\pi n y)\right|^p dy \\
\geq \frac{M^p}{2^p} \int_0^{1/100M} 1 dy \\
= \frac{M^{p-1}}{2^p 100}.
\end{aligned} \tag{2.39}$$

Hence we are done.

□

Lemma 2.3 *For any $\varepsilon > 0$, there exists a natural number n_0 such that $n^\varepsilon > \log n$ for all $n \geq n_0$.*

Proof Let $\varepsilon > 0$ be given. We will show that the sequence $\{\log n/n^\varepsilon\}_{n\in\mathbb{N}}$ converges to zero. Observe that

$$\lim_{n\to\infty} \frac{\log n}{n^\varepsilon} = \frac{\infty}{\infty}.$$

By using L'Hôpital's Rule, we have

$$\lim_{n\to\infty} \frac{\log n}{n^\varepsilon} = \lim_{n\to\infty} \frac{1/n}{\varepsilon n^{\varepsilon-1}} = \frac{1}{\varepsilon n^\varepsilon} = 0.$$

Let $\delta = 1$. Then there exits $n_0 \in \mathbb{N}$ such that

$$\log n \leq n^\varepsilon \quad \text{for all} \quad n \geq n_0.$$

Thus we are done. □

Remark 2.4 *An inequality $\log^2 n < n^\varepsilon$ is also true with the same conditions that the previous lemma has.*

Remark 2.5 *For all $\varepsilon > 0$ we have $\log n \in O(n^\varepsilon)$.*

The following lemmas on the Poisson process allow us to make rigorous the heuristic that if we have a Poisson process $\{N(t)\}_{t\geq 0}$ with an intensity 1 then $N(j)$ takes integer values between $j - \sqrt{j}$ and $j + \sqrt{j}$.

Lemma 2.4 *Let N be a Poisson process of intensity 1, and let $m \in \mathbb{N}$. Let $0 < \lambda \leq \sqrt{m}$. Then*

$$\mathbb{P}[|N(m) - m| > \lambda\sqrt{m}] \leq 2e^{-\lambda^2/4}.$$

Proof We start with

$$\begin{aligned}\mathbb{P}[|N(m) - m| > \lambda\sqrt{m}] &= \mathbb{P}[N(m) - m > \lambda\sqrt{m}] + \mathbb{P}[m - N(m) > \lambda\sqrt{m}] \\ &= \mathbf{I} + \mathbf{II}.\end{aligned}$$

We first deal with $\mathbf{I}$. Let $t > 0$.

$$\begin{aligned}\mathbf{I} = \mathbb{P}[t(N(m) - m) > t\lambda\sqrt{m}] &= \mathbb{P}[e^{t(N(m)-m)} > e^{t\lambda\sqrt{m}}] \\ &= \mathbb{P}[e^{t(N(m)-m)-t\lambda\sqrt{m}} > 1] \\ &\leq \mathbb{E}[e^{t(N(m)-m)-t\lambda\sqrt{m}}] \\ &= e^{-t\lambda\sqrt{m}-mt}\mathbb{E}[e^{tN(m)}].\end{aligned}$$

This last average value is indeed the moment generating function of $N(m)$, and can be calculated by using the third property of Poisson processes to be $e^{m(e^t-1)}$. Thus we have

$$\mathbf{I} \leq e^{-t\lambda\sqrt{m}-m+m(e^t-1)} \tag{2.40}$$

Notice that for $0 < t \leq 1$ we have

$$\begin{aligned}e^t = 1 + t + \frac{t^2}{2}\Big[1 + \frac{t}{3} + \frac{t^2}{4\cdot 3} + \cdots\Big] &\leq 1 + t + \frac{t^2}{2}\Big[1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\Big] \\ &= 1 + t + \frac{t^2}{2}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n \\ &= 1 + t + \frac{3}{4}t^2.\end{aligned} \tag{2.41}$$

Plugging (2.41) into (2.40) yields

$$\mathbf{I} \leq e^{-t\lambda\sqrt{m}+m(e^t-t-1)} \leq e^{-t\lambda\sqrt{m}+\frac{3}{4}mt^2}, \qquad 0 < t \leq 1.$$

By taking $t = \lambda/\sqrt{m}$, $\mathbf{I}$ is bounded by

$$e^{-\frac{\lambda}{\sqrt{m}}\lambda\sqrt{m}+\frac{3}{4}m\frac{\lambda^2}{m}} = e^{-\lambda^2/4}. \tag{2.42}$$

By following the exact same steps we also have

$$\mathbf{II} \leq e^{-t\lambda\sqrt{m}+m(e^{-t}-1+t)}. \tag{2.43}$$

Notice that for $0 < t \leq 1$

$$e^t = 1 - t + \frac{t^2}{2} - \Big[\frac{t^3}{3!} - \frac{t^4}{4!}\Big] - \Big[\frac{t^5}{5!} - \frac{t^6}{6!}\Big] \cdots \leq 1 - t + \frac{t^2}{2} \tag{2.44}$$

where we utilized the fact that every expression inside parentheses is positive.

Plugging (2.44) into (2.43) gives

$$\mathbf{II} \leq e^{-t\lambda\sqrt{m}+m\frac{t^2}{2}}.$$

Choosing $t = \lambda/\sqrt{m}$, $\mathbf{II}$ is bounded by

$$e^{-\frac{\lambda}{\sqrt{m}}\lambda\sqrt{m}+m\frac{\lambda^2}{2m}} = e^{-\lambda^2/2}. \tag{2.45}$$

Combining (2.42) and (2.45) the final bound is $2e^{-\lambda^2/4}$, and this completes the proof. □

We now present two lemmas both of which give nontrivial upper bounds to the probability of a Poisson process with intensity 1.

Lemma 2.5 *Let $\{N(t)\}_{t\geqslant 0}$ be a Poisson counting process with intensity* 1. *Let $a \in \mathbb{N}$. Then*

$$\sup_{t\geq 0} \mathbb{P}[N(t) = a] \leq \frac{1}{\sqrt{2\pi a}}.$$

Proof We have

$$\sup_{t\geq 0} \mathbb{P}[N(t) = a] = \sup_{t\geq 0} e^{-t}\frac{t^a}{a!} = \frac{1}{a!}\sup_{t\geq 0} t^a e^{-t}.$$

Our purpose is to find the supremum of a smooth nonnegative function $\beta(t) = t^a e^{-t}$ on $[0, \infty)$. This function vanishes at the endpoints of this interval, that is,

$$\beta(0) = 0^a e^0 = 0 \quad \text{and} \quad \lim_{t\to\infty} t^a e^{-t} = 0.$$

So to find the supremum we differentiate and set it equal to zero.

$$0 = \frac{d}{dt}(t^a e^{-t}) = at^{a-1}e^{-t} - t^a e^{-t} = t^{a-1}e^{-t}(a - t).$$

Thus supremum is attained at $t = a$, and is $a^a e^{-a}$. By a precise version of Stirling's formula due to (Robbins, 1955, 26), $n \in \mathbb{N}$ we have

$$\frac{1}{\sqrt{2\pi n}}e^{-\frac{1}{12n}} \leq \frac{1}{n!}\cdot\frac{n^n}{e^n} \leq \frac{1}{\sqrt{2\pi n}}e^{-\frac{1}{12n+1}} \leq \frac{1}{\sqrt{2\pi n}}. \tag{2.46}$$

By using (2.46) we obtain

$$\sup_{t\geq 0} \mathbb{P}[N(t) = a] = \frac{1}{a!}\sup_{t\geq 0} t^a e^{-t} \leq \frac{1}{a!}\frac{a^a}{e^a} \leq \frac{1}{\sqrt{2\pi a}}.$$

This concludes the proof.

□

Lemma 2.6 *Let $t \geq 0$. Then*

$$\sup_{a\in\mathbb{Z}_+} \mathbb{P}[N(t) = a] = \mathbb{P}[N(t) = \lfloor t\rfloor] \leq \min\{1, \frac{1}{\sqrt{2\pi\lfloor t\rfloor}}\}.$$

Proof Observe that

$$\mathbb{P}[N(t) = a] = e^{-t}\frac{t^a}{a!}.$$

Thus the supremum over a depends only on the fraction $t^a/a!$. As

$$\frac{t^{a+1}}{a+1!} = \frac{t}{a+1}\frac{t^a}{a!},$$

as long as $a+1 \leq \lfloor t \rfloor$ this fraction increases, and then for $a+1 > \lfloor t \rfloor$ it decreases. Therefore the supremum is attained at $\lfloor t \rfloor$. By applying Lemma 2.5 we have

$$\sup_{a\in\mathbb{Z}_+} \mathbb{P}[N(t) = a] \leq \frac{t^{\lfloor t \rfloor}}{e^t \lfloor t \rfloor !} \leq \frac{1}{\sqrt{2\pi \lfloor t \rfloor}}$$

for $t \geq 1$. Since probability can not be bigger than one we have

$$\sup_{a\in\mathbb{Z}_+} \mathbb{P}[N(t) = a] \leq 1$$

for $0 \leq t < 1$. This finishes the proof.

□

In the proof Theorem 1.8, we will need several lemmas. First we start with the following lemma related to independent random variables.

Lemma 2.7 *Let $X_1, \ldots, X_n$ be independent random variables, $m_1 + \ldots + m_k = n$ and $f_1, \ldots, f_k$ be Borel measurable functions of $m_1, ..., m_k$ variables respectively. Then the random variables*

$$Y_1 = f_1(X_1, ..., X_{m_1}), Y_2 = f_2(X_{m_1+1}, \ldots, X_{m_1+m_2}), \ldots, Y_k = f(X_{m_1+\ldots+m_{k-1}+1}, \ldots, X_n)$$

are independent.

One can find the proof of Lemma 2.7 in (Koralov and Sinai, 2007).

The next lemma enables us to split up random variables into two groups in a way that the sums over these groups are independent.

Lemma 2.8 *Let $\{X_t\}_{t\geq 0}$ be an independent increment process. Let $n \geq 2$ be an integer, and let $(j_i, k_i),\ 1 \leq i \leq n$ be nonempty open intervals. Let $\sigma_1 \cup \sigma_2$ be a partition of indices $\{1, 2, \ldots, n\}$ into two nonempty subsets. Suppose no interval indexed by σ_1 intersects the* *intervals indexed by σ_2 and vice versa. Then the random variables*

$$\sum_{i\in\sigma_1} X_{k_i} - X_{j_i}, \qquad \sum_{i\in\sigma_2} X_{k_i} - X_{j_i}$$

are independent.

Proof We start with listing $j_i, k_i, i \in \sigma_1$ in increasing order, and let $t_1 \leq t_2 \leq \ldots \leq t_{2|\sigma_1|}$ be this list. For the intervals (t_i, t_{i+1}), $1 \leq i \leq 2|\sigma_1| - 1$, let $i \in \beta_1 \subseteq \{1, 2, \ldots, \}$ represent indices where (t_i, t_{t+1}) is contained within one of (j_i, k_i), $i \in \sigma_1$. Then notice that we have

$$\sum_{i\in\sigma_1} X_{k_i} - X_{j_i} = \sum_{i\in\beta_1} d_i(X_{t_{i+1}} - X_{_{t_i}}) \tag{2.47}$$

for some natural numbers d_i. Since we may have more than one interval whose lengths are the same, the coefficients d_i appears in the summation above. We follow this same process for indices in σ_2 to obtain (t_i, t_{i+1}), $i \in \beta_2$, that is we have

$$\sum_{i\in\sigma_2} X_{k_i} - X_{j_i} = \sum_{i\in\beta_2} c_i(X_{t_{i+1}} - X_{_it_i}) \tag{2.48}$$

for some numbers c_i. From (2.47) and (2.48) we see that the intervals indexed by β_1,β_2 are all disjoint. So keeping in mind the independent increment property we obtain our result by applying Lemma 2.7. □

In order to perform calculation of probabilities in Theorem 1.8 we are in need of the next two lemmas which are analogues of Lemmas 2.5,2.6 for a linear combination of Poisson random variables.

Lemma 2.9 *Let N_i be independent random variables of Poisson distribution with mean $\mu_i > 0$, and $d_i \in \mathbb{N}$ where $1 \leq i \leq n$. Let $\mu := \max_{1\leq i\leq n} \mu_i$. Then for $a \in \mathbb{Z}_+$*

$$\mathbb{P}[\sum_{i=1}^{n} d_i N_i = a] \leq \min\{1, \frac{1}{\sqrt{2\pi\lfloor\mu\rfloor}}\}.$$

Proof Without loss of generality assume $\mu_n = \mu$. Observe that

$$\mathbb{P}[\sum_{i=1}^{n} d_i N_i = a] = \sum_{\substack{a_1+a_2+\ldots+a_n=a\\ a_i\in\mathbb{Z}_+}} \mathbb{P}[N_i = a_i/d_i,\ 1 \leq i \leq n].$$

Since N_i are independent random variables, where $1 \leq i \leq n$ the last term equals

$$\sum_{\substack{a_1+a_2+\ldots+a_n=a \\ a_i \in \mathbb{Z}_+}} \prod_{i=1}^{n-1} \mathbb{P}[N_i = a_i/d_i]$$

By using the assumption $\mu_n = \mu$ and Lemma 2.6 we have

$$\leq \min\{1, \frac{1}{\sqrt{2\pi\lfloor\mu\rfloor}}\} \sum_{\substack{a_1+a_2+\ldots+a_n=a \\ a_i \in \mathbb{Z}_+}} \prod_{i=1}^{n-1} \mathbb{P}[N_i = a_i/d_i].$$

This allows us to sum over $a_i,\ 1 \leq i \leq n$

$$\leq \min\{1, \frac{1}{\sqrt{2\pi\lfloor\mu\rfloor}}\} \sum_{\substack{a_i=0 \\ 1\leq i\leq n-1}}^{a} \prod_{i=1}^{n-1} \mathbb{P}[N_i = a_i/d_i] = \min\{1, \frac{1}{\sqrt{2\pi\lfloor\mu\rfloor}}\} \prod_{i=1}^{n-1} \sum_{a_i=0}^{a} \mathbb{P}[N_i = a_i/d_i].$$

Since every sum in the last term less than or equal 1 we finally have

$$\leq \min\{1, \frac{1}{\sqrt{2\pi\lfloor\mu\rfloor}}\}.$$

This finishes the proof. □

Lemma 2.10 *Let $(j_i, k_i),\ 1 \leq i \leq n$ be nonempty open intervals, and let $\{N(t)\}_{t\geq 0}$ be a Poisson process of intensity 1. Let m be an index at which $k_i - j_i$ becomes maximum. Then for any $a \in \mathbb{Z}_+$*

$$\mathbb{P}[\sum_{i=1}^{n} N(k_i) - N(j_i) = a] = \min\{1, \frac{1}{\sqrt{2\pi\lfloor(k_m - j_m)/2n\rfloor}}\}.$$

Proof We start with applying the process expressed in Lemma 2.8 to obtain the disjoint intervals $(t_i, t_{i+1}),\ i \in \beta$ for a set of indices $\beta \subseteq \{1, 2, \ldots 2n-1\}$ that allow us to write

$$\sum_{i=1}^{n} N(k_i) - N(j_i) = \sum_{i\in\beta} d_i[N(t_{i+1}) - N(t_i)]$$

for some natural numbers d_i. Observe that the union of (t_i, t_{i+1}), $i \in \beta$ is the same as the union of (j_i, k_i), $1 \le i \le n$, and hence at least one (t_i, t_{i+1}), $i \in \beta$ has length at least $(k_m - j_m)/2n$. Let $\mu := \max_{i\in\beta}(t_{i+1} - t_i)$. Then we have by Lemma 2.9

$$\begin{aligned}
\mathbb{P}[\sum_{i=1}^{n} N(k_i) - N(j_i) = a] &= \mathbb{P}[\sum_{i\in\beta} d_i[N(t_{i+1}) - N(t_i)] = a] \\
&\le \min\{1, \frac{1}{\sqrt{2\pi\lfloor\mu\rfloor}}\} \\
&\le \min\{1, \frac{1}{\sqrt{2\pi\lfloor(k_m - j_m)/2n\rfloor}}\}.
\end{aligned}$$

This finishes the proof. □

The following two lemmas are very useful and they will be used in the proofs of Theorems 1.3,1.8 respectively.

Lemma 2.11 *Let* $10 > C \ge 1$.

i) If $x > e^{50}$, *then*

$$|x - y| \le C\sqrt{x\log x} \implies |x - y| \le 2C\sqrt{y\log y}.$$

ii) If $y > e^{50}$, $x \ge 1$ *then*

$$|x - y| \ge 2C\sqrt{x\log x} \implies |x - y| \ge C\sqrt{y\log y}.$$

Proof For the first statement observe that $y \ge x - C\sqrt{x\log x} \ge x/2$, thus $x \le 2y$. Since $\sqrt{x\log x}$ is an increasing function, we have $\sqrt{x\log x} \le \sqrt{2y\log 2y} \le 2\sqrt{y\log y}$. This completes the proof of the first statement.

As for the second statement, if $y \le 2x$, due to the same reason $\sqrt{y\log y} \le \sqrt{2x\log 2x} \le 2\sqrt{x\log x}$. If $y \ge 2x$, then $y - x \ge y/2$, and since $C\sqrt{y\log y} \le y/2$. This completes the proof of the second statement.

□

Lemma 2.12 *Let* $A \subseteq \mathbb{N}$ *be a finite set, and let* $a \in \mathbb{R}$, $b \in \mathbb{N}$ *satisfy* $0 < a \le b \le \min A$. *Let* $d \ge 2$ *be an integer. Let* $\Phi : [a, \infty) \to (0, \infty)$ *be a decreasing function. Then*

$$\sum_{\substack{j,k\in A\\ j<k}} \Phi(k^d - j^d) \le \sum_{b\le j<k\le b+|A|-1} \Phi(k^d - j^d).$$

Proof Let us first list the elements of A into a strictly increasing sequence $a_1, a_2, \ldots, a_{|A|}$. Then for any $1 \le j < k \le |A|$ we have $a_{j+1} - a_j \ge 1$. This implies

$$a_k - a_j = \sum_{i=j}^{k-1} a_{i+1} - a_i \ge \sum_{i=j}^{k-1} 1 = k - j. \tag{2.49}$$

Let $j = 1$. By using (2.49) and the assumption $0 < a \le b \le \min A = a_1$ we obtain

$$a_k \ge a_1 + k - 1 \ge b + k - 1.$$

Combining all of these

$$\begin{aligned}
a_k^d - a_j^d &= (a_k - a_j)\Big(a_k^{d-1} + a_k^{d-2}a_j + \cdots + a_k a_j^{d-2} + a_j^{d-1}\Big)\\
&= (a_k - a_j)\sum_{i=0}^{d-1} a_k^{d-1-i} a_j^i\\
&\ge (k - j)\sum_{i=0}^{d-1} (b + k - 1)^{d-1-i}(b + j - 1)^i\\
&= (b + k - 1)^d - (b + j - 1)^d.
\end{aligned}$$

Since $d \ge 2$ the term in the second line above can not be less than $2b$. This implies that both $a_k^d - a_j^d$ and $(b + k - 1)^d - (b + j - 1)^d$ are in the domain of Φ. Since the function Φ is decreasing, this enables us to obtain

$$\begin{aligned}
\sum_{\substack{j,k\in A\\ j<k}} \Phi(k^d - j^d) = \sum_{\substack{1\le j,k\le |A|\\ j<k}} \Phi(a_k^d - a_j^d) &\le \sum_{1\le j<k\le |A|} \Phi((b + k - 1)^d - (b + j - 1)^d)\\
&= \sum_{b\le j<k\le b+|A|-1} \Phi(k^d - j^d),
\end{aligned}$$

where we used the fact that $a_k^d - a_j^d \ge (b + k - 1)^d - (b + j - 1)^d$ to pass from the first line to the second line. $\square$

CHAPTER 3

HARDY-LITTLEWOOD MAJORANT PROBLEM RANDOMIZED VIA STOCHASTIC PROCESSES

In this chapter, our aim is to give exhaustive proofs of Theorem 1.1, Theorem 1.2, Theorem 1.3, Theorem 1.4 and Theorem 1.5 respectively. For this purpose, we will often apply auxiliary results.

