



Application of the G'/G -expansion method to Kawahara type equations using symbolic computation

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ABSTRACT

In this paper, Kawahara type equations are selected to illustrate the effectiveness and simplicity of the G'/G -expansion method. With the aid of a symbolic computation system, three types of more general traveling wave solutions (including hyperbolic functions, trigonometric functions and rational functions) with free parameters are constructed. Solutions concerning solitary and periodic waves are also given by setting the two arbitrary parameters, involved in the traveling waves, as special values.

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1. Introduction

Searching exact and explicit solutions of nonlinear partial differential equations is of vital importance in applied mathematical sciences and it becomes one of the most exciting and extremely active areas of the research. Because, it is well-known that all nonlinear partial equations can be separated essentially on two parts: the integrable partial differential equations and non-integrable ones. The first type, i.e. the integrable equations has infinite number of the exact solutions. The most well-known equations among them are Korteweg–de Vries equation, Sine–Gordon equation, Kawahara type equations, nonlinear Schrödinger equation, Boussinesq equations and the list can be expanded with other basic integrable equations but it is not our purpose to give all list. Nonlinear partial differential equations with some exact solutions or without exact solutions are assumed to be in the class of non-integrable partial differential equations and they may need special treatment to obtain their solutions due to the form of the nonlinear differential equation and the pole of its solution. Burger–Huxley equation, Fisher equation, Fitzhugh–Nagumo equation, Ginzburg–Landau equation can be mentioned as well-known non-integrable partial differential equations among them all.

In the last few decades great progress was made in the development of methods for obtaining exact solutions of nonlinear equations but the progress achieved is not adequate. Because, from our point of view, there is no single best method to obtain exact solutions of nonlinear differential equations of both type and each method have its merits and deficiencies depending on the researchers experience and the sympathy to the method utilized. Moreover, it can be said that all these methods are problem dependant, namely some methods work well with certain problems but others not. Therefore, it is rather significant to apply some well-known methods in the literature to nonlinear partial differential equations which are not solved with that method to search possibly new exact solutions or to verify the existing solutions with different approach.

Recently, there have been many effective and convenient methods for solving nonlinear equations in the literature. The essential part of these methods are based on the proposal that the special functions that one takes to expand the exact solution are the general solution of simpler ordinary differential equation with less order than the original

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differential equation with eminent solution (differential equations with constant coefficients, Riccati equation, the equations in support of elliptic function, etc.). The variant of these methods are auxiliary equation methods [1–4], generalized Riccati expansion method [5], F -expansion method [6], mapping method [7], elliptic function method [8], exp-function method [9–12] and so on.

Very recently, Wang and Zhang [13] pioneered a new direct method, the so-called G'/G -expansion method, to search for traveling wave solutions of nonlinear evolution equations (NLEEs). In their remarkable study [13], Wang and Zhang successfully obtained more traveling wave solutions of four NLEEs. Later, to improve the work made in [13], some important studies on the generalizations and the extensions of the G'/G -expansion method have been presented by the authors [14–31] in the open literature.

The G'/G -expansion method is based on the explicit linearization of nonlinear differential equations for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. The computations are performed with a computer algebra system (CAS) such as *Mathematica* to deduce the solutions of the nonlinear equations in an explicit form. The solution process of the method is direct, effective and convenient due to solving the auxiliary equation of second-order differential equation with constant coefficients.

The present paper is motivated by the desire to use the G'/G -expansion method to seek more general form exact and explicit solutions of Kawahara type equations which may be important to explain some physical phenomena.

2. The G'/G -expansion method

We assume that the given nonlinear partial differential equation for $u(x, t)$ to be in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \quad (1)$$

where P is a polynomial in its arguments, which includes nonlinear terms and the highest order derivatives. Then using the transformation $u(x, t) = u(\xi)$, $\xi = kx + wt$ we reduce Eq. (1) to the ordinary differential equation

$$Q(u, u_\xi, u_{\xi\xi}, \dots) = 0, \quad (2)$$

and we look for its solution $u(\xi)$ in the polynomial form

$$u(\xi) = \sum_{i=0}^N a_i \left(\frac{G'}{G} \right)^i, \quad (3)$$

where $G = G(\xi)$ and a_i are constants to be determined, N is a positive integer which is determined by the homogeneous balancing method and $G(\xi)$ is the solution of the auxiliary linear second order ordinary differential equation

