

Non-gravitating scalars and spacetime compactification

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Abstract

We discuss role of partially gravitating scalar fields, scalar fields whose energy–momentum tensors vanish for a subset of dimensions, in dynamical compactification of a given set of dimensions. We show that the resulting spacetime exhibits a factorizable geometry consisting of usual four-dimensional spacetime with full Poincaré invariance times a manifold of extra dimensions whose size and shape are determined by the scalar field dynamics. Depending on the strength of its coupling to the curvature scalar, the vacuum expectation value (VEV) of the scalar field may or may not vanish. When its VEV is zero the higher-dimensional spacetime is completely flat and there is no compactification effect at all. On the other hand, when its VEV is nonzero the extra dimensions get spontaneously compactified. The compactification process is such that a bulk cosmological constant is utilized for curving the extra dimensions.

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1. Introduction

The pioneering work of Kaluza [1], which states that Einstein's gravity and electromagnetism can be unified into five-dimensional general relativity if the extra dimension is barred from affecting laws of physics, has given rise to, via Klein's compactification approach [2,3], a wealth of higher-dimensional theories motivated by various physical phenomena (see the review [4]). For instance, a unified quantum-theoretic description of gravity and other forces of nature (i.e., supergravity and superstring theories) cannot be formulated without introducing extra dimensions (see, e.g., [5]). Moreover, extra dimensions can provide a viable solution to gauge hierarchy problem of the standard electroweak theory with no need to low-energy supersymmetry [6]. In these and many other applications the extra dimensions are assumed to roll up to form a sufficiently small space. The characteristic size of the extra dimensions can vary from Planck length up to a few mm, as allowed by the present experimental bounds.

The aforementioned scheme is incomplete in that it is necessary to explain how or why extra dimensions differed so

markedly in size and topology from the ordinary four dimensions. In other words, it is necessary to find the dynamical mechanism that leads to compactification of the extra dimensions, that is, the macroscopic four-dimensional spacetime times the compact manifold of extra dimensions should be an energetically-preferred solution of the higher-dimensional Einstein equations. This has been accomplished by utilizing higher-curvature gravity [7] and by coupling Einstein gravity to matter in a judicious way. The latter leads to spontaneous compactification of extra dimensions as was first pointed out in [8]. Spontaneous compactification has been realized with Yang–Mills fields [9], antisymmetric tensor fields [10], sigma model fields [11,12], and conformally-coupled scalars [13]. In each case, components of the Ricci tensor are balanced by those of the stress tensor, and depending on the structure of the latter a subset of dimensions are compactified.

In this work we are interested in dynamical compactification induced by scalar fields. The role of scalars in dynamical compactification process was first analyzed in [11,12] where a D -dimensional minimally-coupled nonlinear sigma model with metric $h_{ij}(\phi)$ ($i, j = 1, \dots, D$) was shown to lead to a dynamical compactification of D extra dimensions provided that sigma model metric is Einstein. In other words, equations of motion for the metric field requires the Ricci tensor \mathcal{R}_{ij} to be proportional to $h_{ij}(\phi)$, and thus, D -dimensional extra space relaxes

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to the geometry of the sigma model. The remaining dimensions x^μ ($\mu = 0, 1, \dots$) span a strictly flat Minkowski space. That this set-up compactifies the extra dimensions y^i becomes especially clear with the ansatz $\phi^i = y^i$ or any function of y^i . In the literature role of scalars in spontaneous compactification was also emphasized in [13] where a strictly-conformal invariant scalar-tensor theory (though conformal invariant Weyl contribution is absent) of gravity in six dimensions is shown to lead to compactification of the two extra dimensions and simultaneously generate Newton's constant spontaneously. The scalar fields can be chosen in varying group representations depending on the desired compactification structure, and the gauge symmetry in four dimensions turns out to be smaller than the isometries of the compact manifold.

Obviously, the space of extra dimensions may [8,9,11,12] or may not [12,14] form a compact space. The extra dimensions can possess negative curvature [15] yet they can still be compact [16]. Moreover, the geometry does not need to be factorizable [17]. In general, shape and topology of the extra space are entirely determined by the mechanism of dynamical compactification.

In this work we discuss yet another compactification mechanism induced by scalar fields. We will show that a single scalar field living in a higher-dimensional spacetime can lead to dynamical compactification of the extra dimensions without inducing a classical cosmological constant when it gravitates only in those dimensions which are to be compactified. For proving this statement, it is necessary to show first that a strictly flat spacetime supports nontrivial scalar field configurations. This we will do in Section 2. The next step is to show the compactification of the extra dimensions into a D -dimensional manifold, and this we will show in Section 3. We will conclude in Section 4.

