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Analytical approaches to the delta-Eddington model of the radiative transfer through vertically inhomogeneous optical depths

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Abstract

Analytical approaches have been developed for one-dimensional monochromatic delta-Eddington radiative transfer equation through a vertically inhomogeneous medium. They are based on the solution of the Riccati equation that arises from the decoupling of the two-stream radiances, and seek to approximate the exponent functions in the solution as opposed to finding the solution as a whole. Depending on the case, Green–Liouville approximation or other techniques presented in this paper are utilized for finding these exponents. Though developed for atmospheric radiative transfer problems applicable to the global climate change modelling, and for non-invasive medical applications on tissue–light interactions, the techniques considered here are quiet general in nature. Hence, they can also be useful in other boundary value problems of the diffusion type that involve linear second order ordinary differential equations with variable coefficients.

Keywords: delta-Eddington method; Two-stream methods; Inhomogeneous optical depth; Green–Liouville; WBB; Riccati equation; Analytical approximations

1. Introduction

Solving an integro-differential equation of the radiative transfer for a light intensity, even in one dimension, becomes a challenge due to the fact that its integral kernel involves all other yet unknown light intensities arising from all angles of scattering. As a consequence, the solutions at best are forced to be approximate in representing these scattering directions as a finite number of primary pre-chosen streams. In a class of solutions that is in general called the n-stream approximations, the two-stream approximations provide the least complicated solution methods. It is quite possible that, depending on the physics of the problem, and the accuracy demands on the particular application, one can obtain acceptable results with the number of streams as little as only the two. Sometimes, as is the case for the general circulation models (GCMs) used in the climate change studies, they are the only feasible approaches: as computational cost is necessarily high due to the fact

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that the dynamics, as well as the radiative, evaporative, convective heating and cooling processes have to be computed over all the vertical and horizontal grids at all time steps (typically amounting to time spans of 100 years from now) in order to predict the future climate for one specific scenario of a climate forcing. Even then all these computations turn out to be still expensive, and simpler than the two-stream radiative parameterizations are to be resorted to in the GCMs. On the other hand, the uncertainties in climate change predictions that utilize GCMs on supercomputers, have arisen a need for better modeling of the atmospheric radiative transfer, especially within clouds and aerosol loaded mediums [1–5]. Though in the past, it had been acceptable to model these clouds as homogenous plane parallel slabs for purposes of climatological comparisons, the emerging trends are towards a better and more detailed investigation and modeling of the internal inhomogeneities of those clouds at least in the vertical dimension [6–8].

In addition, the new developments in the bio-medical research is towards the non-invasive techniques that utilize the absorption-scattering measurements of light at appropriate wavelengths on tissues or in vivo skin. For example, the light–biological tissue interaction research advances on the detection of bilirubin concentrations of newly born infants [9], blood glucose level detection [10,11], tissue PH measurements [12], and the cancerous tissue search and detection with a promising future [13]. For the tissues and the skin, it is often hard to exactly determine the optical parameters to begin with, for a number of reasons, including the obvious one that they are race, sex, age and health dependent and have a wide range of variability over the organisms [9,14,15]. While the models used for the light tissue interactions are limited to accuracies comparable with the measurements, they are still to be the best among the possible ones. Two-stream approximations are again the most suitable candidates in this respect.

The final form of any two-stream radiative transfer equation in vertically inhomogeneous medium results in a boundary value problem for a second order ordinary differential equation (ODE) with variable coefficients. The traditional approach in atmospheric radiative transfer is to consider the medium as composed of thin homogeneous slabs, that is, in terms of a series of second order ODEs with constant coefficients. For each of these slabs the solution is simple, analytical and thus found straightforwardly. The system of solutions and their first order derivatives are joined together at the inter slab boundaries, and the complete solution is obtained through processing the related matrix (by inversion, elimination, LU decomposition, relaxation, etc.). However, in this paper on the vertically inhomogeneous problems, where the coefficients of the second order ODEs are not constants, we aim to get the desired solutions as analytical as possible, or semi-analytically at the worst, while at the same time keeping the complexity of solutions at the lowest point.

The primary objective of this study is to investigate and further the application of the approximate analytical or semi-analytical solution techniques to the boundary value problems of the diffusion type, where the two-stream approximations serve us well in providing examples for second order ODEs with variable coefficients. We show that the Liouville–Green (or also known as the WBB approximation) can be readily applied to some regions of our problems, and beyond that region we can build other analytical approximations. Our secondary objective is to demonstrate the degree of success and the range of applicability of the techniques that we propose, by using the radiative transfer problems relevant to the climate change and biomedical research, and the Schroedinger equation as employed in quantum mechanical tunneling phenomena; since final form of the transfer equations in all these fields fall into the same broader category known as the diffusion equation.

In order to better illustrate our approach, we focus on an analytical model based on the delta-Eddington solution [16] to the monochromatic radiative transfer equation through an inhomogeneous optical depth. This model is one dimensional in the vertical, the horizontal inhomogeneities and temporal variations being averaged out. The reason for our preference to work with the delta-Eddington approximation is that it has been shown to be the most accurate among all the two-stream approximations [17].

For details, we focus on the steady-state equation of monochromatic radiative transfer at a fixed wave number, which can be given as [18]

$$-\mathrm{d}I/(\beta_{\mathrm{e}}\,\mathrm{d}s) = I - J, \tag{1}$$

where β_{e} is the extinction coefficient of the medium, $\vec{s}$ is the position and (for $\mathrm{d}s = \mathrm{d}z/\mu$ the viewing direction is along $\mathrm{d}\vec{s}$, and μ is the cosine of the angle between the direction of the outgoing beam and the upward z-axis), the light intensity, I (also known as radiance, and defined as the energy of the light per unit solid angle, per unit time, per unit area that is perpendicular to its direction of propagation) is located at $\vec{s}$. J is the source term at the position $\vec{s}$, for a plane parallel medium (meaning homogeneous or made homogenous by averaging in the horizontal direction and horizontally extending to infinity as if bounded by two infinite planes from above and below). At the point $\vec{s}$, the source term is given by

$$J = \frac{\varpi_0}{4\pi}\int_{-1}^{1} d\mu' \int_0^{2\pi} d\phi' \cdot P(\mu,\phi,\mu',\phi') \cdot I(\vec{s};\mu',\phi') + (1-\varpi_0)\cdot B(T(\vec{s})), \tag{2}$$

where $B(T(\vec{s}))$ is the Planck's black body function at the temperature T of the point $\vec{s}$ (or the altitude z) [18], ϖ_0 is the single scattering albedo (the ratio of scattering optical path to the extinction optical path, as given in the denominator of the right-hand side of Eq. (1)), μ' is the cosine of the angle between the z-axis and the incoming radiance, and ϕ' is the azimuthal angle for the incoming radiance in a pre-chosen spherical coordinate system. Also the phase function that determines the angular distribution of the scattering light off from the point $\vec{s}$ is given by $P(\mu,\phi,\mu',\phi')$, with the primed variables standing for the incoming directions over which the designated integral is to be performed. Since the medium is vertically inhomogeneous, all variables depend on the altitude. As mentioned above, it is this integral form of the source term that mandates approximations in the solutions as the streams, and yielding better accuracies with higher number of the streams.

The radiative transfer equation (Eq. (1) with Eq. (2)) is an integro-differential equation that traces the light intensity at any desired direction within an infinitesimal volume of known shape and position, accounting for the loss terms of absorption and scattering, and gain terms due to emission if present, and the multiple scattering gains from the environs that scatter within this volume element, finally emerging out in the desired direction. In this equation the dependent variable I is the radiance, which is a multi-variable function that depends on both the position and the viewing direction at that position, comprising total of five variables (considering only the majority of cases that are at steady-state and have practical significance, otherwise time would be the sixth variable). In the approximations, the dependency on the viewing direction can be represented by a finite set of discrete radiance function I's at a given position, facilitating replacement of the integral term with a summation over these I's. The simplest approach is to consider only two radiances, or two streams, one at the front and the other one at the back with respect to the incidence direction.

However, this simplicity comes with a price that we have to deal with negative reflectances and transmittances for optically thin problems in all two-stream solutions (for this well documented shortcoming, see Tables 6.2 and 6.3 on pp. 189–191 in Liou [18]. This is rectified in higher streams: for example as little as four streams yield physically more acceptable (energy conserving) solutions [18], but the order of ODE then becomes at least four. On the other hand, in the tissue–light interactions and in the climate change research, this shortcoming is much offset with the statistically averaged nature of the needed solutions, leaving the two-stream models as the first choice due to their directness and simplicity. For example, by definition a climatological value is to be averaged at least for a decade over a given location on Earth, cancelling all random errors of few percent during the averaging process. If the field of application is selected in such a way that the accuracy of the other parameters and uncertainties are of the same magnitude as that of the model results, the simplicity and ensuing efficiency of use come as trade offs. Examples of these fields, as mentioned above are climate research, tissue–light research, and some atmospheric remote sensing operations such as CEPEX and INDOEX field experiments of the California Space Institute at La Jolla, CA, though for very detailed and narrow field focused remote sensing operations, more accurate, and elaborate models are preferred (such as discrete ordinate radiative transfer: DISORT [19], adding doubling [18]).

If the optimum angles of these representative radiances are selected in accordance with a two-point Gauss–Legendre quadrature formula (see DeVries [20] for the details of the Gauss–Legendre quadrature), the errors arising from having only two streams can be minimized, though not fully eliminated. Although the Gauss–Legendre quadrature used in selecting these streams is numerical in origin, it preserves the analytical character of the equations. Generalizations to multiple or n-streams had been developed; a source code named DISORT was originally constructed from the Ph.D. thesis of Stamnes. It has been widely used, and now publicly available for electronic distribution [19,21].

An alternative and relatively more accurate approach among the two-stream approximations category is known as the Eddington solution (along with its even more accurate modification known as the delta-Eddington)

[17,18]. This approach treats the source integral fully analytically by representing the radiance as a superposition of a symmetric and asymmetric parts, much as in a moment expansion with respect to μ, where I linearly depends on μ, and resulting in two fluxes, one upward and the other in downward direction. Due to the azimuthal symmetry of I within the plane parallel medium, the source integral does not depend on the azimuthal angle, ϕ; thus the ϕ dependent term in the truncated phase function does not appear explicitly. Also the phase function is expanded in Legendre polynomials, containing only the zeroth and first order terms, though this truncation to two terms is still the leading cause of negative reflectances and transmissivities apparently violating the conservation of energy for thin mediums. Here we focus on the two-stream approximations which are though not very accurate, still adequate for atmospheric radiation applications primarily pertinent to climate change research [22], and some medical applications.