3.1. Hardy-Littlewood majorant problem randomized via stationary processes

Proof of Theorem 1.1 Let the distribution of the random variables in our process be given by $\mu_{X_j}(k) = \mu(k) = d_k$. We will see that (1.12) and (1.13) follow from (1.11). Clearly for every $\omega \in \Omega$ by considering a neighborhood of $y = 0$ we have

$$\Big\| \sum_{j\in A} e^{2\pi i y X_j(\omega)} \Big\|_\infty = |A|. \tag{3.1}$$

From this our theorem follows for $p = \infty$. Let $1 \leq p < \infty$. From (3.1), we also have

$$\Big\| \sum_{j\in A} e^{2\pi i y X_j(\omega)} \Big\|_p \leq \Big\| \sum_{j\in A} e^{2\pi i y X_j(\omega)} \Big\|_\infty \leq |A|,$$

$$\mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y X_j} \Big\|_p^p \leq |A|^p.$$

This finishes one direction of (1.11). For the other direction of (1.11) with $p = 2$ we have

$$\begin{aligned} \mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y X_j} \Big\|_2^2 = \mathbb{E} \int_{\mathbb{T}} \Big| \sum_{j\in A} e^{2\pi i y X_j} \Big|^2 dy &= \mathbb{E} \int_{\mathbb{T}} \sum_{i,j\in A} e^{2\pi i y (X_j - X_i)} dy \\ &= \mathbb{E} \sum_{j\in A} \sum_{i\in A} \int_{\mathbb{T}} e^{2\pi i y (X_j - X_i)} dy \end{aligned} \tag{3.2}$$

For a fixed ω the integral in the last expression is 1 if $X_j(\omega) = X_i(\omega)$ and 0 otherwise. We define

$$U(\omega, k) := |\{j \in A : X_j(\omega) = k\}| = \sum_{j\in A} \mathbb{I}_{\{X_j=k\}}(\omega), \tag{3.3}$$

where $\mathbb{I}$ denotes the indicator function. Therefore we have

$$\mathbb{E}U(\omega, k) = \sum_{j\in A} \mathbb{E}\,\mathbb{I}_{\{X_j=k\}}(\omega) = \sum_{j\in A} \mathbb{P}\{X_j = k\} = |A|d_k. \tag{3.4}$$

The Cauchy-Schwarz inequality yields $\mathbb{E}U^2(\omega, k) \geq [\mathbb{E}U(\omega, k)]^2 = |A|^2 d_k^2$, and applying this we continue from (3.2) as follows

$$= \mathbb{E}\sum_{j\in A} U(\omega, X_j(\omega)) = \mathbb{E}\sum_{k\in\mathbb{Z}} U^2(\omega, k) = \sum_{k\in\mathbb{Z}} \mathbb{E}U^2(\omega, k) \geq |A|^2 \sum_{k\in\mathbb{Z}} d_k^2. \tag{3.5}$$

This finishes (1.11) for $p = 2$. For $2 < p < \infty$:

$$\begin{aligned}
\mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y X_j}\Big\|_p^p = \mathbb{E}\int_{\mathbb{T}}\Big|\sum_{j\in A} e^{2\pi i y X_j}\Big|^p dy &\overset{\text{by (2.1)}}{\geq} \mathbb{E}\Big[\int_{\mathbb{T}}\Big|\sum_{j\in A} e^{2\pi i y X_j}\Big|^2 dy\Big]^{p/2} \\
&\geq \Big[\mathbb{E}\int_{\mathbb{T}}\Big|\sum_{j\in A} e^{2\pi i y X_j}\Big|^2 dy\Big]^{p/2} \\
&\geq |A|^p\Big[\sum_{k\in\mathbb{Z}} d_k^2\Big]^{p/2}.
\end{aligned} \tag{3.6}$$

For $1 \leq p < 2$ we again benefit from the same ideas:

$$\begin{aligned}
\mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y X_j}\Big\|_2^2 &= \mathbb{E}\int_{\mathbb{T}}\Big|\sum_{j\in A} e^{2\pi i y X_j}\Big|^2 dy \\
&\leq \mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y X_j(\omega)}\Big\|_\infty^{2-p} \int_{\mathbb{T}}\Big|\sum_{j\in A} e^{2\pi i y X_j}\Big|^p dy \\
&= |A|^{2-p}\mathbb{E}\int_{\mathbb{T}}\Big|\sum_{j\in A} e^{2\pi i y X_j}\Big|^p dy.
\end{aligned} \tag{3.7}$$

Thus

$$\mathbb{E}\Big\|\sum_{j\in A}e^{2\pi iyX_j}\Big\|_p^p\geq|A|^p\sum_{k\in\mathbb{Z}}d_k^2. \tag{3.8}$$

This finishes the proof of (1.11). To obtain (1.12) it is sufficient to observe that for any ω and any sequence $\{a_j\}_{j\in A}$ with $|a_j|\leq 1$

$$\Big\|\sum_{j\in A}a_je^{2\pi iyX_j(\omega)}\Big\|_\infty\leq\Big\|\sum_{j\in A}e^{2\pi iyX_j(\omega)}\Big\|_\infty\leq|A|, \tag{3.9}$$

and therefore for any $1\leq p<\infty$

$$\Big\|\sum_{j\in A}a_je^{2\pi iyX_j(\omega)}\Big\|_p^p\leq\Big\|\sum_{j\in A}a_je^{2\pi iyX_j(\omega)}\Big\|_\infty^p\leq|A|^p. \tag{3.10}$$

We use this and (3.9) to conclude that for any $1\leq p\leq\infty$

$$\mathbb{E}\sup_{|a_j|\leq 1}\Big\|\sum_{j\in}a_je^{2\pi iyX_j}\Big\|_p^p\leq|A|^p,\qquad\mathbb{E}\sup_{|a_j|\leq 1}\Big\|\sum_{j\in A}a_je^{2\pi iyX_j}\Big\|_\infty\leq|A|. \tag{3.11}$$

Then using (1.11) we obtain (1.12).

To deal with (1.13) we need to introduce a new method that is more general than the one we used to prove (1.11), in that it is applicable to any real valued stationary process and not just to integer valued ones. But as a tradeoff it gives worse bounds. Let $\kappa>0$ be small, and let K be such that

$$\sum_{|k|\leq K}d_k>1-\kappa. \tag{3.12}$$

Therefore

$$\mathbb{E}\left[\sum_{j\in A}\mathbb{I}_{\{|X_j|\leq K\}}\right]\geq(1-\kappa)|A|. \tag{3.13}$$

Then for the set

$$\Omega_\kappa:=\{\omega\in\Omega:\sum_{j\in A}\mathbb{I}_{\{|X_j|\leq K\}}\geq\frac{9}{10}|A|\} \tag{3.14}$$

we have $\mathbb{P}(\Omega_\kappa)\geq 1-10\kappa$. To see this, we consider the following argument. For $\omega\in\Omega_\kappa$, we have

$$\begin{aligned}\frac{9|A|}{10}+\sum_{j\in A}\mathbb{I}_{\{|X_j|>K\}}&\leq\sum_{j\in A}\mathbb{I}_{\{|X_j|\leq K\}}+\sum_{j\in A}\mathbb{I}_{\{|X_j|>K\}}\\&=|A|.\end{aligned}$$

Then

$$\frac{9|A|}{10} + \sum_{j\in A} \mathbb{I}_{\{|X_j|>K\}} \le |A|,$$

$$\sum_{j\in A} \mathbb{I}_{\{|X_j|>K\}} \le \frac{|A|}{10}.$$

By Markov's inequality and the fact $\mathbb{E}[\sum_{j\in A} \mathbb{I}_{\{|X_j|>K\}}] \le |A|\kappa$, we have

$$\begin{aligned}\mathbb{P}\left(\omega \in \Omega \setminus \Omega_\kappa : \sum_{j\in A} \mathbb{I}_{\{|X_j|>K\}} > \frac{|A|}{10}\right) &\le \mathbb{E}\left[\sum_{j\in A} \mathbb{I}_{\{|X_j|>K\}}\right]\frac{10}{|A|} \\ &\le |A|\kappa\frac{10}{|A|} \\ &= 10\kappa.\end{aligned} \tag{3.15}$$

From (3.15), we obtain

$$\begin{aligned}\mathbb{P}\left(\omega \in \Omega_\kappa : \sum_{j\in A} \mathbb{I}_{\{|X_j|>K\}} \le \frac{|A|}{10}\right) &= 1 - \mathbb{P}\left(\omega \in \Omega \setminus \Omega_\kappa : \sum_{j\in A} \mathbb{I}_{\{|X_j|>K\}} > \frac{|A|}{10}\right) \\ &\ge 1 - 10\kappa.\end{aligned}$$

Now we observe that for $\omega \in \Omega_\kappa$ and $0 \le y \le 1/100K$ we have

$$\begin{aligned}\Big|\sum_{j\in A} e^{2\pi iyX_j(\omega)}\Big| &\ge \Big|\sum_{|X_j(\omega)|\le K} e^{2\pi iyX_j(\omega)}\Big| - \Big|\sum_{|X_j(\omega)|>K} e^{2\pi iyX_j(\omega)}\Big| \\ &\ge \Big|\sum_{|X_j(\omega)|\le K} \cos 2\pi yX_j(\omega)\Big| - \frac{|A|}{10} \\ &\ge \frac{9|A|}{20} - \frac{|A|}{10} > \frac{|A|}{4}.\end{aligned} \tag{3.16}$$

For any $\omega \in \Omega_\kappa$, and $p \ge 1$ we apply this last result to obtain

$$\begin{aligned}\Big\|\sum_{j\in A} e^{2\pi iyX_j(\omega)}\Big\|_p^p = \int_{\mathbb{T}}\Big|\sum_{j\in A} e^{2\pi iyX_j(\omega)}\Big|^p dy &\ge \int_0^{1/100K}\Big|\sum_{j\in A} e^{2\pi iyX_j(\omega)}\Big|^p dy \\ &\ge \int_0^{1/100K} \frac{|A|^p}{4^p} dy \\ &= \frac{|A|^p}{4^p 100K}\end{aligned} \tag{3.17}$$

From this we have

$$\mathbb{P}\Big[\Big\|\sum_{j\in A} e^{2\pi iyX_j(\omega)}\Big\|_p^p \geq \frac{|A|^p}{4^p 100K}\Big] \geq 1-10\kappa. \tag{3.18}$$

So, if $|A|$ is larger than a constant depending only on κ, ε, p we have

$$\mathbb{P}\Big[|A|^\varepsilon\Big\|\sum_{j\in A} e^{2\pi iyX_j(\omega)}\Big\|_p^p \geq |A|^{p+\varepsilon/2}\frac{|A|^{\varepsilon/2}}{4^p 100K} \geq |A|^{p+\varepsilon/2}\Big] \geq 1-10\kappa. \tag{3.19}$$

The expression above yields

$$\mathbb{P}\Big[|A|^\varepsilon\Big\|\sum_{j\in A} e^{2\pi iyX_j(\omega)}\Big\|_p^p \geq \sup_{|a_j|\leq 1}\Big\|\sum_{j\in A} a_j e^{2\pi iyX_j(\omega)}\Big\|_p^p\Big] \geq 1-10\kappa. \tag{3.20}$$

Therefore taking limits and considering that κ is arbitrary we obtain the desired result.

□

3.2. Hardy-Littlewood majorant problem randomized via Poisson processes

Proof of Theorem 1.2

As before we will see that (1.15) and (1.16) follow from (1.14). Clearly for every $\omega\in\Omega$ by considering a neighborhood of $y=0$ we have

$$\Big\|\sum_{j=1}^{M} e^{2\pi iyN(j)(\omega)}\Big\|_\infty = M. \tag{3.21}$$

From this our theorem follows for $p=\infty$ immediately. The claim (1.14) for $p=2$ is settled by observing that

$$\begin{aligned}\mathbb{E}\Big\|\sum_{j=1}^{M} e^{2\pi iyN(j)(\omega)}\Big\|_2^2 = \mathbb{E}\sum_{j,k=1}^{M}\int_{\mathbb{T}} e^{2\pi iy[N(j)(\omega)-N(k)(\omega)]}dy &= \sum_{j,k=1}^{M}\mathbb{P}[N(|j-k|)=0] \\ &= \sum_{j,k=1}^{M} e^{-|j-k|},\end{aligned}$$

and

$$M \le \sum_{j,k=1}^{M} e^{-|j-k|} = M + 2\sum_{j=1}^{M-1}(M-j)e^{-j} \le M + 2M\sum_{j=1}^{M-1} e^{-j} \le 3M. \tag{3.22}$$

For $2 < p < \infty$ we have

$$\begin{aligned}
\Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)(\omega)} \Big\|_p^p &= \int_{\mathbb{T}} \Big| \sum_{j=1}^{M} e^{2\pi i y N(j)(\omega)} \Big|^p dy \\
&\le \Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)(\omega)} \Big\|_\infty^{p-2} \int_{\mathbb{T}} \Big| \sum_{j=1}^{M} e^{2\pi i y N(j)(\omega)} \Big|^2 dy \\
&\le M^{p-2} \Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)(\omega)} \Big\|_2^2.
\end{aligned} \tag{3.23}$$

Taking expectation of both sides of (3.23) we obtain

$$\mathbb{E}\Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)(\omega)} \Big\|_p^p \le M^{p-2}\mathbb{E}\Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)(\omega)} \Big\|_2^2 \le 3M^{p-1}. \tag{3.24}$$

This concludes one direction of (1.14).

For the other direction, let $B_M = \{\omega \in \Omega |\ N(M) > 2M\}$. We have by Lemma 2.4

$$\mathbb{P}(B_M) = \mathbb{P}(\{\omega \in \Omega |\ N(M) - M > \sqrt{M}\sqrt{M}\}) \le 2e^{-M/4}.$$

Thus for ω except these, and $1 \le p < \infty$ we have

$$\begin{aligned}
\Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)(\omega)} \Big\|_p^p &\ge \int_0^{1/100M} \Big| \sum_{j=1}^{M} e^{2\pi i y N(j)(\omega)} \Big|^p dy \\
&\ge \int_0^{1/100M} \Big| \sum_{j=1}^{M} \cos 2\pi y N(j)(\omega) \Big|^p dy \\
&\ge M^{p-1}/100 \cdot 2^p.
\end{aligned} \tag{3.25}$$

Hence

$$\mathbb{E}\Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)} \Big\|_p^p \ge M^{p-1}/100 \cdot 2^{p+1}.$$

This settles (1.14) completely.

Observe that for any ω and any sequence $\{a_j\}_{j=1}^M$, $|a_j| \le 1$ we have

$$\Big\| \sum_{j=1}^{M} a_j e^{2\pi i y N(j)(\omega)} \Big\|_2^2 \le \Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)(\omega)} \Big\|_2^2. \tag{3.26}$$

To show (1.15), we have by (3.26)

$$\begin{aligned} \Big\| \sum_{j=1}^{M} a_j e^{2\pi i y N(j)(\omega)} \Big\|_p^p &\le \Big\| \sum_{j=1}^{M} a_j e^{2\pi i y N(j)(\omega)} \Big\|_\infty^{p-2} \Big\| \sum_{j=1}^{M} a_j e^{2\pi i y N(j)(\omega)} \Big\|_2^2 \\ &\le M^{p-2} \Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)(\omega)} \Big\|_2^2. \end{aligned} \tag{3.27}$$

Using (3.27) and (1.14) we obtain

$$\mathbb{E} \sup_{|a_j| \le 1} \Big\| \sum_{j=1}^{M} a_j e^{2\pi i y N(j)(\omega)} \Big\|_p^p \le 3M^{p-1} \lesssim \mathbb{E} \Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)(\omega)} \Big\|_p^p.$$

This concludes proof of (1.15).

Now our aim is to show (1.16).

$$\mathbb{E} \sup_{|a_j| \le 1} \Big\| \sum_{j=1}^{M} a_j e^{2\pi i y N(j)(\omega)} \Big\|_p^p \le 3M^{p-1}$$

implies that

$$\mathbb{P}\Big[\sup_{|a_j| \le 1} \Big\| \sum_{j=1}^{M} a_j e^{2\pi i y N(j)(\omega)} \Big\|_p^p \ge 3M^{p-1} \log(M+1) \Big] \le 1/\log(M+1).$$

Therefore using (3.25) and Lemma 2.3 except for a set of probability at most $2e^{-M/4} + 1/\log(M+1)$

$$\sup_{|a_j| \le 1} \Big\| \sum_{j=1}^{M} a_j e^{2\pi i y N(j)(\omega)} \Big\|_p^p \le 300 \cdot 2^p M^{\varepsilon} \Big\| \sum_{j=1}^{M} e^{2\pi i y N(j)(\omega)} \Big\|_p^p,$$

and this proves (1.16).

$\square$

Proof of Theorem 1.3 We first show (1.17) for $p = 2$.

$$\begin{aligned}
\mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_2^2 = \mathbb{E} \sum_{j,k\in A} \int_{\mathbb{T}} e^{2\pi i y [N(j^d)-N(k^d)]} dy &= \sum_{j,k\in A} \mathbb{P}[N(j^d) = N(k^d)] \\
&= \sum_{j,k\in A} \mathbb{P}[N(|j^d - k^d|) = 0] \\
&= \sum_{j,k\in A} e^{-|j^d-k^d|}.
\end{aligned}$$

This can be bounded from both sides

$$\begin{aligned}
|A| \leq \sum_{j,k\in A} e^{-|j^d-k^d|} \leq |A| + 2 \sum_{j\in A} \sum_{\substack{k\in A \\ k<j}} e^{k^d-j^d} &\leq |A| + 2 \sum_{j\in A} \sum_{k=1}^{(j-1)^d} e^{k-j^d} \\
&\leq |A| + 2 \sum_{j\in A} e^{-j^d} \frac{[e^{(j-1)^d+1} - e]}{e-1} \\
&\leq |A| + 4 \sum_{j\in A} e^{(j-1)^d - j^d} \\
&\leq |A| + 4 \sum_{j\in A} e^{-d(j-1)^{d-1}} \\
&\leq |A| + 4.
\end{aligned}$$

To see how we pass from the third line to the fourth line above, let us consider the following argument. Let $\alpha(x) = -x^d$ be a function on $[j-1, j]$. By the Mean Value Theorem there exists $c \in [j-1, j]$ such that

$$(j-1)^d - j^d = -(j - j + 1)dc^{d-1} \leq -d(j-1)^{d-1}.$$

This yields (1.17) for $p = 2$. We now turn to (1.17) for $p = 4$. From this (1.17) follows for all $2 \leq p \leq 4$. By the inequality (2.1) and $p = 2$ case, the left hand side of (1.17) can be estimated as follows.

$$\mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y N(j)(\omega)}\Big\|_p \geq \mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y N(j)(\omega)}\Big\|_2$$
$$\geq |A|^{1/2}.$$

Now we will focus on the right hand side of (1.17). We begin with some reductions that will be useful for us later on. If $|A| \leq 3e^{100}$, then our result is immediate. If $|A| > 3e^{100}$, then we let $A_0 = A \cap [1, e^{100}]$. By Minkowski's inequality, we obtain

$$\begin{aligned}
\mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y N(j^d)}\Big\|_4^4 &\leq \mathbb{E}\Big\|\sum_{j\in A_0} e^{2\pi i y N(j^d)} + \sum_{j\in A/A_0} e^{2\pi i y N(j^d)}\Big\|_4^4 \\
&\leq \mathbb{E}\Big[\Big\|\sum_{j\in A_0} e^{2\pi i y N(j^d)}\Big\|_4 + \Big\|\sum_{j\in A/A_0} e^{2\pi i y N(j^d)}\Big\|_4\Big]^4 \\
&\leq 15\Big[\mathbb{E}\Big\|\sum_{j\in A_0} e^{2\pi i y N(j^d)}\Big\|_4^4 + \mathbb{E}\Big\|\sum_{j\in A/A_0} e^{2\pi i y N(j^d)}\Big\|_4^4\Big] \\
&\leq 15\Big[|A_0|^4 + \mathbb{E}\Big\|\sum_{j\in A\setminus A_0} e^{2\pi i y N(j^d)}\Big\|_4^4\Big].
\end{aligned}$$

If we had our result for sets that contains no element in $[1, e^{100}]$ and have more elements than e^{100}, then applying it we would obtain

$$15\Big[|A_0|^4 + C|A \setminus A_0|^2\Big] \leq 15(C + e^{200})|A|^2.$$

So we may assume that our set A contains no element in $[1, e^{100}]$ and have more elements than e^{100}.