2. Partially gravitating scalar fields

Let us consider a real scalar field ϕ living in a $(4 + D)$ -dimensional spacetime with coordinates $z^A = (x^\mu, y^i)$ where $\mu = 0, \dots, 3$ and $i = 1, \dots, D$. Keeping the gravitational sector minimal, the most general action integral takes the form

$$S = \int d^{4+D}z \sqrt{-g} \left\{ \frac{1}{2} M_\star^{D+2} \mathcal{R} - \frac{1}{2} g^{AB} \partial_A \phi \partial_B \phi - \frac{1}{2} \zeta \mathcal{R} \phi^2 - V(\phi) \right\}, \quad (1)$$

where we have adopted $(-1, +1, \dots, +1)$ metric signature, and denoted the curvature scalar by \mathcal{R} and fundamental scale of gravity by M_\star . There is no symmetry principle¹ for avoiding direct coupling of ϕ to the curvature scalar, namely, a scalar field should always exhibit $\zeta \mathcal{R} \phi^2$ type interaction with Ricci scalar. Note that the scalar field theory in (1) exhibits conformal

invariance when $V(\phi) \propto \phi^{4+D}$ and $\zeta = \zeta_{4+D}$, where

$$\zeta_{4+D} = \frac{D+2}{4(D+3)} \quad (2)$$

which equals $1/6$ for $D = 0$ and $1/4$ for $D = \infty$. This property, however, is of little use since conformal invariance is explicitly broken by the Einstein–Hilbert term, and full invariance cannot be achieved unless it is replaced by a term quadratic in Weyl tensor [19,20].

The action (1) attains its extremum when scalar and metric fields obey their equations of motion

$$\mathcal{R}_{AB} = \frac{\mathcal{T}_{AB}(\phi)}{M_\star^{D+2} - \zeta \phi^2}, \quad (3)$$

$$\square \phi = \zeta R \phi + V'(\phi), \quad (4)$$

where prime denotes differentiation with respect to ϕ . Here, the source term for the Ricci tensor is given by

$$\begin{aligned} \mathcal{T}_{AB}(\phi) &= T_{AB} - \frac{1}{D+2} g_{AB} g^{CD} T_{CD} \\ &= \partial_A \phi \partial_B \phi - \zeta \nabla_A \nabla_B \phi^2 \\ &\quad + \frac{1}{D+2} (2V(\phi) - \zeta \square \phi^2) g_{AB}, \end{aligned} \quad (5)$$

where

$$\begin{aligned} T_{AB} &= \partial_A \phi \partial_B \phi - g_{AB} \left(\frac{1}{2} g^{CD} \partial_C \phi \partial_D \phi + V(\phi) \right) \\ &\quad + \zeta (g_{AB} \square - \nabla_A \nabla_B) \phi^2 \end{aligned} \quad (6)$$

is the stress tensor of $\phi(z^A)$. The second line at right-hand side follows from direct coupling of ϕ to Ricci scalar, and it remains nonvanishing even in the flat limit [21]. One can show that $\nabla^A T_{AB} = -\zeta \mathcal{G}_{CB} \nabla^C \phi^2$ where

$$\mathcal{G}_{AB} = \mathcal{R}_{AB} - \frac{1}{2} \mathcal{R} g_{AB} \quad (7)$$

is the Einstein tensor. It is clear that $T_{AB} + \zeta \phi^2 \mathcal{G}_{AB}$ is a conserved tensor source in agreement with the Bianchi identity.

The Einstein equations for the Ricci tensor, Eq. (3), guarantee that if \mathcal{T}_{AB} vanishes for a certain range of its indices so does \mathcal{R}_{AB} for the same index ranges. When \mathcal{R}_{AB} vanishes for a range of indices the metric tensor on that block will be assumed to be η_{AB} . Therefore, a given range of indices for which $\mathcal{R}_{AB} = 0$ will be interpreted to form a flat subspace in a $(4 + D)$ -dimensional spacetime. In the opposite case, when \mathcal{T}_{AB} is nonvanishing for a range of indices so is \mathcal{R}_{AB} , and precise forms of the metric and scalar field are determined from a self-consistent solution of (3) and (4). In the following we discuss under what conditions \mathcal{T}_{AB} possesses specific texture zeroes.

2.1. Nongravitating scalar field

We start our analysis by considering first a completely nongravitating scalar, i.e., we impose $\mathcal{T}_{AB} = 0$ for all $A = (\mu, i)$ and $B = (v, j)$. This implies that \mathcal{R}_{AB} vanishes for all A, B so that metric tensor may be assumed to take the form $\eta_{AB} =$

¹ The Goldstone bosons generated by spontaneously broken continuous symmetries is an exception. They do not couple to the curvature scalar directly [18,19].

