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Reconsidering extra time-like dimensions

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Abstract. In this study we reconsider the phenomenological problems related to tachyonic modes in the context of extra time-like dimensions. First we reconsider a lower bound on the size of extra time-like dimensions and improve on the conclusion in the literature. Next we discuss the issues of spontaneous decay of stable fermions through tachyonic decays and disappearance of fermions due to tachyonic contributions to their self-energies. We find that the tachyonic modes due to extra time-like dimensions are less problematic than the tachyonic modes in the usual 4-dimensional setting because the most troublesome Feynman diagrams are forbidden once the conservation of momentum in the extra time-like dimensions is imposed.

Extra spatial coordinates have been considered thoroughly in recent years. A glance at ArXiv shows that there are hundreds of papers on extra dimensions published in the last five years and almost all of them being wholly or mainly on spatial extra dimensions. From the theoretical point of view the scarcity of studies involving extra time-like dimensions [1–7] is mainly due to the existence of tachyonic modes in such models, which are problematic because of the violation of causality and unitarity and the lack of an adequate field theoretic description of tachyonic fields [2], while from the phenomenological point of view the most serious problems are the extremely small empirical lower bound in literature on the size(s) of extra time-like dimensions [8], the spontaneous decay of stable particles induced by negative energy tachyons [2, 9], and the imaginary self-energy for charged fermions induced by tachyonic photon modes, which, in turn, seems to cause the disappearance of the fermion into nothing in a very short time [2]. In this study we will focus on the phenomenological difficulties and try to see whether one may moderate the phenomenological problems mentioned above in the hope that a thoroughly consistent formulation of the field theory of tachyons and their interactions with the usual particles may be formulated in future (if tachyons exist at all). The first phenomenological problem that will be considered here is the extremely small lower bound derived from the lower bound on the lifetime of the proton [8]. In the light of this extremely small lower bound on the size of extra time-like dimension(s), in the order of a tenth of the Planck scale, either one should dare to employ such (unnaturally) small dimension(s) or should use brane models where our physical world is a brane with an infinitesimal width in the extra time-like direction [2] or a scheme where tachyonic modes are not allowed to be produced [5, 6]. A possible relaxation of the bound on the size of extra time dimension(s) would give more freedom to the model constructions with extra time-like dimension(s). So we reconsider the lower bound obtained from the lower bound on the proton lifetime and the calculation of a tree level Feynman diagram. We find that the calculation leads to no bound on the size of extra time-like dimensions. In fact we just repeat the calculations in [8], except that we notice the fact that there is a cutoff momentum in the Fourier transform. In other words the difference between our result and the original study results from the naive application of the Fourier transform in [8] to get the non-relativistic potential corresponding to the scattering of protons inside a nucleus by tachyonic photon modes. In the original study the effect of tachyonic modes on fermion self-energies are neglected and no cutoff was taken, the integration is from minus infinity to plus infinity in the momenta, while one should take a cutoff corresponding to the maximum momentum available to the protons inside the nucleus. One obtains the same result as the one obtained in [8] when one lets the cutoff momentum go to infinity and neglects the self-energy contributions. Next we consider the problems of the spontaneous decay of the particles through release of negative energy tachyons and the imaginary mass induced through self-energy diagrams of fermions. We argue that these problems may be evaded by imposing conservation of momentum in the extra time direction provided that the standard model particles are identified as the zero modes of the Kaluza–Klein tower (that is, the standard identification).

First consider the tree level diagram for the electromagnetic scattering of two protons inside a nucleus [10, 11] (see Fig. 1).

