OPERATIONS ON PROPER CLASSES RELATED TO SUPPLEMENTS

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ABSTRACT

OPERATIONS ON PROPER CLASSES RELATED TO SUPPLEMENTS

The purpose of this study is to understand the properties of the operations $+$, $\underline{\circ}$, and $*$ defined on classes of short exact sequences and apply them to the proper classes related to supplements. The operation $\overline{\circ}$ on classes of short exact sequences is introduced and it is proved that the class of extended weak supplements is the result of the operation $\overline{\circ}$ applied to two classes one of which is the class of splitting short exact sequences. Using the direct sum of proper classes defined by R. Alizade, G. Bilhan and A. Pancar, a direct sum decomposition for quasi-splitting short exact sequences over the ring of integers is obtained. Closures of classes of short exact sequences along with the one studied by C. P. Walker, N. Hart and R. Alizade are defined over an integral domain. It is shown that these classes are proper when the underlying class is proper and they are related to the operation $+$. The closures of proper classes related to supplements are described in terms of Ivanov classes. Closures for modules over an integral domain are also defined and it is proved that submodules of torsion-free modules have unique closures. A closure for classes of short exact sequences is defined over an associative ring with identity and it is proved that this closure is proper when the underlying class is proper. Results shows that the operation $+$ and closures of splitting short exact sequences plays an important role on the closures of proper classes.

ÖZET

TÜMLEYENLERLE İLGİLİ ÖZ SINIFLAR ÜZERİNDE İŞLEMLER

Bu çalışmanın amacı kısa tam dizi sınıfları üzerinde tanımlanmış $+$, $\underline{\circ}$, ve $*$ işlemlerinin özelliklerini anlamak ve bu işlemleri tümleyenlerle ilgili öz sınıflara uygulamaktır. Kısa tam dizi sınıfları üzerinde $\overline{\circ}$ işlemi tanımlanmış ve genişletilmiş zayıf tümleyenler sınıfının, $\overline{\circ}$ işleminin, biri parçalanan kısa tam dizilerin sınıfı olmak üzere, iki sınıfa uygulanmasının sonucu olduğu kanıtlanmıştır. R. Alizade, G. Bilhan ve A. Pancar tarafından tanımlanan, öz sınıfların dik toplamı kullanılarak, tam sayılar halkası üzerinde yarı-parçalanan kısa tam diziler sınıfı için bir dik toplam ayrışması elde edilmiştir. C. P. Walker, N. Hart ve R. Alizade tarafından çalışılan dahil olmak üzere kısa tam dizi sınıflarının kapanışları tamlık bölgeleri üzerinde tanımlanmıştır. Altında yatan sınıf öz sınıf olduğunda, bu kapanışların öz sınıf olduğu ve $+$ işlemiyle ilişkili olduğu gösterilmiştir. Tümleyenlerle ilgili öz sınıfların kapanışları İvanov sınıfları cinsinden belirtilmiştir. Tamlık bölgeleri üzerinde modüllerin kapanışları da tanımlanmış ve burulmasız modüllerin altmodüllerinin kapanışlarının tek olduğu kanıtlanmıştır. Birleşmeli ve birimli halkalar üzerinde kısa tam dizi sınıfları için bir kapanış tanımlanmış ve altında yatan sınıf öz sınıf olduğunda bu kapanışın öz sınıf olduğu kanıtlanmıştır. Elde edilen sonuçlar, $+$ işleminin ve parçalanan kısa tam diziler sınıfının kapanışlarının, öz sınıfların kapanışları üzerinde önemli rol oynadıklarını göstermiştir.

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LIST OF SYMBOLS AND ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
$\mathbb{Z}$, $\mathbb{Z}^+$	the ring of integers, the set of all positive integers
$\mathbb{Q}$	the field of rational numbers
G^1	the first Ulm subgroup of abelian group G: $G^1 = \bigcap_{n=1}^{\infty} nG$
R-module	*left* R-module
R-$\mathcal{M}od$	the category of *left* R-modules
$\mathcal{A}b = \mathbb{Z}$-$\mathcal{M}od$	the category of abelian groups ($\mathbb{Z}$-modules)
$\mathrm{Hom}_R(M, N)$	all R-module homomorphisms from M to N
$\mathrm{Ker}\, f$	the kernel of the map f
$\mathrm{Im}\, f$	the image of the map f
$T(M)$	the torsion submodule of the R-module M for an integral domain R: $T(M) = \{m \in M \mid rm = 0 \text{ for some } 0 \neq r \in R\}$
$M[k]$	the submodule $\{m \in M \mid km = 0 \text{ for some } 0 \neq k \in R\}$ of the R-module M for an integral domain R
$\mathrm{Soc}\, M$	the socle of the R-module M
$\mathrm{Rad}\, M$	the radical of the R-module M
$\mathcal{B}$	the class of bounded R-modules
$\mathcal{B}_r$	the class of modules bounded by a power of r for $0 \neq r \in R$
$\mathcal{S}m$	the class of small modules
$\langle \mathcal{E} \rangle$	the smallest proper class containing the class $\mathcal{E}$ of short exact sequences
$\mathcal{P}$	a proper class of R-modules
$\hat{\mathcal{P}}$	the set $\{\mathbb{E} \mid r\mathbb{E} \in \mathcal{P} \text{ for some } 0 \neq r \in R\}$ for a proper class $\mathcal{P}$
$\hat{\mathcal{P}}_r$	the set $\{\mathbb{E} \mid r^t\mathbb{E} \in \mathcal{P} \text{ for some nonnegative integer } t\}$ for a proper class $\mathcal{P}$ and for $0 \neq r \in R$
$\pi(\mathcal{P})$	all $\mathcal{P}$-projective modules
$\pi^{-1}(\mathcal{M})$	the proper class of R-modules projectively generated by a class $\mathcal{M}$ of R-modules
$\iota(\mathcal{P})$	all $\mathcal{P}$-injective modules