$$G'' + \lambda G' + \mu G = 0, \quad (4)$$

where $G' = \frac{dG}{d\xi}$, $G'' = \frac{d^2G}{d\xi^2}$, λ and μ are constants to be determined later. Substituting (3) into Eq. (2) with the aid of a CAS, we determine λ , μ and a_i . Depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, the solutions of Eq. (4) can be readily found. As a result, exact and explicit traveling wave solutions to the given nonlinear partial differential Eq. (1) can be derived immediately, see [13] for more details.

3. The Kawahara equation

Let us consider the so-called Kawahara equation

$$u_t + \alpha uu_x + \beta u_{xxx} - \gamma u_{xxxx} = 0, \quad (5)$$

where α , β , and γ are nonzero arbitrary constants. Eq. (5), proposed first by Kawahara [32] in 1972, occurs in the theory of shallow water waves and plays an important role in the modeling of many physical phenomena such as plasma waves, magneto-acoustic waves, see [33–35] and the references therein. The existence and uniqueness of solutions are obtained by Shuangping and Shuangbin [36]. Now, substituting $u(x, t) = u(\xi)$, $\xi = kx + wt$ in Eq. (5) and integrating the resulting ordinary differential equation once, one obtains

$$wu + \frac{\alpha k}{2} u^2 + \beta k^3 u'' - \gamma k^5 u^{(4)} + d = 0, \quad (6)$$

where prime denotes the derivative with respect to ξ and d is an integration constant. Now, we make an ansatz (3) together with (4) for the solution of Eq. (6) and thus balancing the highest derivative term $u^{(4)}$ with the nonlinear term u^2 in Eq. (6) yields the leading order $N = 4$. Therefore, we can write the solution of Eq. (6) in the form

$$u = a_0 + a_1 \left(\frac{G'}{G} \right) + a_2 \left(\frac{G'}{G} \right)^2 + a_3 \left(\frac{G'}{G} \right)^3 + a_4 \left(\frac{G'}{G} \right)^4. \quad (7)$$

By (4) and (7) we derive the formulas for u^2 , u'' , and $u^{(4)}$ as follows:

$$u^2 = a_0^2 + 2a_0a_1\left(\frac{G'}{G}\right) + \dots + 2a_3a_4\left(\frac{G'}{G}\right)^7 + a_4^2\left(\frac{G'}{G}\right)^8, \quad (8)$$

$$u'' = \lambda\mu a_1 + 2\mu^2 a_2 + (\lambda^2 a_1 + 2\mu a_1 + 6\lambda\mu a_2 + 6\mu^2 a_3)\left(\frac{G'}{G}\right) + \dots + (12a_3 + 36\lambda a_4)\left(\frac{G'}{G}\right)^5 + 20a_4\left(\frac{G'}{G}\right)^6, \quad (9)$$

$$u^{(4)} = \lambda^3\mu a_1 + 8\lambda\mu^2 a_1 + 14\lambda^2\mu^2 a_2 + 16\mu^3 a_2 + 36\lambda\mu^3 a_3 + 24\mu^4 a_4 \\ + (\lambda^4 a_1 + 22\lambda^2\mu a_1 + 16\mu^2 a_1 + 30\lambda^3\mu a_2 + 120\lambda\mu^2 a_2 + 150\lambda^2\mu^2 a_3 + 120\mu^3 a_3 + 240\lambda\mu^3 a_4)\left(\frac{G'}{G}\right) \\ + \dots + (360a_3 + 2640\lambda a_4)\left(\frac{G'}{G}\right)^7 + 840a_4\left(\frac{G'}{G}\right)^8, \quad (10)$$