The scattering cross section corresponding to Fig. 1 may be obtained from the scattering amplitude of elastic

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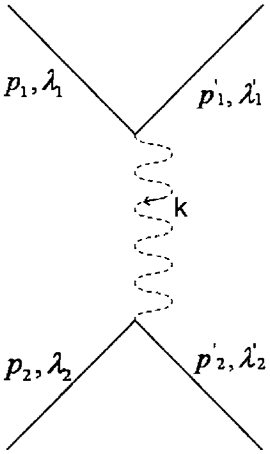

Fig. 1. The Feynman diagram for the scattering of two protons with the initial 4-momenta and the spins, p_1, p_2 and λ_1, λ_2, and the final 4-momenta and the spins p_1', p_2' and λ_1', λ_2'. The *wavy line* denotes the tachyonic Kaluza–Klein modes of photon

fermion–fermion scattering. The differential cross section for elastic fermion–fermion scattering is related to scattering amplitude T by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |T|^2 = \frac{1}{2p_{10}2p_{20}2p_{10}'2p_{20}'}|M|^2\,, \tag{1}$$

where M is the matrix element given by

$$\begin{aligned} M = &\frac{e^2}{4\pi^2}\overline{u}(p_1', \lambda_1')\gamma_\mu u(p_1\,, \lambda_1) \\ &\times \frac{1}{k^2 + m_n^2 + \mathrm{i}0}\,\overline{u}(p_2', \lambda_2')\gamma^\mu u p_2, \lambda_2)\,, \end{aligned} \tag{2}$$

where

$$m_n^2 = \frac{n^2}{L^2}\,, \tag{3}$$

and the u are 4-component Dirac spinors; the γ_μ are the usual gamma matrices. One should also include the exchange scattering where $p_1' \leftrightarrow p_2'$, $\lambda_1' \leftrightarrow \lambda_2'$, but we are only interested in the order of magnitude results and the crossed term of (2) gives a similar contribution as (2) itself and does not alter the conclusion. So it is sufficient to consider (2). In the non-relativistic limit [10] the 0-component of the proton 4-momenta p_{01}, p_{02} and the photon 4-momentum transfer k are approximated by

$$\begin{aligned} p_0 &\simeq m + \frac{\mathbf{p}^2}{2m} - \frac{\mathbf{p}^4}{8m^3} \\ k^2 &= (p_1 - p_1')^2 = (p_{10} - p_{10}')^2 - (\mathbf{p}_1 - \mathbf{p}_1')^2 \\ &= \frac{\left(\mathbf{p}_1^2 - \mathbf{p}_1'^2\right)^2}{4m} - \mathbf{k}^2\,, \\ \frac{1}{\sqrt{2p_0}}u_{(p,\lambda)} &= \sqrt{\frac{m+p_0}{2p_0}}\begin{pmatrix} \chi(\lambda) \\ \frac{\mathbf{p}\cdot\sigma}{m+p_0}\chi(\lambda) \end{pmatrix} \\ &\simeq \begin{pmatrix} \left(1 - \frac{\mathbf{p}^2}{4m^2}\right)\chi(\lambda) \\ \frac{1}{2m}\mathbf{p}\cdot\sigma\chi(\lambda) \end{pmatrix}\,. \end{aligned} \tag{4}$$

Hence in the strict non-relativistic limit (i.e. $p_0 = m$, $1 - \frac{\mathbf{p}^2}{4m^2} = 1$) T becomes

$$\begin{aligned} T = &\frac{e^2}{4\pi^2}\chi^\dagger(\lambda_1')\,\chi(\lambda_1)\frac{1}{|\mathbf{k}|^2 - m_n^2}\chi^\dagger(\lambda_2')\chi(\lambda_2) \\ &|\mathbf{k}| < |\mathbf{R}| = R\,, \end{aligned} \tag{5}$$

$$\gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}\,, \qquad \gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\,,$$

where we have introduced the cutoff R which should be taken in the order of the momentum corresponding to the binding energy of the nucleus. This cutoff is explicitly written in (5) because $k^2 \simeq -|\mathbf{k}|^2$ is not enough to indicate that T in (5) is the non-relativistic expression since the photon is off-shell in the propagator and one may take $k^2 \simeq -|\mathbf{k}|^2$ for relativistic values of $|\mathbf{k}|$ as well provided that $k_0 \ll |\mathbf{k}|$. In other words the strict non-relativistic limit implies $k^2 = -|\mathbf{k}|$ but $k^2 \simeq -|\mathbf{k}|^2$ does not necessarily imply the strict non-relativistic limit. Therefore the explicit expression of the conservation is not true; that is, $|\mathbf{k}| < |\mathbf{R}|$ is necessary. In non-relativistic quantum mechanics the scattering amplitude for the elastic scattering of a particle from a potential V, in the Born approximation may be written as [10–12]