$\iota^{-1}(\mathcal{M})$	the proper class of R-modules injectively generated by a class $\mathcal{M}$ of R-modules
$\overline{k}(\mathcal{M})$	the proper class coprojectively generated by a class $\mathcal{M}$ of R-modules
$\underline{k}(\mathcal{M})$	the proper class coinjectively generated by a class $\mathcal{M}$ of R-modules
$\mathrm{Ext}_R(C,A) = \mathrm{Ext}^1_R(C,A)$	the set of all equivalence classes of short exact sequences starting with the R-module A and ending with the R-module C
$\mathrm{Text}(C,A)$	the set $\{\mathbb{E} \in \mathrm{Ext}(C,A) \mid r\mathbb{E} \equiv 0$ for some $0 \neq r \in R\}$ of equivalence classes of short exact sequences of abelian groups
$\mathrm{Pext}(C,A)$	the set of all equivalence classes of pure-exact sequences starting with the group A and ending with the group C
$\mathrm{Next}(C,A)$	the set of all equivalence classes of neat-exact sequences starting with the group A and ending with the group C
$\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$	the proper class of pure-exact sequences of abelian groups
	For the category R-$\mathcal{M}$od, the following classes are defined:
$\mathcal{S}plit$	the smallest proper class consisting only of *splitting* short exact sequences in the category R-$\mathcal{M}od$
$\mathcal{A}bs$	the largest proper class consisting of *all* short exact sequences in the category R-$\mathcal{M}od$
$\mathcal{C}ompl$	the proper class of complements in the category R-$\mathcal{M}od$
$\mathcal{S}uppl$	the proper class of supplements in the category R-$\mathcal{M}od$
$\mathcal{N}eat$	the proper class of neats in the category R-$\mathcal{M}od$
$\mathcal{C}o\text{-}\mathcal{N}eat$	the proper class of coneats in the category R-$\mathcal{M}od$
$\mathcal{S}$	the class of κ-exact sequences in the category R-$\mathcal{M}od$
$\mathcal{SB}$	the class of β-exact sequences in the category R-$\mathcal{M}od$
$\mathcal{WS}$	the class of weak supplements in the category R-$\mathcal{M}od$
$\overline{\mathcal{WS}}$	the class of extend weak supplements in the category R-$\mathcal{M}od$
$\mathcal{D}$	the class of torsion-splitting short exact sequences in the category R-$\mathcal{M}od$
$i(\mathcal{M},\mathcal{J})$	the least proper class for which every module from the class $\mathcal{M}$ of modules is coprojective and every module from the class $\mathcal{J}$ of modules is coinjective

$\cong$	isomorphic
$\leq$	submodule
$\ll$	small (=superfluous) submodule
$\trianglelefteq$	essential submodule

CHAPTER 1

INTRODUCTION

Throughout this work, R is an associative ring identity unless otherwise stated and all modules are unital left R-modules. Restrictions on the ring R, if there are any, are given at the beginning of the chapter or the section. We use Ext instead of Ext^1_R when there is no ambiguity. Definitions not given here can be found in the books (Wisbauer 1991, Anderson and Fuller 1992, Hungerford 1974, Mac Lane 1963, Fuchs 1970).

In this thesis, we study the operations $+$, $\circ$ and $*$ on proper classes defined by A. Pancar in (Pancar 1997) and apply them to the proper classes related to supplements.

In Chapter 2, we introduce some basic information about the alternative definition of the functor Ext and supplements in module theory. $\mathrm{Ext}_R(C, A) = \mathrm{Ext}^1_R(C, A)$ can be viewed as the class of short exact sequences starting with A and ending with C (see (Fuchs 1970, §50)). With the aid of the sum called Baer sum, it turns out that $\mathrm{Ext}_R(C, A)$ is a group for all R-modules A and C, and it is called the group of extensions of A by C. Results on properties of the group of extensions can be found in the books (Fuchs 1970) and (Mac Lane 1963). Supplement submodules and some generalizations were investigated by R. Alizade, E. Büyükaşık, A. I. Generalov, D. Keskin,C. Lomp, P. Smith, R. Tribak, H. Zöschinger and many others. The main results about supplements in module theory can be found in the books (Wisbauer 1991, Ch. 8, §41) and (Clark et al. 2006, Ch. 4).

In Chapter 3, we give the definition of a proper class (see (Buchsbaum 1959) and (Sklyarenko 1978)). The result (Nunke 1963, Theorem 1.1) that is used many times in this thesis is also mentioned, and it is claimed that an e-functor $\mathcal{F}$ gives a proper class if the composition of two epimorphisms (monomorphisms) from $\mathcal{F}$ also belongs to $\mathcal{F}$. The class $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$ of pure-exact sequences and the class $\mathcal{D}_{\mathbb{Z}\text{-}\mathcal{M}od}$ of torsion-splitting short exact sequences are important examples of proper classes in the category of abelian groups. The proper class generated by a class of short exact sequences is defined by A. Pancar as the least proper class that contains the given class or equivalently the intersection of proper classes that contains the given class (see (Pancar 1997)). If $\mathcal{M}$ is a given class of $R\text{-}\mathcal{M}od$ for an additive functor $T(M, \cdot) : R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$, the class of exact triples $\mathbb{E}$ such that $T(M, \mathbb{E})$ is exact forms a proper class. This result is helpful in the definition of projectively and injectively generated proper classes. At the end of this part of the thesis, we give two results from (Alizade 1985) which give the characterization of coprojectively and coinjectively generated classes that are dual to the characterization of projectively and injectively generated proper classes.

In Chapter 4, proper classes related to complements and supplements are studied. The question if the composition of a subfunctor of the identity with the functor Ext gives a proper class is answered with a counterexample as the composition of the functors Soc and Ext does not give a proper class in the case of abelian groups. Following Zöschinger, a short exact sequence $\mathbb{E}: 0 \longrightarrow A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$ of $\text{Ext}_R(C, A)$ is called κ-exact if $\text{Im} f$ has a supplement in B, i.e. a minimal element in the set $\{V \subseteq B | V + \text{Im} f = B\}$. The corresponding class of short exact sequences is denoted by $\mathcal{S}$. The elements of the class $\mathcal{SB} \subseteq \mathcal{S}$ are defined with an extra condition that $V \cap \text{Im} f$ is bounded. Similarly, the elements of the classes $\mathcal{S}mall$ and $\mathcal{WS}$ are defined with $\text{Im} f$ being small in B and having a weak supplement in B, that is a submodule U of B with $\text{Im} f + U = B$ and $\text{Im} f \cap U \ll B$, respectively. We showed that over a Noetherian integral domain of Krull dimension 1, the class $\mathcal{SB}$ is proper and it coincides with the class coinjectively generated by bounded modules (see (Demirci 2008, Proposition 4.3)). Over a hereditary ring, the proper class generated by the classes $\mathcal{S}$, $\mathcal{S}mall$ and $\mathcal{WS}$ coincides with the proper class $\overline{\mathcal{WS}}$ which consists of all images of $\mathcal{WS}$-elements of $\text{Ext}(C', A)$ under $\text{Ext}(g, 1_A) : \text{Ext}(C', A) \longrightarrow \text{Ext}(C, A)$ for all homomorphisms $g : C \longrightarrow C'$ (see (Alizade et al. 2012, Corollary 3.13)). It is shown in the same work that over a hereditary ring the class $\overline{\mathcal{WS}}$ is coinjectively generated by the class of small modules (see (Alizade et al. 2012, Proposition 4.13)). A homomorphism $g : C' \longrightarrow C$ is called *coneat* if for every decomposition $g = \beta \circ \alpha$, where β is a small epimorphism, β is an isomorphism. In the case of abelian groups, coneat homomorphisms give a necessary condition for the elements of the class $\text{Ext}_{\mathcal{S}}(C, A)$ to be a subgroup of $\text{Ext}(C, A)$.