Substituting (7)–(10) into (6), and setting the coefficients of $(G'/G)^i$, $(i = 0, 1, \dots, 8)$ to zero, we obtain the system of non-linear algebraic equations for $a_0, a_1, a_2, a_3, a_4, \lambda$, and μ :

$$0: \quad d + wa_0 + \frac{1}{2}k\alpha a_0^2 + k^3\beta\lambda\mu a_1 - k^5\gamma\lambda^3\mu a_1 - 8k^5\gamma\lambda\mu^2 a_1 + 2k^3\beta\mu^2 a_2 - 14k^5\gamma\lambda^2\mu^2 a_2 - 16k^5\gamma\mu^3 a_2 \\ - 36k^5\gamma\lambda\mu^3 a_3 - 24k^5\gamma\mu^4 a_4 = 0, \\ 1: \quad wa_1 + k^3\beta\lambda^2 a_1 - k^5\gamma\lambda^4 a_1 + 2k^3\beta\mu a_1 - 22k^5\gamma\lambda^2\mu a_1 - 16k^5\gamma\mu^2 a_1 + k\alpha a_0 a_1 + 6k^3\beta\lambda\mu a_2 - 30k^5\gamma\lambda^3\mu a_2 \\ - 120k^5\gamma\lambda\mu^2 a_2 + 6k^3\beta\mu^2 a_3 - 150k^5\gamma\lambda^2\mu^2 a_3 - 120k^5\gamma\mu^3 a_3 - 240k^5\gamma\lambda\mu^3 a_4 = 0, \\ 2: \quad 3k^3\beta\lambda a_1 - 15k^5\gamma\lambda^3 a_1 - 60k^5\gamma\lambda\mu a_1 + \frac{1}{2}k\alpha a_1^2 + wa_2 + 4k^3\beta\lambda^2 a_2 - 16k^5\gamma\lambda^4 a_2 + 8k^3\beta\mu a_2 - 232k^5\gamma\lambda^2\mu a_2 + 136k^5\gamma\mu^2 a_2 \\ + k\alpha a_0 a_2 + 15k^3\beta\lambda\mu a_3 - 195k^5\gamma\lambda^3\mu a_3 - 660k^5\gamma\lambda\mu^2 a_3 + 12k^3\beta\mu^2 a_4 - 660k^5\gamma\lambda^2\mu^2 a_4 - 480k^5\gamma\mu^3 a_4 = 0, \\ 3: \quad 2k^3\beta a_1 - 50k^5\gamma\lambda^2 a_1 - 40k^5\gamma\mu a_1 + 10k^3\beta\lambda a_2 - 130k^5\gamma\lambda^3 a_2 - 440k^5\gamma\lambda\mu a_2 + k\alpha a_1 a_2 + wa_3 + 9k^3\beta\lambda^2 a_3 \\ - 81k^5\gamma\lambda^4 a_3 + 18k^3\beta\mu a_3 - 1062k^5\gamma\lambda^2\mu a_3 - 576k^5\gamma\mu^2 a_3 + k\alpha a_0 a_3 + 28k^3\beta\lambda\mu a_4 - 700k^5\gamma\lambda^3\mu a_4 - 2240k^5\gamma\lambda\mu^2 a_4 \\ 4: \quad -60k^5\gamma\lambda a_1 + 6k^3\beta a_2 - 330k^5\gamma\lambda^2 a_2 - 240k^5\gamma\mu a_2 + \frac{1}{2}k\alpha a_2^2 + 21k^3\beta\lambda a_3 - 525k^5\gamma\lambda^3 a_3 - 1680k^5\gamma\lambda\mu a_3 + k\alpha a_1 a_3 \\ + wa_4 + 16k^3\beta\lambda^2 a_4 - 256k^5\gamma\lambda^4 a_4 + 32k^3\beta\mu a_4 - 3232k^5\gamma\lambda^2\mu a_4 - 1696k^5\gamma\mu^2 a_4 + k\alpha a_0 a_4 = 0, \\ 5: \quad -24k^5\gamma a_1 - 336k^5\gamma\lambda a_2 + 12k^3\beta a_3 - 1164k^5\gamma\lambda^2 a_3 - 816k^5\gamma\mu a_3 + k\alpha a_2 a_3 + 36k^3\beta\lambda a_4 - 1476k^5\gamma\lambda^3 a_4 \\ - 4608k^5\gamma\lambda\mu a_4 + k\alpha a_1 a_4 = 0, \\ 6: \quad -120k^5\gamma a_2 - 1080k^5\gamma\lambda a_3 + \frac{1}{2}k\alpha a_3^2 + 20k^3\beta a_4 - 3020k^5\gamma\lambda^2 a_4 - 2080k^5\gamma\mu a_4 + k\alpha a_2 a_4 = 0, \\ 7: \quad -360k^5\gamma a_3 - 2640k^5\gamma\lambda a_4 + k\alpha a_3 a_4 = 0, \\ 8: \quad -840k^5\gamma a_4 + \frac{1}{2}k\alpha a_4^2 = 0.$$