$$T(\mathbf{k}) = \frac{1}{(2\pi)^2}\int \mathrm{d}^3\mathbf{x}\mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{x}}\chi^\dagger(\lambda_1')\chi^\dagger(\lambda_2')V(\mathbf{x})\chi(\lambda_1)\chi(\lambda_2)\,. \tag{6}$$

After comparing (5) and (6) one notices that

$$f(|\mathbf{k}|) = \int \mathrm{d}^3\mathbf{x}\mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{x}}V(\mathbf{x})\,, \tag{7}$$

where

$$f(|\mathbf{k}|) = \begin{cases} \frac{e^2}{|\mathbf{k}|^2 - m_n^2} & \text{for } |\mathbf{k}| \le R\,, \\ 0 & \text{elsewhere}\,. \end{cases} \tag{8}$$

$V(\mathbf{x})$ is obtained as the Fourier transform of $f(|\mathbf{k}|)$ as

$$\begin{aligned} V(\mathbf{x}) &= \frac{e^2}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}}}{|\mathbf{k}|^2 - m_n^2} \\ &= \frac{e^2}{(2\pi)^3}\int_0^R \frac{|\mathbf{k}|^2\,\mathrm{d}k}{|\mathbf{k}|^2 - \frac{n^2}{L^2}} \\ &\quad\times \int_0^\pi \exp\{\mathrm{i}|\mathbf{k}|r\cos\theta\}\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi \\ &= \frac{e^2}{2\mathrm{i}(2\pi)^2 r}\int_{-R}^R \frac{k\,\mathrm{d}k}{k^2 - \frac{n^2}{L^2}}\{\exp(\mathrm{i}kr) - \exp(-\mathrm{i}kr)\} \\ &= \frac{e^2}{\mathrm{i}(2\pi)^2\,r}\int_{-R}^R \frac{k\cdot\exp(\mathrm{i}kr)}{k^2 - \frac{n^2}{L^2}}\mathrm{d}k\,. \end{aligned} \tag{9}$$

We take the wave function of two protons inside a nucleus to be

$$\Psi = \frac{\sqrt{m_\pi^3}}{\sqrt{\pi}}\mathrm{e}^{-m_\pi r}\,, \tag{10}$$

where m_π denotes the mass of pions. Then the decay width is obtained as

$$\Gamma = \mathrm{Im}\langle\Psi|V(r)|\Psi\rangle\,. \tag{11}$$

The evaluation of $\Gamma = \langle\Psi|V(r)|\Psi\rangle$ is done in the appendix and is found to be

$$\begin{aligned}\langle\Psi|V(r)|\Psi\rangle = \mathrm{i}\frac{e^2\,m_\pi^3}{\pi^2}\Bigg[&\frac{2m\beta}{(m^2-\beta^2)^2}\ln\left(\frac{\beta+R}{\beta-R}\right)\\&-\frac{(m^2+\beta^2)}{(m^2-\beta^2)^2}\ln\left(\frac{m+R}{m-R}\right)\\&-\frac{2mR}{(m^2-\beta^2)(m^2-R^2)}\Bigg]\,,\end{aligned} \tag{12}$$

where $m = 2\mathrm{i}m_\pi$, $\beta = \frac{n}{L}$. The result in (12) is the decay width corresponding to the nth mode Kaluza–Klein tower. One should sum over $n = 1, 2, ...$ to get the total contribution of tachyonic modes to the decay width. This may be easily approximated, in the case where $m_\pi \ll \beta$ and $R \ll \beta$ [8] (which is the case for protons inside nuclei), by