In Chapter 5, the operations on proper classes are studied. The result of operation $+$ is always a proper class while it need not be in the case of $\underline{\circ}$, $\overline{\circ}$ and $*$ in general. The operations $+$ and $*$ are commutative and the result of operation $*$ is always a subgroup of $\text{Ext}(C, A)$. Definition of the direct sum of two proper classes are given using the operation $+$. An infinite direct sum decomposition is obtained for the class of quasi-splitting short exact sequences over the ring of integers. In the case of abelian groups, the sum of the proper classes projectively generated and coprojectively generated by a class of groups, which is closed under direct sums, is the class $\mathcal{A}bs$ all short exact sequences, provided that the projectively generated class is projective. It is also proved that this sum is direct when the generating class of abelian groups consists of reduced abelian groups (see

(Alizade et al. 1997, Theorem 4)).

Chapter 6 is devoted to the study of closures of classes of short exact sequences. In the case of abelian groups, the class $\hat{\mathcal{P}} = \{E|\, kE \in \mathcal{P}$ for some $0 \neq k \in R\}$ was studied by C. P. Walker for $\mathcal{P} = \mathcal{S}plit$, by N. Hart for $\mathcal{P} = \mathcal{P}ure$ and $\mathcal{P} = \mathcal{D}$, and by R. Alizade for $\mathcal{P} = \mathcal{S}plit$, where $\hat{\mathcal{S}plit}$ was denoted by *Text* since $\mathrm{Ext}_{\hat{\mathcal{S}plit}}(C, A) = T(\mathrm{Ext}(C, A))$, the torsion part of $\mathrm{Ext}(C, A)$, and for every proper class $\mathcal{P}$ (see (Walker 1964), (Alizade 1986) and (Hart 1974)). In the case of abelian groups, the operations $+$, $\underline{\circ}$ and $*$ give the same proper class when applied to the classes $\hat{\mathcal{S}plit}$ and $\mathcal{D}$ (see (Pancar 1997, Theorem 4.2)). A similar definition is used for modules over an integral domain R, and it is proved that for R-modules $M \leq N$, $\hat{M}$ is the unique closure of $M + T(N)$ in N, where $T(N)$ is the torsion part of N. For a class $\mathcal{P}$ of short exact sequences over an integral domain R and for all $0 \neq r \in R$, the class $\hat{\mathcal{P}}_r = \{E|\, r^t E \in \mathcal{P}$ for some nonnegative integer $t\}$, which is a closure of $\mathcal{P}$, is proved to be proper when the underlying class is proper, and its relation with the operation $+$ is shown. The class $i(\mathcal{M}, \mathcal{J})$ for classes $\mathcal{M}$ and $\mathcal{J}$ of R-modules is introduced for an integral domain R as the least proper class of short exact sequences for which every module from $\mathcal{M}$ is coprojective and every module from $\mathcal{J}$ is coinjective (see (Ivanov 1978), (Alizade 1986)). The relation between the class $i(\mathcal{M}, \mathcal{J})$ and the closures of the classes $\mathcal{SB}$ and $\overline{\mathcal{WS}}$ is proved. The class $r^t\mathcal{P}$ for the class $\mathcal{P}$, which is included in the class $\mathcal{P}$ when $\mathcal{P}$ is proper, is introduced over a principal ideal domain R for $0 \neq r \in R$ and for every nonnegative integer t. It is proved that $r^t\mathcal{P}$ is proper when the class $\mathcal{P}$ is proper and $\hat{\mathcal{P}}_r = \mathcal{P}$. In the second section of this chapter, a closure $\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$ for a class $\mathcal{P}$ is introduced for a compatible pair of classes $\mathcal{F}$ and $\mathcal{G}$ of homomorphisms over an associative ring with an identity element (see § 6.2), and it is proved to be proper when the underlying class $\mathcal{P}$ is proper. A relation between the class $\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$ and the operation $+$ is also given at the end of this section.

CHAPTER 2

PRELIMINARIES

This Chapter will consist of preliminary information about the group of extensions by short exact sequences and supplements in module theory. One can find further information and missing proofs in the books (Fuchs 1970), (Vermani 2003) and (Mac Lane 1963) about the group of extensions, in the books (Anderson and Fuller 1992), (Kasch 1982) and (Wisbauer 1991) about supplements and module theory.

2.1. Extensions as Short Exact Sequences

Given the R-modules A and C, the extension B of A by C can be visualized as a short exact sequence

$$0 \longrightarrow A \xrightarrow{\mu} B \xrightarrow{\nu} C \longrightarrow 0\,, \tag{2.1}$$

where μ is a monomorphism and ν is an epimorphism with kernel $\mu(A)$. Then there is a category $\mathscr{E}$ in which the objects are the short exact sequences and a morphism between two short exact sequences $\mathbb{E}$ and $\mathbb{E}'$ is defined as a triple (α, β, γ) of module homomorphisms such that the diagram

$$\begin{array}{ccccccccc} \mathbb{E} : 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ \mathbb{E}' : 0 & \longrightarrow & A' & \xrightarrow{\mu'} & B' & \xrightarrow{\nu'} & C' & \longrightarrow & 0 \end{array} \tag{2.2}$$

has commutative squares.

The extensions $\mathbb{E}$ and $\mathbb{E}'$ with $A = A'$, $C = C'$ are said to be *equivalent*, denoted by $\mathbb{E} \equiv \mathbb{E}'$, if there is a morphism $(1_A, \beta, 1_C)$, where $\beta : B \to B'$ is an isomorphism.