Solving the above system, we get the solution set

$$\left\{ \begin{array}{l} d = \frac{w^2}{2k\alpha} - \frac{648k\beta^4}{28561\alpha\gamma^2}, \quad \mu = -\frac{\beta}{52k^2\gamma} + \frac{\lambda^2}{4}, \quad a_0 = \frac{69k\beta^2 - 169w\gamma - 2730k^3\beta\gamma\lambda^2 + 17745k^5\gamma^2\lambda^4}{169k\alpha\gamma}, \\ a_1 = \frac{840(-k^2\beta\lambda + 13k^4\gamma\lambda^3)}{13\alpha}, \quad a_2 = \frac{840(-k^2\beta + 39k^4\gamma\lambda^2)}{13\alpha}, \quad a_3 = \frac{3360k^4\gamma\lambda}{\alpha}, \quad a_4 = \frac{1680k^4\gamma}{\alpha}. \end{array} \right. \quad (11)$$

Now, substituting the solution set (11) into (7) and taking the solutions of Eq. (4) into account; we obtain a more general form hyperbolic function traveling wave solution to Eq. (5) as

$$u_1(x, t) = \frac{105\beta^2}{169\alpha\gamma} \left(\frac{C_1 \cosh\left(\frac{1}{2}\sqrt{\frac{\beta}{13k^2\gamma}}(kx + wt)\right) + C_2 \sinh\left(\frac{1}{2}\sqrt{\frac{\beta}{13k^2\gamma}}(kx + wt)\right)}{C_1 \sinh\left(\frac{1}{2}\sqrt{\frac{\beta}{13k^2\gamma}}(kx + wt)\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{\frac{\beta}{13k^2\gamma}}(kx + wt)\right)} \right)^4 \\ - \frac{210\beta^2}{169\alpha\gamma} \left(\frac{C_1 \cosh\left(\frac{1}{2}\sqrt{\frac{\beta}{13k^2\gamma}}(kx + wt)\right) + C_2 \sinh\left(\frac{1}{2}\sqrt{\frac{\beta}{13k^2\gamma}}(kx + wt)\right)}{C_1 \sinh\left(\frac{1}{2}\sqrt{\frac{\beta}{13k^2\gamma}}(kx + wt)\right) + C_2 \cosh\left(\frac{1}{2}\sqrt{\frac{\beta}{13k^2\gamma}}(kx + wt)\right)} \right)^2 - \frac{w}{k\alpha} + \frac{69\beta^2}{169\alpha\gamma}, \quad (12)$$

where $\gamma\beta > 0$, C_1 and C_2 are arbitrary constants; a more general form trigonometric function traveling wave solution to Eq. (5) as

$$u_2(x, t) = \frac{105\beta^2}{169\alpha\gamma} \left(\frac{-C_1 \sin\left(\frac{1}{2}\sqrt{\frac{-\beta}{13k^2\gamma}}(kx + wt)\right) + C_2 \cos\left(\frac{1}{2}\sqrt{\frac{-\beta}{13k^2\gamma}}(kx + wt)\right)}{C_1 \cos\left(\frac{1}{2}\sqrt{\frac{-\beta}{13k^2\gamma}}(kx + wt)\right) + C_2 \sin\left(\frac{1}{2}\sqrt{\frac{-\beta}{13k^2\gamma}}(kx + wt)\right)} \right)^4 + \frac{210\beta^2}{169\alpha\gamma} \left(\frac{-C_1 \sin\left(\frac{1}{2}\sqrt{\frac{-\beta}{13k^2\gamma}}(kx + wt)\right) + C_2 \cos\left(\frac{1}{2}\sqrt{\frac{-\beta}{13k^2\gamma}}(kx + wt)\right)}{C_1 \cos\left(\frac{1}{2}\sqrt{\frac{-\beta}{13k^2\gamma}}(kx + wt)\right) + C_2 \sin\left(\frac{1}{2}\sqrt{\frac{-\beta}{13k^2\gamma}}(kx + wt)\right)} \right)^2 - \frac{w}{k\alpha} + \frac{69\beta^2}{169\alpha\gamma}, \quad (13)$$