$$\begin{aligned}\langle\Psi|V(r)|\Psi\rangle &\simeq \mathrm{i}\frac{e^2\,m_\pi^3}{\pi^2}\Bigg[\frac{4mR}{\beta^3(\beta-R)}\\&\quad-\frac{1}{\beta^2}\ln\left(\frac{m+R}{m-R}\right)+\frac{2mR}{\beta^2(m^2-R^2)}\Bigg]\,,\\ \sum_{n=1}^{\infty}\langle\Psi|V(r)|\Psi\rangle &\simeq \mathrm{i}\frac{e^2 m_\pi^3 L^2}{\pi^2}\Bigg[\zeta(3)\frac{4mRL}{(\beta-R)}\\&\quad-\zeta(2)\ln\left(\frac{m+R}{m-R}\right)+\zeta(2)\frac{2mR}{(m^2-R^2)}\Bigg]\,,\end{aligned} \tag{13}$$

where L denotes the radius of the extra time-like dimension, ζ denotes the Riemann zeta function (defined by $\zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n}$) and $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(3) \simeq 1.2$. After examining (12) or (13) one notices that $\langle\Psi|V(r)|\Psi\rangle$ is real if $\beta > R$ (which is the most natural choice). Otherwise it means that the tachyonic photon masses are in the order of MeV. (In fact one obtains the result of [8] when one lets $R \to \infty$.) In other words the tachyonic photon modes cannot lead to decay of the proton through processes given in Fig. 1 unless the size of the extra dimension is larger than nuclear sizes. However this does not imply that tachyonic modes cannot induce spontaneous decay of protons once the size(s) of the extra time-like dimension(s) are taken smaller than the nuclear sizes. There are other contributions which may induce spontaneous decay of protons although the size(s) of the extra time-like dimension(s) are taken smaller than nuclear sizes. Such a possible contribution is induced through fermion self-energies as discussed in the second next paragraph. Inspection of (14) reveals that the rate of spontaneous decay of a proton (or quark) is much larger than the one that would be induced by the process given in Fig. 1. Moreover fermion self-energy diagrams would induce an imaginary part for the pion self-energy, and hence for its mass. This, in turn, would make the pion mass in (12) complex. So there would be an imaginary contribution to (12)

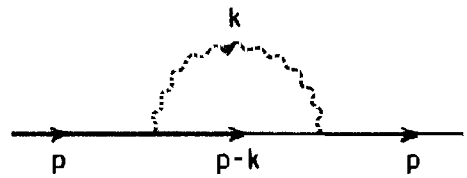

Fig. 2. The Feynman diagram for the contribution of a photonic tachyon to fermion self-energy. The *wavy line* denotes the tachyonic Kaluza–Klein modes of the photon, and the *solid line* denotes the fermion

even in the case $R < \beta$, that is, even in the case that the sizes of the extra dimension(s) are much smaller than the nuclear sizes. So we will impose conservation of momentum in the extra time-like dimensions in the paragraph after the next paragraph to forbid fermion self-energy diagrams with tachyonic photons. In that way the processes similar to Fig. 1 are forbidden as well as the processes in Fig. 2. One may wonder if the calculation of that process (given in (12)) is unnecessary or redundant once conservation of momentum is imposed in the extra dimensions. In fact it is not. The result of (12) gives more flexibility in model building. For example, one may consider a process similar to the one given in Fig. 1, where one of the incoming and one of the outgoing protons are replaced by their tachyonic Kaluza–Klein counterparts. (These modes may be produced in the early universe in models where quarks are allowed to propagate in the extra time-like dimensions.) Such processes are not forbidden by conservation of momentum (in the extra time-like dimensions) and their decay would be of the same form as (12) provided that the wave functions for protons and their tachyonic counterparts have the same form as (10). So the reality of (12) is important in the discussion of the stability of protons in the presence of tachyonic modes.