If A is a fixed R-module, for a homomorphism $\gamma : C' \to C$, to the extension $\mathbb{E}$ in (2.2), there is a pullback square

$$\begin{array}{ccccccccc} & & & & B' & \xrightarrow{\nu'} & C' & & \\ & & & & {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0 \end{array} \tag{2.3}$$

for some B', β and ν'. By properties of pullback, ν' is epic (since ν is epic), and $\operatorname{Ker}\nu' \cong \operatorname{Ker}\nu \cong A$, hence there is a monomorphism $\mu' : A \to B'$ (i.e. $\mu' a = (\mu a,\, 0) \in B'$ if B' is defined to be a submodule of $B \oplus C'$) such that the diagram

$$\begin{array}{lccccccccc} \mathbb{E}\gamma : & 0 & \longrightarrow & A & \xrightarrow{\mu'} & B' & \xrightarrow{\nu'} & C' & \longrightarrow & 0 \\ & & & \Vert & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ \mathbb{E} : & 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0 \end{array} \tag{2.4}$$

with exact rows and pullback right square commutes. The top row is an extension of A by C' which we denote by $\mathbb{E}\gamma$. Notice that $\gamma^* = (1_A, \beta, \gamma)$ is a morphism $\mathbb{E}\gamma \to \mathbb{E}$ in $\mathscr{E}$.

Next let C be fixed and for a given $\alpha : A \to A'$, let B' be defined by the pushout square

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & & & \\ & & A' & \xrightarrow{\mu'} & B' & & & & \end{array} \tag{2.5}$$

Here μ' is a monomorphism, and if B' is defined as the quotient module $(A' \oplus B)/H$ where H is the submodule of $A \oplus B$ consisting of elements of the form $(\mu(a), -\alpha(a))$ for $a \in A$, then $\nu' : B' \longrightarrow C$ defined by $\nu'((a', b) + H) = \nu(b)$ for $(a', b) \in A' \oplus B$, makes the diagram

$$\begin{array}{lccccccccc} \mathbb{E} : & 0 & \longrightarrow & A & \xrightarrow{\mu} & B & \xrightarrow{\nu} & C & \longrightarrow & 0 \\ & & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \Vert & & \\ \alpha\mathbb{E} : & 0 & \longrightarrow & A' & \xrightarrow{\mu'} & B' & \xrightarrow{\nu'} & C & \longrightarrow & 0 \end{array} \tag{2.6}$$

with exact rows commutative. The bottom row of this diagram is an extension of A' by C which we denote by $\alpha\mathbb{E}$. Here $\alpha_* = (\alpha, \beta, 1_C)$ is a morphism $\mathbb{E} \to \alpha\mathbb{E}$ in $\mathscr{E}$.

With $\alpha : A \to A'$ and $\gamma : C' \to C$, we have the important associative law

$$\alpha(\mathbb{E}\gamma) \equiv (\alpha\mathbb{E})\gamma. \tag{2.7}$$

In order to describe the group operation in the language of short exact sequences, we make use of the diagonal map $\Delta_G : g \mapsto (g, g)$ and the codiagonal map $\nabla_G : (g_1, g_2) \mapsto g_1 + g_2$ of a module G. If we understand by the *direct sum* of two extensions

$$\mathbb{E}_i : \qquad 0 \longrightarrow A_i \xrightarrow{\mu_i} B_i \xrightarrow{\nu_i} C_i \longrightarrow 0 \qquad (i = 1, 2) \tag{2.8}$$

the extension

$$\mathbb{E}_1 \oplus \mathbb{E}_2 : 0 \longrightarrow A_1 \oplus A_2 \xrightarrow{\mu_1 \oplus \mu_2} B_1 \oplus B_2 \xrightarrow{\nu_1 \oplus \nu_2} C_1 \oplus C_2 \longrightarrow 0\,, \tag{2.9}$$

then we have :

Theorem 2.1 ((Mac Lane 1963), Ch. III, Theorem 2.1) *For given R-modules A and C, the set* $\mathrm{Ext}_R(C, A)$ *of all congruence classes of extensions of A by C is an abelian group under the binary operation which assigns to the congruence classes of extensions* $\mathbb{E}_1$ *and* $\mathbb{E}_2$*, the congruence class of the extension*

$$\mathbb{E}_1 + \mathbb{E}_2 = \nabla_A(\mathbb{E}_1 \oplus \mathbb{E}_2)\Delta_C. \tag{2.10}$$

The class of the split extension $0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$ *is the zero element of this group, while the inverse of any* $\mathbb{E}$ *is the extension* $(-1_A)\mathbb{E}$*. For homomorphisms* $\alpha : A \longrightarrow A'$ *and* $\gamma : C' \longrightarrow C$*, one has*

$$\alpha(\mathbb{E}_1 + \mathbb{E}_2) \equiv \alpha\mathbb{E}_1 + \alpha\mathbb{E}_2, \qquad (\mathbb{E}_1 + \mathbb{E}_2)\gamma \equiv \mathbb{E}_1\gamma + \mathbb{E}_2\gamma, \tag{2.11}$$

$$(\alpha_1 + \alpha_2)\mathbb{E} \equiv \alpha_1\mathbb{E} + \alpha_2\mathbb{E}, \qquad \mathbb{E}(\gamma_1 + \gamma_2) \equiv \mathbb{E}\gamma_1 + \mathbb{E}\gamma_2. \tag{2.12}$$

The equivalences in (2.11) and (2.12) express the fact that $\alpha_* : \mathbb{E} \mapsto \alpha\mathbb{E}$ and $\gamma^* : \mathbb{E} \mapsto \mathbb{E}\gamma$ are group homomorphisms

$$\alpha_* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C, A'), \qquad \gamma^* : \mathrm{Ext}_R(C, A) \to \mathrm{Ext}_R(C', A), \tag{2.13}$$

and that $(\alpha_1 + \alpha_2)_* = (\alpha_1)_* + (\alpha_2)_*$ and $(\gamma_1 + \gamma_2)^* = (\gamma_1)^* + (\gamma_2)^*$ for $\alpha_1, \alpha_2 : A \longrightarrow A'$, $\gamma_1, \gamma_2 : C' \longrightarrow C$.

Lemma 2.1 (see (Mac Lane 1963), Ch. III, Lemma 1.6) Ext_R *is an additive bifunctor on R-$\mathcal{M}od \times R$-$\mathcal{M}od$ to $\mathcal{A}b$ which is contravariant in the first and covariant in the second variable.*

For the rest of this work, we will use Ext instead of Ext_R^1 and we will denote the equivalence class of the short exact sequence E by just E. Since we are working on the category of left R-modules, we will not use any subscript to indicate the category we are dealing with unless it is necessary.

2.2. Supplements and Supplemented Modules

This section includes definitions and some results about supplements and supplemented modules. See (Wisbauer 1991, Ch. 8, §41) for more information about supplements and supplemented modules.