$(-1, 1, \dots, 1)$, as mentioned before.² It is convenient to nullify first \mathcal{T}_{AB} for $A \neq B$. These equations receive contributions from the first line of the second equality in (5) only, and they enforce ϕ to have the form

$$\phi = \psi^{-\frac{2\zeta}{1-4\zeta}}, \quad (8)$$

where ψ is another real scalar field. Then vanishing of the diagonal entries of \mathcal{T}_{AB} further determines ψ to be a second order polynomial in z^A , and $V(\phi)$ to be a function of ϕ only. Consequently, one finds for $\phi(z)$

$$\phi(z) \equiv \phi_0(z) = \left(\frac{\tilde{a}}{2} \eta^{AB} z_A z_B + \eta^{AB} z_A \tilde{p}_B + \tilde{b} \right)^{-\frac{2\zeta}{1-4\zeta}}, \quad (9)$$

where \tilde{a} , \tilde{b} and \tilde{p}_A are constants of integration. For $\phi(z)$ to take this rather specific form its self-interaction potential must equal

$$V(\phi_0) = 16\tilde{a}(D+3) \frac{\zeta^2}{(1-4\zeta)^2} (\zeta - \zeta_{4+D}) \phi_0^{\frac{1}{2\zeta}} + 2(\eta^{AB} \tilde{p}_A \tilde{p}_B - 2\tilde{a}\tilde{b}) \frac{\zeta^2}{(1-4\zeta)^2} \phi_0^{\frac{1-2\zeta}{\zeta}} \quad (10)$$

which explicitly depends on the parameters of (9). Consequently, for \mathcal{T}_{AB} to vanish the scalar field itself does not need to vanish; all that is required is to devise a self-interaction potential (10) on the specific solution (9) for $\phi(z)$. One notices that this nongravitating nontrivial field configuration arises thanks to the ζ dependent terms in \mathcal{T}_{AB} or equivalently the non-minimal coupling of ϕ to the curvature scalar. Indeed, when $\zeta \rightarrow 0$ the scalar field reduces to a constant and $V(\phi) \rightarrow 0$, which is a trivial configuration.

It is not hard to see that (9) and (10) also nullify T_{AB} , the true energy–momentum tensor of ϕ in (6). Actually, this coincidence is expected since the Einstein tensor vanishes whenever the Ricci tensor vanishes. The fact that a nonminimally coupled scalar field possesses a nontrivial configuration despite its vanishing T_{AB} has recently been discussed in [22], and field and potential solutions in (9) and (10) have already been obtained therein. The solution for ϕ in (9) represents a shock wave propagation. The wave front is spherical for $\tilde{p}_A = 0$ and planar for $\tilde{a} = 0$. When $\zeta = \zeta_{4+D}$ the first term in potential drops out, and the second term becomes proportional to $\phi_0^{-(D+4)}$, which is precisely what is required by conformal invariance [19,20].

An interesting property of the potential function (10) is that its minimum varies with ζ . Indeed, for $\zeta > \zeta_{4+D}$ it is minimized at $\phi = 0$ whereas its minimum occurs at

$$\bar{\phi} = \left(\frac{(D+3)(\zeta - \zeta_{4+D})}{2\zeta - 1} \frac{4\tilde{a}}{\eta^{AB} \tilde{p}_A \tilde{p}_B - 2\tilde{a}\tilde{b}} \right)^{\frac{2\zeta}{1-4\zeta}} \quad (11)$$

when $\zeta < \zeta_{4+D}$ and $\eta^{AB} \tilde{p}_A \tilde{p}_B - 2\tilde{a}\tilde{b} > 0$. In this sense the conformal value of ζ represents a threshold point below and

above which the lowest energy configuration for $V(\phi_0)$ drastically changes.

So far we have discussed only the solution of $\mathcal{T}_{AB} = 0$ with no mention of the equation of motion of ϕ in (4). Actually, the field configuration (9) with $V(\phi)$ given in (10) automatically satisfies (4). This observation is correct for all parameter ranges; in particular, at the two possible minima of the potential: $\phi = 0$ and $\phi = \bar{\phi}$. Therefore, it is not necessary to require ϕ to take nonconstant values as claimed in [22].

2.2. Partially gravitating scalar field

In this section we discuss cases where ϕ gravitates only in a subset of dimensions. The construction of completely nongravitating scalar above will serve as a useful guide for our analysis. We will look for metric and scalar field configurations in agreement with the following \mathcal{T}_{AB} texture:

$$\mathcal{T}_{\mu\nu}(\phi) = 0, \quad (12)$$

$$\mathcal{T}_{\mu j}(\phi) = \mathcal{T}_{i\nu}(\phi) = 0, \quad (13)$$

$$\mathcal{T}_{ij}(\phi) \neq 0, \quad (14)$$

where $\mathcal{T}_{ij}(\phi)$ determines topology and shape of the extra space via (3). As mentioned before, when \mathcal{T}_{AB} vanishes for a certain range of indices so does the Ricci tensor. This, however, is not a trivial condition when ϕ gravitates in a subset of dimensions only. To clarify this point consider, for instance, the constraint (12). It guarantees that $\mathcal{R}_{\mu\nu} = 0$; however, it cannot guarantee, even with $g_{\mu\nu} = \eta_{\mu\nu}$, that the quartet (x_0, x_1, x_2, x_3) forms a flat space. The reason is that $\nabla_\mu \nabla_\nu \phi^2 = \partial_\mu \partial_\nu \phi^2$ if and only if the connection coefficients, Γ_{BC}^A , satisfy $\Gamma_{\mu\nu}^A = 0$ for all (A, μ, ν) . This is guaranteed if $g_{\mu j}$ and $g_{i\nu}$ depend only on the extra dimensions. On the other hand, considering $T_{\mu j}$ and $T_{i\nu}$, one finds that $\nabla_\mu \nabla_i = \partial_\mu \partial_i$ if $g_{\mu j}$ and $g_{i\nu}$ both are constants with respect to all coordinates x^A , and if g_{ij} depends only on the extra dimensions. These flatness conditions on different groups of coordinates implies that the metric tensor g_{AB} must conform to structure of \mathcal{T}_{AB} in (12)–(14):