We start with converting our sum

$$
\begin{aligned}
\mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y N(j^d)}\Big\|_4^4 &= \mathbb{E}\int_{\mathbb{T}}\Big|\sum_{j\in A} e^{2\pi i y N(j^d)}\Big|^4 dy \\
&= \mathbb{E}\int_{\mathbb{T}}\Big|\sum_{j\in A} e^{2\pi i y N(j^d)}\Big|^2\Big|\sum_{j\in A} e^{2\pi i y N(j^d)}\Big|^2 dy \\
&= \mathbb{E}\sum_{j_1,j_2,k_1,k_2\in A}\int_{\mathbb{T}} e^{2\pi i y[N(j_1^d)-N(k_1^d)]}e^{2\pi i y[N(j_2^d)-N(k_2^d)]}dy \\
&= \mathbb{E}\sum_{j_1,j_2,k_1,k_2\in A}\int_{\mathbb{T}} e^{2\pi i y[N(j_1^d)+N(j_2^d)-N(k_1^d)-N(k_2^d)]}dy \\
&= \sum_{j_1,j_2,k_1,k_2\in A}\mathbb{P}[N(j_1^d)+N(j_2^d)=N(k_1^d)+N(k_2^d)].
\end{aligned}
$$

The set A^4 to which any quadruple (j_1, j_2, k_1, k_2) belongs can be written as follows

$$
A^4 = A_{11} \cup A_{12} \cup A_{21} \cup A_{22} \cup A_3.
$$

Here $A_{ab}, a, b \in \{1, 2\}$ is the set of quadruples (j_1, j_2, k_1, k_2) for which $j_a = k_b$. The set A_3 contains those elements of A^4 that is in none of these $A_{ab}, a, b \in \{1, 2\}$. We have

$$
\begin{aligned}
\sum_{(j_1,j_2,k_1,k_2)\in A_{11}}\mathbb{P}[N(j_1^d)+N(j_2^d)=N(k_1^d)+N(k_2^d)] &= \sum_{j_1,j_2,k_2\in A}\mathbb{P}[N(j_2^d)=N(k_2^d)] \\
&= |A|\sum_{j,k\in A}\mathbb{P}[N(j^d)=N(k^d)],
\end{aligned}
$$

and this last sum has been estimated above. For A_{12}, A_{21}, A_{22} via the same argument we get the same result. Therefore it remains to handle A_3. We decompose

$$
A_3 = A_{31} \cup A_{32} \cup A_{33} \cup A_{34},
$$

where

$$A_{31} : \{(j_1, j_2, k_1, k_2) \in A_3 : j_1 > k_1,\ j_2 > k_2\}$$

$$A_{32} : \{(j_1, j_2, k_1, k_2) \in A_3 : j_1 > k_1,\ j_2 < k_2\}$$

$$A_{33} : \{(j_1, j_2, k_1, k_2) \in A_3 : j_1 < k_1,\ j_2 > k_2\}$$

$$A_{34} : \{(j_1, j_2, k_1, k_2) \in A_3 : j_1 < k_1,\ j_2 < k_2\}$$

Summing over A_{31}, A_{34} is easy, and we first handle these.

$$\sum_{A_{31}} \mathbb{P}[N(j_1^d) + N(j_2^d) = N(k_1^d) + N(k_2^d)] = \sum_{A_{31}} \mathbb{P}[N(j_1^d) - N(k_1^d) = N(k_2^d) - N(j_2^d)].$$

As $j_2 > k_2$, and $N(t)$ is increasing we must have $N(j_1^d) - N(k_1^d) \geq 0$ and $N(k_2^d) - N(j_2^d) \leq 0$. Therefore for these to be equal they must both be zero. Hence

$$\leq \sum_{A_{31}} \mathbb{P}[N(j_1^d) - N(k_1^d) = 0] \leq |A|^2 \sum_{\substack{j,k \in A \\ k<j}} \mathbb{P}[N(j^d) - N(k^d) = 0].$$

We estimated this last sum above to be bounded by 2. The set A_{34} is handled the same way.

We now move on to A_{32}. This, together with its symmetric counterpart A_{33}, represent the most important, generic cases. We have

$$\sum_{A_{32}} \mathbb{P}[N(j_1^d) + N(j_2^d) = N(k_1^d) + N(k_2^d)] = \sum_{A_{32}} \mathbb{P}[N(j_1^d) - N(k_1^d) = N(k_2^d) - N(j_2^d)].$$

We decompose $A_{32} = A_{321} \cup A_{322}$, in the first of which are contained those (j_1, j_2, k_1, k_2) for which the intervals $(k_1, j_1), (j_2, k_2)$ are disjoint, and in the second those for which they intersect. We first sum over A_{321} which can be written as

$$\begin{aligned}
&\sum_{A_{321}} \mathbb{P}[N(j_1^d) - N(k_1^d) = N(k_2^d) - N(j_2^d)] \\
&= \sum_{A_{321}} \sum_{a=0}^{\infty} \mathbb{P}[N(j_1^d) - N(k_1^d) = a = N(k_2^d) - N(j_2^d)] \\
&= \sum_{a=0}^{\infty} \sum_{A_{321}} \mathbb{P}[N(j_1^d) - N(k_1^d) = a]\mathbb{P}[N(k_2^d) - N(j_2^d) = a] \\
&\le \sum_{a=0}^{\infty} \sum_{\substack{j_1,k_1\in A \\ j_1>k_1}} \sum_{\substack{j_2,k_2\in A \\ k_2>j_2 \\ (k_1,j_1)\cap(j_2,k_2)=\emptyset}} \mathbb{P}[N(j_1^d) - N(k_1^d) = a]\mathbb{P}[N(k_2^d) - N(j_2^d) = a] \\
&\le \sum_{a=0}^{\infty} \Big[\sum_{\substack{j,k\in A \\ j<k}} \mathbb{P}[N(k^d - j^d) = a]\Big]^2.
\end{aligned}$$

Now our aim is to estimate the inner sum independent of a, and then use this and Fubini's Theorem to reach the desired estimate. If we manage to show that the inner sum is bounded by a constant $C(A)$ that may depend on A but is independent of a, then we have

$$\begin{aligned}
\sum_{a=0}^{\infty} \Big[\sum_{\substack{j,k\in A \\ j<k}} \mathbb{P}[N(k^d - j^d) = a]\Big]^2 &\le C(A) \sum_{a=0}^{\infty} \sum_{\substack{j,k\in A \\ j<k}} \mathbb{P}[N(k^d - j^d) = a] \\
&= C(A) \sum_{\substack{j,k\in A \\ j<k}} \sum_{a=0}^{\infty} \mathbb{P}[N(k^d - j^d) = a] \\
&= C(A) \sum_{\substack{j,k\in A \\ j<k}} \sum_{a=0}^{\infty} \frac{(k^d - j^d)^a}{a!} e^{-(k^d - j^d)} \\
&= C(A) \sum_{\substack{j,k\in A \\ j<k}} e^{(k^d - j^d)} e^{-(k^d - j^d)} \\
&= C(A) \sum_{\substack{j,k\in A \\ j<k}} 1 \\
&\le C(A)|A|^2.
\end{aligned}$$

Let us begin with a crude estimate. For $a \in \mathbb{N}$ we have by Lemma 2.5

$$\sum_{\substack{j,k\in A \\ j<k}} \mathbb{P}[N(k^d - j^d) = a] \le |A|^2 \sup_{t\ge 0} \mathbb{P}[N(t) = a] \le \frac{|A|^2}{\sqrt{2\pi a}}.$$

Observe that the inner sum is bounded by a positive constant for $a \geq |A|^4$. This is also true for $a = 0$ since we already have

$$\sum_{\substack{j,k\in A\\ j<k}} \mathbb{P}[N(k^d - j^d) = 0] \leq 2.$$

We want to obtain this same property, if possible, for all $a \geq 0$. Thus we may assume $1 \leq a < |A|^4$. Our strategy is to decompose the inner sum to two sums over sets A_1, A_2. For this purpose we define the function $f(x) := 4\sqrt{x \log x}$ for $x \geq 1$. Then define

$$\begin{aligned} A_1 &:= \{(j,k) \in A^2 : j < k,\ |a - (k^d - j^d)| \geq 2f(a)\} \\ A_2 &:= \{(j,k) \in A^2 : j < k\} \setminus A_1. \end{aligned}$$

We note that if $a < e^{90}$ then A_2 is empty, and therefore we may assume that a is large when summing over that set.

We first consider the sum over A_1. By Lemma 2.11 we have $|a - (k^d - j^d)| \geq f(k^d - j^d)$ and by Lemma 2.4

$$\begin{aligned} \mathbb{P}[N(k^d - j^d) = a] &\leq \mathbb{P}[|N(k^d - j^d) - (k^d - j^d)| = |a - (k^d - j^d)|] \\ &\leq \mathbb{P}[|N(k^d - j^d) - (k^d - j^d)| \geq f(k^d - j^d)] \\ &= \mathbb{P}[|N(k^d - j^d) - (k^d - j^d)| \geq 4\sqrt{k^d - j^d \log(k^d - j^d)}] \\ &\leq 2e^{-16\log(k^d - j^d)/4} \\ &= 2e^{-4\log(k^d - j^d)} \\ &= 2(k^d - j^d)^{-4}. \end{aligned}$$

Therefore

$$\sum_{A_1} \mathbb{P}[N(k^d - j^d) = a] \leq 2\sum_{j<k}(k^d - j^d)^{-4} \leq 2\sum_{j}\sum_{k>j} k^{-4} \leq 1. \tag{3.28}$$

We now focus on summing over the set A_2. By Lemma 2.5 we have

$$\sum_{A_2} \mathbb{P}[N(j^d - k^d) = a] \leq \frac{1}{\sqrt{\pi}} a^{-1/2} |A_2| \leq a^{-1/2} |A_2|.$$

So our aim is to estimate the cardinality of A_2. We can rewrite A_2 as

$$A_2 = \{(j,k) : j < k, \ (j^d + a - 2f(a)))^{\frac{1}{d}} < k < (j^d + a + 2f(a))^{\frac{1}{d}}\}.$$

Let $g(x) = x^{1/d}$ be a function on the closed interval $[j^d, j^d + a + 2f(a)]$. By the Mean Value Theorem, there exists $c \in [j^d, j^d + a + 2f(a)]$ such that

$$\begin{aligned}
(j^d + a + 2f(a))^{\frac{1}{d}} - j &= (j^d + a + 2f(a) - j^d) \frac{c^{\frac{1-d}{d}}}{d} \\
&\leq [a + 2f(a)] \frac{j^{1-d}}{d} \\
&= a \left[1 + \frac{8 \log a}{\sqrt{a}} \right] \frac{j^{1-d}}{d} \\
&\leq a j^{1-d}.
\end{aligned}$$

When $j > a^{1/(d-1)}$ this final expression is less than 1. Thus for these j we have no pair (j,k) in A_2. Similarly let $h(x) = x^{1/d}$ be a function on the closed interval $[j^d + a - 2f(a), j^d + a + 2f(a)]$. By the Mean Value theorem, there exists $c \in [j^d + a - 2f(a), j^d + a + 2f(a)]$ such that

$$\begin{aligned}
(j^d + a + 2f(a))^{\frac{1}{d}} - (j^d + a - 2f(a))^{\frac{1}{d}} &= (j^d + a + 2f(a) - j^d - a + 2f(a)) \frac{c^{\frac{1}{d}-1}}{d} \\
&\leq 4f(a) \frac{(j^d + a - 2f(a))^{\frac{1}{d}-1}}{d} \\
&\leq 4f(a)(a - 2f(a))^{\frac{1}{d}-1} \frac{1}{d} \\
&= 4f(a) a^{\frac{1}{d}-1} \left[1 - \frac{8 \log a}{\sqrt{a}} \right]^{\frac{1}{d}-1} \frac{1}{d} \\
&\leq 4f(a) a^{\frac{1}{d}-1} \\
&= 16 \log a a^{\frac{1}{d}-1} a^{\frac{1}{2}} \\
&\leq 16 a^{\frac{1}{d}-\frac{1}{2}} \log a.
\end{aligned}$$

Observe that, the final expression is less than one for $d \geq 3$. Thus there can be at most 1 solution for every $j \leq a^{1/(d-1)}$, which yields $|A_2| \leq a^{1/2}$.

Now we concentrate on the case $d = 2$. this method of counting solutions for each fixed j is too crude, and instead we will count them for each fixed value of $b = k - j$. Since when b is fixed knowing j immediately gives k, all we need to do is to count j. Since we have $k^2 - j^2 = b^2 + 2jb$, elements of A_2 satisfy

$$|b^2 + 2jb - a| < 2f(a) \implies \frac{a}{2b} - \frac{b}{2} - \frac{f(a)}{b} < j < \frac{a}{2b} - \frac{b}{2} + \frac{f(a)}{b}.$$

Therefore possible number of solutions j for fixed b is bounded by

$$1 + \frac{a}{2b} - \frac{b}{2} + \frac{f(a)}{b} - \frac{a}{2b} + \frac{b}{2} + \frac{f(a)}{b} = 1 + \frac{2f(a)}{b}.$$

Since $j \geq 1$, we must have

$$0 < \frac{a}{2b} - \frac{b}{2} + \frac{f(a)}{b} \implies b^2 < a + 2f(a) < 2a \implies b < \sqrt{2a}.$$

So summing the number of solutions over b

$$\begin{aligned}
|A_2| \leq \sum_{b=1}^{\lfloor \sqrt{2a} \rfloor} 1 + \frac{2f(a)}{b} &\leq \sqrt{2a} + 2f(a)\left(1 + \int_1^{\sqrt{2a}} \frac{1}{b} db\right) \\
&= \sqrt{2a} + 2f(a)\left[1 + \log b\Big|_1^{\sqrt{2a}}\right] \\
&< \sqrt{2a} + 2f(a)\log a \\
&< 10\sqrt{a}\log^{3/2} a.
\end{aligned}$$

By this and our assumption $a \leq |A|^4$ we obtain

$$\sum_{A_2} \mathbb{P}[N(j^d - k^d) = a] \leq a^{-1/2} 10\sqrt{a}\log^{3/2} a \leq 10\log^{3/2} a \leq 80\log^{3/2}|A| \tag{3.29}$$

Combining (3.28) and (3.29) we have

$$\sum_{\substack{j,k\in A\\ j<k}} \mathbb{P}[N(k^d - j^d) = a] \le \sum_{A_1} \mathbb{P}[N(k^d - j^d) = a] + \sum_{A_2} \mathbb{P}[N(j^d - k^d) = a]$$
$$\lesssim \begin{cases} 1 + \log^{3/2}(1 + |A|) \quad \text{if} \quad d = 2 \\ 1 \quad \text{if} \quad d \ge 3. \end{cases}$$

This concludes the sum over A_{321}.

We finally consider the sum over A_{322}. We partitioned this set into four:

$$\begin{aligned}
A_{3221} &:= \{(j_1, j_2, k_1, k_2) \in A_{322} : [k_1, j_1] \subseteq (j_2, k_2)\} \\
A_{3222} &:= \{(j_1, j_2, k_1, k_2) \in A_{322} : [j_2, k_2] \subseteq (k_1, j_1)\} \\
A_{3223} &:= \{(j_1, j_2, k_1, k_2) \in A_{322} : k_1 < j_2 < j_1 < k_2\} \\
A_{3224} &:= \{(j_1, j_2, k_1, k_2) \in A_{322} : j_2 < k_1 < k_2 < j_1\}.
\end{aligned}$$

First two and the last two are handled in the same way so we will only consider A_{3221}, A_{3223}. On A_{3221} the condition $[k_1, j_1] \subseteq (j_2, k_2)$ means

$$\sum_{A_{3221}} \mathbb{P}[N(j_1^d) + N(j_2^d) = N(k_1^d) + N(k_2^d)] = \sum_{A_{32}} \mathbb{P}[N(j_2^d) - N(k_1^d) = N(k_2^d) - N(j_1^d)].$$

Since $j_2 < k_1$, but $k_2 > j_1$, and since $(j_2, k_1) \cap (j_1, k_2) = \emptyset$ we must have

$$\begin{aligned}
&= \sum_{A_{32}} \mathbb{P}[N(j_2^d) - N(k_1^d) = 0 = N(k_2^d) - N(j_1^d)] \\
&= \sum_{A_{32}} \mathbb{P}[N(k_1^d) - N(j_2^d) = 0]\mathbb{P}[N(k_2^d) - N(j_1^d) = 0] \\
&\le \Big[\sum_{k>j} \mathbb{P}[N(k^d) - N(j^d) = 0] \Big]^2.
\end{aligned}$$

We estimated the sum inside the square by 2. So this case is bounded by just 4. On A_{3223} the condition $k_1 < j_2 < j_1 < k_2$ means $(k_1, j_2) \cap (j_1, k_2) = \emptyset$, and again we reduce to the case A_{321}. Hence we obtain (1.17) for $p = 4$.

We simply have by the fact that $|a_j| \leq 1$ $j \in \mathbb{N}$

$$\Big\| \sum_{j\in A} a_j e^{2\pi i y N(j^d)} \Big\|_{2n}^{2n} \leq \Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_{2n}^{2n} \tag{3.30}$$

for any $n \in \mathbb{N}$. By using (3.30) and (1.17), we have

$$\mathbb{E} \sup_{|a_j|\leq 1} \Big\| \sum_{j\in A} a_j e^{2\pi i y N(j^d)} \Big\|_4^4 \lesssim \begin{cases} C_\epsilon |A|^\varepsilon \mathbb{E} \Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_4^4 & \text{for } d = 2 \\ \mathbb{E} \Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_4^4 & \text{for } d \geq 3. \end{cases}$$

This yields (1.18) for $p = 4$. Using the inequality (2.1) completes the proof of (1.18) for all $p \in [2, 4]$.

Now we are ready to show (1.19). By the inequality (2.1), we have

$$\Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_p^p \geq \Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_2^p \geq |A|^{p/2}$$

for any $p \geq 2$. Therefore

$$\begin{aligned} &\mathbb{P}\Big[\sup_{|a_j|\leq 1} \Big\| \sum_{j\in A} a_j e^{2\pi i y N(j^d)} \Big\|_p^p \geq |A|^\varepsilon \Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_p^p \Big] \\ \leq &\mathbb{P}\Big[\sup_{|a_j|\leq 1} \Big\| \sum_{j\in A} a_j e^{2\pi i y N(j^d)} \Big\|_p^p \geq |A|^{\frac{p}{2}+\varepsilon} \Big]. \end{aligned}$$

By Markov's inequality and (1.18)

$$\begin{aligned} \leq |A|^{-\frac{p}{2}-\varepsilon} \mathbb{E} \sup_{|a_j|\leq 1} \Big\| \sum_{j\in A} a_j e^{2\pi i y N(j^d)} \Big\|_p^p &\leq C_\varepsilon |A|^{\varepsilon/2} |A|^{p/2} A|^{-p/2} |A|^{-\varepsilon} \\ &\leq C_\varepsilon |A|^{-\varepsilon/2}. \end{aligned}$$

By taking limits, we obtain

$$\lim_{|A|\to\infty} \mathbb{P}\Big[\sup_{|a_j|\le 1} \Big\| \sum_{j\in A} a_j e^{2\pi i y N(j^d)} \Big\|_p^p \ge |A|^{\varepsilon} \Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_p^p \le \lim_{|A|\to\infty} C_\varepsilon |A|^{-\varepsilon/2}$$
$$= 0.$$

Hence we are done.

□

3.3. Hardy-Littlewood majorant problem randomized via random walks

Proof of Theorem 1.4 As before we will see that (1.21) and (1.22) follow from (1.20). Clearly for every $\omega \in \Omega$ by considering a neighborhood of $y = 0$ we have

$$\Big\| \sum_{j=1}^{M} e^{2\pi i y R(j)(\omega)} \Big\|_\infty = M. \tag{3.31}$$

From this our theorem follows for $p = \infty$ immediately.

For (1.14) with $p = 2$ we have

$$\begin{aligned}\mathbb{E}\Big\| \sum_{j=1}^{M} e^{2\pi i y R(j)} \Big\|_2^2 &= \sum_{j,k=1}^{M} \mathbb{P}[R(|j-k|) = 0] = M + 2\sum_{j>k}^{M} \mathbb{P}[R(j-k) = 0] \\ &= \sum_{j=1}^{\lfloor (M-1)/2 \rfloor} (M-2j)\mathbb{P}[R(2j) = 0] \\ &\underset{\text{by (2.4)}}{=} \sum_{j=1}^{\lfloor (M-1)/2 \rfloor} (M-2j)\binom{2j}{j} 2^{-2j},\end{aligned}$$

and this can be estimated by using a precise version of Stirling formula (Robbins, 1955, 26)

$$
\begin{aligned}
&= \sum_{j=1}^{\lfloor (M-1)/2 \rfloor} (M-2j)\frac{(2j)!}{(j!)^2}2^{-2j} \le \sum_{j=1}^{\lfloor (M-1)/2 \rfloor} (M-2j)\frac{[(2j)^{2j}/e^{2j}]\sqrt{2\pi(2j)}e^{\frac{1}{24j}}}{(j^j/e^j)^2(2\pi j)e^{\frac{2}{12j+1}}}2^{-2j} \\
&\le \sum_{j=1}^{\lfloor (M-1)/2 \rfloor} (M-2j)\frac{2^{2j}}{\sqrt{\pi j}}2^{-2j} \\
&\le M + \int_1^{M/2} \frac{M-2x}{\sqrt{x}}dx \qquad (3.32) \\
&= M + \left(2M\sqrt{x} - \frac{4}{3}x^{3/2}\right)\Big|_1^{M/2} \\
&\le M + \frac{\sqrt{8}}{3}M^{3/2} \\
&\le 2M^{3/2}.
\end{aligned}
$$

For $2 < p < \infty$, we have

$$
\begin{aligned}
\Big\| \sum_{j=1}^{M} e^{2\pi i y R(j)(\omega)} \Big\|_p^p &= \int_{\mathbb{T}} \Big| \sum_{j=1}^{M} e^{2\pi i y R(j)(\omega)} \Big|^p dy \\
&\le \Big\| \sum_{j=1}^{M} e^{2\pi i y R(j)(\omega)} \Big\|_\infty^{p-2} \int_{\mathbb{T}} \Big| \sum_{j=1}^{M} e^{2\pi i y R(j)(\omega)} \Big|^2 dy \qquad (3.33) \\
&\le M^{p-2} \int_{\mathbb{T}} \Big| \sum_{j=1}^{M} e^{2\pi i y R(j)} \Big|^2 dy.
\end{aligned}
$$

Taking expectation of both sides of (3.33) we obtain

$$
\mathbb{E}\Big\| \sum_{j=1}^{M} e^{2\pi i y R(j)(\omega)} \Big\|_p^p \le M^{p-2}\mathbb{E}\Big\| \sum_{j=1}^{M} e^{2\pi i y R(j)(\omega)} \Big\|_2^2 \le 2M^{p-1/2}. \qquad (3.34)
$$

This concludes one direction of (1.20).