2. The solution methods

In the following, we survey and present only those details of the delta-Eddington method as we emphasize the difference of our approach from the original development. For the full derivation details of the delta-Eddington integro-differential equation in a general case of both collimated and diffuse radiances through a vertically homogeneous medium, the reader is referred to the references by Joseph et al., Lenoble, Liou and Coakley [16–18,22]. Through inhomogeneous or homogeneous paths, the solution for the direct beam is easily obtained and given as

$$I_d(\tau,\mu) = \pi \cdot F_{\mathrm{S}} \cdot \exp[-(1-\varpi_0\cdot f)\cdot \tau/|\mu_0|], \tag{3}$$

where μ_0 is the negative of the cosine of the incoming solar beam angle with respect to the vertically up z-axis, and $\tau(z) = \int_z^\infty \beta_{\mathrm{e}}(z)\cdot \mathrm{d}z$ is the optical depth as measured from a given altitude upwards, and F_{S} is the direct solar beam intensity (or radiance). Then from here on, we shall proceed with the equation of the diffuse beam only, hereafter denoted by $I(\tau,\mu)$ for simplicity and it is given as

$$\begin{aligned}\mu\frac{\mathrm{d}I(\tau,\mu)}{\mathrm{d}\tau} &= I(\tau,\mu) - \frac{\varpi_0}{2}\int_{-1}^{1}\mathrm{d}\mu'\cdot P(\mu,\mu')\cdot I(\tau,\mu') \\ &\quad - \frac{\varpi_0}{2}\int_{-1}^{1}\mathrm{d}\mu'\cdot P_{\mathrm{NF}}(\mu,\mu')\cdot I_{\mathrm{dir}}(\tau,\mu')\cdot(1-\varpi_0)\cdot B(T(\tau)),\end{aligned} \tag{4}$$

where I_{dir} is the direct beam intensity, and P_{NF} is the non-forward part of the phase function. The forward part is

$$P_{\mathrm{f}}(\mu,\mu') = 4\cdot\pi\cdot f\cdot\delta(\mu-\mu') \tag{5}$$

and the non-forward part is

$$P_{\mathrm{NF}}(\mu,\mu') = (1-f)\cdot(1+3\cdot\tilde{g}\cdot\mu\cdot\mu'), \tag{6}$$

where $\tilde{g} = (g-f)/(1-f)$ is the new asymmetry parameter for the delta-Eddington case in terms of the original asymmetry parameter g (equal to spherical average of μ), and the forward fraction f is commonly taken as $f = g^2$ for optimum atmospheric cloud radiation applications. Through an inhomogeneous medium, single scattering albedo and asymmetry parameter depend on the position and hence on the optical depth τ. Skipping the arduous derivation details, we present the integro-differential equation for the diffuse radiance $I(\tau,\mu)$:

$$\begin{aligned}\mu\frac{\mathrm{d}I(\tau,\mu)}{\mathrm{d}\tau} &= (1-\varpi_0\cdot f)\cdot I(\tau,\mu) - \frac{\varpi_0\cdot(1-f)}{2}\int_{-1}^{1}\mathrm{d}\mu'\cdot(1+3\cdot\tilde{g}\cdot\mu\cdot\mu')\cdot I(\tau,\mu') - \frac{\varpi_0\cdot(1-\varpi_0\cdot f)}{4} \\ &\quad \cdot F_{\mathrm{S}}\cdot\exp[-(1-\varpi_0\cdot f)\cdot\tau|\mu_0|]\cdot(1-3\cdot\tilde{g}\cdot\mu\cdot|\mu_0|) - (1-\varpi_0)\cdot B(T(\tau)).\end{aligned} \tag{7}$$

In the established delta-Eddington approach, the above diffuse radiance $I(\tau,\mu)$ is expressed as a combination of an isotropic and non-isotropic parts (I_0 and I_1, respectively), and through an integration over the cosine of the zenith angle, one second order ordinary differential equation is obtained for each of these parts. However, in this paper we like to demonstrate an alternative approach through a separation of variables like method (see Arfken [23] for the details of separation of variables method):

$$I(\tau,\mu) = Z(\tau)\cdot M(\mu,\tau), \tag{8}$$

where the Z function depends on only τ, but the angular function M, depends on both μ and τ. Since we are pursuing a two-stream solution, it would suffice to express M as a linear function of μ as in

$$M(\mu,\tau) = 1 + m(\tau)\cdot\mu. \tag{9}$$

After performing the mathematical operations indicated in Eq. (7), we obtain the following expression:

$$\mu\cdot Z' + \mu^2\cdot m\cdot Z' + \mu^2\cdot m'\cdot Z = Z\cdot b + \mu\cdot m\cdot a\cdot Z - (p + q\cdot\mu), \tag{10}$$

where primes denote the derivatives taken with respect to τ, and

$$a = 1 - g\cdot\varpi_0, \tag{11}$$

$$b = 1 - \varpi_0, \tag{12}$$

while the source contributions are

$$p = \frac{\varpi_0\cdot(1-f)}{4}\cdot F_S\cdot\exp[-(1-\varpi_0\cdot f)\cdot\tau|\mu_0|] + B(T(\tau))\cdot(1-\varpi_0), \tag{13}$$

$$q = -\frac{3\cdot\varpi_0\cdot(1-f)}{4}\cdot F_S\cdot\tilde{g}\cdot|\mu_0|\cdot\exp[-(1-\varpi_0\cdot f)\cdot\tau/|\mu_0|] \tag{14}$$

and all these are functions of τ. The Z part of the radiance for which we derive the ordinary differential equation depends only on the optical depth τ; hence the above separation of variables trial turns out to be successful.

Again in the original delta-Eddington approach the Eq. (10) and its μ-fold is integrated (averaged) over μ, instead here we will force the above equation to satisfy the two streams at the Gauss–Legendre quadrature points $\mu_1 = 1/\sqrt{3}$ and $\mu_2 = -1/\sqrt{3}$. By once subtracting and adding the resultant two equations, and defining a new variable as $\mathrm{d}u = a\cdot\mathrm{d}\tau$, we obtain the following two independent first order ordinary differential equations:

$$Z' - m\cdot Z = -q/a, \tag{15}$$

$$Z' - \frac{\gamma^2 - m'}{m}Z = -\frac{3\cdot p}{m\cdot a}, \tag{16}$$

where $\gamma = \sqrt{3}\cdot\sqrt{b/a}$ with $\sqrt{b/a}$ as the similarity parameter, and all derivatives are now with respect to the new variable u, which is well known in the asymptotic radiation theory literature as the scaled optical thickness [24]. From this point on, all functions and derivatives will be in terms of the variable u unless otherwise stated. If we define $Y = m\cdot Z$, Eq. (15) becomes: $Z' - Y = -q/a$; and Eq. (16) becomes: $Y' - \gamma^2\cdot Z = -3\cdot p/a$. Here Z corresponds to I_0, and Y corresponds to I_1 of the original delta-Eddington method. By taking the derivative of the first equation and substituting Y' from the second one, we can show that one of these differential equations can be converted to a second order one:

$$Z'' - \gamma^2\cdot Z = R, \tag{17}$$

where $R = -(q/a)' - 3\cdot p/a$, and it can be used in conjunction with any of these two new first order differential equations (Eq. (15) or Eq. (16)). Through the above definition of γ and u, we have gained the advantage of preventing the occurrence of Z' term in Eq. (17). We will show that an analytical approximation to the solution of this equation is possible, and we will give its form under various ranges of the atmospheric cloud and biological tissue optical parameters. Though we will present the solutions for the sourceless cases, the particular solution related to the source term can be added later on either through an integrating factor method or a variation of parameters method. The homogeneous part of this second order ordinary differential equation is

$$Z''_H - \gamma^2\cdot Z_H = 0 \tag{18}$$

and it can be solved by analytical approximations (or by numerical methods if desired) after we factorize it with K to be determined by the non-linear first order ordinary differential equation (Eq. (20)) in the following form:

$$(D+K)\cdot(D-K)\cdot Z_{\rm H}=0 \tag{19}$$

with $D = \mathrm{d}/\mathrm{d}u$, resulting in a Riccati type equation for a new unknown function K:

$$K^2=\gamma^2-K'. \tag{20}$$

The homogeneous solution for $Z(u)$ can be found by

$$Z_{\rm H}(u)=C_a\exp\left[\int_0^u K(\alpha)\,\mathrm{d}\alpha\right]+C_b\exp\left[\int_0^u K(\alpha)\,\mathrm{d}\alpha\right]\int_0^u \exp\left[-2\int_0^w K(\alpha)\,\mathrm{d}\alpha\right]\mathrm{d}w \tag{21}$$

with two constants C_a and C_b, which are to be determined from the given boundary conditions. The particular solution can be obtained as

$$Z_{\rm P}(u)=\exp\left[\int_0^u K(\alpha)\,\mathrm{d}\alpha\right]\int_0^u \exp\left[-2\int_0^v K(\beta)\,\mathrm{d}\beta\right]\left\{\int_0^v R(w)\exp\left[\int_0^w K(\eta)\,\mathrm{d}\eta\right]\right\}\mathrm{d}v$$

or using the integration by parts technique this can also be shown to be identical to

$$\begin{aligned}Z_{\rm P}(u)=&\exp\left[\int_0^u K(\alpha)\,\mathrm{d}\alpha\right]\cdot\left\{\int_0^u \exp\left[-2\int_0^v K(\beta)\cdot\mathrm{d}\beta\right]\mathrm{d}v\right\}\\ &\cdot\int_0^u\left\{R(w)\cdot\exp\left[\int_0^w K(\eta)\cdot\mathrm{d}\eta\right]\right\}\cdot\mathrm{d}w-\exp\left[\int_0^u K(\alpha)\,\mathrm{d}\alpha\right]\\ &\cdot\int_0^u\left\{R(w)\cdot\exp\left[\int_0^v K(\beta)\,\mathrm{d}\beta\right]\cdot\left[\int_0^v \exp\left[-2\int_0^w K(\eta)\cdot\mathrm{d}\eta\right]\mathrm{d}w\right]\right\}\mathrm{d}v,\end{aligned} \tag{22}$$

which is the same as the solution to be obtained by a variation of parameters technique (see Arfken [23, Chapter 20] for more information on variation of parameters technique).