$$g_{\mu\nu} = \eta_{\mu\nu}, \quad (15)$$

$$g_{\mu j} = g_{i\nu} = 0, \quad (16)$$

$$g_{ij} = g_{ij}(\vec{y}) \quad (17)$$

which exhibits a block-diagonal structure as it should for extra coordinates $\{y^i\}$ to be compactified, i.e., decoupled from the rest. With this structure for the metric tensor, the source term of the Ricci tensor $\mathcal{R}_{\mu\nu}$ in four dimensions takes the form

$$\mathcal{T}_{\mu\nu}(\phi) = \partial_\mu \phi \partial_\nu \phi - \zeta \partial_\mu \partial_\nu \phi^2 + \frac{1}{D+2} (2V_{\text{new}}(\phi) - \zeta \eta^{\alpha\beta} \partial_\alpha \partial_\beta \phi^2) \eta_{\mu\nu} \quad (18)$$

as follows from (5). Hence, as seen from four dimensions, the self-interaction potential of ϕ is not the original one $V(\phi)$, but

$$V_{\text{new}}(\phi) = V(\phi) - \frac{1}{2} \zeta g^{ij} \nabla_i \nabla_j \phi^2 \quad (19)$$

which involves derivatives of ϕ^2 with respect to extra coordinates $\{y^i\}$. For $\mathcal{T}_{\mu\nu}(\phi)$ to vanish, first of all, the scalar field

² The purely nongravitating scalar field configuration discussed in this subsection is not new at all; it has been shown to exist already in [22] where one can find a more detailed description of the solution of $T_{AB} = 0$. This subsection is included here for completeness of the discussions.

must have the special form

$$\phi(z) \equiv \phi_0(z) = \left(\frac{a}{2} \eta^{\mu\nu} x_\mu x_\nu + \eta^{\mu\nu} x_\mu p_\nu + b \right)^{-\frac{2\zeta}{1-4\zeta}} \quad (20)$$

in analogy with (9) derived in Section 2.1. Here, in principle, all the parameters a , b and p_μ are functions of the extra coordinates $\{y^i\}$, and their mass dimensions are $2 - (1 - 4\zeta)(D + 2)/4\zeta$, $-(1 - 4\zeta)(D + 2)/4\zeta$ and $1 - (1 - 4\zeta)(D + 2)/4\zeta$, respectively. The scalar field configuration (20) describes a shock wave propagation in four dimensions at each point $\{y^i\}$ of the extra space.

Having $\phi(z)$ obeying to (20) is not sufficient for nullifying all components of $\mathcal{T}_{\mu\nu}$, however. Indeed, for $\mathcal{T}_{\mu\nu}$ to vanish completely the self-interaction potential felt by $\phi_0(z)$ must have the special form

$$\begin{aligned} \tilde{V}(\phi_0) = & 8a(D+6) \frac{\zeta^2}{(1-4\zeta)^2} (\zeta - \zeta_{\text{crit}}) \phi_0^{\frac{1}{2\zeta}} \\ & + 2(\eta^{\mu\nu} \tilde{p}_\mu \tilde{p}_\nu - 2ab) \frac{\zeta^2}{(1-4\zeta)^2} \phi_0^{\frac{1-2\zeta}{\zeta}} \end{aligned} \quad (21)$$

which is to be contrasted with the potential function (10) of purely non-gravitating scalar field discussed in Section 2.1. The most important difference between the two potentials comes from replacement of ζ_{4+D} in (10) by

$$\zeta_{\text{crit}} = \frac{(D+4)}{4(D+6)} \quad (22)$$

which ranges from $1/6$ at $D = 0$ to $1/4$ at $D = \infty$. These two critical ζ values, ζ_{crit} and ζ_{4+D} , agree at $D = 0$ and $D = \infty$, but behave differently in between. Clearly, ζ_{crit} arises from $1/(D+2)$ factor in (18), and the two potentials (10) and (21) coincide when $D = 0$. In other words, (18) is not the true stress tensor of a scalar field living in four-dimensions; it is just cross section of the stress tensor of a scalar field living in $(4+D)$ upon four-dimensional subspace. It is with the special solution (21), i.e., it is with

$$V_{\text{new}}(\phi_0) = \tilde{V}(\phi_0) \quad (23)$$

which holds on $\phi(z) = \phi_0(z)$ that all ten components of $\mathcal{T}_{\mu\nu}$ and hence those of $\mathcal{R}_{\mu\nu}$ vanish with a strictly flat metric $\eta_{\mu\nu}$.