Let U be a submodule of an R-module M. If there exists a submodule V of M minimal with respect to the property $M = U + V$ then V is called a *supplement* of U in M.

A submodule K of an R-module M is called *superfluous* or *small* in M, written $K \ll M$, if, for every submodule $L \subseteq M$, the equality $K + L = M$ implies $L = M$. The following lemma is used frequently while studying supplements.

Lemma 2.2 *V is a supplement of U in M if and only if $U + V = M$ and $U \cap V \ll V$.*

The properties of supplements are given in the next proposition.

Proposition 2.1 ((Wisbauer 1991), 41.1) *Let U, $V \subseteq M$ and V be a supplement of U in M.*

1. If $W + V = M$ for some $W \subseteq U$, then V is a supplement of W.

2. If M is finitely generated, then V is also finitely generated.

3. If U is a maximal submodule of M, then V is cyclic and $U \cap V = \operatorname{Rad} V$ is a (the unique) maximal submodule of V.
4. If $K \ll M$, then V is a supplement of $U + K$.
5. If $K \ll M$, then $V \cap K \ll V$ and $\operatorname{Rad} V = V \cap \operatorname{Rad} M$.
6. If $\operatorname{Rad} M \ll M$, then U is contained in a maximal submodule of M.
7. If $L \subseteq U$, then $(V + L)/L$ is a supplement of U/L in M/L.
8. If $\operatorname{Rad} M \ll M$ or $\operatorname{Rad} M \subseteq U$ and $p : M \longrightarrow M/\operatorname{Rad} M$ is the canonical epimorphism, then $M/\operatorname{Rad} M = p(U) \oplus p(V)$.

Let M be a module. If every submodule of M has a supplement in M, then M is called a *supplemented module*. Artinian modules and semisimple modules are examples of supplemented modules. The ring $\mathbb{Z}$ of integers as a module over itself is an example to show that every module need not be supplemented.

For the properties of supplemented modules, we have the following proposition from the book (Wisbauer 1991).

Proposition 2.2 ((Wisbauer 1991), 41.2) *Let M be an R-module.*
1. Let U and V be submodules of M such that U is supplemented and $U + V$ have a supplement in M, then V has a supplement in M.
2. If $M = M_1 + M_2$ with M_1 and M_2 supplemented, then M is also supplemented.
3. If M is supplemented, then $M/\operatorname{Rad} M$ is semisimple.

CHAPTER 3

PROPER CLASSES

3.1. Proper Classes

In this part of the thesis, the definition of proper classes along with some important examples already known will be given.

Let $\mathcal{P}$ be a class of short exact sequences of R-modules and R-module homomorphisms. If a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{3.1}$$

belongs to $\mathcal{P}$, then f is said to be a $\mathcal{P}$-*monomorphism*, and g is said to be a $\mathcal{P}$-*epimorphism* (both are said to be $\mathcal{P}$-*proper*, and the short exact sequence is said to be a $\mathcal{P}$-*proper* short exact sequence.). The class $\mathcal{P}$ is said to be *proper* (in the sense of Buchsbaum) if it satisfies the following conditions (see (Buchsbaum 1959), (Mac Lane 1963, Ch. 12, §4) and (Sklyarenko 1978, Introduction)):

P-1) If a short exact sequence E is in $\mathcal{P}$, then $\mathcal{P}$ contains every short exact sequence isomorphic to E .

P-2) $\mathcal{P}$ contains all splitting short exact sequences.

P-3) The composite of two $\mathcal{P}$-monomorphisms is a $\mathcal{P}$-monomorphism if this composite is defined.

P-3') The composite of two $\mathcal{P}$-epimorphisms is a $\mathcal{P}$-epimorphism if this composite is defined.

P-4) If g and f are monomorphisms, and $g \circ f$ is a $\mathcal{P}$-monomorphism, then f is a $\mathcal{P}$-monomorphism.

P-4') If g and f are epimorphisms, and $g \circ f$ is a $\mathcal{P}$-epimorphism, then g is a $\mathcal{P}$-epimorphism.

One of the most important examples for proper classes in abelian groups is $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$. It is the class of all short exact sequences (3.1) of abelian groups and abelian group homomorphisms such that $\mathrm{Im}(f)$ is a pure subgroup of B, where a subgroup A of a group B is *pure* in B if $A \cap nB = nA$ for all integers n (see (Fuchs 1970, §26-30) for the important notion of purity in abelian groups). The short exact sequences in $\mathcal{P}ure_{\mathbb{Z}\text{-}\mathcal{M}od}$ are called *pure-exact sequences* of abelian groups. The corresponding subgroup of $\mathrm{Ext}(C, A)$ is denoted by $\mathrm{Pext}(C, A)$. The following Theorem gives the structure of $\mathrm{Pext}(C, A)$ in terms of subgroups of $\mathrm{Ext}(C, A)$.

Theorem 3.1 (see (Fuchs 1970), Theorem 53.3) *For all abelian groups A and C,* $\mathrm{Pext}(C, A)$ *coincides with the first Ulm subgroup of* $\mathrm{Ext}(C, A)$*, i.e.*

$$\mathrm{Pext}(C, A) = \mathrm{Ext}(C, A)^1 = \bigcap_{n \in \mathbb{Z}^+} n\,\mathrm{Ext}(C, A). \tag{3.2}$$

The smallest proper class of R-modules consists only of *splitting* short exact sequences of R-modules which we denote by $\mathcal{S}plit$. The largest proper class of R-modules consists of *all* short exact sequences of R-modules which we denote by $\mathcal{A}bs$ (*absolute purity*).

A subfunctor $\mathcal{F}$ of Ext such that $\mathcal{F}(C, A)$ is a subgroup of $\mathrm{Ext}(C, A)$ is called an *e-functor* (see (Butler and Horrocks 1961)). By (Nunke 1963, Theorem 1.1), an *e-functor* $\mathcal{F}$ of Ext gives a proper class if it satisfies one of the properties *P-3)* and *P-3')*. This result enables us to define a proper class in terms of subfunctors of Ext.

For a proper class $\mathcal{P}$ of R-modules, call a submodule A of a module B a $\mathcal{P}$-submodule of B, if the inclusion monomorphism $i_A : A \to B$, $i_A(a) = a$, $a \in A$, is a $\mathcal{P}$-monomorphism.