We now want to obtain a lower bound. As opposed to the Poisson process, the Random walk is not increasing, therefore it is not enough to consider $R(M)$. We turn around this difficulty using the fact that the random walk is a martingale. By Doob's martingale inequality

$$
\begin{aligned}
\mathbb{E} \sup_{1 \le j \le M} |R(j)|^2 &\le \left(\frac{2}{2-1} \right)^2 \mathbb{E}|R(M)|^2 \\
&= 4\mathbb{E}|R(M)|^2 \\
&= 4\mathbb{E}\left(\sum_{j=1}^{M} X_j \right)^2 \\
&= 4\mathbb{E}\left(\sum_{j=1}^{M} X_j^2 + 2 \sum_{1 \le i \le j \le M} X_i X_j \right) \\
&= 4 \sum_{j=1}^{M} \mathbb{E}(X_j^2) + 8 \sum_{1 \le i \le j \le M} \mathbb{E}(X_j)\mathbb{E}(X_i) \\
&= 4M + 0 \\
&= 4M
\end{aligned}
\tag{3.35}
$$

where we use $\mathbb{E}(X_j^2) = 1$ and $\mathbb{E}(X_j) = 0$ for $j = 1, 2, \ldots, M$. Then let us consider the following argument. By Markov's inequality, we have

$$
\begin{aligned}
\mathbb{P}\left(\sup_{1 \le j \le M} |R(j)|^2(\omega) \ge 16M \right) &\le \frac{\mathbb{E} \sup_{1 \le j \le M} |R(j)|^2}{16M} \\
&\le 4M/16M \\
&\le 1/4.
\end{aligned}
\tag{3.36}
$$

For ω except these

$$
\sup_{1 \le j \le M} |R(j)|(\omega) \le 4M^{1/2}, \tag{3.37}
$$

and thus

$$
\begin{aligned}
\Big\| \sum_{j=1}^{M} e^{2\pi i y R(j)(\omega)} \Big\|_p^p &\ge \int_0^{1/200M^{1/2}} \Big| \sum_{j=1}^{M} e^{2\pi i y R(j)(\omega)} \Big|^p dy \\
&\ge \int_0^{1/200M^{1/2}} \Big| \sum_{j=1}^{M} \cos 2\pi y R(j)(\omega) \Big|^p dy \\
&\ge M^{p-1/2}/200 \cdot 2^p.
\end{aligned}
$$

Therefore

$$
\mathbb{E} \Big\| \sum_{j=1}^{M} e^{2\pi i y R(j)} \Big\|_p^p \ge M^{p-1/2}/100 \cdot 2^{p+2}.
$$

This concludes (1.20).

To show (1.21), observe that for any ω and any sequence $\{a_j\}_{j=1}^M$, $|a_j| \le 1$ we have

$$\begin{aligned}\Big\|\sum_{j=1}^M a_j e^{2\pi i y R(j)(\omega)}\Big\|_p^p &\le \Big\|\sum_{j=1}^M a_j e^{2\pi i y R(j)(\omega)}\Big\|_\infty^{p-2}\Big\|\sum_{j=1}^M a_j e^{2\pi i y R(j)(\omega)}\Big\|_2^2 \\ &\le M^{p-2}\Big\|\sum_{j=1}^M e^{2\pi i y N(j)(\omega)}\Big\|_2^2.\end{aligned} \tag{3.38}$$

Using (3.38) and (1.20) we obtain

$$\mathbb{E}\sup_{|a_j|\le 1}\Big\|\sum_{j=1}^M a_j e^{2\pi i y R(j)(\omega)}\Big\|_p^p \le 2M^{p-1/2} \lesssim \mathbb{E}\Big\|\sum_{j=1}^M e^{2\pi i y R(j)(\omega)}\Big\|_p^p.$$

This concludes the proof of (1.21).

We have with probability at most $1/\log^2(M+1)$

$$\sup_{1\le j\le M}|R(j)|^2(\omega) \ge 4M\log^2(M+1). \tag{3.39}$$

To see this, we follow the same steps as we do in (3.36).

$$\begin{aligned}\mathbb{P}\Big(\sup_{1\le j\le M}|R(j)|^2(\omega) \ge 4M\log^2(M+1)\Big) &\le \frac{\mathbb{E}\sup_{1\le j\le M}|R(j)|^2}{4M\log^2(M+1)} \\ &\le 4M/4M\log^2(M+1) \\ &\le 1/\log^2(M+1).\end{aligned} \tag{3.40}$$

For ω except these

$$\sup_{1\le j\le M}|R(j)|(\omega) \le 2M^{1/2}\log(M+1), \tag{3.41}$$

and thus

$$\begin{aligned}\Big\|\sum_{j=1}^M e^{2\pi i y R(j)(\omega)}\Big\|_p^p &\ge \int_0^{1/200M^{1/2}\log(M+1)}\Big|\sum_{j=1}^M e^{2\pi i y R(j)(\omega)}\Big|^p dy \\ &\ge \int_0^{1/200M^{1/2}\log(M+1)}\Big|\sum_{j=1}^M \cos 2\pi y R(j)(\omega)\Big|^p dy \\ &\ge M^{p-1/2}/200\cdot 2^p\log(M+1).\end{aligned}$$

Now we aim to show (1.13).

$$\mathbb{E} \sup_{|a_j|\le 1} \Big\| \sum_{j=1}^{M} a_j e^{2\pi i y R(j)(\omega)} \Big\|_p^p \le 2M^{p-1/2}$$

implies that

$$\mathbb{P}\Big[\sup_{|a_j|\le 1} \Big\| \sum_{j=1}^{M} a_j e^{2\pi i y R(j)(\omega)} \Big\|_p^p \ge 2M^{p-1/2} \log(M+1) \Big] \le 1/\log(M+1).$$

Therefore for a set probability at least $1 - 3/\log(M+1)$ we have

$$100 \cdot 2^{p+2} \log^2(M+1) \Big\| \sum_{j=1}^{M} e^{2\pi i y R(j)(\omega)} \Big\|_p^p \ge \sup_{|a_j|\le 1} \Big\| \sum_{j=1}^{M} a_j e^{2\pi i y R(j)(\omega)} \Big\|_p^p.$$

By Remark 2.4, we have

$$100 \cdot 2^{p+2} C M^{\varepsilon} \Big\| \sum_{j=1}^{M} e^{2\pi i y R(j)(\omega)} \Big\|_p^p \ge \sup_{|a_j|\le 1} \Big\| \sum_{j=1}^{M} a_j e^{2\pi i y R(j)(\omega)} \Big\|_p^p.$$

This proves (1.22).

□

3.4. Perturbation of a wide class of sparse sets

Proof of Theorem 1.5 Again simply

$$\mathbb{E} \Big\| \sum_{j\in A} e^{2\pi i y N(j)} \Big\|_\infty = |A|.$$

We now show (1.24) for $p = 2$.

$$\mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y N(j)}\Big\|_2^2 = \int_{\mathbb{T}}\Big|\sum_{j\in A} e^{2\pi i y N(j)}\Big|^2 dy = \int_{\mathbb{T}}\sum_{j,k\in A} e^{2\pi i y[N(j)-N(k)]}dy$$
$$= \sum_{j,k\in A}\mathbb{P}[N(|j-k|)=0]$$
$$= \sum_{j,k\in A} e^{-|j-k|}.$$

We estimate this last term from both above and below as:

$$|A| \le \sum_{j,k\in A} e^{-|j-k|} = |A| + 2\sum_{k\in A}\sum_{\substack{j\in A\\ j<k}} e^{j-k} \le |A| + 2\sum_{k\in A}\sum_{j=1}^{k-1} e^{j-k}$$
$$\le |A| + 2\sum_{k\in A} e^{-k}\int_1^k e^j dj$$
$$\le |A| + 2\sum_{k\in A} e^{-k}e^k$$
$$= |A| + 2\sum_{k\in A} 1$$
$$= 3|A|.$$

Here we note that if for example $A = \Lambda_{D,k}\cap[0,M]$, then any element a of A of the form $1+\sum_{j=1}^{k-1} d_j D^j$ there exists $b\in A$ with $b = a-1$. Also if a is of the form $3+\sum_{j=1}^{k-1} d_j D^j$, there exists $b\in A$ with $b = a-2$. Therefore for large $|A|$

$$\sum_{k\in A}\sum_{\substack{j\in A\\ j<k}} e^{j-k} \ge \frac{|A|-2}{3}[e^{-1}+e^{-2}] \gtrsim |A|.$$

Thus for such sets the contribution of this nondiagonal term is as much as the contribution of the diagonal term, as opposed to the situation we encountered in estimates over powers. Consequently all claims of the theorem for $p = 2$ follows.

We turn to showing (1.24) for $p = 4$. From this (1.24) follows for all $2 \le p \le 4$ immediately. We estimate the $\gtrsim$ part of (1.24) by using the inequality (2.1) and $p = 2$

case as

$$|A|^{p/2} \leq \mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y N(j)}\Big\|_2^p \leq \mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y N(j)}\Big\|_p^p.$$

Thus we focus on the $\lesssim$ part. We make some reductions that help us later. We may assume that A contains no element in $[0, e^{100}]$, have more elements than e^{100}, and whenever $j, k \in A,\ k > j$ we have $k - j \geq e^{100}$. For if we have our result under these assumptions, then for a generic set $A \subset \mathbb{Z}_+$ we can do the following decomposition. For this let us partition the set A into sets as follows.

$$A_{-1} := A \cap [0, e^{100}], \qquad A_n = (A \setminus A_{-1}) \cap (\lceil e^{100} \rceil \mathbb{Z} + n), \qquad 0 \leq n < \lceil e^{100} \rceil. \tag{3.42}$$

Some $A_n,\ -1 \leq n < \lceil e^{100} \rceil$ may have more elements than e^{100}, let N_1 be the set of these n, and let N_2 be the set of the other n, including $n = -1$. By Minkowski's inequality

$$\begin{aligned}
\mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y N(j)}\Big\|_4^4 &= \mathbb{E}\Big\| \sum_{n\in N_1}\sum_{j\in A_n} e^{2\pi i y N(j)} + \sum_{n\in N_2}\sum_{j\in A_n} e^{2\pi i y N(j)}\Big\|_4^4 \\
&\leq \mathbb{E}\Big[\Big\| \sum_{n\in N_1}\sum_{j\in A_n} e^{2\pi i y N(j)}\Big\|_4 + \Big\| \sum_{n\in N_2}\sum_{j\in A_n} e^{2\pi i y N(j)}\Big\|_4\Big]^4 \\
&\leq 15\Big[\mathbb{E}\Big\| \sum_{n\in N_1}\sum_{j\in A_n} e^{2\pi i y N(j)}\Big\|_4^4 + \mathbb{E}\Big\| \sum_{n\in N_2}\sum_{j\in A_n} e^{2\pi i y N(j)}\Big\|_4^4\Big] \\
&\leq 15e^{404}\Big[\sum_{n\in N_1}\mathbb{E}\Big\| \sum_{j\in A_n} e^{2\pi i y N(j)}\Big\|_4^4 + \sum_{n\in N_2}\mathbb{E}\Big\| \sum_{j\in A_n} e^{2\pi i y N(j)}\Big\|_4^4\Big].
\end{aligned}$$

Applying our result under the assumptions above to $A_n,\ n \in N_1$, and estimating sums over $A_n,\ n \in N_2$ trivially

$$\begin{aligned}
\leq 15e^{404}\Big[C\sum_{n\in N_1}|A_n|^2 + \sum_{n\in N_2}|A_n|^4\Big] &\leq 15e^{404}\Big[\sum_{n\in N_1}C|A_n|^2 + e^{200}\sum_{n\in N_2}|A_n|^2\Big] \\
&\leq 15e^{404}[C + e^{200}]|A|^2.
\end{aligned}$$

So proving our result under these assumptions is enough.

We begin with

$$\mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y N(j)}\Big\|_4^4 = \sum_{j_1, j_2, k_1, k_2 \in A} \mathbb{P}[N(j_1) + N(j_2) = N(k_1) + N(k_2)].$$

As before, the set A^4 can be written as $A^4 = A_{11} \cup A_{12} \cup A_{21} \cup A_{22} \cup A_3$, and for any $A_{ij}, i, j = 1, 2$ we have

$$\begin{aligned}\sum_{A_{ij}} \mathbb{P}[N(j_1) + N(j_2) = N(k_1) + N(k_2)] &= |A| \sum_{j,k \in A} \mathbb{P}[N(j) = N(k)] \\ &= |A| \sum_{j,k \in A} \mathbb{P}[N(|j-k|) = 0] \\ &\leq 3|A|^2.\end{aligned}$$

Thus from these sets we have $\approx |A|^2$ contribution.

We now decompose $A_3 = A_{31} \cup A_{32} \cup A_{33} \cup A_{34}$ as before. Because we lack the good bounds on the nondiagonal sum in $p = 2$ case, unlike the powers case, estimating the sets A_{31}, A_{34} is now more difficult. Without loss of generality we can just concentrate on A_{31}.

$$\sum_{A_{31}} \mathbb{P}[N(j_1) + N(j_2) = N(k_1) + N(k_2)] = \sum_{A_{31}} \mathbb{P}[N(j_1) - N(k_1) = N(k_2) - N(j_2)].$$

As $j_2 > k_2$, and $N(t)$ is increasing we must have $N(j_1) - N(k_1) \geq 0$ and $N(k_2) - N(j_2) \leq 0$. Therefore for these to be equal they must both be zero. Hence

$$= \sum_{A_{31}} \mathbb{P}[N(j_1) - N(k_1) = 0 = N(j_2) - N(k_2)].$$

At this stage if we just ignore one of these equation and crudely estimate by

$$\leq |A|^2 \sum_{\substack{j,k \in A \\ j > k}} \mathbb{P}[N(j) - N(k) = 0] \leq 3|A|^3$$

$|A|^3$ is a bound that is not good enough. Therefore we cannot follow this approach, and must employ more delicate analysis similar in vein to the one we deploy for sets A_{32}, A_{33}.

We decompose

$$A_{311} := \{(j_1, j_2, k_1, k_2) \in A_{31} : (k_1, j_1) \cap (k_2, j_2) = \emptyset\}$$
$$A_{312} := \{(j_1, j_2, k_1, k_2) \in A_{31} : (k_1, j_1) \cap (k_2, j_2) \neq \emptyset\}$$

For A_{311} we have

$$\begin{aligned}
&\sum_{A_{311}} \mathbb{P}[N(j_1) - N(k_1) = 0 = N(j_2) - N(k_2)] \\
=&\sum_{A_{311}} \mathbb{P}[N(j_1) - N(k_1) = 0]\mathbb{P}[N(j_2) - N(k_2) = 0] \\
\leq&\Big[\sum_{\substack{j,k\in A \\ j>k}} \mathbb{P}[N(j) - N(k) = 0]\Big]^2 \\
\leq&9|A|^2,
\end{aligned}$$

where we used the estimation

$$\sum_{\substack{j,k\in A \\ j>k}} \mathbb{P}[N(j) - N(k) = 0] \leq 3|A|$$

in the third line above. In A_{312} we have two possibilities, either $j_2 - k_2 \geq j_1 - k_1$ or $j_2 - k_2 < j_1 - k_1$. Without loss of generality we may assume the first. Then

$$\begin{aligned}
&\sum_{\substack{A_{312} \\ j_2-k_2\geq j_1-k_1}} \mathbb{P}[N(j_1) - N(k_1) = 0 = N(j_2) - N(k_2)] \\
\leq &\sum_{\substack{A_{312} \\ j_2-k_2\geq j_1-k_1}} \mathbb{P}[N(j_2) - N(k_2) = 0].
\end{aligned}$$

The conditions that (k_1, j_1) must intersect (k_2, j_2), and $j_2 - k_2 \geq j_1 - k_1$ forces j_1, k_1 to be within an interval of length $3(j_2 - k_2) - 2$, that is there are $3(j_2 - k_2) - 1$ integers to choose them from. By using the sparsity condition on our set A the last sum above can be bounded by

$$\leq \sum_{\substack{j_2,k_2\in A\\ j_2>k_2}} \mathbb{P}[N(j_2-k_2)=0]3C_A(j_2-k_2)^{\alpha}3C_A(j_2-k_2)^{\alpha} = 9C_A^2 \sum_{\substack{j_2,k_2\in A\\ j_2>k_2}} e^{-(j_2-k_2)}(j_2-k_2)^{2\alpha}$$
$$\leq 9C_A^2|A|^2 \sup_{x\geq 0} xe^{-x}$$
$$\leq 9C_A^2|A|^2.$$

Therefore contribution from A_{31}, A_{34} is $\lesssim_{C_A} |A|^2$.

We are ready to move on to A_{32}. This, together with its symmetric counterpart A_{33}, represent the most important, generic cases. We decompose $A_{32} = A_{321} \cup A_{322}$ as before. The sum over A_{321} can be written as

$$\sum_{A_{321}} \mathbb{P}[N(j_1)+N(j_2)=N(k_1)+N(k_2)] = \sum_{A_{321}} \mathbb{P}[N(j_1)-N(k_1)=N(k_2)-N(j_2)]$$
$$\leq \sum_{a=0}^{\infty} \Big[\sum_{\substack{j,k\in A\\ j<k}} \mathbb{P}[N(k-j)=a]\Big]^2,$$

and use the ideas introduced above, but this is not sufficient. For the extra variable a introduced to relate j_1, j_2, k_1, k_2 to each other leads to inefficiencies. Instead we will relate these variables to each other directly with an inequality. We further decompose A_{321} into the set of quadruples A_{3211} with $k_2 - j_2 \geq j_1 - k_1$, which we consider without loss of generality, and the remaining ones comprising A_{3212} . We define $f(x) := 4\sqrt{x\log x}$ for $x \geq 1$. To each quadruple (j_1, j_2, k_1, k_2) we consider, we assign three events

$$\Omega_* := \{\omega \in \Omega : N(j_1)(\omega) - N(k_1)(\omega) = N(k_2)(\omega) - N(j_2)(\omega)\}, \tag{3.43}$$

$$\Omega_1 := \{\omega \in \Omega : |N(j_1)(\omega) - N(k_1)(\omega) - (j_1-k_1)| < f(k_2-j_2)\}, \tag{3.44}$$

$$\Omega_2 := \{\omega \in \Omega : |N(k_2)(\omega) - N(j_2)(\omega) - (k_2-j_2)| < f(k_2-j_2)\}. \tag{3.45}$$

By using these sets it allows us to write

$$\Omega_* \subseteq [\Omega_* \cap \Omega_1 \cap \Omega_2] \cup \Omega_1^c \cup \Omega_2^c,$$

and thus also

$$\mathbb{P}[\Omega_*] \leq \mathbb{P}[\Omega_* \cap \Omega_1 \cap \Omega_2] + \mathbb{P}[\Omega_1^c] + \mathbb{P}[\Omega_2^c].$$

We have by Lemma 2.4

$$\begin{aligned}
\mathbb{P}[\Omega_1^c] &= \mathbb{P}[\{\omega \in \Omega : |N(j_1)(\omega) - N(k_1)(\omega) - (j_1 - k_1)| \geq f(k_2 - j_2)\}] \\
&= \mathbb{P}[\{\omega \in \Omega : |N(j_1)(\omega) - N(k_1)(\omega) - (j_1 - k_1)| \geq 4\sqrt{(k_2 - j_2)\log(k_2 - j_2)}\}] \\
&\leq \mathbb{P}[\{\omega \in \Omega : |N(j_1)(\omega) - N(k_1)(\omega) - (j_1 - k_1)| \geq 4\sqrt{(k_1 - j_1)\log(k_2 - j_2)}\}] \\
&\leq 2e^{\frac{-16\log(k_2 - j_2)}{4}} \\
&= 2(k_2 - j_2)^{-4},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}[\Omega_2^c] &= \mathbb{P}[\{\omega \in \Omega : |N(j_2)(\omega) - N(k_2)(\omega) - (j_2 - k_2)| \geq f(k_2 - j_2)\}] \\
&= \mathbb{P}[\{\omega \in \Omega : |N(k_2)(\omega) - N(j_2)(\omega) - (j_2 - k_2)| \geq 4\sqrt{(k_2 - j_2)\log(k_2 - j_2)}\}] \\
&\leq 2e^{\frac{-16\log(k_2 - j_2)}{4}} \\
&= 2(k_2 - j_2)^{-4},
\end{aligned}$$

that is, the contribution of these two terms is harmless. For the main term $\Omega_* \cap \Omega_1 \cap \Omega_2$ we first observe that if this set contains even one ω, we have by (3.44), (3.45) and triangle inequality

$$\begin{aligned}
&|(k_2 - j_2) - (j_1 - k_1)| \\
=&\left|(k_2 - j_2) - (j_1 - k_1) + N(j_1)(\omega) - N(k_1)(\omega) - [N(k_2)(\omega) - N(j_2)(\omega)]\right| \\
\leq&|N(j_1)(\omega) - N(k_1)(\omega) - (j_1 - k_1)| + |N(k_2)(\omega) - N(j_2)(\omega) - (k_2 - j_2)| \\
<&2f(k_2 - j_2).
\end{aligned} \tag{3.46}$$