where $\gamma\beta < 0$, C_1 and C_2 are arbitrary constants.

In particular, if we take $C_2 \neq 0$, $C_1^2 < C_2^2$, in (12) and (13), then we derive a formal solitary wave solution to Eq. (5) as

$$u_3(x, t) = \frac{105\beta^2}{169\alpha\gamma} \tanh^4\left(\frac{1}{2}\sqrt{\frac{\beta}{13k^2\gamma}}(kx + wt) + \xi_0\right) - \frac{210\beta^2}{169\alpha\gamma} \tanh^2\left(\frac{1}{2}\sqrt{\frac{\beta}{13k^2\gamma}}(kx + wt) + \xi_0\right) - \frac{w}{k\alpha} + \frac{69\beta^2}{169\alpha\gamma}, \quad (14)$$

where $\gamma\beta > 0$ and $\xi_0 = \tanh^{-1}(C_1/C_2)$; a periodic wave solution to Eq. (5) as

$$u_4(x, t) = \frac{105\beta^2}{169\alpha\gamma} \cot^4\left(\frac{1}{2}\sqrt{\frac{-\beta}{13k^2\gamma}}(kx + wt) + \xi_0\right) + \frac{210\beta^2}{169\alpha\gamma} \cot^2\left(\frac{1}{2}\sqrt{\frac{-\beta}{13k^2\gamma}}(kx + wt) + \xi_0\right) - \frac{w}{k\alpha} + \frac{69\beta^2}{169\alpha\gamma}, \quad (15)$$

where $\gamma\beta < 0$ and $\xi_0 = \tan^{-1}(C_1/C_2)$.

Now, we compare our results with others: (1) Yusufoglu and Bekir's results [37]: first we take $\alpha = \beta = \gamma = 1$ in (5). Then if we take $\xi_0 = 0$, $k = \alpha$, $w = -\alpha\beta$ in the new form of (14), our result will be the same as the first expression of (4.6) in [37]. Moreover, by taking $C_1 \neq 0$, $C_1^2 > C_2^2$, and $\xi_0 = \tanh^{-1}(C_2/C_1)$ in our more general result (12) and following the same approach leads to the second expression of (4.6) in [37]. (2) Wazwaz's results [38]: first we take $\gamma \rightarrow -\gamma$ in (5). Then if we take $\xi_0 = 0$, $k = 1$, $w = -\frac{36\beta^2}{169}$ in the new form of (14), our result will be the same as (33) in [38]. Moreover, by a simple manipulation, we can get (34) in [38] from our more general result (12). We observed that (35) and (36) in [38] can be derived from our results (13) and (15). Similarly, it can be assured that (37)–(40) in [38] can be obtained from our more general solutions (12) and (13).

4. The modified Kawahara equation

Next, we consider the modified Kawahara equation

$$u_t + \alpha u^2 u_x + \beta u_{xxx} - \gamma u_{xxxx} = 0, \quad (16)$$

where α , β , and γ are nonzero arbitrary constants. This equation is also called the singularly perturbed KdV equation. Letting $u(x, t) = u(\xi)$, $\xi = kx + wt$ in (16) and integrating the resulting ordinary differential equation once, we get

$$wu + \frac{\alpha k}{3} u^3 + \beta k^3 u'' - \gamma k^5 u^{(4)} + d = 0, \quad (17)$$

where prime denotes the derivative with respect to ξ and d is an integration constant. Balancing the terms $u^{(4)}$ and u^3 in the Eq. (17) yields the leading order $N = 2$. Therefore, we can seek the solution of (17) in the form