Let $T(M, \cdot) : R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ be an additive functor (covariant or contravariant), left or right exact and depending on an R-module M. If $\mathcal{M}$ is a given class of modules of this category, we denote by $t^{-1}(\mathcal{M})$ the class $\mathcal{P}$ of short exact sequences E such that $T(M,\ E)$ is exact for all $M \in \mathcal{M}$.

The following Lemma can be found in (Sklyarenko 1978) and for a proof see (Demirci 2008, Lemma 3.1).

Lemma 3.1 $\mathcal{P} = t^{-1}(\mathcal{M})$ *is a proper class.*

Let $t(\mathcal{P})$ be the class of all modules M for which the triples $T(M, E)$ are exact for all $E \in \mathcal{P}$. As we can take the functors Hom or $\otimes$ for T, $t(\mathcal{P})$ and $t^{-1}(\mathcal{P})$ leads us to projectively, injectively or flatly generated proper classes.

Over an integral domain R, a short exact sequence $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is called *torsion-splitting* if $E\tau$ splits for the injection $\tau : T(C) \to C$. The class of torsion-splitting short exact sequences will be denoted by $\mathcal{D}$ as it was in (Pancar 1997). If C is torsion-free, or if E is splitting, then E is trivially torsion-splitting. More information about torsion-splitting exact sequences can be found in (Fuchs 1970, §58). We give one of the important results here.

Proposition 3.1 ((Fuchs 1970), 58.3) *In the case of abelian groups, the exact sequence* $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *is torsion-splitting if and only if it is an element of the maximal divisible subgroup of* $\mathrm{Ext}(C, A)$.

Let $\mathcal{E}$ be a class of short exact sequences. The smallest proper class containing $\mathcal{E}$ is said to be *generated by* $\mathcal{E}$ and denoted by $\langle\mathcal{E}\rangle$ (see (Pancar 1997)).

Since the intersection of any family of proper classes is proper, for a class $\mathcal{E}$ of short exact sequences

$$\langle\mathcal{E}\rangle = \bigcap \Big\{\mathcal{P} : \mathcal{E} \subseteq \mathcal{P} ; \mathcal{P} \text{ is a proper class}\Big\}. \tag{3.3}$$

For more information about proper classes generated by a class of short exact sequences see (Pancar 1997).

3.2. Objects of a Proper Class

Take a short exact sequence

$$E : 0 \longrightarrow A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \longrightarrow 0 \tag{3.4}$$

of R-modules and R-module homomorphisms.

An R-module M is said to be *projective with respect to the short exact sequence E*, or *with respect to the epimorphism g* if any of the following equivalent conditions holds:

1. every diagram

$$\begin{array}{cccccccccc} E: & 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & & & & & {\scriptstyle\tilde{\gamma}}\nwarrow & \uparrow{\scriptstyle\gamma} & & \\ & & & & & & & M & & \end{array} \tag{3.5}$$

   where the first row is E and $\gamma : M \longrightarrow C$ is an R-module homomorphism can be embedded in a commutative diagram by choosing an R-module homomorphism $\tilde{\gamma} : M \longrightarrow B$; that is, for every homomorphism $\gamma : M \longrightarrow C$, there exits a homomorphism $\tilde{\gamma} : M \longrightarrow B$ such that $g \circ \tilde{\gamma} = \gamma$.

2. The sequence

$$\mathrm{Hom}(M, E) : \quad 0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C) \longrightarrow 0$$

   is exact.

Dually, an R-module M is said to be *injective with respect to the short exact sequence* E, or *with respect to the monomorphism* f if any of the following equivalent conditions holds:

1. every diagram

$$\begin{array}{cccccccccc} E: & 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & & {\scriptstyle\alpha}\downarrow & \swarrow{\scriptstyle\tilde{\alpha}} & & & & & \\ & & & M & & & & & & \end{array} \tag{3.6}$$

   where the first row is E and $\alpha : A \longrightarrow M$ is an R-module homomorphism can be embedded in a commutative diagram by choosing an R-module homomorphism $\tilde{\alpha} : B \longrightarrow M$; that is, for every homomorphism $\alpha : A \longrightarrow M$, there exists a homomorphism $\tilde{\alpha} : B \longrightarrow M$ such that $\tilde{\alpha} \circ f = \alpha$.

2. The sequence

$$\text{Hom}(E, M): \quad 0 \longrightarrow \text{Hom}(C, M) \xrightarrow{g^*} \text{Hom}(B, M) \xrightarrow{f^*} \text{Hom}(A, M) \longrightarrow 0$$

is exact.

An R-module M is said to be $\mathcal{P}$*-projective* (resp. $\mathcal{P}$*-injective*) if it is projective (resp. injective) with respect to all short exact sequences in $\mathcal{P}$. The relative projectiveness (resp. injectiveness) of M is equivalent to the requirement that $\text{Ext}^1_{\mathcal{P}}(M, B) = 0$, for every R-module B (resp. $\text{Ext}^1_{\mathcal{P}}(A, M) = 0$, for every R-module A). Denote all $\mathcal{P}$-projective (resp. $\mathcal{P}$-injective) modules by $\pi(\mathcal{P})$ (resp. $\iota(\mathcal{P})$).

In a proper class $\mathcal{P}$, there need not be a $\mathcal{P}$-epimorphism from some $\mathcal{P}$-projective module to a given R-module A. For this reason, in general, it is not possible to define the functor $\text{Ext}^1_{\mathcal{P}}$ by using the derived functor of the functor Hom. However, the alternative definition of $\text{Ext}^1_{\mathcal{P}}$ may be used in this case.

For a proper class $\mathcal{P}$ and R-modules A, C, denote by $\text{Ext}^1_{\mathcal{P}}(C, A)$ or shortly by $\text{Ext}_{\mathcal{P}}(C, A)$, the equivalence classes of all short exact sequences in $\mathcal{P}$ which start with A and end with C. This turns out to be a subgroup of Ext(C, A) and a bifunctor $\text{Ext}_{\mathcal{P}}$: $R\text{-}\mathcal{M}od \times R\text{-}\mathcal{M}od \longrightarrow \mathcal{A}b$ is obtained which is a subfunctor of Ext.

A class $\mathcal{P}$ of R-modules is said to have *enough projectives* if for every module A we can find a $\mathcal{P}$-epimorhism from some $\mathcal{P}$-projective module P to A. A class $\mathcal{P}$ of R-modules is said to have *enough injectives* if for every module B we can find a $\mathcal{P}$-monomorphism from B to some $\mathcal{P}$-injective module J. A proper class $\mathcal{P}$ of R-modules with enough projectives [enough injectives] is also said to be a *projective proper class* [resp. *injective proper class*].