We assumed that these boundary conditions were given at the $u_0 = 0$ and $u = u^*$ points, though, in general, u_0 need not to be zero. $K(u)$ can be computed numerically from Eq. (20) as the integral of $(\gamma^2 + K^2)$ term with a pre-chosen $K(0)$. The solution $Z_{\rm H}(u)$ does not depend on the choice of $K(0)$; but some values may lead to diverging $K(u)$ solutions, and they cannot be useful as a solution to $Z_{\rm H}(u)$. Eq. (21) looks cumbersome: even if $K(u)$ can be analytically integrable in some simpler cases, depending on the second integral in the term with the coefficient C_b; the solution $Z_{\rm H}(u)$ may or may not be analytically evaluated.

In Eq. (18), if we proceed following the methods given by Schelkunoff and Stephenson [25,26], where the solution for K is $K \simeq \gamma - \gamma'/2\gamma$, we get

$$Z_{\rm H}(u)=C_1\cdot\exp\left[\int_0^u \gamma(v)\cdot\mathrm{d}v\right]\Big/\sqrt{\gamma(u)}+C_2\cdot\exp\left[-\int_0^u \gamma(v)\cdot\mathrm{d}v\right]\Big/\sqrt{\gamma(u)} \tag{23}$$

with C_1 and C_2 as the new integration constants. Their method is based on defining a new variable as $\mathrm{d}t = \gamma(u)\cdot\mathrm{d}u$, eliminating the $Z'_{\rm H}(u)$ derivative in the subsequent differential equation, and then neglecting all other terms except unity in the coefficient of $Z_{\rm H}(u)$, provided that

$$|\gamma'|<\gamma^2. \tag{24}$$

However this condition on γ' may not be satisfied in all cases: therefore we will present new methods that covers these cases in the latter part of this paper. When the above condition is satisfied, it means that inhomogeneity is nowhere extreme since derivative of γ depends on the inhomogeneity of the optical path. The particular solution in Eq. (22) can also be re-expressed in terms of the Liouville–Green solution $K \simeq \gamma - \gamma'/2\gamma$ as

$$\begin{aligned}Z_{\rm P}(u)=&\exp\left[\int_0^u \gamma(v)\,\mathrm{d}v\right]\Big/\left[2\sqrt{\gamma(u)}\right]\cdot\int_0^u\left\{\exp\left[-\int_0^v \gamma(w)\cdot\mathrm{d}w\right]\Big/\sqrt{\gamma(v)}\right\}R(v)\,\mathrm{d}v\\ &-\exp\left[-\int_0^u \gamma(v)\,\mathrm{d}v\right]\Big/\left[2\sqrt{\gamma(u)}\right]\cdot\int_0^u\left\{\exp\left[\int_0^v \gamma(w)\,\mathrm{d}w\right]\Big/[\sqrt{\gamma(v)}]\right\}R(v)\,\mathrm{d}v.\end{aligned} \tag{25}$$

The above method with $K \simeq \gamma - \gamma'/2\gamma$ is known as the Liouville–Green approximation or WKB approximation due to its wide application in quantum mechanics by Wentzel, Kramers and Brioullion [27]. It has been forgotten and rediscovered by several applied mathematicians [25]. One variant is given by Stephenson and Radmore [26] in the form of a perturbation solution. Still another approach is to iterate on the Eq. (20), and is given by Merzbacher [27] for the $(n+1)$th iteration: $K_{n+1} = +\sqrt{\gamma^2 - K_n'}$.

We notice that if we repeat the procedure used in obtaining Eq. (23) many times by defining $\mathrm{d}u_{n+1} = \gamma_n \cdot \mathrm{d}u_n$ at each nth step, as long as $|\mathrm{d}\gamma_n/\mathrm{d}u_n| < \gamma_n^2$ is satisfied (but after some steps it may not be satisfied; the functional $\Delta(\gamma)$, also called the Schwartzian derivative [26] defined as $\Delta(\gamma) = (\gamma'/2\gamma)' - (\gamma'/2\gamma)^2$ may not be negligible), we get a better approximation to Eq. (23).

In the following, we will work on the solutions to the homogenous parts of Eqs. (15) and (16) which are

$$Z_{\mathrm{H}}' - m \cdot Z_{\mathrm{H}} = 0, \tag{26}$$

$$Z_{\mathrm{H}}' - \frac{\gamma^2 - m'}{m} Z_{\mathrm{H}} = 0 \tag{27}$$

and we will directly solve them for m. By equating the coefficients of Z_{H} in the above equations, we obtain a duplication of the Eq. (20), this time in terms of m, where $m \equiv K$:

$$m^2 = \gamma^2 - m', \tag{28}$$

which is to be used along with Eq. (25). It readily admits an approximate analytical solution if the function γ, whose behavior depends on the inhomogeneity along the optical path, satisfies the condition, $|\gamma'| < \gamma^2$, everywhere in the domain of the solution. Under these circumstances, we shall demonstrate that $m \equiv \gamma - \gamma'/2\gamma$.

By inspecting Eq. (28), we see that when the inequality in Eq. (24) holds, m is nearly equal to γ. Hence we begin with $m \equiv \gamma + \varepsilon_1$, where ε_1 is a perturbation on the function γ. Inserting this into Eq. (27) and neglecting ε_1' and ε_1^2 terms, since we expect that they are smaller than the others, we obtain: $\varepsilon_1 = -\gamma'/2\gamma$. This last result validates our neglecting of ε_1' and ε_1^2 terms. Instead of neglecting these terms, if we would desire a better approximation, we would begin with $m \simeq \gamma - \gamma'/2\gamma + \varepsilon_2$, and continue in the same fashion by neglecting ε_2' and ε_2^2, and $\varepsilon_2 \cdot \gamma'/\gamma$ terms to obtain $\varepsilon_2 = [(\gamma'/\gamma)^2/8 - (\gamma'/\gamma)'/4]/\gamma$. Higher order terms can be obtained continuing this way, and this method gives the same terms as in the exponential perturbation method given in Stephenson [26]. Additionally a more systematic approach would be by taking $m = m_0 + \lambda \cdot m_1 + \lambda^2 \cdot m_2 + \cdots$ and considering Eq. (28) as $m^2 = \gamma^2 - \lambda \cdot m'$ beforehand, which corresponds to the a priori knowledge that the inequality in Eq. (24) holds. Here the λ parameter serves as the ordering tool, and at the end of the perturbation process it is set equal to unity.

Up to here, we have just demonstrated a variant of the perturbation approximation of Stephenson [26], and we started with a Riccati [28] equation (Eq. (28)). We are proposing this solution for the atmospheric radiative transfer use in climate change computations as long as the inequality of Eq. (24) holds, where the scattering occurs in a one-dimensionally inhomogeneous medium, such as in clouds or aerosol loaded atmospheres. In this form however, Eq. (24) restricts its utility to a cloud where the single scattering albedo ϖ_0 is not close to unity; or indirectly b, or equivalently γ is not close to zero, and γ' is not very large. This condition can be realized in the long wave (infrared and near infrared) absorption and emission of the water clouds and relatively thin clouds with significant water vapor content and atmospheric infrared active trace gas contents, since in the long wave range, water clouds have smaller single scattering albedos that are around 0.5. When we consider the atmospheric gas and water vapor together which have zero single scatter albedos, the volume averaged single scatter albedo of the atmospheric cloud layer turns out to be even less than the pure cloud case we considered in the examples below. The conditions of this nature mostly occur at moist lower altitudes. For water and ice clouds that scatter in the visible wavelengths, analytical radiation modeling through inhomogeneous optical paths becomes more challenging and from here on we set out to address it also.

In the Riccati equation as given by Eq. (28), we only need its two distinct solutions (any two among the many possible ones) for the solution of the original second order ODE (Eq. (18)). We are actually interested in the solutions for $I(u,\mu)$, through these two solutions of $m(u)$ in Eqs. (26) and (28). From Eqs. (8) and (9), we can form the complete homogeneous solution as

$$I_{\mathrm{H}}(u,\mu) = C_1 \cdot Z_{\mathrm{H}_1}(u) \cdot M_1(u,\mu) + C_2 \cdot Z_{\mathrm{H}_2}(u) \cdot M_2(u,\mu), \tag{29}$$

where $Z_{H_1}(u) = \exp[\int_0^u m_1(v)\cdot \mathrm{d}v]$, $Z_{H_2}(u) = \exp[\int_0^u m_2(v)\cdot \mathrm{d}v]$, $M_1(u,\mu) = 1 + m_1(u)\cdot\mu$, and $M_2(u,\mu) = 1 + m_2(u)\cdot\mu$.

Now we show that the Liouville–Green approach is a special case of the above factorization and Riccati solution approach of tackling the second order ODEs with variable coefficients; and it allows us to write the general solution in the same manner as would be for a second order ODE with constant coefficients. Using Eq. (28) twice and denoting the two solutions as m_1 and m_2, we get an identity: namely $m_1^2 - m_2^2 = -(m_1 - m_2)'$. In other words we get $j = -k'/k$, where $2k = m_1 - m_2$ and $2j = m_1 + m_2$. This helps us to re-express m_1 and m_2 in terms of k and its derivative k'; thus we obtain:

$$K_1 = m_1 = k - k'/2k \quad \text{and} \quad K_2 = m_2 = -k - k'/2k. \tag{30}$$

If we can compute k, we will have a solution whose accuracy depends on the type and degree of the approximations employed. The formulas listed in Eq. (30) give us an idea regarding the form of the function k that satisfies Eq. (28), even when γ goes to zero (a turning point problem), or γ' is too large. They actually show that the Liouville–Green method is one of the approximations where $k \cong \gamma$, provided that the inequality in Eq. (24) holds. We can write the general solution as $Z(u) = C_1 Z_{H_1}(u) + C_2 Z_{H_2}(u) + Z_{H_2}(u)\cdot\int_{u_0}^u Z_{H_1}(v)R(v)/W(v)\,\mathrm{d}v - Z_{H_1}(u)\cdot\int_{u_0}^u Z_{H_2}(v)R(v)/W(v)\,\mathrm{d}v$ where the homogeneous solutions are $Z_{H_1}(u) = \exp[\int_{u_0}^u m_1(v)\cdot\mathrm{d}v]$ and $Z_{H_2}(u) = \exp[\int_{u_0}^u m_2(v)\cdot\mathrm{d}v]$ with $W(v)$, the Wronskian that is equal to $-2k(u_0)$. In order to see that this last form is exactly the same for a second order ODE with constant coefficients, first we absorb the constant $-k(u_0)$ into the coefficients C_1 and C_2, and substitute m_1 and m_2 from Eq. (30): we obtain the combination of Eqs. (23) and (25) for the general solution, this time in terms of k instead of γ, demonstrating the generality of the factorization plus Riccati approach for the second order ODEs.