$$u = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \quad (18)$$

for which we can easily calculate the derivatives u'' and $u^{(4)}$ from (9) and (10) by setting $a_3 = 0$ and $a_4 = 0$. Substituting (18) into (17) under the consideration of Eq. (4), and setting the coefficients of $(G'/G)^i$, ($i = 0, 1, \dots, 6$) to zero, we obtain the system of nonlinear algebraic equations for a_0 , a_1 , a_2 , λ , and μ :

$$0: \quad d + wa_0 + \frac{1}{3}k\alpha a_0^3 + k^3\beta\lambda\mu a_1 - k^5\gamma\lambda^3\mu a_1 - 8k^5\gamma\lambda\mu^2 a_1 + 2k^3\beta\mu^2 a_2 - 14k^5\gamma\lambda^2\mu^2 a_2 - 16k^5\gamma\mu^3 a_2 = 0,$$

$$1: \quad wa_1 + k^3\beta\lambda^2 a_1 - k^5\gamma\lambda^4 a_1 + 2k^3\beta\mu a_1 - 22k^5\gamma\lambda^2\mu a_1 - 16k^5\gamma\mu^2 a_1 + k\alpha a_0^2 a_1 + 6k^3\beta\lambda\mu a_2 - 30k^5\gamma\lambda^3\mu a_2 - 120k^5\gamma\lambda\mu^2 a_2 = 0,$$

$$2: \quad 3k^3\beta\lambda a_1 - 15k^5\gamma\lambda^3 a_1 - 60k^5\gamma\lambda\mu a_1 + k\alpha a_0 a_1^2 + w a_2 + 4k^3\beta\lambda^2 a_2 - 16k^5\gamma\lambda^4 a_2 + 8k^3\beta\mu a_2 - 232k^5\gamma\lambda^2\mu a_2 - 136k^5\gamma\mu^2 a_2 + k\alpha a_0^2 a_2 = 0,$$

$$3: \quad 2k^3\beta a_1 - 50k^5\gamma\lambda^2 a_1 - 40k^5\gamma\mu a_1 + \frac{1}{3}k\alpha a_1^3 + 10k^3\beta\lambda a_2 - 130k^5\gamma\lambda^3 a_2 - 440k^5\gamma\lambda\mu a_2 + 2k\alpha a_0 a_1 a_2 = 0,$$

$$4: \quad -60k^5\gamma\lambda a_1 + 6k^3\beta a_2 - 330k^5\gamma\lambda^2 a_2 - 240k^5\gamma\mu a_2 + k\alpha a_1^2 a_2 + k\alpha a_0 a_2^2 = 0,$$

$$5: \quad -24k^5\gamma a_1 - 336k^5\gamma\lambda a_2 + k\alpha a_1 a_2^2 = 0,$$

$$6: \quad -120k^5\gamma a_2 + \frac{1}{3}k\alpha a_2^3 = 0.$$

Solving the above system, we get the solution sets:

$$\left\{ \begin{aligned} d &= \frac{k(\mp\beta^3 \pm 15k^4\beta\gamma^2(\lambda^2 - 4\mu)^2 \pm 50k^6\gamma^3(\lambda^2 - 4\mu)^3)}{15\sqrt{10}\sqrt{\alpha}\gamma^{3/2}}, w = -\frac{k(\beta^2 + 15k^4\gamma^2(\lambda^2 - 4\mu)^2)}{10\gamma}, \\ a_0 &= \frac{\mp\beta + 5k^2\gamma(\lambda^2 + 8\mu)}{\sqrt{10}\sqrt{\alpha}\sqrt{\gamma}}, a_1 = \pm \frac{6\sqrt{10}k^2\sqrt{\gamma}\lambda}{\sqrt{\alpha}}, a_2 = \pm \frac{6\sqrt{10}k^2\sqrt{\gamma}}{\sqrt{\alpha}}. \end{aligned} \right. \tag{19}$$