The following propositions give the relation between projective (resp. injective) modules with respect to a class $\mathcal{E}$ of short exact sequences and with respect to the proper class $< \mathcal{E} >$ generated by $\mathcal{E}$.

Proposition 3.2 ((Pancar 1997), Propositions 2.3 and 2.4)

(a) $\pi(\mathcal{E}) = \pi(< \mathcal{E} >)$.

(b) $\iota(\mathcal{E}) = \iota(< \mathcal{E} >)$.

An R -module C is said to be $\mathcal{P}$*-coprojective* if *every* short exact sequence of R-modules and R-module homomorphisms of the form

$$E : 0 \longrightarrow A' \longrightarrow B' \longrightarrow C \longrightarrow 0 \tag{3.7}$$

ending with C is in the proper class $\mathcal{P}$. An R-module A is said to be $\mathcal{P}$*-coinjective* if *every* short exact sequence of R-modules and R-module homomorphisms of the form

$$E : 0 \longrightarrow A \longrightarrow B'' \longrightarrow C' \longrightarrow 0 \tag{3.8}$$

starting with A is in the proper class $\mathcal{P}$.

Using the subfunctor $\mathrm{Ext}_{\mathcal{P}}$ of Ext, the $\mathcal{P}$-projectives, $\mathcal{P}$-injectives, $\mathcal{P}$-coprojectives and $\mathcal{P}$-coinjectives are simply described in terms of the subgroup $\mathrm{Ext}_{\mathcal{P}}(C, A) \leq \mathrm{Ext}(C, A)$ being 0 or the whole of $\mathrm{Ext}(C, A)$:

1. An R-module C is $\mathcal{P}$*-projective* if and only if $\mathrm{Ext}_{\mathcal{P}}(C, A) = 0$ for all R-modules A.
2. An R-module C is $\mathcal{P}$*-coprojective* if and only if $\mathrm{Ext}_{\mathcal{P}}(C, A) = \mathrm{Ext}(C, A)$ for all R-modules A.
3. An R-module A is $\mathcal{P}$*-injective* if and only if $\mathrm{Ext}_{\mathcal{P}}(C, A) = 0$ for all R-modules C.
4. An R-module A is $\mathcal{P}$*-coinjective* if and only if $\mathrm{Ext}_{\mathcal{P}}(C, A) = \mathrm{Ext}(C, A)$ for all R-modules C.

3.3. Projectively Generated Proper Classes

For a given class $\mathcal{M}$ of modules, denote by $\pi^{-1}(\mathcal{M})$ the class of all short exact sequences E of R-modules and R-module homomorphisms such that $\mathrm{Hom}(M, E)$ is exact for all $M \in \mathcal{M}$, that is,

$$\pi^{-1}(\mathcal{M}) = \{E \in \mathcal{A}bs \,|\, \mathrm{Hom}(M, E) \text{ is exact for all } M \in \mathcal{M}\}. \tag{3.9}$$

$\pi^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-projective, and it is called the proper class *projectively generated* by $\mathcal{M}$.

Proof This is a consequence of Lemma 3.1. Take $T(M, \cdot) = \mathrm{Hom}(M, \cdot)$. □

Proposition 3.3 *Let $\mathcal{P}$ be a proper class and $\mathcal{M}$ a class of modules. Then we have*

1. $\mathcal{P} \subseteq \pi^{-1}(\pi(\mathcal{P}))$,
2. $\mathcal{M} \subseteq \pi(\pi^{-1}(\mathcal{M}))$,
3. $\pi(\mathcal{P}) = \pi(\pi^{-1}(\pi(\mathcal{P})))$,
4. $\pi^{-1}(\mathcal{M}) = \pi^{-1}(\pi(\pi^{-1}(\mathcal{M})))$.

For a proper class $\mathcal{P}$, $\pi^{-1}(\pi(\mathcal{P}))$ is called the *projective closure* of $\mathcal{P}$ and it always contains $\mathcal{P}$.

3.4. Injectively Generated Proper Classes

For a given class $\mathcal{M}$ of modules, denote by $\iota^{-1}(\mathcal{M})$ the class of all short exact sequences E of R-modules and R-module homomorphisms such that $\mathrm{Hom}(E, M)$ is exact for all $M \in \mathcal{M}$, that is,

$$\iota^{-1}(\mathcal{M}) = \{E \in \mathcal{A}bs \mid \mathrm{Hom}(E, M) \text{ is exact for all } M \in \mathcal{M}\}. \tag{3.10}$$

$\iota^{-1}(\mathcal{M})$ is the largest proper class $\mathcal{P}$ for which each $M \in \mathcal{M}$ is $\mathcal{P}$-injective which is called the proper class *injectively generated* by $\mathcal{M}$.

Proof This is a consequence of Lemma 3.1. Take $T(M, \cdot) = \mathrm{Hom}(\cdot, M)$. □

3.5. Coprojectively and Coinjectively Generated Proper Classes

Let $\mathcal{M}$ and $\mathcal{J}$ be classes of modules over some ring R. The smallest proper class $\overline{k}(\mathcal{M})$ (resp. $\underline{k}(\mathcal{J})$) for which all modules in $\mathcal{M}$ (resp. $\mathcal{J}$) are coprojective (resp. coinjective) is said to be coprojectively (resp. coinjectively) generated by $\mathcal{M}$ (resp. $\mathcal{J}$).

Theorem 3.2 ((Alizade 1985), Theorem 1) *Let $\mathcal{M}$ be a class of modules closed under extensions. Consider the class $\mathcal{R}$ of exact triples, defined as:*

$$\text{Ext}_{\mathcal{R}}(C, A) = \bigcup_{M,\alpha} \text{Im}\left\{ \text{Ext}(M, A) \xrightarrow{\alpha^*} \text{Ext}(C, A)\right\} \tag{3.11}$$

over all $M \in \mathcal{M}$ and all homomorphisms $\alpha : C \longrightarrow M$. Then exact triples $0 \longrightarrow A \longrightarrow X \longrightarrow C \longrightarrow 0$ belonging to $\text{Ext}_{\mathcal{R}}(C, A)$*, form a proper class, and $\mathcal{R}$ coincides with $\overline{k}(\mathcal{M})$.*

Theorem 3.3 ((Alizade 1985), Theorem 2) *Let $\mathcal{L}$ be a class of modules closed under extensions. Consider the class $\mathcal{R}$ of exact triples, defined as:*

$$\text{Ext}_{\mathcal{R}}(C, A) = \bigcup_{I,\alpha} \text{Im}\left\{ \text{Ext}(C, I) \xrightarrow{\alpha_*} \text{Ext}(C, A)\right\} \tag{3.12}$$

over all $I \in \mathcal{L}$ and all homomorphisms $\alpha : I \longrightarrow A$. Then exact triples $0 \longrightarrow A \longrightarrow X \longrightarrow C \longrightarrow 0$ belonging to $\text{Ext}_{\mathcal{R}}(C, A)$*, form a proper class, and $\mathcal{R}$ coincides with $\underline{k}(\mathcal{L})$.*

For more information about coprojectively and coinjectively generated proper classes see (Alizade 1985) and (Alizade 1986).