One might think that the scattering of high energy free protons (e.g. in cosmic rays) through processes similar to the one given in Fig. 1 may change the bound given above. The cross section in that case can be directly found from (5) and is seen to be real. So the decay width of a two free nucleon system due to a process similar to Fig. 1 is zero. One may notice this fact without doing the calculation of the corresponding decay width explicitly. The decay width, i.e. the imaginary part of $\langle\Psi|V(r)|\Psi\rangle$, is due to the mixture of the arguments of the real exponent in Ψ and the complex exponential in $V(r)$. If one takes Ψ to be the wave function of two free protons (which is expressed by a complex exponential) then in the evaluation of $\langle\Psi|V(r)|\Psi\rangle$ the overall complex exponentials cancel and $\langle\Psi|V(r)|\Psi\rangle$ results in a real number, so it has no imaginary component. In other words the decay width of two free protons due to tachyonic photon modes is always zero. However for confined particles one may expect a wave function of the form of (10), which results in a non-zero decay width. Hence the quarks inside the nucleons may give such a non-zero decay width. On the other hand we do not know the wave functions of quarks inside nucleons, so it is impossible to obtain an exact lower bound on the size of extra time-like dimensions by considering the quarks inside nucleons. However one may expect this wave function not to be drastically different from (10). In that case one would expect the lower bound on the size of extra dimensions to be in the order of (cutoff momentum)$^{-1}$, that is, $\mathcal{O}(\frac{1}{1\,\text{GeV}})$. In the same way one may put a still smaller lower limit if the quarks are made of composites of some other particles (preons). If this generalization is reliable then one may relate the lower limit on the size of the extra time-like dimension(s) and the binding energy. In this case one may speculate that, if an extra time-like dimension of the size much larger than the (Planck mass)$^{-1}$ is discovered, then it excludes the possibility of stable bound states with energies much higher than the inverse of the size of the extra time-like dimension.

Next we consider the problem of the spontaneous decay of a particle (e.g. an electron) into a tachyon and the original particle, and the problem of an imaginary mass contribution to the stable fermions (e.g. electron or proton) through self-energy diagrams involving a tachyon. The decay of a particle (say an electron) into another electron and a negative energy tachyonic photon is kinematically allowed. It is difficult to identify these negative energy tachyons with anti-tachyons because negative energy tachyons may be made to be of positive energy by a simple Lorentz boost [9]. So the result of such decays can be catastrophic because the kinematics allows large negative values to occur for the energy of such a tachyon, and such a large negative value energy destabilizes the whole vacuum. However, once we identify the tachyon with the Kaluza–Klein mode of the photon in the extra time dimension, this decay becomes impossible since (at least in the transient time till the formation of the standing waves) there will be a nonzero net momentum flow in the extra time direction due to the tachyon and there is no other momentum to balance it.

The problem of the imaginary contribution to the masses of stable fermions through self-energy diagrams involving tachyons can be avoided in the same way, i.e. by imposing the conservation of momentum corresponding to the extra time-like dimension. Without taking this conservation into account, the contribution of the self-energy diagram given in Fig. 2 to the fermion mass (in the Pauli–Villars regularization scheme) is of the form

$$\delta m \propto \frac{e^2\,m}{4\pi^2}\ln\frac{\mu^2-\Lambda^2}{\mu^2}\,, \quad \mu^2>0\,, \tag{14}$$

where m, e, μ and Λ stand for the fermion mass, the electric charge of the fermion, the mass of the tachyonic photon, the Pauli–Villars regularization cutoff scale, respectively, and we have modified the propagator of the tachyonic photon mode (in the Pauli–Villars regularization scheme) by

$$\frac{1}{k^2-\mu^2}\to\Big(\frac{1}{k^2-\mu^2}\Big)\frac{\Lambda^2-\mu^2}{k^2-\mu^2+\Lambda^2}\,. \tag{15}$$