So the set $\Omega_* \cap \Omega_1 \cap \Omega_2$ is empty for quadruples that does not satisfy this relation. For quadruples satisfying this relation, we estimate the probability of the set as follows.

$$\begin{aligned}
\mathbb{P}[\Omega_* \cap \Omega_1 \cap \Omega_2] \leq \mathbb{P}[\Omega_*] &= \sum_{a=0}^{\infty} \mathbb{P}[N(j_1) - N(k_1) = a = N(k_2) - N(j_2)] \\
&= \sum_{a=0}^{\infty} \mathbb{P}[N(j_1 - k_1) = a]\mathbb{P}[N(k_2 - j_2) = a] \\
&\leq \frac{1}{\sqrt{2\pi(k_2 - j_2)}} \sum_{a=0}^{\infty} \mathbb{P}[N(j_1 - k_1) = a] \\
&= \frac{1}{\sqrt{k_2 - j_2}}.
\end{aligned}$$

where we passed from the second line to the third line by using Lemma (2.6). Combining all of these we can write

$$\begin{aligned}
&\sum_{A_{3211}} \mathbb{P}[N(j_1) + N(j_2) = N(k_1) + N(k_2)] \\
&= \sum_{A_{3211}} \mathbb{P}[N(j_1) - N(k_1) = N(k_2) - N(j_2)] \\
&\leq \sum_{A_{3211}} \mathbb{P}[\Omega_* \cap \Omega_1 \cap \Omega_2] + \mathbb{P}[\Omega_1^c] + \mathbb{P}[\Omega_2^c] \\
&\leq \sum_{\substack{A_{3211} \\ |(k_2 - j_2) - (j_1 - k_1)| < 2f(k_2 - j_2)}} \frac{1}{\sqrt{k_2 - j_2}} + \sum_{A_{3211}} 4(k_2 - j_2)^{-4}.
\end{aligned}$$

The second sum is easy to estimate:

$$\begin{aligned}
\sum_{A_{3211}} 4(k_2 - j_2)^{-4} &\leq \sum_{A_{3211}} 4(k_2 - j_2)^{-2}(j_1 - k_1)^{-2} \\
&\leq \sum_{\substack{j_2, k_2 \in A \\ k_2 > j_2}} \sum_{\substack{j_1, k_1 \in A \\ j_1 > k_1}} 4(k_2 - j_2)^{-2}(j_1 - k_1)^{-2} \\
&\leq 4\Big[\sum_{\substack{j,k \in A \\ k > j}} (k - j)^{-2}\Big]^2.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}\sum_{\substack{j,k\in A\\ k>j}}(k-j)^{-2} \leq \sum_{k\in A}\sum_{1<j<k}(k-j)^{-2} &< \sum_{k\in A}\sum_{n=1}^{\infty}\frac{1}{n^2}\\ &= \sum_{k\in A}\frac{\pi^2}{6}\\ &< 2|A|.\end{aligned}$$

Therefore the second sum is bounded by $16|A|^2$.

Now we concentrate on the first sum. Clearly it is bounded by

$$\sum_{\substack{j_2,k_2\in A\\ k_2>j_2}}\frac{1}{\sqrt{k_2-j_2}}|\{(j_1,k_1)\in A^2 : |(k_2-j_2)-(j_1-k_1)| < 2f(k_2-j_2)\}|.$$

Thus it remains to estimate the cardinality of the set within the summation. The condition $|(k_2-j_2)-(j_1-k_1)| < 2f(k_2-j_2)$ implies that for a fixed $k_1 \in A$ the element j_1 lies in the following interval

$$\{(j_1,k_1)\in A^2 : -2f(k_2-j_2)+(k_2-j_2)+k_1 < j_1 < 2f(k_2-j_2)+(k_2-j_2)+k_1\}.$$

This shows that the radius of the interval that contains j_1 is $2f(k_2-j_2)$. By the sparsity condition on our set A, there are at most $C_A(2f(k_2-j_2))^{\alpha}$ elements j_1 for each $k_1 \in A$. Thus we obtain

$$\begin{aligned}&\leq C_A\sum_{\substack{j_2,k_2\in A\\ k_2>j_2}}\frac{|A|}{\sqrt{k_2-j_2}}2\log(k_2-j_2)^{\frac{\alpha}{2}}(k_2-j_2)^{\frac{\alpha}{2}}\\ &= 2C_A|A|\sum_{\substack{j_2,k_2\in A\\ k_2>j_2}}(\log k_2-j_2)^{\frac{\alpha}{2}}(k_2-j_2)^{\frac{\alpha-1}{2}}.\end{aligned}$$

We can estimate this sum with a dyadic decomposition as follows.

$$
\begin{aligned}
&= 2C_A|A| \sum_{n=0}^{\infty} \sum_{\substack{j,k\in A \\ 2^n \le k-j<2^{n+1}}} (\log k_2 - j_2)^{\frac{\alpha}{2}} (k_2 - j_2)^{\frac{\alpha-1}{2}} \\
&\le 2C_A|A| \sum_{n=0}^{\infty} \sum_{\substack{j,k\in A \\ 2^n \le k-j<2^{n+1}}} ((n+1)\log 2)^{\frac{\alpha}{2}} 2^{n\frac{\alpha-1}{2}}.
\end{aligned}
$$

To go further we need to estimate the cardinality of $\{(j,k) \in A^2 : 2^n \le k - j < 2^{n+1}\}$. For a pair (j,k) to be in this set, for a fixed j we must have k within the interval $[j+2^n, j+2^{n+1})$. By sparsity of A, this means for a fixed j we can have at most $C_A 2^{n\alpha}$ values of k. Therefore

$$
\begin{aligned}
&\le 2C_A^2|A| \sum_{j\in A} \sum_{n=0}^{\infty} (n+1)^{\frac{\alpha}{2}} 2^{n\frac{3\alpha-1}{2}} \\
&\le 2C_A^2|A|^2 \sum_{n=0}^{\infty} (n+1)^{\frac{\alpha}{2}} 2^{n\frac{3\alpha-1}{2}}.
\end{aligned}
$$

By the ratio test this sum converges to a positive constant C_α provided that $\alpha < 1/3$. This concludes the estimation of the sum over A_{321} with a constant that depends only on C_A, α We finally consider the sum over A_{322}.

We partitioned this set into four:

$$
\begin{aligned}
A_{3221} &:= \{(j_1, j_2, k_1, k_2) \in A_{322} : [k_1, j_1] \subseteq (j_2, k_2)\} \\
A_{3222} &:= \{(j_1, j_2, k_1, k_2) \in A_{322} : [j_2, k_2] \subseteq (k_1, j_1)\} \\
A_{3223} &:= \{(j_1, j_2, k_1, k_2) \in A_{322} : k_1 < j_2 < j_1 < k_2\} \\
A_{3224} &:= \{(j_1, j_2, k_1, k_2) \in A_{322} : j_2 < k_1 < k_2 < j_1\}.
\end{aligned}
$$

First two and the last two are handled in the same way so we will only consider A_{3221}, A_{3223}. On A_{3221} the condition $[k_1, j_1] \subseteq (j_2, k_2)$ means

$$
\sum_{A_{3221}} \mathbb{P}[N(j_1) + N(j_2) = N(k_1) + N(k_2)] = \sum_{A_{3221}} \mathbb{P}[N(j_2) - N(k_1) = N(k_2) - N(j_1)].
$$

Since $j_2 < k_1$, but $k_2 > j_1$, and since $(j_2, k_1) \cap (j_1, k_2) = \emptyset$ we must have

$$\begin{aligned}
&= \sum_{A_{3221}} \mathbb{P}[N(j_2) - N(k_1) = 0 = N(k_2) - N(j_1)] \\
&= \sum_{A_{3221}} \mathbb{P}[N(k_1 - j_2) = 0]\mathbb{P}[N(k_2 - j_1) = 0] \\
&\leq \Big[\sum_{k>j} \mathbb{P}[N(k-j) = 0] \Big]^2.
\end{aligned}$$

We estimated the sum inside the square by $|A|$. So contribution from here is at most $|A|^2$.

On A_{3223} the condition $k_1 < j_2 < j_1 < k_2$ means $(k_1, j_2) \cap (j_1, k_2) = \emptyset$, and again we reduce to the case A_{321}. Hence we obtain (1.24) for $p = 4$.

We simply have by the fact that $|a_j| \leq 1$ $j \in \mathbb{N}$

$$\Big\| \sum_{j \in A} a_j e^{2\pi i y N(j)} \Big\|_{2n}^{2n} \leq \Big\| \sum_{j \in A} e^{2\pi i y N(j)} \Big\|_{2n}^{2n} \tag{3.47}$$

for any $n \in \mathbb{N}$. By using (3.47) and (1.24), we have

$$\mathbb{E} \sup_{|a_j| \leq 1} \Big\| \sum_{j \in A} a_j e^{2\pi i y N(j)} \Big\|_4^4 \lesssim \mathbb{E} \Big\| \sum_{j \in A} e^{2\pi i y N(j)} \Big\|_4^4.$$

Therefore we obtain

$$\begin{aligned}
\mathbb{E} \sup_{|a_j| \leq 1} \Big\| \sum_{j \in A} a_j e^{2\pi i y N(j)} \Big\|_p^p &\leq \mathbb{E} \sup_{|a_j| \leq 1} \Big\| \sum_{j \in A} a_j e^{2\pi i y N(j)} \Big\|_4^p \\
&\leq \mathbb{E} \Big\| \sum_{j \in A} e^{2\pi i y N(j)} \Big\|_4^p \\
&\lesssim |A|^{p/2} \\
&\lesssim \mathbb{E} \Big\| \sum_{j \in A} e^{2\pi i y N(j)} \Big\|_p^p.
\end{aligned}$$

Now we are ready to show (1.26). By the inequality (2.1), we have

$$\Big\| \sum_{j \in A} e^{2\pi i y N(j)} \Big\|_p^p \geq \Big\| \sum_{j \in A} e^{2\pi i y N(j)} \Big\|_2^p \geq |A|^{p/2}$$

for any $p \geq 2$. Therefore

$$\begin{aligned}
&\mathbb{P}\Big[\sup_{|a_j|\leq 1}\Big\|\sum_{j\in A} a_j e^{2\pi i y N(j)}\Big\|_p^p \geq |A|^{\varepsilon}\Big\|\sum_{j\in A} e^{2\pi i y N(j)}\Big\|_p^p\Big] \\
\leq &\mathbb{P}\Big[\sup_{|a_j|\leq 1}\Big\|\sum_{j\in A} a_j e^{2\pi i y N(j)}\Big\|_p^p \geq |A|^{\frac{p}{2}+\varepsilon}\Big].
\end{aligned}$$

By Markov's inequality and (1.25)

$$\begin{aligned}
\leq |A|^{-\frac{p}{2}-\varepsilon}\mathbb{E}\sup_{|a_j|\leq 1}\Big\|\sum_{j\in A} a_j e^{2\pi i y N(j)}\Big\|_p^p &\leq C_\varepsilon |A|^{p/2}|A|^{-p/2}|A|^{-\varepsilon} \\
&\leq C_\varepsilon |A|^{-\varepsilon}.
\end{aligned}$$

By taking limits, we obtain

$$\begin{aligned}
\lim_{|A|\to\infty}\mathbb{P}\Big[\sup_{|a_j|\leq 1}\Big\|\sum_{j\in A} a_j e^{2\pi i y N(j)}\Big\|_p^p \geq |A|^{\varepsilon}\Big\|\sum_{j\in A} e^{2\pi i y N(j)}\Big\|_p^p &\leq \lim_{|A|\to\infty} C_\varepsilon |A|^{-\varepsilon} \\
&= 0.
\end{aligned}$$

Hence we are done.

□

CHAPTER 4

PERTURBATION OF POWERS

In this chapter, our aim is to give detailed proofs of Theorem 1.6, Theorem 1.7 and Theorem 1.8, respectively.

4.1. Arithmetic progressions of larger step size

Proof of Theorem 1.6 Simply for any ω

$$\Big\|\sum_{j=1}^{M} e^{2\pi i y N(jM^r)(\omega)}\Big\|_\infty = M.$$

Let us consider the $p = 2$ case.

$$\begin{aligned} M \le \mathbb{E}\Big\|\sum_{j=1}^{M} e^{2\pi i y N(jM^r)}\Big\|_2^2 = \mathbb{E}\sum_{j,k=1}^{M}\int_{\mathbb{T}} e^{2\pi i y[N(jM^r)-N(kM^r)]}dy &= \sum_{j,k=1}^{M}\mathbb{P}[N(jM^r)-N(kM^r)] \\ &= \sum_{j,k=1}^{M} e^{-|j-k|M^r} \\ &= M + 2\sum_{k=2}^{M} e^{-kM^r}\sum_{j<k} e^{jM^r}. \end{aligned}$$

Then

$$\begin{aligned} \sum_{k=2}^{M} e^{-kM^r}\sum_{j<k} e^{jM^r} &= \sum_{k=2}^{M} e^{-kM^r}\sum_{j=1}^{k-1}(e^{M^r})^j \\ &\le \sum_{k=2}^{M} e^{-kM^r}\frac{[e^{kM^r}-e^{M^r}]}{e^{M^r}-1} \\ &\le 2\sum_{k=2}^{M} e^{-kM^r}e^{(k-1)M^r} \\ &= 2\sum_{k=2}^{M} e^{-M^r} \\ &\le 2(M-1)e^{-M^r}. \end{aligned}$$

Finally we have the bound $M + 4(M-1)e^{-M^r} \leq M + 2\min\{M, C_r\}$, where C_r is a constant that depends only on r.

Now armed with these we move on to the case $p = 4$, where we hope to start to see effects of randomization stronger.

$$\mathbb{E}\Big\| \sum_{j=1}^{M} e^{2\pi i y N(jM^r)} \Big\|_4^4 = \sum_{\substack{j_i, k_i = 1 \\ i=1,2}}^{M} \mathbb{P}[N(j_1 M^r) + N(j_2 M^r) = N(k_1 M^r) + N(k_2 M^r).$$

Let $A := \{1, 2, 3 \ldots, M\}$ and we decompose the set A^4 to which any quadruple (j_1, j_2, k_1, k_2) belongs as in the proof of Theorem 1.3,

$$A^4 = A_{11} \cup A_{12} \cup A_{21} \cup A_{22} \cup A_3,$$

and all except A_3 are dealt with easily as before. For example on A_{11}

$$\sum_{A_{11}} \mathbb{P}[N(j_2 M^r) = N(k_2 M^r)] = M \sum_{j,k=1}^{M} \mathbb{P}[N(|j-k|M^r) = 0] = M \sum_{j,k=1}^{M} e^{-|j-k|M^r}$$

which as above gives a $\approx M^2$ term. Each of the $A_{ij},\ i, j = 1, 2$ above gives the same contribution. We thus proceed to A_3. We decompose

$$A_3 = A_{31} \cup A_{32} \cup A_{33} \cup A_{34}$$

as in the proof of Theorem 1.3, and A_{31}, A_{34} can be dispatched utilizing exactly the same ideas. For example the sum on A_{31} is bounded by

$$M^2 \sum_{j>k} \mathbb{P}[N(jM^r) - N(kM^r) = 0] = M^2 \cdot 2Me^{-M^r} \leq C_r M^{3-r}$$

where Cr is the same as the above one. The set A_{34} also give the same contribution. Due to this contribution of these two will be dominated by the contribution of other sets.

We now move on to A_{32}. We decompose $A_{32} = A_{321} \cup A_{322}$, in the first of which are contained those (j_1, j_2, k_1, k_2) for which the intervals $(k_1, j_1), (j_2, k_2)$ are disjoint, and in the second those for which they intersect. The sum over A_{321} is written as

$$\sum_{a=0}^{\infty}\Big[\sum_{k>j}\mathbb{P}[N((k-j)M^r)=a]\Big]^2 = \left(\sum_{a\le M^{2r}} + \sum_{a>M^{2r}}\right)\Big[\sum_{k>j}\mathbb{P}[N((k-j)M^r)=a]\Big]^2 = \mathbf{I}+\mathbf{II}.$$

We estimate the inner sum. Let $b = k - j$. Then

$$\sum_{k>j}\mathbb{P}[N((k-j)M^r)=a] = \sum_{b=1}^{M-1}(M-b)\mathbb{P}[N(bM^r)=a] = \frac{M^{ar}}{a!}\sum_{b=1}^{M-1}(M-b)b^a e^{-bM^r}$$

As $g(b) = b^a e^{-bM^r}$ for $b \ge 0$ is a function that increases up to a supremum and then decreases, this can be estimated by

$$\le \frac{M^{1+ar}}{a!}\sum_{b=1}^{M-1} b^a e^{-bM^r} \le \frac{M^{1+ar}}{a!}2\sup_{b\ge 0} b^a e^{-bM^r} + \frac{M^{1+ar}}{a!}\int_0^{\infty} b^a e^{-bM^r}db.$$

The supremum is attained when $b = aM^{-r}$. Now our aim is to estimate the integral above. We apply the change of variables. Let $b = M^r u$. Then $db = M^r du$.

$$\begin{aligned}\int_0^{\infty} b^a e^{-bM^r}db = \int_0^{\infty}\frac{u^a}{M^{ra}}e^{-u}M^{-r}du &= M^{-r-ra}\int_0^{\infty}u^a e^{-u}\\ &= M^{-r-ra}\Gamma(a+1)\\ &= M^{-r-ra}a!,\end{aligned}$$

where we use the property $\Gamma(a+1) = a!,\ a \in \mathbb{N}$ of the Gamma function. Plugging these and using Stirling's formula we have

$$= 2\frac{Ma^a}{a!e^a} + \frac{M^{1+ar}}{a!}\frac{a!}{M^{r(a+1)}} \le \frac{M}{\sqrt{a}} + M^{1-r}.$$

In the summation range of **I** this last sum is bounded by $2M/\sqrt{a}$, and thus

$$\mathbf{I} = \sum_{a \le M^{2r}} \Big[\sum_{k>j} \mathbb{P}[N((k-j)M^r) = a]\Big]^2 \le \Big[\sum_{k>j} \mathbb{P}[N((k-j)M^r) = 0]\Big]^2 + \sum_{1 \le a \le M^{2r}} \frac{4M^2}{a}$$
$$\le C_r^2 + 4M^2(1 + 2r \log M).$$

In the summation range of **II** the term M^{1-r} dominates, hence

$$\begin{aligned}
\mathbf{II} = \sum_{a>M^{2r}} \Big[\sum_{k>j} \mathbb{P}[N((k-j)M^r) = a]\Big]^2 &\le 2M^{1-r} \sum_{a>M^{2r}} \sum_{k>j} \mathbb{P}[N((k-j)M^r) = a] \\
&= 2M^{1-r} \sum_{k>j} \sum_{a>M^{2r}} \mathbb{P}[N((k-j)M^r) = a] \\
&\le 2M^{1-r} \sum_{k>j} 1 \\
&\le 2M^{1-r} M^2 \\
&\le 2M^{3-r}.
\end{aligned}$$

Thus we conclude the estimate on A_{321}.

As in the previous proof, this case of non intersection is the generic case, and intersection cases contained in A_{322} reduce to previous cases. The set A_{33} is similar to A_{32}. We thus obtain the upper bound $\lesssim_r M^2 \log M + M^{3-r}$.

Now we give an estimation for the lower bound. By the inequality (2.1) we have

$$M^2 \lesssim \mathbb{E}\Big\| \sum_{j=1}^{M} e^{2\pi i y N(jM^r)} \Big\|_2^4 \lesssim \mathbb{E}\Big\| \sum_{j=1}^{M} e^{2\pi i y N(jM^r)} \Big\|_4^4. \tag{4.1}$$

This gives M^2 term. Let $\omega \notin B = \{\omega \mid N(M^{r+1}) \ge 2M^{r+1}\}$ and $0 < y < 1/100M^{r+1}$. Then

$$\begin{aligned}
\Big\| \sum_{j=1}^{M} e^{2\pi i y N(jM^r)} \Big\|_4^4 \ge \int_0^{1/100M^{r+1}} \Big| \sum_{j=1}^{M} \cos 2\pi y N(jM^r)(\omega) \Big|^4 dy &\ge \frac{M^4}{1600 M^{r+1}} \\
&= \frac{M^{3-r}}{1600}.
\end{aligned}$$

Hence we have

$$\mathbb{E}\Big\|\sum_{j=1}^{M} e^{2\pi i y N(jM^r)}\Big\|_4^4 \geq \frac{M^{3-r}}{3200}. \tag{4.2}$$

By combining (4.1) and (4.2) we obtain

$$M^2 + M^{3-r} \lesssim \mathbb{E}\Big\|\sum_{j=1}^{M} e^{2\pi i y N(jM^r)}\Big\|_4^4.$$

This completes the proof of Theorem 1.6.