Now, substituting the solution sets (19) into (18) and taking the solutions of Eq. (4) into account; we obtain more general form hyperbolic function traveling wave solutions to Eq. (16) as

$$u_{1,2}(x, t) = \mp \frac{\sqrt{10}\gamma k^2(\lambda^2 - 4\mu)}{\sqrt{\alpha}} \left(1 - \frac{3}{2} \left(\frac{C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)}{C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)} \right)^2 \right) \mp \frac{\beta}{\sqrt{10\alpha\gamma}}, \tag{20}$$

where $\lambda^2 - 4\mu > 0$, C_1 and C_2 are arbitrary constants, and $\xi = kx - \frac{k(\beta^2 + 15k^4\gamma^2(\lambda^2 - 4\mu)^2)}{10\gamma}t$; more general form trigonometric function traveling wave solutions to Eq. (16) as

$$u_{3,4}(x, t) = \mp \frac{\sqrt{10}\gamma k^2(\lambda^2 - 4\mu)}{\sqrt{\alpha}} \left(1 + \frac{3}{2} \left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)} \right)^2 \right) \mp \frac{\beta}{\sqrt{10\alpha\gamma}}, \tag{21}$$

where $\lambda^2 - 4\mu < 0$, C_1 and C_2 are arbitrary constants, and $\xi = kx - \frac{k(\beta^2 + 15k^4\gamma^2(\lambda^2 - 4\mu)^2)}{10\gamma}t$; more general form rational function traveling wave solutions to Eq. (16) as

$$u_{5,6}(x, t) = \mp \frac{\beta}{\sqrt{10\alpha\lambda}} \pm \frac{6\sqrt{10}\gamma k^2 C_1^2}{\sqrt{\alpha} \left(C_1 k \left(x - \frac{\beta^2}{10\gamma}t \right) + C_2 \right)^2}, \tag{22}$$

where C_1 and C_2 are arbitrary constants. Meanwhile, (22) yields new rational function solutions for Eq. (16) which have not been reported anywhere else to the best of our knowledge.

In particular, if we take $C_2 \neq 0, C_1^2 < C_2^2$, then (20) leads formal solitary wave solutions to Eq. (16) as

$$u_{7,8}(x, t) = \mp \frac{\sqrt{10}\gamma k^2(\lambda^2 - 4\mu)}{\sqrt{\alpha}} \left(1 - \frac{3}{2} \tanh^2 \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + \xi_0 \right) \right) \mp \frac{\beta}{\sqrt{10\alpha\gamma}}, \tag{23}$$

where $\lambda^2 - 4\mu > 0$, $\xi = kx - \frac{k(\beta^2 + 15k^4\gamma^2(\lambda^2 - 4\mu)^2)}{10\gamma}t$, $\xi_0 = \tanh^{-1}(C_1/C_2)$, and (15) gives periodic wave solutions to Eq. (16) as

$$u_{9,10}(x, t) = \mp \frac{\sqrt{10}\gamma k^2(\lambda^2 - 4\mu)}{\sqrt{\alpha}} \left(1 + \frac{3}{2} \cot^2 \left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi + \xi_0 \right) \right) \mp \frac{\beta}{\sqrt{10\alpha\gamma}}, \tag{24}$$

where $\lambda^2 - 4\mu < 0$, $\xi = kx - \frac{k(\beta^2 + 15k^4\gamma^2(\lambda^2 - 4\mu)^2)}{10\gamma}t$, and $\xi_0 = \tan^{-1}(C_1/C_2)$.

As in the previous case, if we compare our results with Yusufoglu and Bekir's results [37] and Wazwaz's results [38], then we see that the results (5.5) and (5.8) in [37] are special cases of our results (20), (21), (23), (24) and so are the results (29)-(32) in [39]. Besides, the rational function solutions (22) are new and not obtained by the methods in [37,38].

5. Conclusion

In the present work, the G'/G -expansion method has been successfully applied to the Kawahara type equations. Solutions in terms of solitary and periodic waves are found in more general forms. For certain choices of the parameters, it is observed

that the previously known solutions can be recovered. New rational function solutions are also presented. The solutions have different physical structures and depend on the real parameters. All solutions obtained in this study have been checked with Mathematica by putting them back into the original equations. We predict that, being more effective and simple, wider classes of exact and explicit solutions to other nonlinear evolution equations arising in applied mathematics can be easily derived by using the G'/G -expansion method.

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