2.1. Approximations on k

In order to approximate k, when γ is close to zero (the Liouville–Green solution will not work, and this case is called a turning point case [29,30]), we need to consider Eq. (28) with m_1 (or alternatively with m_2), since both depend on k, giving $(k - k'/2k)^2 = \gamma^2 - (k - k'/2k)'$ which can be written as a non-linear second order ordinary differential equation,

$$k^2 - \gamma^2 = (k'/2k)' - (k'/2k)^2 \tag{31}$$

or in the alternative form

$$4k^4 - 4k^2\cdot\gamma^2 = 2k\cdot k'' - 3\cdot(k')^2 \tag{32}$$

also in the Picard approximate integral form

$$k(u) \simeq k(0) + \frac{k'(0)}{k(0)}\cdot\int_0^u k(v)\cdot\mathrm{d}v + 4\int_0^u k(v)\cdot\mathrm{d}v\int_0^u \{k^2(w) + [k'(w)/2k(w)]^2 - \gamma^2(w)\}\cdot\mathrm{d}w. \tag{33}$$

If we start with an initial guess function $k(u)$ and its value at $u = 0$, $k(0)$, we can employ the Picard iterative integration procedure [29] until we obtain an acceptably accurate solution. Since we are interested in solutions that are at the vicinity of the turning point (meaning $\gamma = 0$), we first solve Eq. (28) in the form $m^2 \simeq -m'$. Again we need two (any two) solutions; one of these solutions, say m_2 is obviously zero. Then, in Eq. (30) with $m_2 = 0$, we get $-k - (k'/2k) = 0$, or $k' = -2k^2$ (which has the same form as m in $m_2' = -m_2^2$) integration of which yields an hyperbolic function in terms of u:

$$k(u) = (1/2)/[1/2k(0) + u] \tag{34}$$

with an arbitrary constant $k(0)$. We also get $m_2 \equiv 2\cdot k$, from which we see that for any number of Picard type iterations in the Eq. (33) with the $\gamma^2 \simeq 0$, we will always get back the same $k(u)$, exactly as given in Eq. (34). Therefore we focus on the rest of the Eq. (33) that contains the small but non-zero γ^2 term. Then

$$k(u) = (1/2)/[1/2k(0) + u] - \int_0^u \frac{\mathrm{d}v}{1/2k(0) + v}\int_0^v \gamma^2(w)\cdot\mathrm{d}w \tag{35}$$

is the expression we get after the first iteration for the solution around a turning point. The first term in the above equation always survives after every iteration. Though $k(0)$ can be arbitrarily chosen, care should be taken that Eq. (34) will not diverge, i.e., $1/2k(0)$ should not equate to any value of u in the range from u_0 to u^* (0 to u^* in this paper). The last term in Eq. (35) contains a double integral. It can either be analytically integrated, depending on the form of the $\gamma^2(u)$ expression, or else it can be analytically approximated by the application of Gauss–Legendre quadrature in the interval from 0 to u, regarding u as a parameter, as shown below. Although we have chosen a two point Gauss–Legendre quadrature for the sake of demonstration, the number of quadrature points can also be chosen larger.

With a two point Gauss–Legendre quadrature, any integral such as $\int_c^x f(v)\,\mathrm{d}v = \int_{-1}^{+1} f(t)\,\mathrm{d}t$ can be approximated, by defining a new variable t in $v = (1+t)\cdot\frac{x-c}{2} + c$ and $\mathrm{d}v = (x-c)\cdot \mathrm{d}t/2$ such that $\int_c^x f(v)\,\mathrm{d}v = \int_{-1}^{+1} f(t)\,\mathrm{d}t = \frac{x-c}{2}\cdot\left\{f\left[\frac{x-c}{2}\cdot\left(1+\frac{1}{\sqrt{3}}\right)+c\right]+f\left[\frac{x-c}{2}\left(1-\frac{1}{\sqrt{3}}\right)+c\right]\right\}$, where $\pm\frac{1}{\sqrt{3}}$ are the quadrature points, and the weights are unity. Similarly a double integration can be converted to three single integrations, each expressible through the Gauss–Legendre approximation: $\int_b^x \int_a^v f(w)\,\mathrm{d}w = x\cdot\int_a^x f(w)\,\mathrm{d}w - b\cdot\int_a^b f(w)\,\mathrm{d}w - \int_b^x v\cdot f(v)\,\mathrm{d}v$, leading a way to express multiple integrals in the same fashion.

For example, the inner integral in Eq. (35) can be approximated as $\Gamma(v) = \int_0^v \gamma^2(w)\,\mathrm{d}w = \frac{x}{2}\left\{\gamma^2\left[\frac{x}{2}\cdot\left(1+\frac{1}{\sqrt{3}}\right)\right]+\gamma^2\left[\frac{x}{2}\cdot\left(1-\frac{1}{\sqrt{3}}\right)\right]\right\}$, then altogether the second term in Eq. (35) can be re-expressed as $\int_0^u \frac{\mathrm{d}v}{1/2k(0)+v}\cdot\int_0^v \gamma^2(w)\,\mathrm{d}w = \int_0^u H(v)\,\mathrm{d}v$, where $H(v) = \frac{\Gamma(v)}{[1/2k(0)+v]}$. Finally we get:

$$k(u) = (1/2)/[1/2k(0)+u] - u\left\{H\left[\frac{u}{2}\cdot\left(1+\frac{1}{\sqrt{3}}\right)\right]+H\left[\frac{u}{2}\left(1-\frac{1}{\sqrt{3}}\right)\right]\right\}. \tag{36}$$

Another case that we need to consider is when $\gamma^2(u)$ is a slowly varying function of u, say around $u = 0$: in other words it is nearly constant in the domain of the solution, but the Schwartzian derivative $\Delta(\gamma)$ is not small. In that case we have the two solutions as $m_1 = \delta m_1 + \gamma$ and $m_2 = \delta m_2 - \gamma$ with new unknowns δm_1 and δm_2 (satisfying $\delta m_1^2 + 2\gamma\cdot\delta m_1 = -\gamma' - (\delta m_1)'$, and $\delta m_2^2 + 2\gamma\cdot\delta m_2 = -\gamma' - (\delta m_2)'$, from Eq. (28)). In a similar fashion in obtaining the Eq. (30), we get $2\delta k\cdot\delta j + 2\gamma\cdot\delta k = -\gamma' - (\delta k)'$ where $2\delta k = \delta m_1 - \delta m_2$ and $2\delta j = \delta m_1 + \delta m_2$. This yields $\delta m_1 = \delta k - (\delta k)'/[2(\delta k+\gamma)] - \gamma'/[2(\delta k+\gamma)]$ and $\delta m_2 = -\delta k - (\delta k)'/[2(\delta k+\gamma)] - \gamma'/[2(\delta k+2\gamma)]$. For δm_1 and δm_2, and consequently for δk to be usable, they should have finite values in the region of concern. If we further examine δm_1 or δm_2 expressions above, leaving γ as a variable, we can consider two cases: (1) γ is very small but $|\gamma'|$ is finite, (2) γ is finite (including zero) but $|\gamma'|$ is too large. Case 1 means $(\delta k)'$ term should balance γ': then it is sufficient to have a finite derivative for δk, and hence a finite δk. In case 2, all the denominators are finite and only $(\delta k)'$ term can balance a very large γ'; but a very large $(\delta k)'$ implies, in turn, a very large δk as u changes farther away from this point. This behavior can only be tolerated for an ignorably small region of the total solution domain and it is not desirable. However in our applications below, we will choose to define $m_1 = \delta m_1 + \gamma(0)$ and $m_2 = \delta m_2 + \gamma(0)$ for $\gamma \neq 0$ but nearly constant ($\gamma \simeq \gamma(0)$) cases so we will not need to consider the behavior of γ'.

In that case we have $\delta m_1 \simeq \delta k - (\delta k)'/[2[\delta k + \gamma(0)]$ and $\delta m_2 \simeq -\delta k - (\delta k)'/[2[\delta k+\gamma(0)]$, or in other words, in term of k, $k = \delta k + \gamma(0)$. Since $2k = m_1 - m_2 = 2\cdot\gamma(0)$, and from Eq. (30) $m_1 = k - k'/2k$, at $u = 0$, we get $k'(0) = 2k^2(0) - 2\cdot m_1(0)\cdot k(0)$. But $m_1(0) \simeq \gamma(0)$, thus we get $k'(0) \simeq 0$. Then we can approximate our initial $k(u)$ once more as $k_0(u) \simeq \gamma(0)$ to be used in the Picard iterative method over Eq. (30) to get $k(u) = k(0) + 2\int_0^u k_0^2(v)\cdot\mathrm{d}v - 2\int_0^u m_{1,0}(v)\cdot k_0(v)\cdot\mathrm{d}v$ after the first iteration. But we also need a good starting form for $m_1(u)$ as in $m_{1,0}(u) = \gamma(0) + \int_0^u[\gamma^2(v) - \gamma^2(0)]\cdot\mathrm{d}v$ which comes from applying the Picard integration on Eq. (27). Altogether we obtain a better approximation for $k(u)$:

$$k(u) = \gamma(0) + \gamma^3(0)\cdot u^2 - 2\gamma(0)\cdot\int_0^u \mathrm{d}v\int_0^v \gamma^2(w)\cdot\mathrm{d}w. \tag{37}$$

In the passing we note that when $\gamma \simeq 0$ we get back Eq. (32). The first term in Eq. (35), when used according to Eqs. (41) and (42), gives the full solution where we may need to employ an appropriate Gauss–Legendre quadrature in the complicated integrals.