$\square$

4.2. Perturbation of powers II, $p = 6$

Proof of Theorem 1.7 By employing $p = 2$ case and the inequality (2.1) the $\gtrsim$ part of (1.35) can be estimated as follows

$$|A|^3 \lesssim \mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y N(j^d)}\Big\|_2^6 \leq \mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y N(j^d)}\Big\|_6^6.$$

Now we focus on the $\lesssim$ part. For $d = 2$ case we utilize $p = 4$ case, and we then obtain

$$\begin{aligned}\mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y N(j^d)}\Big\|_6^6 &\leq \Big\|\sum_{j\in A} e^{2\pi i y N(j^d)}\Big\|_\infty^2 \mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y N(j^d)}\Big\|_4^4 \\ &\lesssim |A|^2|A|^2\log^2(1+|A|) \\ &= |A|^4\log^2(1+|A|).\end{aligned}$$

Thus we suppose $d \geq 3$. We may suppose that A contains no element in $[1, e^{100}]$, and have more elements than e^{100}. It follows immediately that whenever $j, k \in A,\ k > j$ we have by the Mean Value Theorem that $k^d - j^d \geq e^{200}$.

We first transform our sum

$$\mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y N(j^d)}\Big\|_6^6 = \sum_{j_1,j_2,j_3,k_1,k_2,k_3\in A} \mathbb{P}\Big[\sum_{i=1}^{3} N(j_i^d) = \sum_{i=1}^{3} N(k_i^d)\Big].$$

The set A^6 to which any vector $(j_1, j_2, j_3, k_1, k_2, k_3)$ belongs can be written as follows

$$A^4 = \Big[\bigcup_{1 \le a, b \le 3} A_{ab} \Big] \cup A_4.$$

Here $A_{ab},\ a, b \in \{1, 2, 3\}$ is the set of vectors $(j_1, j_2, j_3, k_1, k_2, k_3)$ for which $j_a = k_b$. The set A_4 contains those elements of A^6 that is in none of these $A_{ab}, a, b \in \{1, 2, 3\}$. We have

$$\begin{aligned} \sum_{A_{33}} \mathbb{P}\Big[\sum_{i=1}^{3} N(j_i^d) = \sum_{i=1}^{3} N(k_i^d) \Big] &= \sum_{j_1, j_2, j_3, k_1, k_2 \in A} \mathbb{P}\Big[\sum_{i=1}^{2} N(j_i^d) = \sum_{i=1}^{2} N(k_i^d) \Big] \\ &= |A| \sum_{j_1, j_2, k_1, k_2 \in A} \mathbb{P}\Big[\sum_{i=1}^{2} N(j_i^d) = \sum_{i=1}^{2} N(k_i^d) \Big]. \end{aligned}$$

This last sum is bounded by

$$|A|^2 \log^2(1 + |A|) \quad \text{and} \quad |A|^2$$

for $d = 2$ and $d \ge 3$, respectively. The same argument gives the same result for other A_{ab}. Thus the contribution from these sets is harmless. Therefore we are left with handling A_4. We consider the set

$$A_{41} =: \{(j_1, j_2, j_3, k_1, k_2, k_3) \in A_4 : j_1 \le j_2 \le j_3,\ \ k_1 \le k_2 \le k_3\}.$$

Observe that there exists different orderings of $j_i, 1 \le i \le 3$, and 6 different orderings of $k_i, 1 \le i \le 3$. Since by simple change of variables these can be transformed into each other, we obtain

$$\sum_{A_4} \mathbb{P}\Big[\sum_{i=1}^{3} N(j_i^d) = \sum_{i=1}^{3} N(k_i^d) \Big] \le 36 \sum_{A_{41}} \mathbb{P}\Big[\sum_{i=1}^{3} N(j_i^d) = \sum_{i=1}^{3} N(k_i^d) \Big].$$

Therefore we focus on this A_{41}. We split this set into

$$A_{411} =: \{(j_1, j_2, j_3, k_1, k_2, k_3) \in A_{41} : j_3 < k_3\}$$
$$A_{412} =: \{(j_1, j_2, j_3, k_1, k_2, k_3) \in A_{41} : k_3 < j_3\}.$$

Since the sets A_{411} and A_{412} are symmetric, it is enough to handle A_{411}. If we further decompose the set A_{411} we have

$$A_{4111} =: \{(j_1, j_2, j_3, k_1, k_2, k_3) \in A_{41} : j_i < k_i \ 1 \le i \le 3\}$$

and $A_{4112} := A_{411} \setminus A_{4111}$. The set A_{4111} is easier to handle than the set A_{4112}. Actually the set A_{4112} is the main contributor. Let us first deal with the easier set A_{4111}. By a simple observation we have

$$\begin{aligned}
\sum_{A_{4111}} \mathbb{P}\Big[\sum_{i=1}^{3} N(j_i^d) = \sum_{i=1}^{3} N(k_i^d)\Big] &= \sum_{A_{4111}} \mathbb{P}\Big[\sum_{i=1}^{2} \underbrace{N(j_i^d) - N(k_i^d)}_{\le 0} = \underbrace{N(k_3^d) - N(j_3^d)}_{\ge 0}\Big] \\
&= \sum_{A_{4111}} \mathbb{P}\Big[\sum_{i=1}^{3} N(k_i^d) - N(j_i^d) = 0\Big] \\
&= \sum_{A_{4111}} \mathbb{P}[N(k_i^d) - N(j_i^d) = 0, \ 1 \le i \le 3].
\end{aligned}$$

Here we need to deal with three cases. In order to describe these let I_i denote the interval (j_i, k_i) for each $1 \le i \le 3$. The first case is $I_3 \cap I_2 = \emptyset$. In this case I_3 cannot intersect I_1 either and we have

$$\begin{aligned}
&\sum_{\substack{A_{4111} \\ I_3 \cap I_2 = \emptyset}} \mathbb{P}[N(k_i^d) - N(j_i^d) = 0, \ 1 \le i \le 3] \\
&= \sum_{\substack{A_{4111} \\ I_3 \cap I_2 = \emptyset}} \mathbb{P}[N(k_3^d - j_3^d) = 0]\mathbb{P}[N(k_i^d) - N(j_i^d) = 0, \ 1 \le i \le 2] \\
&= \sum_{\substack{j,k \in A \\ k > j}} \mathbb{P}[N(k^d - j^d) = 0] \cdot \sum_{\substack{j_1, j_2, k_1, k_2 \in A \\ k_1 > j_1, \ k_2 > j_2}} \mathbb{P}[N(k_i^d) - N(j_i^d) = 0, \ 1 \le i \le 2].
\end{aligned}$$

In the second chapter where we estimated the $p = 4$ case, we estimated the first sum by 2 and the second sum by $2|A|^2$. Thus the contribution from this term is harmless. Moreover this bound can easily be improved to a constant . The second case is $I_1 \cap I_2 = \emptyset$, and by the same arguments we obtain the same bounds. The third case is when I_2 intersects both I_1, I_3. In this case

$$\begin{aligned}
\sum_{\substack{A_{4111} \\ I_1\cap I_2\neq\emptyset \\ I_3\cap I_2\neq\emptyset}} \mathbb{P}[N(k_i^d) - N(j_i^d) = 0,\ 1 \le i \le 3] &= \sum_{\substack{A_{4111} \\ I_1\cap I_2\neq\emptyset \\ I_3\cap I_2\neq\emptyset}} \mathbb{P}[N(k_3^d) - N(j_1^d) = 0] \\
&\le \sum_{\substack{j_1,k_3\in A \\ j_1<j_2,j_3,k_1,k_2<k_3}} \mathbb{P}[N(k_3^d - j_1^d) = 0] \\
&\le \underbrace{\sum_{\substack{j_1,k_3\in A \\ j_1<k_3}} (k_3 - j_1)^4 \mathbb{P}[N(k_3^d - j_1^d) = 0]}_{I},
\end{aligned}$$

where we utilized

$$\sum_{j_1<k_i<k_3} 1 < (k_3 - j_1) \quad \text{for} \quad i = 1, 2,$$

and

$$\sum_{j_1<j_i<k_3} 1 < (k_3 - j_1) \quad \text{for} \quad j = 2, 3$$

to pass from the second line to the third line. I can be estimated by

$$\begin{aligned}
\sum_{\substack{j,k\in A \\ j<k}} (k-j)^4 e^{-(k^d - j^d)} \le \sum_{\substack{j,k\in A \\ j<k}} (k-j)^4 e^{-(k^2 - j^2)} &\le \sum_{\substack{j,k\in A \\ j<k}} (k-j)^4 e^{-(k-j)} e^{(k-j)(1-(k+j))} \\
&\le \sum_{\substack{j,k\in A \\ j<k}} (k-j)^4 e^{-(k-j)} e^{1-(k+j)}.
\end{aligned}$$

The final sum is bounded by

$$\begin{aligned}
\Big[\sup_{x\ge0} x^4 e^{-x}\Big] \cdot \sum_{\substack{j,k\in A \\ j<k}} e^{1-(k+j)} \le \Big[\sup_{x\ge0} x^4 e^{-x}\Big] \cdot \sum_{\substack{j,k\in A \\ j<k}} e e^{-k} e^{-j} &\le 5e\Big[\sum_{j\in\mathbb{N}} e^{-j}\Big]^2 \\
&\le 5e.
\end{aligned}$$

To reach the desired result in the above we used the estimations

$$\sup_{x\geq 0} x^4 e^{-x} \leq 5 \quad \text{and} \quad \sum_{j\in\mathbb{N}} e^{-j} = \frac{1}{e-1} < 1.$$

This finishes the estimation of A_{4111}.

We now move on to A_{4112}. Within A_{411} we already assumed $j_3 < k_3$. So there are four options that emerge from relations of j_1, k_1 and j_2, k_2. We already covered one of these in A_{4111}. Then there remains three cases:

CASE I. $j_1 > k_1,\ j_2 > k_2$.
This case itself is separated into two
CASE Ia.$k_1 \leq k_2 < j_1 \leq j_2 \leq j_3 < k_3$
CASE Ib. $k_1 < j_1 < k_2 < j_2 \leq j_3 < k_3$

CASE II. $j_1 > k_1,\ j_2 < k_2$.
This case also separates into two
CASE IIa. $k_1 < j_1 \leq j_2 \leq j_3 < k_2 \leq k_3$
CASE IIb. $k_1 < j_1 \leq j_2 < k_2 < j_3 < k_3$

CASE III. $j_1 < k_1,\ j_2 > k_2$.
This means $j_1 < k_1 \leq k_2 < j_2 \leq j_3 < k_3$.

We will say that the pair j_i, k_i have positive orientation if $j_i < k_i$ and negative orientation if $j_i > k_i$. In all of these five subcases we have positive orientation for one or two values of $1 \leq i \leq 3$ and for the remaining values of i we have negative orientation. We observe that within A_{4112} the open intervals obtained by the pairs in one orientation cannot intersect the pairs in the other orientation. To see this suppose that the open intervals (j_a, k_a) (k_b, j_b) intersect. This implies $j_a < j_b$ and hence $a < b$. But it also implies $k_b < k_a$ and thus $b < a$ contradicting the first implication.

With this observation we will prove **CASE Ia.** Since this property is shared in all cases their proofs will follow from the same arguments. So proofs of these other cases will not be explicitly written.

Let A_{41121} be the set of vectors satisfying **CASE Ia.** We have

$$\sum_{A_{41121}} \mathbb{P}\Big[\sum_{i=1}^{3} N(j_i^d) = \sum_{i=1}^{3} N(k_i^d)\Big] = \sum_{A_{41121}} \mathbb{P}\Big[\sum_{i=1}^{2} N(j_i^d) - N(k_i^d) = N(k_3^d) - N(j_3^d)\Big].$$

We define $f(x) := 4\sqrt{x \log x}$ for $x \geq 1$. Our approach will be the same one which we used in the proof of Theorem 1.5. For this purpose, to each vector in A_{41121} we assign the events

$$\Omega_* := \Big\{\sum_{i=1}^{2} N(j_i^d) - N(k_i^d) = N(k_3^d) - N(j_3^d)\Big\},$$

$$\Omega_i := \Big\{\Big|N(k_i^d)(\omega) - N(j_i^d)(\omega) - (k_i^d - j_i^d)\Big| < \max_{1 \leq n \leq 3} f(|k_n^d - j_n^d|)\Big\} \qquad 1 \leq i \leq 3.$$

Let m be an index that maximizes $|k_i^d - j_i^d|$, since f is an increasing function, it also maximizes $f(|k_i^d - j_i^d|)$. By using these sets we can write

$$\mathbb{P}[\Omega_*] \subseteq \mathbb{P}[\Omega_* \cap \Omega_1 \cap \Omega_2 \cap \Omega_3] \cup \mathbb{P}[\Omega_1^c] \cup \mathbb{P}[\Omega_2^c] \cup \mathbb{P}[\Omega_3^c],$$

and this yields

$$\mathbb{P}[\Omega_*] \leq \mathbb{P}[\Omega_* \cap \Omega_1 \cap \Omega_2 \cap \Omega_3] + \mathbb{P}[\Omega_1^c] + \mathbb{P}[\Omega_2^c] + \mathbb{P}[\Omega_3^c].$$

We have by Lemma 2.4

$$\begin{aligned}
\mathbb{P}[\Omega_1^c] &= \mathbb{P}\Big[\Big\{\Big|N(k_1^d)(\omega) - N(j_1^d)(\omega) - (k_1^d - j_1^d)\Big| \geq f(|k_m^d - j_m^d|)\Big\}\Big] \\
&= \mathbb{P}\Big[\Big\{\Big|N(k_1^d)(\omega) - N(j_1^d)(\omega) - (k_1^d - j_1^d)\Big| \geq 4\sqrt{|k_m^d - j_m^d| \log |k_m^d - j_m^d|}\Big\}\Big] \\
&\leq \mathbb{P}\Big[\Big\{\Big|N(k_1^d)(\omega) - N(j_1^d)(\omega) - (k_1^d - j_1^d)\Big| \geq 4\sqrt{|k_1^d - j_1^d|}\sqrt{\log |k_m^d - j_m^d|}\Big\}\Big] \\
&\leq 2e^{\frac{-16 \log |k_m - j_m|}{4}} \\
&= 2|k_m - j_m|^{-4},
\end{aligned}$$

$$
\begin{aligned}
\mathbb{P}[\Omega_2^c] &= \mathbb{P}\Big[\Big\{\Big|N(k_2^d)(\omega) - N(j_2^d)(\omega) - (k_2^d - j_2^d)\Big| \geq f(|k_m^d - j_m^d|)\Big\}\Big] \\
&= \mathbb{P}\Big[\Big\{\Big|N(k_2^d)(\omega) - N(j_2^d)(\omega) - (k_2^d - j_2^d)\Big| \geq 4\sqrt{|k_m^d - j_m^d| \log |k_m^d - j_m^d|}\Big\}\Big] \\
&\leq \mathbb{P}\Big[\Big\{\Big|N(k_2^d)(\omega) - N(j_2^d)(\omega) - (k_2^d - j_2^d)\Big| \geq 4\sqrt{|k_2^d - j_2^d|}\sqrt{\log |k_m^d - j_m^d|}\Big\}\Big] \\
&\leq 2e^{\frac{-16 \log |k_m - j_m|}{4}} \\
&= 2|k_m - j_m|^{-4},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{P}[\Omega_3^c] &= \mathbb{P}\Big[\Big\{\Big|N(k_3^d)(\omega) - N(j_3^d)(\omega) - (k_3^d - j_3^d)\Big| \geq f(|k_m^d - j_m^d|)\Big\}\Big] \\
&= \mathbb{P}\Big[\Big\{\Big|N(k_3^d)(\omega) - N(j_3^d)(\omega) - (k_3^d - j_3^d)\Big| \geq 4\sqrt{|k_m^d - j_m^d| \log |k_m^d - j_m^d|}\Big\}\Big] \\
&\leq \mathbb{P}\Big[\Big\{\Big|N(k_3^d)(\omega) - N(j_3^d)(\omega) - (k_3^d - j_3^d)\Big| \geq 4\sqrt{|k_3^d - j_3^d|}\sqrt{\log |k_m^d - j_m^d|}\Big\}\Big] \\
&\leq 2e^{\frac{-16 \log |k_m - j_m|}{4}} \\
&= 2|k_m - j_m|^{-4}
\end{aligned}
$$

The contribution of these terms are small. For the main term $\Omega_* \cap \Omega_1 \cap \Omega_2 \cap \Omega_3$ if this set is not empty we have by triangle inequality

$$
\begin{aligned}
&\Big|(k_3^d - j_3^d) - (j_1^d - k_1^d) - (j_2^d - k_2^d)\Big| \\
&\quad = \Big|\sum_{i=1}^{2} N(j_i^d) - N(k_i^d) - [N(k_3^d) - N(j_3^d)] + (k_3^d - j_3^d) - (j_1^d - k_1^d) - (j_2^d - k_2^d)\Big| \\
&= \Big|N(j_1^d) - N(k_1^d) - (j_1^d - k_1^d) + N(j_2^d) - N(k_2^d) - (j_2^d - k_2^d) + (k_3^d - j_3^d) - N(k_3^d) - N(j_3^d)\Big| \\
&\leq \Big|N(j_1^d) - N(k_1^d) - (j_1^d - k_1^d)\Big| + \Big|N(j_2^d) - N(k_2^d) - (j_2^d - k_2^d)\Big| + \Big|N(k_3^d) - N(j_3^d) - (k_3^d - j_3^d)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\leq f(|k_m^d - j_m^d|) + f(|k_m^d - j_m^d|) + f(|k_m^d - j_m^d|) \\
&= 3f(|k_m^d - j_m^d|).
\end{aligned}
$$

Then we have

$$
\Big|(k_3^d - j_3^d) - (j_1^d - k_1^d) - (j_2^d - k_2^d)\Big| \leq 3f(|k_m^d - j_m^d|). \tag{4.3}
$$

Thus the set $\Omega_* \cap \Omega_1 \cap \Omega_2 \cap \Omega_3$ is empty if vectors do not satisfy this relation. For vectors in A_{411211}, the probability of this set is equal to

$$\sum_{a=\lfloor k_3^d - j_3^d - f(|k_m^d - j_m^d|)\rfloor}^{\lceil k_3^d - j_3^d + f(|k_m^d - j_m^d|)\rceil} \mathbb{P}\Big[\sum_{i=1}^{2} N(j_i^d) - N(k_i^d) = a = N(k_3^d) - N(j_3^d)\Big]$$
$$= \sum_{a=\lfloor k_3^d - j_3^d - f(|k_m^d - j_m^d|)\rfloor}^{\lceil k_3^d - j_3^d + f(|k_m^d - j_m^d|)\rceil} \mathbb{P}\Big[\sum_{i=1}^{2} N(j_i^d) - N(k_i^d) = a\Big]\mathbb{P}[N(k_3^d - j_3^d) = a].$$

By Lemma 2.6 this is less than or equal to

$$\sum_{a=\lfloor k_3^d - j_3^d - f(|k_m^d - j_m^d|)\rfloor}^{\lceil k_3^d - j_3^d + f(|k_m^d - j_m^d|)\rceil} \mathbb{P}\Big[\sum_{i=1}^{2} N(j_i^d) - N(k_i^d) = a\Big]\frac{1}{\sqrt{k_3^d - j_3^d}}.$$

Then summing over a yields

$$\sum_{a=\lfloor k_3^d - j_3^d - f(|k_m^d - j_m^d|)\rfloor}^{\lceil k_3^d - j_3^d + f(|k_m^d - j_m^d|)\rceil} \mathbb{P}\Big[\sum_{i=1}^{2} N(j_i^d) - N(k_i^d) = a\Big]\frac{1}{\sqrt{k_3^d - j_3^d}} \le \frac{1}{\sqrt{k_3^d - j_3^d}}.$$

Combining all of these we can write

$$\sum_{A_{41121}} \mathbb{P}\Big[\sum_{i=1}^{2} N(j_i^d) - N(k_i^d) = N(k_3^d) - N(j_3^d)\Big]$$
$$\le \sum_{A_{41121}} \mathbb{P}[\Omega_* \cap \Omega_1 \cap \Omega_2 \cap \Omega_3] + \mathbb{P}[\Omega_1^c] + \mathbb{P}[\Omega_2^c] + \mathbb{P}[\Omega_3^c]$$
$$\le \sum_{A_{411211}} \frac{1}{\sqrt{k_3^d - j_3^d}} + \sum_{A_{41121}} 6|k_m^d - j_m^d|^{-4}.$$