Having determined under what conditions $\mathcal{T}_{\mu\nu}$ vanishes, we now look for implications of (13). Obviously, vanishing of $\mathcal{T}_{\mu j}$ and $\mathcal{T}_{i\nu}$ is guaranteed if $\phi_0(z)$ in (20) does not involve mixed terms of x^μ and y^i . In other words, the parameters a , ζ and p_μ must be global constants yet $b = b(\vec{y})$. The dependence of b on extra dimensions is rather general; all that is needed is to satisfy equations of motion self-consistently. For future reference, taking $a > 0$ and $p_\mu p^\mu - 2ab(\vec{y}) > 0$, one notes that $\tilde{V}(\phi_0)$ is minimized at $\phi_0 = 0$ for $1/4 > \zeta > \zeta_{\text{crit}}$, and at $\phi_0 = \bar{\phi}$ with

$$\bar{\phi} = \left(\frac{(D+6)(\zeta - \zeta_{\text{crit}})}{2\zeta - 1} \frac{2a}{\eta^{\mu\nu} p_\mu p_\nu - 2ab(\vec{y})} \right)^{\frac{2\zeta}{1-4\zeta}} \quad (24)$$

for $\zeta < \zeta_{\text{crit}}$. Clearly, unless the shock wave propagation in four dimensions is a spherical one, $a \neq 0$, this very minimum of $\tilde{V}(\phi_0)$ is neither possible nor meaningful.

Finally, we analyze implications of a finite \mathcal{T}_{ij} . By construction, \mathcal{T}_{ij} does not vanish and hence extra space experiences a nontrivial curving. On the field configuration (20) for which $\mathcal{T}_{\mu\nu}$, $\mathcal{T}_{i\nu}$ and $\mathcal{T}_{\mu j}$ vanish identically, equations of motion for the metric tensor and ϕ_0 take the form

$$\mathcal{R}_{ij} = \frac{\mathcal{T}_{ij}(\phi_0)}{M_\star^{D+2} - \zeta \phi_0^2}, \quad (25)$$

$$\begin{aligned} g^{ij} \nabla_i \nabla_j \phi_0 \\ = \zeta \mathcal{R} \phi_0 + V'(\phi_0) - \tilde{V}'(\phi_0) - aD \frac{\zeta}{1-4\zeta} \phi_0^{\frac{1-2\zeta}{\zeta}}, \end{aligned} \quad (26)$$

where the Ricci tensor is sourced by

$$\mathcal{T}_{ij}(\phi_0) = \partial_i \phi_0 \partial_j \phi_0 - \zeta \nabla_i \nabla_j \phi_0^2 - \frac{4a\zeta^2}{1-4\zeta} \phi_0^{\frac{1}{2\zeta}} g_{ij} \quad (27)$$

which require ϕ to possess the specific solution ϕ_0 given in (20). A simultaneous solution of (25) and (26) completely determines the curvature scalar

$$\begin{aligned} \mathcal{R} = \frac{1}{M_\star^{D+2}} \left\{ \left(2 - \frac{1}{\zeta} \right) (\tilde{V}(\phi_0) - V(\phi_0)) \right. \\ \left. + \phi_0 (\tilde{V}'(\phi_0) - V'(\phi_0)) + aD\zeta \phi_0^{\frac{1}{2\zeta}} \right\} \end{aligned} \quad (28)$$

which is a measure of the degree to which the extra space is curved.

Having worked out the question of under what conditions a bulk scalar in $4+D$ dimensions gravitates only in a subgroup of dimensions, we now turn to a discussion of the role and nature of the self-interaction potential $V_{\text{new}}(\phi)$ of $\phi(z)$. First of all, $V_{\text{new}}(\phi)$ is the scalar potential felt by a generic scalar field when the higher-dimensional metric obtains the block diagonal structure in (15)–(17). In other words, it refers to part of the action density when all derivatives with respect to x_μ are dropped. In fact, it is not more than a rearrangement of the terms involving derivatives with respect to extra dimensions so that action density looks like a four-dimensional one to facilitate analysis of $\mathcal{T}_{\mu\nu} = 0$. In particular, $V_{\text{new}}(\phi)$ has nothing to do with the effective potential one would obtain by integrating out degrees of freedom associated with extra dimensions. It is neither a four-dimensional effective potential in the common sense of the word nor a $(4+D)$ -dimensional effective potential; it is a local function of coordinates, and by taking the specific form $\tilde{V}(\phi)$, it directly participates in flattening of the four-dimensional spacetime and in curving of the extra space via the equations of motion (25) and (26). To stress again, $\tilde{V}(\phi)$ is just an analog of (10), and mathematically it is highly useful since its extrema in (24) will feature in the next section when we discuss compactification of the extra dimensions.