By definition $\Lambda>\mu$, so (14) results in an imaginary contribution of the form

$$\mathrm{i}\frac{e^2\,m}{4\pi}\,, \tag{16}$$

which is independent of μ and Λ and essentially equal to the width of the spontaneous decay of the fermion through the release of a tachyonic photon. This result is extremely problematic because it predicts a decay rate for the fermion comparable to the decay width of hadronic resonances and moreover the result in (16) may be multiplied by a large number because the number of Kaluza–Klein modes is about $\frac{\Lambda}{\mu_0}$ where μ_0 is the mass of the first Kaluza–Klein mode and Λ is at most of the order of the Planck mass. However if we require conservation of the momentum in the extra time direction (at least in the transient time till the formation of standing waves), then the usual fermions (i.e. Kaluza–Klein zero modes of fermions) can only radiate the usual photons (i.e. Kaluza–Klein zero modes of photons) and the contribution to the fermion self-energies given by Fig. 2 is absent and hence the problem is removed. In other words, the contribution of a tachyonic photon to the electron mass (as given in Fig. 2) results in extremely problematic results if the tachyonic mode is not due to an extra time dimension. On the other hand the diagram in Fig. 2 is forbidden (hence the problem is removed) if one considers the tachyon to be due to an extra time dimension and requires the conservation of momentum corresponding to this dimension.

In this study we have re-examined some phenomenological difficulties due to tachyonic photon modes in the study of extra time-like dimension(s). We have shown that the lower bound on the size of extra time dimension(s) due to the lower bound on the lifetime of the proton may be relaxed, and moreover the presence of tachyons related to the extra time dimension(s) is not as problematic as the tachyons in the usual 4-dimensional picture. Although we believe that we have made some progress in the phenomenological viability of extra time-like dimensions there are still some points to be studied further. We hope that this study will facilitate more freedom in model building in future studies.

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Appendix

In this appendix we give the details of the evaluation of the integral given in (12). We have

$$\begin{aligned}&\int_{-R}^{R}\frac{k\,\mathrm{d}k}{k^2-\frac{n^2}{L^2}}\int_0^\infty r\mathrm{e}^{(\mathrm{i}k-2m_\pi)r}\,\mathrm{d}r\int_0^\pi\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\\ &\quad=4\pi\int_{-R}^{R}\frac{k\,\mathrm{d}k}{k^2-\frac{n^2}{L^2}}\int_0^\infty r\mathrm{e}^{(\mathrm{i}k-2m_\pi)r}\,\mathrm{d}r\\ &\quad=-4\pi\int_{-R}^{R}\frac{k\,\mathrm{d}k}{(k^2-\frac{n^2}{L^2})(k+2\,\mathrm{i}\,m_\pi)^2}\,.\end{aligned} \tag{A.1}$$

The denominator of the integral may be written as

$$\begin{aligned}&\frac{1}{(k+\beta)(k-\beta)(k+m-\epsilon)(k+m+\epsilon)}\\ &\quad=\frac{1}{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}\,,\end{aligned} \tag{A.2}$$

where

$$\begin{aligned} &m = 2\mathrm{i}m_\pi\,, \quad \beta = \frac{n}{L} \\ &x = k\,, \quad x_1 = -\beta\,, \quad x_2 = \beta\,, \\ &x_3 = -m+\epsilon\,, \quad x_4 = -m-\epsilon\,. \end{aligned} \tag{A.3}$$

We use the identity

$$\frac{1}{(x-x_1)(x-x_2)} = \frac{1}{x_1-x_2}\left[\frac{1}{x-x_1} - \frac{1}{x-x_2}\right] \tag{A.4}$$

to write (A.2) as

$$\begin{aligned} &\frac{1}{(x-x_1)(x-x_2)(x-x_3)(x-x_4)} \\ &= \frac{1}{x_1-x_2}\left\{\frac{1}{(x_1-x_3)(x_1-x_4)}\cdot\frac{1}{x-x_1} - \frac{1}{(x_2-x_3)(x_2-x_4)}\cdot\frac{1}{x-x_2}\right\} \\ &+ \frac{1}{x_3-x_4}\left\{\frac{1}{(x_1-x_3)(x_2-x_3)}\cdot\frac{1}{x-x_3} - \frac{1}{(x_1-x_4)(x_2-x_4)}\cdot\frac{1}{x-x_4}\right\}. \end{aligned} \tag{A.5}$$