The second sum is easy to handle. Since $|k_m^d - j_m^d|^{-1} \le |k_i^d - j_i^d|^{-1}$ for $1 \le i \le 3$ we then have

$$\begin{aligned}\sum_{A_{41121}} 6|k_m^d - j_m^d|^{-4} &\le 6 \sum_{A_{41121}} \prod_{i=1}^{3} |k_i^d - j_i^d|^{-\frac{4}{3}} \\ &\le 6 \sum_{\substack{j_3,k_3\in A\\ k_3>j_3}} \sum_{\substack{j_2,k_2\in A\\ j_2>k_2}} \sum_{\substack{j_1,k_1\in A\\ j_1>k_1}} \prod_{i=1}^{3} |k_i^d - j_i^d|^{-\frac{4}{3}} \\ &= 6\Big[\sum_{\substack{j,k\in A\\ k>j}} (k^d - j^d)^{-\frac{4}{3}}\Big]^3.\end{aligned}$$

But we have

$$\begin{aligned}\sum_{\substack{j,k\in A\\ k>j}} (k^d - j^d)^{-\frac{4}{3}} \le \sum_{k\in A} \sum_{1<j<k} (k^2 - j^2)^{-\frac{4}{3}} &= \sum_{k\in A} \sum_{1<j<k} (k+j)^{-\frac{4}{3}} (k-j)^{-\frac{4}{3}} \\ &< \sum_{k\in A} k^{-\frac{4}{3}} \sum_{1<j<k} (k-j)^{-\frac{4}{3}} \\ &\le C_{\frac{4}{3}}^2,\end{aligned}$$

where

$$C_{4/3} := \sum_{j\in\mathbb{N}} j^{-4/3}.$$

Thus the second sum is bounded by an absolute constant. We now try to estimate the first sum. Clearly it is bounded by

$$\sum_{\substack{j_3,k_3\in A\\ k_3>j_3}} \frac{1}{\sqrt{k_3^d - j_3^d}} |A_{411211}^{j_3,k_3}|.$$

where the set $A_{411211}^{j_3,k_3}$ is the set of vectors in A_{411211} with fixed j_3, k_3. This set lies within

$$\{(j_1, k_1, j_2, k_2) \in A^4 : j_1 > k_1,\ j_2 > k_2,\ \text{(4.3) holds}\}. \tag{4.4}$$

The condition (4.3) implies the condition

$$\left|(k_3^d - j_3^d) - (j_1^d - k_1^d) - (j_2^d - k_2^d)\right| < 3f(k_1^d - j_1^d) < 3f(2(k_1^d - j_1^d)) < 6f(k_3^d - j_3^d). \tag{4.5}$$

To see this, for $m = 1$

$$(j_1^d - k_1^d) + (j_2^d - k_2^d) - 3f(|k_1^d - j_1^d|) < (k_3^d - j_3^d).$$

This implies that

$$(j_1^d - k_1^d) < k_3^d - j_3^d + 3f(|k_1^d - j_1^d|) < 2(k_3^d - j_3^d).$$

Similarly for $m = 2$

$$(j_1^d - k_1^d) + (j_2^d - k_2^d) - 3f(|k_2^d - j_2^d|) < (k_3^d - j_3^d).$$

this also implies that

$$(j_2^d - k_2^d) < k_3^d - j_3^d + 3f(|k_2^d - j_2^d|) < 2(k_3^d - j_3^d).$$

We again reach (4.5) for $m = 2$. Therefore (4.4) is the subset of the set

$$\{(j_1, k_1, j_2, k_2) \in A^4 : j_1 > k_1,\ j_2 > k_2,\ \text{(4.5) holds}\}.$$

If we crudely estimate the number of pairs (j_2, k_2), the cardinality of this set can be estimated by

$$\frac{|A|^2}{2} \sup_C \left|\{(j_1, k_1) \in A^2 : j_1 > k_1,\ |j_1^d - k_1^d + C - (k_3^d - j_3^d)| < 6f(k_3^d - j_3^d)\}\right|,$$

where C is a natural number less than $(k_3^d - j_3^d) + 6f(k_3^d - j_3^d)$. This bound on C is due to the condition (4.5). This can be simplified to bounding

$$\frac{|A|^2}{2} \sup_{D \le C \le D^2} \left|\{(j_1, k_1) \in A^2 : j_1 > k_1,\ |j_1^d - k_1^d - C| < D\}\right|,$$

where C is an integer in $(-6f(k_3^d - j_3^d), k_3^d - j_3^d)$ and $D = 6f(k_3^d - j_3^d)$. Clearly the maximum is attained when $C \geq 6f(k_3^d - j_3^d)$. By Theorem 2.11 this supremum is bounded by $6f^{2/d}(k_3^d - j_3^d)$. So combining all of these and using Remark 2.5 we obtain for small $\varepsilon > 0$,

$$\begin{aligned}\sum_{\substack{j_3,k_3\in A\\ k_3>j_3}} \frac{1}{\sqrt{k_3^d - j_3^d}} |A_{411211}^{j_3,k_3}| &\leq 3|A|^2 \sum_{\substack{j_3,k_3\in A\\ k_3>j_3}} \frac{f^{\frac{2}{d}}(k_3^d - j_3^d)}{\sqrt{k_3^d - j_3^d}} \\ &\leq 12|A|^2 \sum_{\substack{j,k\in A\\ k>j}} \frac{\log^{\frac{1}{d}}(k^d - j^d)}{(k^d - j^d)^{\frac{1}{2}-\frac{1}{d}}} \\ &\leq C_\varepsilon |A|^2 \sum_{\substack{j,k\in A\\ k>j}} (k^d - j^d)^{\frac{1}{d}-\frac{1}{2}+\varepsilon}.\end{aligned}$$

Then by Lemma 2.12 the last sum is bounded by

$$\begin{aligned}\sum_{\substack{1\leq j,k\leq|A|\\ j<k}} (k^d - j^d)^{\frac{1}{d}-\frac{1}{2}+\varepsilon} &\leq \sum_{1\leq k\leq|A|} k^{(d-1)(\frac{1}{d}-\frac{1}{2}+\varepsilon)} \sum_{1\leq j<k} (k-j)^{\frac{1}{d}-\frac{1}{2}+\varepsilon} \\ &\leq \sum_{1\leq k\leq|A|} k^{(d-1)(\frac{1}{d}-\frac{1}{2}+\varepsilon)} \sum_{1\leq j<k} j^{\frac{1}{d}-\frac{1}{2}+\varepsilon} \\ &\leq C_{d,\varepsilon} \sum_{1\leq k\leq|A|} k^{(d-1)(\frac{1}{d}-\frac{1}{2}+\varepsilon)} k^{\frac{1}{d}+\frac{1}{2}+\varepsilon} \\ &\leq C_{d,\varepsilon} \sum_{1\leq k\leq|A|} k^{2-\frac{d}{2}+d\varepsilon}.\end{aligned}$$

This last sum is bounded by

$$\leq \begin{cases} |A|^{3/2+\varepsilon} \text{ for } d = 3 \\ |A|^{1+\varepsilon} \text{ for } d = 4 \\ |A| \text{ for } d \geq 5. \end{cases} \tag{4.6}$$

We finally have

$$\sum_{\substack{j_3,k_3\in A\\ k_3>j_3}} \frac{1}{\sqrt{k_3^d - j_3^d}} |A_{411211}^{j_3,k_3}| \lesssim_{d,\varepsilon} \begin{cases} |A|^{7/2+\varepsilon} \text{ for } d = 3 \\ |A|^{3+\varepsilon} \text{ for } d = 4 \\ |A|^3 \text{ for } d \geq 5 \end{cases}$$

This finishes the proof.

□

4.3. Perturbation of powers III, $p = 2n$

Proof of Theorem 1.8 We already proved the cases $p = 2, 4, 6$ with Theorems 1.3,1.7. Thus we may suppose that $p \geq 8$, and so $d \geq n \geq 3$. We may also suppose that A contains no element in $[1, e^{100d}]$, and have more elements than e^{100d}. If $j, k \in A$, $k > j$ then we have by the Mean Value Theorem that $k^d - j^d \geq e^{100d(d-1)}$.

Let $\mathbf{j} = (j_1, j_2, \cdots, j_n)$ and $\mathbf{k} = (k_1, k_2, \cdots, k_n)$ denote vectors in A^n. We first convert our sum

$$\mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_{2n}^{2n} = \sum_{(\mathbf{j},\mathbf{k})\in A^n} \mathbb{P}\Big[\sum_{i=1}^{n} N(j_i^d) = \sum_{i=1}^{n} N(k_i^d) \Big].$$

We decompose $A^{2n} = A^* \cup A^{**}$, where A^{**} contains those vectors $(\mathbf{j}, \mathbf{k})$ that have $j_a = k_b$ for some entries $1 \leq a, b, \leq n$. The main contribution comes from A^*, and A^{**} can easily be coped with induction.

We let A_{ab}^{**} for some fixed $1 \leq a, b \leq n$ denote the subset of A^{**} consisting of vectors $(\mathbf{j}, \mathbf{k})$ with $j_a = k_b$. For the base step $n = 1$ we have by Theorem 1.3

$$\mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_2^2 \lesssim_p |A|.$$

Suppose that our statement is true for $n - 1$, that is, we have

$$\mathbb{E}\Big\| \sum_{j\in A} e^{2\pi i y N(j^d)} \Big\|_{2(n-1)}^{2(n-1)} \lesssim_p |A|^{n-1}.$$

Then after reordering, relabeling and applying the inductive hypothesis we obtain

$$\begin{aligned}\sum_{A_{ab}^{**}} \mathbb{P}\Big[\sum_{i=1}^{n} N(j_i^d) = \sum_{i=1}^{n} N(k_i^d)\Big] &= |A| \sum_{(\mathbf{j}',\mathbf{k}')\in A^{2(n-1)}} \mathbb{P}\Big[\sum_{i=1}^{n-1} N(j_i^d) = |A| \sum_{i=1}^{n-1} N(k_i^d)\Big] \\ &= |A|\mathbb{E}\Big\|\sum_{j\in A} e^{2\pi i y N(j^d)}\Big\|_{2(n-1)}^{2(n-1)} \\ &\lesssim_p |A|^n.\end{aligned}$$

Thus we are just left with handling A^*. We consider the set

$$A_1^* =: \{(\mathbf{j},\mathbf{k}) \in A^{2n} : j_1 \le j_2 \le \ldots \le j_n, \;\; k_1 \le k_2 \le \ldots \le k_n\}.$$

Observe that since there are $n!$ different orderings of for j_i, and similarly for k_i and since each ordering can be turned into each other by a relabeling, we have

$$\sum_{A^*} \mathbb{P}\Big[\sum_{i=1}^{n}(j_i^d) = \sum_{i=1}^{n} N(k_i^d)\Big] \le (n!)^2 \sum_{A_1^*} \mathbb{P}\Big[\sum_{i=1}^{n} N(j_i^d) = \sum_{i=1}^{n} N(k_i^d)\Big].$$

The pair j_i, k_i is said to have positive orientation if $j_i < k_i$ and negative orientation if $j_i > k_i$. For every vector $(\mathbf{j},\mathbf{k}) \in A_1^*$ this splits up pairs or indices $1 \le i \le n$ of the vector into two sets $\sigma_+(\mathbf{j},\mathbf{k})$ and $\sigma_-(\mathbf{j},\mathbf{k})$. Observe that open intervals $(j_a, k_a) \in \sigma_+(\mathbf{j},\mathbf{k})$ and $(k_b, j_b) \in \sigma_-(\mathbf{j},\mathbf{k})$ cannot intersect. For intersection implies $j_a < j_b$ and hence $a < b$, but it also implies $k_b < k_a$ and thus $b < a$. Now armed with this observation we proceed.

$$\begin{aligned}&\sum_{A_1^*} \mathbb{P}\Big[\sum_{i=1}^{n} N(j_i^d) = \sum_{i=1}^{n} N(k_i^d)\Big] \\ =&\sum_{A_1^*} \mathbb{P}\Big[\sum_{i\in\sigma_-(\mathbf{j},\mathbf{k})} N(j_i^d) - N(k_i^d) = \sum_{i\in\sigma_+(\mathbf{j},\mathbf{k})} N(k_i^d) - N(j_i^d)\Big].\end{aligned} \tag{4.7}$$

Notice that $\sigma_+(\mathbf{j},\mathbf{k}), \sigma_-(\mathbf{j},\mathbf{k})$ may be empty, in this case the sum over that set is taken to be zero. Let us define $f(x) := 4\sqrt{nx\log x}$ for $x \ge 1$. To every vector in A_1^* we assign the events

$$\Omega_* := \Big\{ \sum_{i\in\sigma_-(\mathbf{j},\mathbf{k})} N(j_i^d) - N(k_i^d) = \sum_{i\in\sigma_+(\mathbf{j},\mathbf{k})} N(k_i^d) - N(j_i^d) \Big\},$$
$$\Omega_i := \Big\{ \big| N(k_i^d)(\omega) - N(j_i^d)(\omega) - (k_i^d - j_i^d) \big| < \max_{1\le l\le n} f(|k_l^d - j_l^d|) \Big\}, \qquad 1 \le i \le n.$$

Let m denote an index that maximizes $|k_i^d - j_i^d|$ and thus $f(|k_i^d - j_i^d|)$. By utilizing these sets we can define $\Omega_{**} := \Omega_* \cap \Omega_1 \cap \Omega_2 \ldots \cap \Omega_n$, and obtain

$$\mathbb{P}[\Omega_*] \le \mathbb{P}[\Omega_{**}] + \sum_{i=1}^{n} \mathbb{P}[\Omega_i^c].$$

We have by Lemma 2.4

$$\begin{aligned}
\mathbb{P}[\Omega_i^c] &= \mathbb{P}\Big[\Big\{ \big| N(k_i^d)(\omega) - N(j_i^d)(\omega) - (k_i^d - j_i^d) \big| \ge f(|k_m^d - j_m^d|) \Big\}\Big] \\
&= \mathbb{P}\Big[\Big\{ \big| N(k_i^d)(\omega) - N(j_i^d)(\omega) - (k_i^d - j_i^d) \big| \ge 4\sqrt{n|k_m^d - j_m^d| \log |k_m^d - j_m^d|} \Big\}\Big] \\
&\le \mathbb{P}\Big[\Big\{ \big| N(k_i^d)(\omega) - N(j_i^d)(\omega) - (k_i^d - j_i^d) \big| \ge 4\sqrt{|k_i^d - j_1^i|}\sqrt{n \log |k_m^d - j_m^d|} \Big\}\Big] \\
&\le 2e^{\frac{-16n \log |k_m - j_m|}{4}} \\
&= 2|k_m - j_m|^{-4n},
\end{aligned}$$

that is, the contribution of these terms are small. As for the main term Ω_{**} we first notice that if this set is nonempty for a vector $(\mathbf{j},\mathbf{k})$ we have by triangle inequality

$$\begin{aligned}
&\Big|\sum_{i\in\sigma_+(\mathbf{j},\mathbf{k})}(k_i^d-j_i^d)-\sum_{i\in\sigma_-(\mathbf{j},\mathbf{k})}(j_i^d-k_i^d)\Big|\\
&=\Big|\sum_{i\in\sigma_+(\mathbf{j},\mathbf{k})}(k_i^d-j_i^d)-\sum_{i\in\sigma_-(\mathbf{j},\mathbf{k})}(j_i^d-k_i^d)+\sum_{i\in\sigma_-(\mathbf{j},\mathbf{k})}[N(j_i^d)-N(k_i^d)]-\sum_{i\in\sigma_+(\mathbf{j},\mathbf{k})}[N(k_i^d)-N(j_i^d)]\Big|\\
&=\Big|\sum_{i\in\sigma_+(\mathbf{j},\mathbf{k})}[(k_i^d-j_i^d)-N(k_i^d)+N(j_i^d)]+\sum_{i\in\sigma_-(\mathbf{j},\mathbf{k})}[N(j_i^d)-N(k_i^d)-(j_i^d-k_i^d)]\Big|\\
&\le\sum_{i\in\sigma_-(\mathbf{j},\mathbf{k})}\big|N(j_i^d)-N(k_i^d)-(j_i^d-k_i^d)\big|+\sum_{i\in\sigma_+(\mathbf{j},\mathbf{k})}\big|k_i^d-j_i^d)-N(k_i^d)+N(j_i^d)\big|\\
&\le\sum_{i\in\sigma_-(\mathbf{j},\mathbf{k})}+\sum_{i\in\sigma_+(\mathbf{j},\mathbf{k})}f(|k_m^d-j_m^d|)\\
&=nf(|k_m^d-j_m^d|).
\end{aligned} \tag{4.8}$$

Therefore the set Ω_{**} is empty for vectors that do not satisfy this relation. Let A_{11}^* be the set of vectors in A_1^* satisfying this relation. Observe that if in a vector $(\mathbf{j},\mathbf{k})$ one of the sets $\sigma_+(\mathbf{j},\mathbf{k}),\sigma_-(\mathbf{j},\mathbf{k})$ is empty, then that vector cannot be in the set A_{11}^*. For vectors in A_{11}^*, assuming $m\in\sigma_+(\mathbf{j},\mathbf{k})$, we compute the probability of the set Ω_{**} as follows. By our observation on non-intersection of pairs in sets σ_-,σ_+, and independent increment property of Poisson processes we have by Lemma 2.8

$$\begin{aligned}
\mathbb{P}[\Omega_{**}]&=\sum_{a\in\mathbb{Z}_+}\mathbb{P}\Big[\sum_{i\in\sigma_-(\mathbf{j},\mathbf{k})}N(j_i^d)-N(k_i^d)=a=\sum_{i\in\sigma_+(\mathbf{j},\mathbf{k})}N(k_i^d)-N(j_i^d)\Big]\\
&=\sum_{a\in\mathbb{Z}_+}\mathbb{P}\Big[\sum_{i\in\sigma_-(\mathbf{j},\mathbf{k})}N(j_i^d)-N(k_i^d)=a\Big]\mathbb{P}\Big[\sum_{i\in\sigma_+(\mathbf{j},\mathbf{k})}N(k_i^d)-N(j_i^d)=a\Big].
\end{aligned}$$

We apply Lemma 2.10 to the sum over σ_+ and sum over a for the sum over σ_-

$$\le\frac{\sqrt{n}}{\sqrt{|k_m^d-j_m^d|}}\underbrace{\sum_{a\in\mathbb{Z}_+}\mathbb{P}\Big[\sum_{i\in\sigma_-(\mathbf{j},\mathbf{k})}N(j_i^d)-N(k_i^d)=a\Big]}_{\le 1}\le\frac{\sqrt{n}}{\sqrt{|k_m^d-j_m^d|}}.$$

If m belongs to $\sigma_-(\mathbf{j},\mathbf{k})$, after using independence we apply Lemma 2.9 to the sum over σ_- and sum over a the other sum to get the same result. Combining all of these we can continue from (4.7)

$$
\begin{aligned}
\le \sum_{A_1^*}\Big[\mathbb{P}[\Omega_{**}] + \sum_{i=1}^n \mathbb{P}[\Omega_i^c]\Big] &= \sum_{A_{11}^*}\mathbb{P}[\Omega_{**}] + \sum_{A_1^*}\Big[\sum_{i=1}^n \mathbb{P}[\Omega_i^c]\Big] \\
&\le \sum_{A_{11}^*}\frac{\sqrt{n}}{\sqrt{|k_m^d - j_m^d|}} + 2n\sum_{A_1^*}|k_m^d - j_m^d|^{-4n}
\end{aligned}
$$

The second sum is easy to handle:

$$
\begin{aligned}
\sum_{A_1^*}|k_m^d - j_m^d|^{-4n} \le \sum_{A_1^*}\prod_{i=1}^n |k_i^d - j_i^d|^{-4} \le 2^n \prod_{i=1}^n \sum_{\substack{j_i,k_i\in A\\ j_i<k_i}} |k_i^d - j_i^d|^{-4}. \\
\le_d 2^n \prod_{i=1}^n \sum_{k_i\in A} k_i^{-4d+4} \sum_{1<j_i<k_i}(k_i - j_i)^{-4} \\
= 2^n \prod_{i=1}^n \sum_{k_i\in A} k_i^{-4d+4} \sum_{1<j_i<k_i} j_i^{-4} \\
\le 2^n \prod_{i=1}^n \sum_{k_i\in A} k_i^{-4d+5},
\end{aligned}
$$

where we used

$$
\begin{aligned}
k^d - j^d &= (k-j)(k^{d-1} + k^{d-2}j + \cdots + kj^{d-2} + j^{d-1}) \\
&\le d(k-j)k^{d-1}.
\end{aligned}
$$

From this we have

$$
2n\sum_{A_1^*}|k_m^d - j_m^d|^{-4n} \le 2^{n+1}n\left[\sum_{k\in A} k^{-4d+5}\right]^n \le 1
$$

We now try to estimate the first sum. We decompose $A_{11}^* = \bigcup_{i=1}^n A_{11i}^*$ where $m = i$ for vectors in A_{11i}^*. Furthermore we decompose $A_{11i}^* = \bigcup_{\substack{j_i,k_i\in A\\ j_i\neq k_i}} A_{11i}^{*j_1k_i}$, where $A_{11i}^{*j_1k_i}$

are vectors in A^*_{11i} for which j_i, k_i are fixed. Then

$$\sum_{A^*_{11}} \frac{1}{\sqrt{|k_m^d - j_m^d|}} \leq \sum_{i=1}^{n} \sum_{A^*_{11i}} \frac{1}{\sqrt{|k_i^d - j_i^d|}} \leq n \sup_{1\leq i\leq n} \sum_{A^*_{11i}} \frac{1}{\sqrt{|k_i^d - j_i^d|}}.$$

Since the estimation does not depend on the choice of r we estimate the last sum for a fixed $i = r$. Then

$$\begin{aligned}\sum_{A^*_{11r}} \frac{1}{\sqrt{|k_r^d - j_r^d|}} &\leq \sum_{\substack{j_r,k_r\in A\\ j_r\neq k_r}} \sum_{A^{*j_rk_r}_{11r}} \frac{1}{\sqrt{|k_r^d - j_r^d|}} \\ &\leq \sum_{j_r,k_r\in A\, j_r\neq k_r} \frac{1}{\sqrt{|k_r^d - j_r^d|}} \left|A^{*j_r k_r}_{11r}\right| = \mathbf{S}.\end{aligned} \tag{4.9}$$

Therefore it remains to estimate the cardinality of $A^{*j_rk_r}_{11r}$. Let $D_i := k_i^d - j_i^d$. Fix any index $1 \leq l \leq n$ with $l \neq i$. Thus the set $A^{*j_rk_r}_{11i}$ lies within the set

$$\left\{(\mathbf{j},\mathbf{k}) \in A^{2n} \middle| \; j_i, k_i \text{ are fixed}, \quad 0 < |k_r^d - j_r^d| < |D_i| \;\; 1 \leq i \leq n, \quad (4.8) \text{ holds}\right\}. \tag{4.10}$$

In order to estimate the cardinality of this set, we just look at the possible choices for pairs j_i, k_i. Since there are $|A|$ possible choices for each of j_i and k_i number of choices for any pair (j_i, k_i) for vectors of this set can be estimated trivially by $|A|^2$. This we call the first method. Alternatively our second method comes from the identity

$$\left|\{(j_i,k_i) \in A^2 \middle| \; 0 < |k_i^d - j_i^d| \leq |D_r| \;\}\right| = 2\left|\{(j_i,k_i) \in A^2 \middle| \; 0 < k_i^d - j_i^d \leq |D_r| \;\}\right|$$

By using Theorem 2.9 with $C = D = |D_r|$ the set above is bounded by $2C_d|D_r|^{2/d}$. Once we fixed $n-1$ pairs, we can use our arithmetic results for the remaining pair. This will be called refined method.