Still another way of approximating the $k(u)$ solution outside and inside the Liouville–Green domain can be obtained by inspecting Eq. (31). If each term on the right-hand side is nearly zero, we get the Liouville–Green solution, but if these terms nearly cancel each other, implying that $k \simeq \gamma$, then, instead we get $k(u) = k(0)/[2k(0)/k(0)' - u]^2$, or more generally

$$k(u) = 1/[k_a + k_b \cdot u]^2 \tag{38}$$

with k_a and k_b as constants to be determined by the linear least square curve fit of this $k(u)$ to $\gamma(u)$ in the domain under consideration by minimizing the error integral $\int_0^u [(k_a + k_b \cdot v) \cdot \sqrt{\gamma(v)} - 1]^2 \, dv$, with respect to k_a and k_b. Equating its partial derivatives with respect to k_a and k_b to zero gives us two linear equations for k_a and k_b. Details of linear least square fit can be found in most elementary statistics textbooks. The above form with $\sqrt{\gamma(v)}$ is preferred so that even if $\gamma(u)$ nears zero the integral or its finite summation approximation will not diverge. The method works well as will be shown in the example on the dermis-blood transition applications.

Finally the last technique that we apply for approximating k is a first order perturbation solution to Eq. (32), which also linearizes it, with $k = \gamma_0 + \lambda \cdot k_1$, and $\gamma^2 = \gamma_0^2 + \lambda(\gamma^2 - \gamma_0^2)$, where λ is the perturbation parameter to be set equal to unity at the end, to obtain

$$k_1'' - 4\gamma_0^2 \cdot k_1 = -2\gamma_0 \cdot (\gamma^2 - \gamma_0^2), \tag{39}$$

where γ_0 is best taken to be the average value of $\gamma(u)$ (as $1/u^* \cdot \int_0^{u^*} \gamma(v) \, dv$) over the domain of consideration. The solution is

$$k(u) = C_a \cdot \exp(-2u) + C_b \cdot \exp(2u) + \gamma_0/2 + k_{1p}, \tag{40}$$

where k_{1p} is the particular solution to $k_1'' - 4\gamma_0^2 \cdot k_1 = -2\gamma_0\gamma^2$, which may be found by the method of undetermined coefficients if γ^2 is given as a polynomial in u (or more generally this polynomial times some combination of cosine, sine or exponential functions of u). In other cases a few term Fourier series approximation to γ^2 can be adequate, given the fact that k_1 itself is one of the terms in the perturbation approximation of k. The boundary conditions on $k(u)$ solution is arbitrary, but given the first order perturbative character of the solution, $k(0)$ and $k(u^*)$ should not differ much from the γ_0. In fact for a symmetrical $\gamma(u)$, choosing $k(0) = k(u^*) = \gamma_0$ appears to be better than other choices. This technique is suitable to the diffusion type problems when γ^2 is given as a symmetric hunch as will be shown in the quantum mechanical tunneling example.

We also employ a first order perturbation for $I_0(u) \equiv Z(u)$ on Eq. (18) for comparison with the above solutions that are approximations direct on the $k(u)$ term. Here we take $Z = Z_0 + \lambda \cdot Z_1$, and $\gamma^2 = \gamma_0^2 + \lambda \cdot (\gamma^2 - \gamma_0^2)$ as before, yielding the two equations for Z_0 and Z_1, as $Z_0 - \gamma_0^2 \cdot Z_0 = 0$ and $Z_1 - \gamma_0^2 \cdot Z_1 = (\gamma^2 - \gamma_0^2) \cdot Z_0$. The prescribed boundary conditions are imposed on the zeroth order solution Z_0, and on the Z_1 solution the boundary values are all taken as zeros. As in the case of the perturbation on $k(u)$, we choose to define γ_0 as the average γ over the domain of consideration. These are ODEs with constant coefficients, but the second one is inhomogeneous and it may not have an exact analytical solution, but can be analytically approximated as for Eq. (39).

Up to here, we have summarized the Liouville–Green technique and developed four other methods for the cases when it would not work: one at and around the turning point (Eqs. (33)–(35)), one at and around a slowly varying γ (Eq. (37)), one with the least square fit (Eq. (38)), and the last one for a symmetrical γ^2 (Eq. (40)). The last two of these solutions can also cover the Liouville–Green range. There is a connection between the near turning point solution and the traditional Liouville–Green solution (which is well away from the turning point) as can be seen by inspecting Eq. (20) or Eq. (28) more closely. In all approximations we are searching for a solution of the kind where $\gamma(u) = \chi(u) \cdot m(u)$. Inserting this form into Eq. (28), we obtain $(1 - \chi^2) \cdot m^2 = -m'$. As χ approaches unity, we obtain a Liouville–Green type solution, which is when γ nearly behaves like a constant. This further means that $m' \approx \gamma' \approx 0$ and the inequality in Eq. (24) holds. Following our steps backward, we can say, if in the domain of the solution, inequality of Eq. (24) holds, we can rely upon the Liouville–Green solution as a legitimate approximation. On the other hand if $\chi \approx 0$, that is if $\gamma \approx 0$, we are near a turning point and then we have $m^2 \approx -m'$, whose approximate solutions are given by one of the above four methods. If we could have a hyperbolic function as given by Eq. (34) even for the Liouville–Green range, that would mean that our γ would have to approximately behave as $(1/2)/[1/2k(0) + u \cdot \epsilon]$, for a small ϵ, since then the inequality in Eq. (24) would still hold. In more complicated problems, more than one case may be encountered: then in order to get the complete solution, it would be best to divide the domain into parts and do a match of the solutions and their derivatives on the division points.

The particular solution to Eq. (17) in terms of the function $k(u)$ can be expressed in a similar fashion to the particular Liouville–Green solution with the only difference that γ is replaced by k:

$$Z_{\mathrm{P}}(u)=\left\{\exp\left[\int_0^u k(v)\,\mathrm{d}v\right]\Big/\left[2\sqrt{k(u)}\right]\right\}\cdot\int_0^u\left\{\exp\left[-\int_0^v k(w)\,\mathrm{d}\omega\right]\Big/\left[\sqrt{k(v)}\right]\right\}R(v)\,\mathrm{d}v$$
$$-\left\{\exp\left[-\int_0^u k(v)\,\mathrm{d}v\right]\Big/\left[2\sqrt{k(u)}\right]\right\}\cdot\int_0^u\left\{\exp\left[\int_0^v k(w)\,\mathrm{d}\omega\right]\Big/\left[\sqrt{k(v)}\right]\right\}R(v)\,\mathrm{d}v, \tag{41}$$

where the form of the function k as given by one of the equations, Eqs. (33)–(40), is adequate for most practical purposes considered in this paper. Furthermore, each of the integrals in the particular solution can be approximated by the Gauss–Legendre quadrature as explained above. The homogeneous solution in terms of $k(u)$ is also:

$$Z_{\mathrm{H}}(u)=\widehat{C}_1\exp\left[\int_0^u k(v)\,\mathrm{d}v\right]\Big/\sqrt{k(u)}+\widehat{C}_2\exp\left[-\int_0^u k(v)\,\mathrm{d}v\right]\Big/\sqrt{k(u)}, \tag{42}$$

where the new constants are $\widehat{C}_1$ and $\widehat{C}_2$. Note that in the Liouville–Green domain $k(u)\simeq\gamma(u)$. The general solution to Eq. (17) can be given as the sum of the above homogeneous solution in Eq. (42) and the particular solution in Eq. (41) as $Z(u)=Z_{\mathrm{H}}(u)+Z_{\mathrm{P}}(u)$.

Now we are able to put together a solution for the delta-Eddington approximation where the actual quantities that are computed are the upward and downward fluxes: $F^{+}(u)=\pi\left[I_0(u)+\frac{2}{3}I_1(u)\right]$ and $F^{-}(u)=\pi\left[I_0(u)-\frac{2}{3}I_1(u)\right]$ respectively, and $I_0\equiv Z$ and $I_1\equiv \mathrm{d}Z/\mathrm{d}u$.

In Earth's atmospheric clouds the single scattering albedo in the visible wave lengths is close to unity, that makes these problem a near the turning point type. Though cloud field experiments and observations are being conducted, regarding the actual vertical optical profiles of all the cloud types, to a great degree our information is still limited [31–36]. If we can speculate and assume a vertical profile of the optical parameters for our examples, saying that an optical profile of a given cloud (at any wavelength) might possibly vary only between the extremes of the known reported cloud types [37–39], we could see that the above four approximations would be adequate in all such cases. Through comparisons with the numerical solutions obtained by standard finite difference differential equation solution techniques, such as a shooting method, we have observed that the correct usage of the equations, Eq. (21) and onward, indeed gives satisfactory results for atmospheric cloud problems as well as some medical applications and diffusion problems, detailed examples of which are given below.

3. Examples from cloud radiation problems

In order to construct some illustrative examples and test the limits of our approximations, we have first inspected the eight cloud type parameterizations of Stephens [37] and cirrus parameterization of Takano and Liou [38,39]. Then we hypothesized that in a cloud as the optical properties changed from cloud top to base, at worst the cloud nature could change from one type to another. This in no ways is a claim on how an actual cloud exhibits a vertical optical profile. The data on the vertical profiles of cloud microphysics is at most not sufficient to construct realistic profiles. We have assumed a vertical extent of 1 km, which is reasonable for most clouds. For demonstration purposes we have assumed that the single scattering albedo ϖ_0, the asymmetry parameter g, and the extinction cross-section β_e (km^{-1}), all change linearly from cloud top to cloud base. We have not included the atmospheric gases and water vapor within our example clouds. If we did so, their speculated change from one type to another would have been moderated by the atmospheric absorption, since atmospheric gases only absorb and have diffuse (Rayleigh) scattering in the visible wavelengths. Under these circumstances it is likely that our example cases are extreme deviations from the reality, toward a harsher test of our solutions, and so they serve us to demonstrate the utility of our models for the radiative effect of clouds in the climate change scenarios as demonstrated below.