In summary, the entire dynamical problem has thus reduced to a self-consistent solution of (25) and (26). The unknowns of the problem are the metric tensor $g_{ij}(\vec{y})$ and $b(\vec{y})$. Once these two parameters are fixed one obtains a precise description of the geometry and shape of the extra space. The terms involving derivatives with respect to x^μ in the original equations of motion (3) and (4) have been eliminated by using the explicit expression of ϕ in (20). It is easy to see that, when

$b(\vec{y}) = \frac{a}{2} \eta^{ij} y_i y_j + \eta^{ij} y_i p_j + b_0$, b_0 being a constant, all components of T_{ij} vanish and entire $(4+D)$ -dimensional spacetime becomes flat, as discussed in detail in Section 2.1. All other forms of $b(\vec{y})$ lead to a nontrivial curving of the extra space. In the next section we will analyze (25) and (26), and discuss their implications for compactification of the extra dimensions.

3. Spacetime compactification

Spontaneous compactification of $(4+D)$ -dimensional spacetime M^{4+D} into a four-dimensional flat spacetime M^4 spanned by the four macroscopic dimensions times a D -dimensional manifold E^D means that $M^4 \otimes E^D$ is an energetically preferred solution compared to M^{4+D} [11,13]. The analysis in Section 2.2 made it clear that flatness of M^4 is governed by $\tilde{V}(\phi)$ not by $V(\phi)$. Indeed, $V(\phi)$ is the self-interaction potential of ϕ in $(4+D)$ dimensions whereas $V_{\text{new}}(\phi)$ is the potential of the same scalar as seen from a four-dimensional perspective (see (18) which has to vanish for flattening the four-dimensional subspace). In this sense, higher-dimensional spacetime configuration consisting of a strictly flat four-dimensional geometry times an extra curved manifold becomes energetically preferable only at those ϕ_0 values for which $\tilde{V}(\phi_0)$ is a minimum.

As follows from Section 2.2, by taking $a > 0$ and $\eta^{\mu\nu} p_\mu p_\nu - 2ab(\vec{y}) > 0$ for definiteness, the scalar potential $\tilde{V}(\phi_0)$ is found to possess two minima: $\phi_0 = 0$ (for $\zeta > \zeta_{\text{crit}}$) and $\phi_0 = \bar{\phi}$ (for $\zeta < \zeta_{\text{crit}}$) given in (24). In the minimum of $\tilde{V}(\phi_0)$ at $\phi_0 = 0$, the scalar field equation (26) is consistently solved if $V(\phi_0) = \tilde{V}(\phi_0)$, i.e., $V(0) = 0$. This, in fact, follows from (19) which implies that $V(\phi_0)$ must be equal to $\tilde{V}(\phi_0)$ for any \vec{y} independent ϕ_0 configuration. With $\phi_0 = 0$ and $V(0) = 0$, Ricci tensor and curvature scalar are found to vanish identically, as follows from (25) and (28). It is clear that the whole picture is consistent since a vanishing ϕ possesses a vanishing \mathcal{T}_{AB} if its potential does also vanish at the field configuration under concern. Consequently, the minimum of $\tilde{V}(\phi_0)$ at $\phi_0 = 0$ represents a Ricci-flat manifold. This, as mentioned at the beginning of Section 2, may be taken to indicate a strictly flat space, i.e., $g_{ij} = \eta_{ij}$. One thus arrives at the conclusion that if $\tilde{V}(\phi_0)$ is minimized at $\phi_0 = 0$ and if $V(0) = 0$ then the resulting spacetime is a $(4+D)$ -dimensional Minkowski spacetime M^{4+D} , i.e., there is no compactification effect at all. The extra space is a strictly flat manifold as the four-dimensional subspace itself.

For $\zeta < \zeta_{\text{crit}}$, the potential $\tilde{V}(\phi_0)$ is minimized at a nonzero ϕ_0 value given in (24). The dynamical equations governing the compactification process are (25) and (26) where now ϕ_0 is replaced by $\bar{\phi}$. All one is to do is to solve dynamical equations for determining $g_{ij}(\vec{y})$ (with $D(D+1)/2$ independent components) and $b(\vec{y})$ in a self-consistent fashion. These two must give a complete description of the shape and topology of the extra space.

We schematically illustrate the two minima and corresponding spacetime structures of $\tilde{V}(\phi)$ in Fig. 1. The overall picture is that as ζ makes a transition from $\zeta > \zeta_{\text{crit}}$ regime to $\zeta < \zeta_{\text{crit}}$ regime the spacetime structure changes from M^{4+D} to $M^4 \otimes E^D$ spontaneously. The topology and shape of the extra

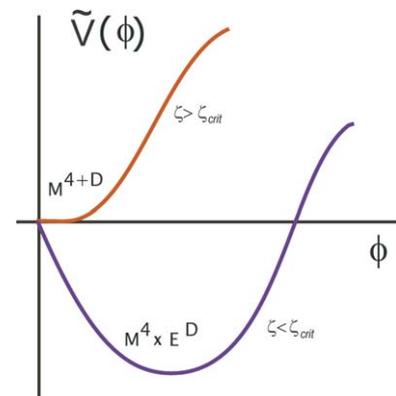


Fig. 1. The two minima of $\tilde{V}(\phi)$ and the corresponding spacetime structures.

space are determined by simultaneous solutions of (25) and (26) for $\phi_0 = \bar{\phi}$, defined in (24).