The second term in (A.5) is

$$\begin{aligned} &\frac{1}{x_3-x_4}\left\{\frac{1}{(x_1-x_3)(x_2-x_3)}\cdot\frac{1}{x-x_3} - \frac{1}{(x_1-x_4)(x_2-x_4)}\cdot\frac{1}{x-x_4}\right\} \\ &= \frac{1}{2\epsilon}\left\{a\frac{1}{x-x_3} - b\frac{1}{x-x_4}\right\} = \frac{1}{2\epsilon}\left\{\frac{(a-b)x+bx_3-ax_4}{(x-x_3)(x-x_4)}\right\}, \end{aligned} \tag{A.6}$$

where

$$a = \frac{1}{(x_1-x_3)(x_2-x_3)}\,, \quad b = \frac{1}{(x_1-x_4)(x_2-x_4)}\,, \tag{A.7}$$

$$\frac{(a-b)x}{x_3-x_4} = \frac{2mx}{\left[(m-\epsilon)^2-\beta^2\right]\left[(m+\epsilon)^2-\beta^2\right]}\,, \tag{A.8}$$

$$\frac{bx_3-ax_4}{x_3-x_4} = -\frac{\beta^2-3m^2-\epsilon^2}{\left[(m-\epsilon)^2-\beta^2\right]\left[(m+\epsilon)^2-\beta^2\right]}\,; \tag{A.9}$$

then (A.6) becomes

$$\begin{aligned} &\frac{1}{x_3-x_4}\left\{\frac{1}{(x_1-x_3)(x_2-x_3)}\cdot\frac{1}{x-x_3} - \frac{1}{(x_1-x_4)(x_2-x_4)}\cdot\frac{1}{x-x_4}\right\} \\ &= \left\{\frac{2mx}{\left[(m-\epsilon)^2-\beta^2\right]\left[(m+\epsilon)^2-\beta^2\right]} - \frac{\beta^2-3m^2-\epsilon^2}{\left[(m-\epsilon)^2-\beta^2\right]\left[(m+\epsilon)^2-\beta^2\right]}\right\} \\ &\cdot \frac{1}{(x-x_3)(x-x_4)}\,. \end{aligned} \tag{A.10}$$

After combining (A.6) and (A.10) and using the explicit values of x_1, x_2, x_3, x_4, and letting $\epsilon \to 0$, one obtains

$$\begin{aligned} \frac{k}{(k^2-\beta^2)(k+m)^2} = &-\frac{1}{2\beta(m-\beta)^2}\frac{k}{k+\beta} \\ &+ \frac{1}{2\beta(m+\beta)^2}\frac{k}{k-\beta} \\ &+ \frac{2m}{(m^2-\beta^2)^2}\frac{k^2}{(k+m)^2} \\ &- \frac{\beta^2-3m^2}{(m^2-\beta^2)^2}\frac{k}{(k+m)^2}\,. \end{aligned} \tag{A.11}$$

The evaluation of the integral (A.1) by the use of (A.11) gives

$$\begin{aligned} \langle\Psi|V(r)|\Psi\rangle = \mathrm{i}\frac{e^2\,m_\pi^3}{\pi^2}\Bigg[&\frac{2m\beta}{(m^2-\beta^2)^2}\ln\left(\frac{\beta+R}{\beta-R}\right) \\ &- \frac{(m^2+\beta^2)}{(m^2-\beta^2)^2}\ln\left(\frac{m+R}{m-R}\right) \\ &- \frac{2m\,R}{(m^2-\beta^2)(m^2-R^2)}\Bigg]\,. \end{aligned} \tag{A.12}$$

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