Our tests are based on the sourceless cases, i.e., there are no direct beam in the visible and no emission in the long wavelengths. We have checked the upward and downward fluxes against a numerical differential equation

Table 1
The optical parameters of the cloud top and base for the hypothetical cases used in the examples

Cloud type	λ	$\beta_e(0)$	$\beta_e(u^*)$	$\varpi_0(0)$	$\varpi_0(u^*)$	$g(0)$	$g(u^*)$
Water	2	124	17	0.85	0.98	0.84	0.79
Cirrus	0.55	2.61	0.17	0.999	0.999	0.84	0.77

The symbols "0" and "u^*" stands for the cloud top and base, respectively. The units for the extinction coefficient, β_e, is in km^{-1}, and the wavenumber λ, is in μm.

Table 2
The profiles of the relevant parameters for the water cloud to cloud transition case at 2 μm

z (km)	u	γ	γ'/γ^2	$\Delta(\gamma)$
0.00	0.00	1.25	−0.007	−0.0003
0.10	3.36	1.21	−0.009	−0.0005
0.20	6.34	1.17	−0.012	−0.0007
0.30	8.96	1.12	−0.017	−0.001
0.40	11.2	1.06	−0.024	−0.002
0.50	13.2	1.00	−0.036	−0.004
0.60	14.8	0.93	−0.057	−0.008
0.70	16.2	0.85	−0.097	−0.019
0.80	17.2	0.76	−0.186	−0.053
0.90	18.0	0.65	−0.442	−0.208
1.00	18.5	0.52	−1.582	−1.577

solver (using a shooting method [40]) results and found that an excellent agreement without the source term is indicative of the reliability of the model even after we could include the source term, since it is expected that the inclusion of the source term would improve the results even more. We assumed that the cloud tops are illuminated with one unit of diffuse radiation (1 W m^{-2}) and the cloud base has no illumination. In this case the upward flux calculated at the top stands for the reflectivity and the downward flux at the base stands for the transmissivity of the cloud as a whole. Absorptivity is ratio of the missing light flux at all the boundaries of the medium to the total incoming flux, due to the conversion of the light energy into the other forms in the medium via inelastic matter–light interactions and collisions between the particles. Also the delta-Eddington method itself, though the best among all two-stream approximations, is still known to yield errors of few percent, especially for thin clouds [18]. In comparing the numerical and analytical results we kept this fact in mind, and we do not expect very accurate results from the either method but a reasonable match between the two. Hence most of the results below are reported only up to a two significant figure level. We also compared the results with those of the first order perturbation solution direct on I_0, and found that our approximate analytical solutions are as good as this I_0 perturbation.

In order to construct hypothetical clouds for our examples, our selection criteria for the vertical change of the cloud properties, was based on the maximum change in $\gamma(u)$ from cloud top to cloud base. We have found in water clouds that the maximum change in $\gamma(u)$ is obtained at 2 μm wavelength, and when our cloud had the optical properties of a Stratus-II at the base and those of a Cumulonimbus at the top. The fact that Stratus-II has 0.05 (g/m^3), and the Cumulonimbus has 2.50 (g/m^3) liquid water content is responsible for this maximum vertical change. We also considered cirrus clouds at long and visible wavelengths; we found that the maximum transitions occur at 5 μm (2200 cm^{-1}) wavelength between the cirrus uncinus and the cold cirrus. But we considered one of the two extremes of the given visible six band center wavelengths in Takano et al. [38,39], namely the one at $\lambda = 0.55$ μm, since it exhibits a turning point case. Table 1 summarizes the cloud type dependent parameters for the hypothetical example cases.

3.1. The water cloud transition example at 2 μm

$\gamma^2(u)$ is fitted versus u as $\gamma^2(u) = 0.934 + 0.472\sqrt{18.735 - u} - 0.0274(18.735 - u) - 0.890$. The profiles of the relevant parameters vs. the vertical z coordinate (measured from top to bottom in terms of km) are given in Table 2. The Schwartzian derivative term is $\Delta(\gamma)$. Tables 3 and 4 below summarize the Liouville–Green

Table 3
The upward and downward diffuse flux results from Liouville–Green approximation vs. a numerical procedure (num) and the perturbation (per) at 2 μm for the water cloud top region

u	F^+_{num}	F^+	F^+_{per}	F^-_{num}	F^-	F^-_{per}
0.00	0.09	0.09	0.06	1.00	1.00	1.00
0.50	0.05	0.05	0.02	0.54	0.53	0.52
1.00	0.03	0.03	0.00	0.29	0.29	0.26
1.50	0.01	0.01	0.00	0.16	0.15	0.13
2.00	0.01	0.01	0.00	0.08	0.08	0.06
2.50	0.00	0.00	0.00	0.05	0.04	0.03
3.00	0.00	0.00	0.00	0.02	0.02	0.01

Table 4
The upward and downward diffuse flux results from Liouville–Green approximation vs. a numerical procedure (num) and the perturbation (per) at 2 μm for the water cloud base region

u	F^+_{num}	F^+	F^+_{per}	F^-_{num}	F^-	F^-_{per}
15.00	0.25	0.25	0.25	1.00	1.00	1.00
15.50	0.17	0.17	0.16	0.64	0.63	0.63
16.00	0.12	0.11	0.11	0.41	0.41	0.41
16.50	0.08	0.07	0.07	0.27	0.27	0.26
17.00	0.06	0.05	0.05	0.18	0.18	0.17
17.50	0.04	0.03	0.03	0.12	0.12	0.11
18.00	0.02	0.01	0.01	0.08	0.08	0.07
18.50	0.00	0.00	0.00	0.06	0.04	0.05

approximation results and compares the fluxes with the numerically computed ones and with the results of the perturbation approach on the original I_0 equation. Since the water cloud is essentially opaque for a thickness of 1 km, we have divided it into five sections of optical thickness of $\Delta u = 3$, plus one $\Delta u = 3.5$ from $u = 15$ to $u = 18.5$. When computing the fluxes for the whole 1 km of the thickness, it is not needed that we go deeper than the first 400 m from each side, since beyond this the downward and the upward fluxes are uniformly zero. Table 4 gives the fluxes for a unit diffuse illumination as in Table 3 but for the lowest region of the cloud. We notice that the Schwartzian term $\Delta(\gamma)$ and the $|\gamma'|/\gamma^2$ term have very high values at the cloud base, however in the Liouville–Green solution, no adverse effects are observed, since this bad behavior is limited to an ignorable part of the region, as had been discussed above. Overall, for this water cloud example, the Liouville–Green approach performs better, and of course more directly and with less mathematical effort, than the I_0 perturbation method.

3.2. The cirrus cloud transition example at 0.55 μm

$\gamma^2(u)$ is fitted as a polynomial in u, $\gamma^2(u) = 0.137 - 0.164u - 1.418u^2 + 6.402u^3 - 13.14u^4 + 12.49u^5 - 4.511u^6$. The profiles of the relevant parameters vs. the vertical coordinate z are given in Table 5. Table 6 below summarizes the results for the numerical approach versus the γ nearly zero approach (Eq. (37) for F_0^+ and F_0^-) approach, a turning point case approach (Eq. (34) for F^+ and F^-), and perturbation on I_0 method. The analytical results in the table below are obtained through a simple $k(u) = 1/2/[1/2k(0) + u]$ term (Eq. (34)) with $k(0)$ set equal to 0.1. When we use the fuller expression for $k(u)$ as given in Eq. (35), we get an almost identical result to this. In this example the simplest $k(u)$ turns out to be adequate; however for thicker clouds, a more elaborate $k(u)$ would be needed. The Schwartzian term is very large at the cloud base, consequently the Liouville–Green approach is not applicable: if used it would lead to high errors (observed in our computations but reported here). Eqs. (37) and (34) methods work as good as the perturbation on I_0 approach; however in this example the preferred approach emerges as Eq. (34), as it is the mathematically the least complicated and the most direct one.

Table 5
The profiles of the relevant parameters for the cirrus cloud to cloud transition case at 0.55 μm

z (km)	u	γ	γ'/γ^2	$\Delta(\gamma)$
0.00	0.00	0.137	−3.81	−0.11
0.10	0.041	0.134	−3.94	−0.20
0.20	0.079	0.131	−4.13	−0.33
0.30	0.115	0.128	−4.40	−0.52
0.40	0.147	0.26	−4.78	−0.83
0.50	0.176	0.124	−5.32	−1.37
0.60	0.201	0.122	−6.12	−2.45
0.70	0.222	0.120	−7.39	−4.97
0.80	0.239	0.118	−9.64	−12.52
0.90	0.250	0.116	−14.6	−48.81
1.00	0.257	0.114	−33.87	−677.5

Table 6
The upward and downward diffuse flux results from a numerical (num) procedure vs. the turning point case solution, the nearly zero approach (0), and the perturbation approach (per) at 0.55 μm for the cirrus case

u	F^+_{num}	F^+	F^+_0	F^+_{per}	F^-_{num}	F^-	F^-_0	F^-_{per}
0.00	0.16	0.16	0.16	0.16	1.00	1.00	1.00	1.00
0.026	0.14	0.14	0.15	0.14	0.98	0.98	0.98	0.98
0.051	0.13	0.13	0.13	0.13	0.97	0.97	0.97	0.97
0.077	0.11	0.11	0.11	0.11	0.95	0.95	0.95	0.95
0.103	0.10	0.10	0.10	0.10	0.93	0.93	0.94	0.93
0.129	0.08	0.08	0.08	0.08	0.92	0.92	0.92	0.92
0.154	0.06	0.06	0.06	0.06	0.90	0.90	0.90	0.90
0.180	0.05	0.05	0.05	0.05	0.89	0.89	0.89	0.89
0.206	0.03	0.03	0.03	0.03	0.87	0.87	0.87	0.87
0.231	0.02	0.02	0.02	0.02	0.85	0.85	0.85	0.85
0.257	0.00	0.00	0.00	0.00	0.84	0.84	0.84	0.84

4. Examples from medical applications

We can also apply the delta-Eddington radiative transfer model to various non-invasive measurements in vitro as well as in vivo biological tissues [41]. As in the cloud examples above, we assumed that the tissue tops are illuminated with one unit of diffuse radiation (1 W m^{-2}) and the base has no illumination

4.1. The human dermis blood transition example at 0.633 μm

Here we present an example on human dermis–blood interaction. At 633 nm, the total extinction coefficient of human dermis is 189.7, the absorption coefficient is 2.7 and scattering coefficient is 187, where the asymmetry parameter is 0.81 in units of cm^{-1} [14], while for the circulating human blood at the same wavelength, they are 775.1, 2.1, 773, and 0.994, respectively [42]. We assumed that the dermis has the thickness of 0.5 mm and the blood level beneath has the same depth, and both fuse together linearly, i.e., the optical properties linearly change form the pure dermis at the top to pure blood at the bottom. The depth versus other optical parameters are given in Table 7 below.