An analytic solution of the topology and shape of the extra space is quite difficult to implement since (25) and (26) exhibit a functional dependence on $b(\vec{y})$ and $b(\vec{y})$ itself depends on $g_{ij}(\vec{y})$ via contraction of the extra coordinates. Therefore, one may eventually need to resort numerical techniques to determine the structure of the extra space. Despite these difficulties in establishing an analytic solution, it may be instructive to analyze certain simple cases by explicit examples.

3.1. Constant curvature space

The simplest $\bar{\phi}$ configuration which admits an analytic solution of (25) and (26) is provided by the ansatz $b(\vec{y}) = b_0$, a completely \vec{y} independent configuration. The equation of motion for $\bar{\phi}$ (26) is satisfied with $V(\bar{\phi}) = \tilde{V}(\bar{\phi})$ as expected from (19). A self-consistent solution of (25), (26) and (28) gives

$$\mathcal{R}_{ij} = \frac{\mathcal{R}}{D} g_{ij} \quad \text{with } \mathcal{R} = \frac{aD}{1-4\zeta} \bar{\phi}^{\frac{1-4\zeta}{2\zeta}}, \quad (29)$$

where vacuum expectation value of the scalar field is fixed via the consistency condition $M_\star^{D+2} = \zeta(1-4\zeta)\bar{\phi}^2$. In other words, the fundamental scale of gravity in $(4+D)$ dimensions, M_\star , fixes the vacuum expectation value of the bulk scalar ϕ_0 which is already designed not to gravitate in the four-dimensional subspace. The integration constants a , b and p_μ in (20) are naturally $\mathcal{O}(M_\star)$ —the only mass scale in the bulk.

In fact, by taking $a = \lambda M_\star^{2 - \frac{1-4\zeta}{4\zeta}(D+2)}$ with λ being a dimensionless constant, one finds

$$\mathcal{R} = \lambda D \zeta^{\frac{4\zeta-1}{4\zeta}} (1-4\zeta)^{-\frac{1}{4\zeta}} M_\star^2 \quad (30)$$

which is completely determined by ζ , D , λ and M_\star . The resulting spacetime topology is obviously $M^4 \otimes E^D$ with E^D being a D dimensional manifold with positive constant curvature. The coordinates $\{y_i\}$ may or may not be compact. The constant $b(\vec{y})$ case under discussion offers an elegant way of solving (25) and (26) and it results in an intuitively simple interpretation of the manifold formed by extra dimensions. Indeed, the self-interaction potential $V(\phi)$, on the partially-gravitating configuration ϕ_0 in (20), gets converted into $\tilde{V}(\phi_0)$ whose minimum

at $\phi_0 = \bar{\phi}$ results in a nontrivial constant-curvature space. In essence, the would-be cosmological term, $V(\bar{\phi})$, as seen from a four-dimensional Poincaré-invariant perspective via (18) is off-loaded and utilized in curving the extra space (in similarity with the mechanism advocated in [23] for solving the cosmological constant problem).

3.2. More general cases

Some further properties of (25) and (26) can be revealed by using an appropriate coordinate system. A suitable setting for such an analysis is provided by the Riemann normal coordinates which are defined by a locally-flat space attached to a point N of the manifold of extra dimensions. The local flatness of the space at (not in any neighborhood of) the point N implies that $\partial_i g_{jk} \equiv 0$ for all $i, j, k = 1, \dots, D$ at N , i.e., all components of the connection coefficients Γ_{jk}^i vanish at N . Clearly, curvature tensors do not need to vanish at N since they involve not only Γ_{jk}^i but also their first derivatives. Consequently, one finds

$$\mathcal{T}_{ij}^{(N)}(\bar{\phi}) = \frac{4\zeta^2}{1-4\zeta} \bar{\phi}^{\frac{1}{2\zeta}} \left(\frac{1-2\zeta}{(D+6)(\zeta-\zeta_{\text{crit}})} \partial_i \partial_j b - a g_{ij} \right) \quad (31)$$

so that \mathcal{R}_{ij} , unlike (29) where it is strictly proportional to g_{ij} , now picks up novel structures generated by $\partial_i \partial_j b$. In other words, it is the \bar{y} dependence of $b(\bar{y})$ that enables \mathcal{R}_{ij} to develop new components not necessarily related to those of the metric field.

Having replaced covariant derivatives with ordinary ones in this particular coordinate system, it is now possible to examine implications of different \bar{y} dependencies of $b(\bar{y})$. If $b(\bar{y})$ exhibits a linear dependence, $b(\bar{y}) = g^{ij} p_i y_j$, then the Ricci tensor turns out to depend on $p'^k x^l \partial_i \partial_j g_{kl}$ which involves curvature tensors rather than the metric tensor itself. When $b(\bar{y})$ is quadratic in \bar{y} , $b(\bar{y}) = (a'/2) y_i y^i$, the Ricci tensor now involves $2a' g_{ij} + a' x^k x^l \partial_i \partial_j g_{kl}$ which again depends on curvature tensors computed at the point N . Consequently, when $b(\bar{y})$ exhibits an explicit \bar{y} dependence the Ricci tensor involves not only the metric tensor itself (as in (29) holding for constant-curvature spaces) but also double derivatives of the metric tensor, i.e., the curvature tensors. More general dependencies are expected to yield more general structures for the geometry and topology of the extra space.