We first start with estimating $\mathbf{S}$ to understand things more clearly. By the second method we have

$$\mathbf{S} \lesssim_{d,n} \sum_{j_r,k_r\in A} \frac{|D_r|^{2(n-1)/d}}{|D_r|^{1/2}} = \sum_{j_r,k_r\in A} |D_r|^{\frac{2(n-1)}{d} - \frac{1}{2}}.$$

Observe that if $d \geq 4n - 4$ then the exponent of $|D_r|$ in the last sum is nonpositive which means the contribution of this term can be controlled by $C_{d,n}|A|^2$. Thus we estimate **S** for $d < 4n-4$.

Now we estimate the cardinality of (4.10) combining all our three methods. We have $n-1$ pairs to estimate in this set. One of them will be estimated by the refined method after all $n-2$ pairs are estimated and fixed by the first and second methods, we fix an index $1 \leq l \leq n$ with $l \neq r$ for this purpose. For the other $n-2$ indices we note that in general the first method gives a better bound than the second, but it is the second method that enables cancellation with the $\sqrt{|k^d - j^d|}$ term of the denominator. So we must use the second method as many times as possible to take advantage of this cancellation, and for the remaining pairs we use the first method. To this end we pick $d' := \lceil \frac{d}{4} - \frac{1}{2} \rceil - 1$ indices apart from r, l. This d' is the largest number of applications for the second method after which the power of $|k^d - j^d|$ still remains nonpositive in the sum. Given the assumption $d < 4n-4$ we also have $d' \leq n-2$. To the remaining $n-2-d'$ pairs we apply the first method. So the cardinality of (4.10) is bounded by

$$\leq C_{d,n}|D_r|^{\frac{2d'}{d}}|A|^{(2n-4-2d')} \sup_{\substack{0<|D_r|\leq|D_i| \\ i\neq r,l}} \left|\left\{(j_l, k_l) \in A^2 \middle| \, j_l \neq k_l, \quad (4.12) \text{ holds}\right\}\right|, \tag{4.11}$$

where

$$\left|k_l^d - j_l^d + \sum_{i\neq l} D_i\right| < nf(|D_r|). \tag{4.12}$$

Therefore only the index l remains and we will apply the refined method for this. Since $-n|D_i| \leq \sum_{r\neq i,l} D_r \leq n|D_i|$ the supremum can be bounded above by

$$\begin{aligned}
&\leq \sup_{|D|\leq n|D_r|} \left|\left\{(j_l, k_l) \in A^2 \middle| \, j_l \neq k_l, \quad \left|k_l^d - j_l^d + D\right| < nf(|D_r|)\right\}\right| \\
&\leq 2 \sup_{|D|\leq n|D_r|} \left|\left\{(j, k) \in A^2 \middle| \, j < k, \quad \left|k^d - j^d - D\right| < nf(|D_r|)\right\}\right| \\
&\leq 2 \sup_{nf(|D_r|)\leq D\leq n|D_r|} \left|\left\{(j, k) \in A^2 \middle| \, j < k, \quad \left|k^d - j^d - D\right| < nf(|D_r|)\right\}\right|.
\end{aligned}$$

To the last line above we can apply our Theorem 2.11 to bound (4.8) by

$$\leq C_d|A|^{2n-4-2d'}|D_r|^{\frac{2d'}{d}} f^{\frac{2}{d}}(|D_r|) \leq C_d|A|^{2n-4-2d'}|D_r|^{\frac{2d'+1}{d}} \log^{\frac{1}{d}} |D_r|.$$

Then we can proceed from (4.9)

$$\begin{aligned}\mathbf{S} &\le C_{d,n}|A|^{2n-4-2d'}\sum_{\substack{j,k\in A\\ j\neq k}}\frac{\log^{\frac{1}{d}}|k^d-j^d|}{\sqrt{2\pi|k^d-j^d|}}|k^d-j^d|^{\frac{2d'+1}{d}}\\ &\le C_{d,n}|A|^{2n-4-2d'}\sum_{\substack{j,k\in A\\ j<k}}\frac{\log^{\frac{1}{d}}(k^d-j^d)}{|k^d-j^d|^{\frac{1}{2}-\frac{2d'+1}{d}}}.\end{aligned}\tag{4.13}$$

The exponent of $|k^d-j^d|$ is

$$\frac{1}{2}-\frac{2d'+1}{d}=\begin{cases}1/d & \text{if} \quad d\equiv 0(\mod 4)\\ 3/2d & \text{if} \quad d\equiv 1(\mod 4)\\ 2/d & \text{if} \quad d\equiv 2(\mod 4)\\ 1/2d & \text{if} \quad d\equiv 3(\mod 4).\end{cases}\tag{4.14}$$

From this we see that the summand in the last sum is a decreasing function of type given in Lemma 2.12 on $[e^2,\infty)$. By Lemma 2.12 we can bound this last sum by

$$\le \sum_{10\le j<k\le|A|+9}\frac{\log^{\frac{1}{d}}|k^d-j^d|}{|k^d-j^d|^{\frac{1}{2}-\frac{2d'+1}{d}}}.$$

Applying once more time to Lemma 2.12 the above expression is bounded by

$$\lesssim_d \log^{\frac{1}{d}}|A|\sum_{1\le j<k\le|A|}|k^d-j^d|^{\frac{2d'+1}{d}-\frac{1}{2}}.$$

By using

$$\begin{aligned}k^d-j^d &= (k-j)(k^{d-1}+k^{d-2}j+\cdots+kj^{d-2}+j^{d-1})\\ &\le d(k-j)k^{d-1},\end{aligned}$$

and applying a change of a variable we bound the last sum by

$$\lesssim_d \sum_{1<k\le|A|}k^{(d-1)(\frac{2d'+1}{d}-\frac{1}{2})}\sum_{1\le j<k}(k-j)^{\frac{2d'+1}{d}-\frac{1}{2}}=\sum_{1<k\le|A|}k^{(d-1)(\frac{2d'+1}{d}-\frac{1}{2})}\sum_{1\le j<k}j^{\frac{2d'+1}{d}-\frac{1}{2}}.$$

Since

$$\sum_{1\le j<k} j^{\frac{2d'+1}{d}-\frac{1}{2}} \le 1 + \int_1^k j^{\frac{2d'+1}{d}-\frac{1}{2}} dj = 1 + \frac{2d}{4d'+d+2} j^{\frac{2d'+1}{d}+\frac{1}{2}}\Big|_1^k \lesssim_d k^{\frac{2d'+1}{d}+\frac{1}{2}},$$

we obtain

$$\lesssim_d \sum_{1<k\le|A|} k^{(d-1)(\frac{2d'+1}{d}-\frac{1}{2})} k^{\frac{2d'+1}{d}+\frac{1}{2}} = \sum_{1<k\le|A|} k^{2d'+2-\frac{d}{2}}. \tag{4.15}$$

The exponent of this last sum is

$$2d'+2-\frac{d}{2} = \begin{cases} 0 & \text{if} \quad d \equiv 0(\mod 4) \\ -1/2 & \text{if} \quad d \equiv 1(\mod 4) \\ -1 & \text{if} \quad d \equiv 2(\mod 4) \\ 1/2 & \text{if} \quad d \equiv 3(\mod 4). \end{cases} \tag{4.16}$$

Therefore the expression on the right-hand side of (4.15) is bounded by

$$\lesssim_d \begin{cases} |A|^{2d'+3-\frac{d}{2}} \log^{1+\frac{1}{d}} |A| & \text{if} \quad d \equiv 2(\mod 4), \\ |A|^{2d'+3-\frac{d}{2}} \log^{\frac{1}{d}} |A| & \text{else}. \end{cases}$$

Plugging this into (4.13)

$$\mathbf{S} \lesssim_{n,d} \begin{cases} |A|^{2n-\frac{d}{2}-1} \log^{1+\frac{1}{d}} |A| & \text{if} \quad d \equiv 2(\mod 4), \\ |A|^{2n-\frac{d}{2}-1} \log^{\frac{1}{d}} |A| & \text{else}. \end{cases}$$

Thus the final bound is

$$\lesssim_{d,n} \begin{cases} \max\{|A|^n, |A|^{2n-\frac{d}{2}-1} \log^{1+\frac{1}{d}}(1+|A|)\} & \text{if} \quad d \equiv 2 \pmod 4, \\ \max\{|A|^n, |A|^{2n-\frac{d}{2}-1} \log^{\frac{1}{d}}(1+|A|)\} & \text{else}. \end{cases}$$

This completes the proof. □

CHAPTER 5

CONCLUSION

This study is mainly concerned with the classical Hardy-Littlewood majorant problem. We consider this problem by randomizing frequencies of exponential sums with stochastic processes such as stationary processes, random walks and the Poisson processes. We prove that the Hardy-Littlewood majorant property remains true up to a negligible probability after randomizing by these processes. B. Green and I. Ruzsa use the Green-Ruzsa set to show the majorant property does not hold for $p = 3$. As opposed to these authors, we randomize a wide class of sparse sets, including the Green-Ruzsa set via Poisson processes and show that the majorant property holds almost surely on these sets. We then explore the impact of randomization on the expected values of the L^2-norm and L^4-norm of an exponential sum with frequencies forming an arithmetic progression with a larger step size. Owing to Theorem 1.2 and Lemma 2.2 we see that there is no notable effect when step size is one. Finally we provide an upper bound for the expected value of the L^n-norms, $n \in 2\mathbb{N}$ of exponential sums whose frequencies are randomized via Poisson processes, and on average this upper bound gives us estimates of the number of solutions to diophantine equations.

REFERENCES

Bachelis, G. F. 1973. On the upper and lower majorant properties in lp (g). *The Quarterly Journal of Mathematics 24*(1), 119–128.

Balister, P., B. Bollobás, R. Morris, J. Sahasrabudhe, and M. Tiba 2020. Flat littlewood polynomials exist. *Ann. Math 192*(3), 977–1004.

Boas, R. 1962. Majorant problems for trigonometric series. *Journal d'Analyse Mathématique 10*(1), 253–271.

Bombieri, E. and J. Pila 1989. The number of integral points on arcs and ovals. *Duke Math. J. 59(2), 337-357*.

Borwein, P. and R. Lockhart 2001. The expected l_p norm of random polynomials. *Proceedings of the American Mathematical Society 129*(5), 1463–1472.

Browning, T. and D. Heath-Brown 2004. Equal sums of three powers. *Invent. Math 157, 553-573*.

Chang, M.-C. 2004. On problems of erdös and rudin. *Journal of Functional Analysis 207*(2), 444–460.

Cilleruelo, J. and A. Granville 2006. Lattice points on circles, squares in arithmetic progressions and sumsets of squares. *arXiv preprint math/0608109*.

Copson, E. 1965. Asymptotic expansions. *Cambridge University Press*.

Daskalakis, L. 2024. Roth's theorem and the hardy–littlewood majorant problem for thin subsets of primes. *Journal of Functional Analysis 286*(5), 110–139.

Dirichlet, G. 1849. Über die bestimmung der mittleren werte der zahlentheorie. *Abh Königl. Preuss. Akad. Wiss,69–83*.

Green, B. 2005. Roth's theorem in the primes. *Annals of mathematics*, 1609–1636.

Green, B. and I. Z. Ruzsa 2004. On the hardy–littlewood majorant problem. In *Mathematical Proceedings of the Cambridge Philosophical Society*, Volume 137, pp. 511–517. Cambridge University Press.

Gressman, P. T., S. Guo, L. B. Pierce, J. Roos, and P.-L. Yung 2023. On the strict majorant property in arbitrary dimensions. *The Quarterly Journal of Mathematics 74*(1), 139–161.

Hardy, G. and J. Littlewood 1935. Notes on the theory of series (xix): a problem concerning majorants of fourier series. *The Quarterly Journal of Mathematics* (1), 304–315.

Hardy, G. H. 1915. On the expression of a number as the sum of two squares. *Quart. J. Math. 46*, 263–283.

Hardy, G. H. 1917. On dirichlet's divisor problem. *Proceedings of the London Mathematical Society 2*(1), 1–25.

Hardy, G. H. and J. E. Littlewood 1920. Some problems of'partitio numerorum i: A new solution of warings problem. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 33–54.

Heath-Brown, D. R. 2002. The density of rational points on curves and surfaces. *Annals of Mathematics 155*(2), 553–598.

Heath-Brown, D. R. 2006. Counting rational points on algebraic varieties. *Lecture notes in mathematics-springer-verlag- 1891*, 51.

Huxley, M. 1993. Exponential sums and lattice points ii. *Proceedings of the London Mathematical Society 3*(2), 279–301.

Huxley, M. N. 1996. *Area, lattice points, and exponential sums*. LMS Monographs 13, Oxford University Press.

Huxley, M. N. 2003. Exponential sums and lattice points iii. *Proceedings of the London Mathematical Society 87*(3), 591–609.

Ivic, A., E. Krätzel, M. Kühleitner, and W. Nowak 2004. Lattice points in large regions and related arithmetic functions: recent developments in a very classic topic. *arXiv preprint math/0410522*.

Jarník, V. 1926. Über die gitterpunkte auf konvexen kurven. *Math Z 24, 500-518*.

Kahane, J.-P. 1985. *Some random series of functions*, Volume 5. Cambridge University Press.

Koralov, L. and Y. G. Sinai 2007. *Theory of probability and random processes*. Springer Science & Business Media.

Krätzel, E. 1969. Mittlere darstellungen natürlicher zahlen als differenz zweier k-ter potenzen. *Acta Arithmetica 16*(2), 111–122.

Krätzel, E. 1988. Lattice points. *Kluwer, Dordrecht-Boston-London*.

Krause, B., M. Mirek, and B. Trojan 2016. On the hardy–littlewood majorant problem for arithmetic sets. *Journal of Functional Analysis 271*(1), 164–181.

Kuba, G. 1993. On sums of two k-th powers of numbers in residue classes ii. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, Volume 63, pp. 87–95. Springer.

Kühleitner, M. 1992. An omega theorem on differences of two squares. *Acta Math. Univ. Comenianae 61*(1), 117–123.

Littlewood, J. 1966. On polynomials $\sum^n \pm z^m$, and $\sum^n e^{\alpha_m i} z^m$, $z = e^{i\theta}$. *J. London Math. Soc. 41*, 367–376.

Loukas, G. 2014. Classical fourier analysis. *Springer*.

Mahler, K. 1936. Note on hypothesis k of hardy and littlewood. *Journal of the London Mathematical Society 1*(2), 136–138.

Malliavin, P. and S. Taniguchi 1997. Analytic functions, cauchy formula, and stationary phase on a real abstract wiener space. *Journal of functional analysis 143*(2), 470–528.

Marmon, O. 2011. Sums and differences of four k th powers. *Monatshefte für Mathematik 164*(1), 55–74.

Mockenhaupt, G. 1996. *Bounds in Lebesgue spaces of oscillatory integral operators*. Ph. D. thesis, Universität-Gesamthochschule-Siegen.

Mockenhaupt, G. and W. Schlag 2009. On the hardy–littlewood majorant problem for random sets. *Journal of Functional Analysis 256*(4), 1189–1237.

Newman, D. J. and J. Byrnes 1990. The l^4 norm of a polynomial with coefficients ± 1. *The American Mathematical Monthly 97*(1), 42–45.

Nowak, W. G. 1995. On sums and differences of two relative prime cubes. *Analysis 15*(4), 325–342.

Nowak, W. G. 1998. On differences of two kth powers of integers. *The Ramanujan Journal 2*, 421–440.

Ogawa, S. 2013. On a stochastic fourier transformation. *Stochastics An International Journal of Probability and Stochastic Processes 85*(2), 286–294.

Paley, R. and A. Zygmund 1930. On some series of functions,(1). In *Mathematical Proceedings of the Cambridge Philosophical Society*, Volume 26, pp. 337–357.

Cambridge University Press.

Revuz, D. and M. Yor 1999. Continuous martingales and brownian motion. *Grundlehren Der Mathematischen Wissenschaften 293*, 198.

Robbins, H. 1955. A remark on stirling's formula. *The American Mathematical Monthly 62*(1), 26–29.

Salberger, P. 2005. Counting rational points on hypersurfaces of low dimension. In *Annales Scientifiques de l'École Normale Supérieure*, Volume 38, pp. 93–115. Elsevier.

Salberger, P. and T. D. Wooley 2010. Rational points on complete intersections of higher degree, and mean values of weyl sums. *Journal of the London Mathematical Society 82*(2), 317–342.

Salem, R. and A. Zygmund 1954. Some properties of trigonometric series whose terms have random signs. *Acta Mathematica, Volume 91, 245-301*.

Sanders, T. 2012. On the bogolyubov–ruzsa lemma. *Analysis & PDE 5*(3), 627–655.

Srinivasan, B. 1963. The lattice point problem of many-dimensional hyperboloids i. *Acta Arithmetica 8*, 153–172.

Srinivasan, B. 1965. The lattice point problem of many dimensional hyperboloids iii. *Mathematische Annalen 160*(4), 280–311.

Stein, E. M. and T. S. Murphy 1993. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Volume 3. Princeton University Press.

Steinhaus, H. 1947. Sur un théoréme de mv jarník. In *Colloquium Mathematicum*, Volume 1, pp. 1–5. Polska Akademia Nauk. Instytut Matematyczny PAN.

Talagrand, M. 1995. A borderline random fourier series. *The Annals of Probability 23(2)*, 776–785.

Taniguchi, S. 1998. On the exponential decay of oscillatory integrals on an abstract wiener space. *journal of functional analysis 154*(2), 424–443.

Van der Corput, J. G. 1920. Über gitterpunkte in der ebene. *Mathematische Annalen 81*(1), 1–20.

Van der Corput, J. G. 1928. Zum teilerproblem. *Mathematische Annalen 98*(1), 697–716.

Vaughan, R. 1997. The hardy-littlewood method. *Cambridge University Press*.

Voronoi, G. 1903. Sur un problème du calcul des fonctions asymptotiques. *J. Math 126, 241-282*.

Wasserman, L. 2013. All of statistics: A concise course in statistical inference. *Springer Science & Business Media*.

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