The fit for $\gamma(u)$ is best obtained as $\gamma(u)^2 = 0.190 - 0.107/(u - 4.314) - 0.0306/(u - 4.314)^2 - 0.005602/(u - 4.314)^3$. The numerical and perturbation on I_0 solutions are given so as to compare with the least square fit approximation on $k(u)$ solutions (from Eq. (38), those with no subscripts) (Table 8).

It is evident from Table 7 that, the Liouville–Green approximation is not suitable for this case, as the Schwartzian derivative grows high towards the pure blood layer. But the lest square approach that seeks $k(u)$ values around the $\gamma(u)$ values and results in a k profile as given in Eq. (38), namely $k(u) = 1/[1.510 - 0.04503u]^2$, is performing as well as the perturbation on I_0 method. Its mathematical price comes

Table 7
The profiles of the relevant parameters for the human dermis to blood transition case at 0.633 μm

z (mm)	u	γ	γ'/γ^2	$\Delta(\gamma)$
0.00	0.00	0.46	0.039	0.000
0.10	0.42	0.46	0.040	0.001
0.20	0.90	0.47	0.044	0.003
0.30	1.41	0.47	0.053	0.005
0.40	1.94	0.48	0.067	0.009
0.50	2.47	0.49	0.093	0.017
0.60	2.96	0.50	0.141	0.037
0.70	3.39	0.53	0.242	0.103
0.80	3.75	0.56	0.505	0.416
0.90	4.01	0.64	1.515	3.619
1.00	4.15	0.97	13.947	438.7

Table 8
The upward and downward diffuse flux results from a numerical (num) procedure vs. the least square fit solution, and the perturbation on I_0 approach (per) at 0.663 μm for the human dermis to blood transition example

u	F^+_{num}	F^+	F^+_{per}	F^-_{num}	F^-	F^-_{per}
0.00	0.51	0.52	0.51	1.00	1.00	1.00
0.41	0.42	0.42	0.41	0.82	0.82	0.82
0.83	0.34	0.34	0.33	0.67	0.68	0.67
1.24	0.27	0.27	0.26	0.55	0.55	0.55
1.66	0.21	0.21	0.21	0.45	0.45	0.44
2.07	0.16	0.16	0.16	0.36	0.36	0.36
2.49	0.12	0.12	0.12	0.29	0.29	0.29
2.90	0.08	0.08	0.08	0.23	0.22	0.23
3.32	0.05	0.05	0.05	0.18	0.17	0.17
3.73	0.02	0.03	0.02	0.13	0.13	0.13
4.15	0.00	0.00	0.00	0.09	0.09	0.09

from numerically evaluating the minimization integrals in computing k_a and k_b, while the perturbation on I_0 method needs to solve the ODE with the inhomogeneous right-hand side, which may still involve some numerical computations if Fourier expansion is utilized, or iterations if some serial expansion is attempted. Both methods cover non-Liouville–Green domains as well as the Liouville–Green domains.

4.2. The human aorta example at 0.633 μm

Another medical example is from the aorta–light interaction. Aorta has three layers named from inside outwardly as intima, media, and adventitia; and they are characterized by different optical parameters. Cheong et al. [14] reports that for the intima, media, and adventitia layers, the total extinction coefficients are 175, 312, and 201, respectively in terms of cm^{-1}; single scatter albedos are 0.977, 0.994, and 0.970, while the asymmetry parameters are 0.85, 0.90, and 0.81, respectively. We assumed that these parameters change according to a quadratic function throughout the assumed 1 mm wall thickness of the aorta. The fit $\gamma(u) = 0.657 - 0.330u + 0.152u^2 - 0.0167u^3$ is adequate for this aorta–light interaction problem (Table 9).

For k found by the γ nearly constant method of near the turning point via Eq. (37), the upward and downward fluxes found with this k is given in Table 10. It is seen that both Eq. (37) approach and the perturbation on I_0 give comparable results. Though not reported here, the least square approach of Eq. (38), also works fairly well. Considering that $\gamma(u)$ has a profile of an upward looking bowl, it is interesting that both Eqs. (37) and (38) can perform well. In the next example, where $\gamma(u)$ has a shape of an upside down bowl, and totally symmetric with respect to its center of geometry, we resort to the direct perturbation on k method (Eq. (40)).

Table 9
The profiles of the relevant parameters for the human aorta at 0.633 μm

z (mm)	u	γ	γ'/γ^2	$\Delta(\gamma)$
0.00	0.00	0.64	−0.38	−0.01
0.10	0.31	0.59	−0.50	−0.12
0.20	0.63	0.53	−0.66	−0.10
0.30	0.95	0.47	−0.73	0.08
0.40	1.27	0.43	−0.42	0.39
0.50	1.59	0.43	0.23	0.39
0.60	1.93	0.46	0.59	0.09
0.70	2.30	0.51	0.54	−0.07
0.80	2.70	0.56	0.39	−0.08
0.90	3.12	0.61	0.27	−0.06
1.00	3.56	0.65	0.20	−0.03

Table 10
The upward and downward diffuse flux results from the near the turning point approach via Eq. (37) vs. a numerical (num) procedure, and the perturbation approach (per) at 0.663 μm for the human aorta example

u	F^+_{num}	F^+	F^+_{per}	F^-_{num}	F^-	F^-_{per}
0.00	0.47	0.47	0.46	1.00	1.00	1.00
0.36	0.40	0.40	0.40	0.82	0.82	0.81
0.71	0.34	0.34	0.33	0.68	0.68	0.68
1.07	0.28	0.28	0.27	0.57	0.57	0.57
1.42	0.22	0.22	0.22	0.48	0.48	0.48
1.78	0.17	0.17	0.17	0.40	0.40	0.40
2.14	0.13	0.13	0.13	0.33	0.33	0.33
2.49	0.09	0.09	0.09	0.27	0.27	0.26
2.85	0.06	0.06	0.05	0.21	0.21	0.21
3.20	0.03	0.03	0.03	0.16	0.16	0.16
3.56	0.00	0.00	0.00	0.12	0.12	0.12

5. Quantum mechanical tunneling example

This last example is what is known in semi-classical quantum mechanics as the electron tunneling through a potential barrier, where the time independent probability amplitude, Ψ, is a real valued function. Its applications are in scanning tunneling microscopy and in manufacturing of nano structures [43]. We adopt the data from Beiser [44, p. 181]: electrons with energies of 1.0 eV are incident on a potential barrier of 10.0 eV through a gap width of 0.50 nm. However we set out to find the distribution of electron flux through the gap for a given value of transmission, say zero. Using the formula $\sqrt{2m\cdot(V-E)}/\hbar$, with the electron mass, $m = 9.11\times 10^{-31}$ kg, and the barrier height $V - E = (10.0 - 1.0)\cdot 1.60\times 10^{-19}$ J/eV, $\hbar = 6.63\times 10^{-34}/2\pi$, we find that the maximum value of γ is 7.5 in the time independent Schroedinger equation in one dimension $\mathrm{d}^2\Psi/\mathrm{d}x^2 - [2m\cdot(V-E)/\hbar^2]\cdot\Psi = 0$. Defining a new variable u, as $\mathrm{d}u = 7.5\mathrm{d}x$, we convert the Schroedinger equation into a simpler diffusion equation with respect to u, $\Psi'' - \gamma^2\cdot\Psi = 0$, where Ψ is the wave function or probability amplitude to whose square the electron flux is proportional. Additionally we assume a parabolic profile for the γ^2, namely, $\gamma^2(u) = u\cdot(7.5-u)/(7.5/2)^2$, such that at the midpoint $u = 7.5/2$, its height is unity and the symmetry axis passes through this point. Consequently Eq. (40) applies, and the first order perturbation on k method is used. We present the probability amplitude profile within the gap in Table 11 below for a zero transmission output.

6. Summary and discussion

Through out the paper we aimed to investigate analytical approaches specifically to the solution of the monochromatic delta-Eddington problem through a plane parallel medium that is optically inhomogeneous

Table 11
The real probability amplitude distribution within the potential barrier for a zero transmission as given above found by the linearization perturbation on k, versus the numerical (num) approach and the direct perturbation on the Schroedinger equation for Ψ (per) results

u	Ψ_{num}	Ψ	Ψ_{per}
0.00	1.00	1.00	1.00
0.75	0.61	0.61	0.60
1.50	0.33	0.32	0.32
2.25	0.16	0.16	0.14
3.00	0.08	0.08	0.06
3.75	0.04	0.04	0.01
4.50	0.02	0.02	0.00
5.25	0.01	0.01	0.00
6.00	0.00	0.00	0.00
6.75	0.00	0.00	0.00
7.50	0.00	0.00	0.00

in the vertical, and in general to the one-dimensional diffusion problem. One of the resulting ordinary differential equations was a second order one with a variable coefficient and the other one was the derivative of that. We have found that if we defined the optical path in terms of u, as defined above, the ordinary differential equation for Z came out in its normal form (Eq. (17)). This form when factorized, led to Eqs. (19), (26) and (27). Instead if we had tried to define the new optical path by $\mathrm{d}v = 3b\,\mathrm{d}\tau$, we would have had a new equation $Y'' - (a/3b)Y = 0$, in terms of v instead of Eq. (18) which then is in terms of u. But in this case as ϖ_0 would go to unity, as it is so at the visible wavelengths, the second term would go to infinity. This means either Y'' must go to infinity or Y must go to zero, or both; none of which is useful as a solution.