In general, irrespective of what coordinate system is chosen $b(\bar{y})$ is a bounded quantity. Therefore, it forces extra dimensions to take values within a hyperboloid. Indeed, a quadratic polynomial dependence for $b(\bar{y})$, for instance, results in

$$\frac{a'}{2} y_i y^i + p'_i y^i + b_0 < \frac{p_\mu p^\mu}{2a} \quad (32)$$

so that extra dimensions are bounded to have a finite size. For a purely quadratic dependence one finds $y_i y^i < p_\mu p^\mu / a a'$ which gives an idea on the maximal size a given dimension y^i can have. However, for more general, in particular, non-polynomial \bar{y} dependencies of $b(\bar{y})$ its bounded nature may not imply any size restriction on the extra space at all. One keeps in mind that all model parameters must eventually return the correct value of Newton's constant in four dimensions:

$\int d^D y \sqrt{-g} = 8\pi G_N M_\star^{D+2}$. This constraint requires the extra space to be of finite volume irrespective of the nature of the manifold [8,9,11–14].

4. Conclusion and future prospects

In this work we have introduced a new method of spontaneous compactification which involves a partially gravitating bulk scalar field. We have systematically constructed first a completely non-gravitating scalar field and then a partially gravitating one. We have examined scalar field configurations and minimum energy configurations in each case. Finally, we have discussed implications of a partially gravitating scalar for spacetime compactification. Our analysis here serves as an existence proof of a novel scalar-induced compactification. In particular, existence of a constant-curvature manifold for extra dimensions, and other novel properties observed in the frame of Riemann normal coordinates are particularly encouraging indications for the fact that a single scalar field, nonminimally coupled to the curvature scalar, can indeed lead to spontaneous compactification of the extra dimensions.

Before concluding, it may be useful to discuss briefly some aspects which have been left untouched in the text. The method of compactification we have discussed can be straightforwardly extended to cases with several scalar fields. However, bulk fields with non-vanishing spin may not always exhibit a physically sensible configuration when $\mathcal{T}_{\mu\nu} = 0$ is imposed (for instance, a fermion $\Psi(x^\mu, y^i)$ acquires a vanishing energy-momentum tensor when $(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu) \Psi = 0$ which comprise the equations of motion but are much wider than them). Therefore, we inherently assume that all fields but $\phi(x^\mu, y^i)$ are long-wavelength modes, and $\phi(x^\mu, y^i)$, a gauge singlet scalar, realizes dynamical compactification at energies $\mathcal{O}(M_\star)$. The low-energy fields disrupt strict flatness of M^4 depending on how their energy scale compares with M_\star .

It is necessary to determine a simultaneous solution of (25) and (26) for having a precise knowledge of the shape and topology of the aimed-at manifold. In particular, these equations cannot be guaranteed to be free of singularities in the extra space. A detailed analysis is expected to shed light on nature of such singularities (see, for instance, [24] for an analysis of the singularities in braneworld scenarios with a self-tuning cosmological term). Moreover, a full account of the spontaneous compactification might require a numerical determination of variables for sample values of the parameters. It will be after such an analysis that one will have detailed information on under what conditions the extra space takes a given shape and topology.

Another important issue is the determination of excitation spectrum about the background geometry we have determined. In other words, it is necessary to determine the graviparticle spectra corresponding to normal modes generated by small oscillations about the background (see [12], for instance). This involves shifts $\eta_{\mu\nu} \rightarrow \eta_{\mu\nu} + h_{\mu\nu}$, $g_{ij} \rightarrow g_{ij} + h_{ij}$, $\phi(x^\mu, y^i) \rightarrow \phi(x^\mu, y^i) + \delta(x^\mu, y^i)$ as well as small but finite values of $g_{\mu j}$ and $g_{i\nu}$. In doing the spectrum analysis, particular care should be paid to the fact that the partially gravitating scalar field configuration in (20) depends explicitly on the metric tensor,

and thus, its variation stems from both $\delta(x^\mu, y^i)$ and variations of the metric components.

One final remark concerns the use of higher curvature gravity. Indeed, higher-curvature gravity theories which generalize Einstein–Hilbert action to a function $f(\mathcal{R}, \square\mathcal{R})$ of the curvature scalar can be mapped, via conformal transformations, into Einstein–Hilbert action plus a scalar field theory [19,25]. In this context, the scalar field theory which facilitates the compactification may be interpreted to have pure gravitational origin, and this may entail possibility of spontaneous compactification via higher curvature gravity.

These aforementioned points summarize some of the important and yet-to-be done aspects of the compactification process advocated in this work. In conclusion, we have shown that a bulk scalar field in $4 + D$ dimensions can lead to a spontaneous compactification of the extra dimensions without inducing a classical cosmological constant when it gravitates only in those dimensions which are to be compactified.

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