In obtaining the two coupled equations from the basic radiative transfer equation, Eq. (7), we have tried a separation of variables like method, and obtained a solution as the multiple of a part that depends on the optical path only, $Z(\tau)$; and the rest depending on the averaged cosine of the zenith angle ($\mp 1/\sqrt{3}$), which represents the angular dependency as a function of the optical path and the angle. Since we are dealing with only two streams, $M(\mu,\tau)$ can only be expanded into two terms as in the original delta-Eddington method, except with an optical path dependent second term, $a(\tau)$ (Eq. (9)). There on, we again diverged from the original approach and obtain two variables, $Z(\tau)$ and $m(\tau)$, by the application of a two point Gauss–Legendre quadrature in a parallel development to non-Eddington two-stream approximations. Upon converting the two coupled first order ordinary differential equations into two decoupled ordinary differential equations we sought for their solutions. But we mainly chose to work on the homogenous (sourceless) part, since the particular solution could be constructed through some known methods such as an integrating factor method, variation of parameters method, or integration of the Wronskian method (Eq. (41)) [23].

The third term in the right-hand side of Eq. (7), the source term in the visible wavelengths, is analytical when, ϖ_0, g, β_{e} are given analytically. In some cases the source term can be given as an approximate analytical expression, for example in the long wavelengths $B(u) = B(0) + u[B(u^*) - B(0)]/[u^* - 0]$ for a cloud layer, where u^* is the maximum value of u [45]. When the source integrals are complicated, we can approximate them analytically through a suitable Gauss–Legendre quadrature. The Gauss–Legendre approximations can be used in computing $k(u)$ if needed.

Eqs. (18) and (28) are the two equivalent homogenous equations, even though they are obtained through different methods. Eq. (18) was found through redefining the dependent variable and factoring out the new equations while Eq. (28) is found directly from the original pair of the first order coupled ordinary differential equations. However Eq. (28) is a Riccati type and its any two solutions, say m_1 and m_2, will lead to the solutions of these two first order equations; one of which is $Z(u) = \widehat{C}_1 \exp[\int_0^u m_1(v)\,\mathrm{d}v] + \widehat{C}_2 \exp[-\int_0^u m_2(v)/\mathrm{d}v]$. At first glance it is not obvious that this equation and Eq. (21) that was obtained through factorization are identical, albeit they have the different constants of integration. However as explained in obtaining Eqs. (30) and (33), this identity emerged out of the fact that, any two solutions of $m^2 = \gamma^2 - m'$ can be recombined to obtain $2k = m_1 - m_2$ and $2j = m_1 + m_2 = -k'/k$ (or equivalently in terms of δk and δj). These new functions convert Eq. (30) into Eq. (42), whose special form is a Liouville–Green solution, Eq. (21). Since earlier we established before Eq. (40), that $2k = 2\delta k + \gamma$, when we chose a very small δk, i.e., $\delta k \approx 0$ to start with in the calculation of

k, we obtained Eq. (23). This establishes Liouville–Green also as a special case of the Picard iterative approach, though in the construction of its solution there is no need for any iteration. At some points in the derivations, we had diverged from the original approaches, but at the end we got the same results while demonstrating that alternative routes were also feasible.

In attacking the cloud radiation problems, the Liouville–Green solution should be the first choice and it should be exploited as far as it can go. Beyond its range of applicability, such as at turning points, we need a working method and we developed four approaches in Eqs. (33)–(40). This method involves the calculation of new functions $k(u)$ or $\delta k(u)$ from the most obvious any two solutions of Eq. (28). When these solutions are not exactly known, we start with the best estimates, and continue through Picard iterative approach. Luckily, in the problems of cloud radiation, it seems as if the first one or two iterations would suffice. The Picard iterative method has been successfully used before in atmospheric radiative transfer problems [29,30,46], albeit in different approaches than ours.

We hypothesized some example clouds whose optical properties would severely change from top to base, and yet we have found that for the water cloud case the Liouville–Green approximation would work satisfactorily. In other cases, where the cloud properties would change even more severely than we presented here, then our four methods involving $k(u)$ or $\delta k(u)$ should be employed. Especially in the turning point problems, the Liouville–Green approximation definitely will not be applicable and our turning point method involving $k(u)$ will successfully work. Though not reported here, we have tested our turning point approximation for negative and imaginary γ^2s and found that it works satisfactorily on both sides of the turning point (negative and positive u) as well. As we have demonstrated in the 0.55 μm cirrus transition example, even the zeroth iteration gave satisfactory results. In computations of $k(u)$, care should be taken not to let $k(u)$ to go or pass through zero ($k(u) = 0$), if not, Eq. (42) would diverge. Since the Riccati equation has theoretically infinite number of solutions, some of which would be suitable for our problem, we can always choose the ones that do not pass through $k(u) = 0$. In doing this we may need to experiment with the arbitrary initial choices of $k(u)$, or $\delta k(u)$ if we are dealing with a γ nearly constant problem. How much nearly constant γ should be depends on how many iterations we will have to do until a satisfactory solution is obtained.

In some problems we may improve the solution by using the Liouville–Green method more than once (stopping when the Schwartzian derivative is large), balancing the extra work needed and the accuracy required. In this study, we have not been able to present a one piece compact formula to cover all ranges. Thus beyond the range of applicability of the Liouville–Green solution, we may need to patch an $k(u)$ or $\delta k(u)$ solution to it. This will be done by matching the fluxes and their first derivatives at the patch points (or surfaces for the clouds).

In the cloud transition examples above, we have compared the analytical results with those of a numerical scheme. If some specific $\gamma^2(u)$ happens to be given in a polynomial form, the solution to the Riccati equation may have a polynomial form if it satisfies some certain conditions [28]. This may not be the case in general, and we would have to solve it step by step, through a series expansion, perturbation, or iterative integration. In choosing Picard iterative integration instead of a series method, we get an advantage of convergence for optical paths that are not too large (if large we need to subdivide the range). If they are too large then we can divide the domain into manageable parts. As we have showed that the abrupt changes in γ (high γ') within a small portion of the domain does not affect the solution Z much; but it would, if we chose to work with a series expansion of γ' and Z. Actually Eq. (18) implies an iteration of the type: $Z_{i+1}(u) = Z_i(0) + Z_i'(0)u + \int_0^u dv \int_0^v \gamma^2(w) Z_i(w)\,dw$, from which we clearly see that the solution depends on the double integral of a term containing γ^2, and not on γ'. Similarly for a γ nearly constant problem, $m_{1,0}(u) \simeq \gamma(0) + \int[\gamma^2(v) - \gamma^2(0)]\,dv$ expression shows that solution m depends on an integral of the γ^2 term. In either case, the solution does not feel the high values of γ', unless it occurs for a considerable part of the domain and eventually steeply raises the value of γ. The Picard iterative method was noted to be insensitive to vertical inhomogeneities [29]. The prime advantage of the Picard iterative approach over the series solution of the Riccati equation is that we do not have to expand $\gamma^2(u)$ into Taylor series where at some point u, it may diverge (becomes singular and goes out of the Liouville–Green range). The iterative integration on the other hand overlooks isolated singularities.

In calculation of $I_1(u)$, some errors are inevitable since we evaluate it as a derivative of $I_0(u)$, which is approximate itself. If we had only one second order ordinary differential equation to solve, ours and the Liouville–Green approximations would perform much better. In general, all the methods presented in this paper are applicable to any second order ordinary differential equation. As a side benefit, these approximate analytical methods can be used reformulating the numerical approaches and help them to converge faster and more accurately. For example one might specifically work around these approximations and compute the numerical solution as a deviation term from those analytical approximations.

For very thick layers the whole solution is dominated by that term in the fluxes that contains the negative exponential. This fact can be used establishing the near zero values for the upward and downward fluxes beyond a certain large optical path even before the computation. In effect the problem can be divided into two regions: one reasonably thick layer as measured from the entrance side of the light and the rest as an optically very thick part. In the optically very thick part the fluxes are practically zero and it may arithmetically (without coupling, as if it is part of the full solution) added to the solution from the other region in constructing the total cloud layer solution. This way one avoids dealing with optical paths that are too large for the approximate solution. In fact, for the water cloud example at 2 μm wavelength, we have observed the applicability of this approach. We have increased the optical path beyond 400 m, and found no change on the fluxes of the top part no matter how far the optical path increased, while the fluxes in the lower part remained zero.

Though we have reported rounded off figures in the examples given above, more precise figures were used in the calculations as it had to be. However considering that the delta-Eddington method is known to produce unphysical values sometimes, especially negative transmissivities for optically thin layers; and since we were demonstrating hypothetical cases of cloud transitions, our rounded off figures do fit the limits of accuracy under these conditions. Additionally, we assumed linearly or quadratically changing profiles for all optical parameters involved. This too should not detract from the generality of the methods: there is nothing in the methods that depends inherently on any type of functional form. But for a very complex profile it would be recommendable to subdivide the domain, rather than iterating or perturbing for a convergence in $k(u)$ or $\delta k(u)$ at once.

A better and more practical method of representing atmospheric cloud radiation processes (absorption-scattering-emission) through more realistic media (inhomogeneous optical paths), would help advance our understanding regarding how clouds affect the climate change scenarios. Recently more interest is being directed towards the interiors of the clouds: ABC (atmospheric brown clouds) and UAV (unmanned aerial vehicle) [47] observations and cloud–aerosol interactions are being investigated [48,49]. Both modelers and data analyzers of cloud interiors will need fast and accurate enough schemes. In addition, these radiative transfer methods can also be utilized in non-invasive medical engineering research. In this paper, we offered some complimentary, analytical, and practical approaches to the inhomogeneous optical depth radiative transfer problem. We tested the methods developed to the limits of current knowledge on the clouds and found them satisfactory. In future as more cloud data becomes available, we aim to extend our modeling capability to include more complicated cases, involving other cloud properties, such as shape, size distribution and extent of the cloud groups statistically. We feel that some part of the answer to problems such as anomalous short wave cloud absorption might lie in the interior inhomogeneity [8,50]. The better modeling and data interpretation then would help us to understand [51,52] and better parameterize the cloud radiative effects in the climate change prediction models [53–55].

It is our hope that, in the near future, as the non-invasive techniques that utilize light absorption-scattering-transmission on in vitro as well as in vivo tissues become more widespread, and as the tissue optical characterizations advance more, our analytical approximations would be of some guidance in developing better numerical and analytical methods.

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