CHAPTER 4

PROPER CLASSES RELATED TO SUPPLEMENTS

In this part of the study, we will give definitions and some properties of the proper classes related to supplements. One can find further information in (Alizade et al. 2012), (Demirci 2008), (Erdoğan 2004) and (Mermut 2004).

4.1. The Classes $\mathcal{C}ompl$, $\mathcal{S}uppl$, $\mathcal{N}eat$ and $\mathcal{C}o\text{-}\mathcal{N}eat$

The class $\mathcal{C}ompl$ consists of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{4.1}$$

such that $\operatorname{Im} f$ is a complement of some submodule K of B, that is $\operatorname{Im} f \cap K = 0$ and K is maximal with respect to this property.

The class $\mathcal{N}eat$ consists of all short exact sequences 4.1 such that every simple R-module is relative projective for it, denoted by

$$\mathcal{N}eat = \pi^{-1}\left\{S \in R\text{-}\mathcal{M}od \mid S \text{ is simple}\right\}. \tag{4.2}$$

The corresponding subgroup of $\operatorname{Ext}(C, A)$ is denoted by $\operatorname{Next}(C, A)$. Over the ring of integers, we have the following result that gives the structure of $\mathcal{N}eat_{\mathbb{Z}\text{-}\mathcal{M}od}$ in terms of the subgroups of $\operatorname{Ext}(C, A)$.

Corollary 4.1 ((Alizade et al. 2004), Corollary 4.3) *For all abelian groups A and C, we have* $\operatorname{Next}(C, A) = \bigcap_p p\operatorname{Ext}(C, A) = F(\operatorname{Ext}(C, A))$*, where p ranges over the prime numbers, and* $F(\operatorname{Ext}(C, A))$ *is the Frattini subgroup of* $\operatorname{Ext}(C, A)$.

Since for an abelian group A, $\operatorname{Rad} A = \bigcap_p pA$, where p ranges over the prime numbers, the question if the composition of a subfunctor of the identity with the functor Ext gives a proper class arises. This question is answered by a counterexample in the case of abelian groups.

Let R be an integral domain. Let $\mathcal{S}ocle$ be the class of short exact sequences defined via the composition of the functors Soc and Ext, that is
$E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \mathcal{S}ocle$ if $E \in \mathrm{Soc}(\mathrm{Ext}(C, A))$.

Proposition 4.1 *$\mathcal{S}ocle$ is not a proper class in the case of abelian groups.*

Proof Suppose that $\mathcal{S}ocle$ is a proper class. The subgroup of $\mathrm{Ext}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z})$ generated by the short exact sequence $E : 0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$, where p is a prime number and $\alpha(1) = p$, is simple since pE is splitting. By (Misina and Skornjakov 1960, Proposition 1.12), $\mathbb{Z}/p\mathbb{Z}$ is $\mathcal{S}ocle$-coprojective. The short exact sequence
$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{f} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{g} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$ where $f(1+p\mathbb{Z}) = p+p^2\mathbb{Z}$ and $g(1+p^2\mathbb{Z}) = 1 + p\mathbb{Z}$ belongs to $\mathcal{S}ocle$ since it ends with $\mathbb{Z}/p\mathbb{Z}$. By (Misina and Skornjakov 1960, Proposition 1.14), $\mathbb{Z}/p^2\mathbb{Z}$ is $\mathcal{S}ocle$-coprojective. However, the short exact sequence
$E' : 0 \longrightarrow \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow 0$, where $\beta(1) = p^2$, does not belong to $\mathrm{Soc}(\mathrm{Ext}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}))$ since the subgroup of $\mathrm{Ext}(\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z})$ generated by E' is a p-group, and $pE' \neq 0$. This leads us to a contradiction. □

The class $\mathcal{S}uppl$, consisting of all short exact sequences 4.1 such that $\mathrm{Im}\, f$ is a supplement of some submodule K of B, is a proper class (see (Clark et al. 2006) or (Erdoğan 2004) for a proof). The properties of *$\mathcal{S}uppl$-coinjective* and *$\mathcal{S}uppl$-coprojective* modules are investigated in (Erdoğan 2004).

Co-$\mathcal{N}eat$ is defined as a dual notion of $\mathcal{N}eat$:

$$Co\text{-}\mathcal{N}eat = \iota^{-1}\left\{M \in R\text{-}\mathcal{M}od \mid \mathrm{Rad}\, M = 0\right\}. \tag{4.3}$$

We have the relations, $\mathcal{C}ompl \subseteq \mathcal{N}eat$ and $\mathcal{S}uppl \subseteq Co\text{-}\mathcal{N}eat$ for an arbitrary ring R. The following result shows a condition under which we have an equality.

Proposition 4.2 ((Mermut 2004), Proposition 5.2.6) *For a Dedekind domain R,*

$$\mathcal{S}uppl \subseteq Co\text{-}\mathcal{N}eat \subseteq \mathcal{N}eat = \mathcal{C}ompl. \tag{4.4}$$

4.2. The κ-Elements of $\text{Ext}(C, A)$

Following Zöschinger we call a short exact sequence

$$E : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{4.5}$$

κ-exact if $\text{Im}\, f$ has a supplement in B. In this case we say that $E \in \text{Ext}(C, A)$ is a *κ-element*, and the set of all κ-elements of $\text{Ext}(C, A)$ will be denoted by $\mathcal{S}$. If $\text{Im}\, f$ has a supplement V in B with $\text{Im}\, f \cap V$ bounded, then $E \in \text{Ext}(C, A)$ is called a *β-element*, and the set of all β-elements of $\text{Ext}(C, A)$ will be denoted by $\mathcal{SB}$.

We denote by $\mathcal{WS}$ the class of short exact sequences 4.5, where $\text{Im}\, f$ has (is) a weak supplement in B, i.e. there is a submodule K of B such that $\text{Im}\, f + K = B$ and $\text{Im}\, f \cap K \ll B$. We denote by $\mathcal{S}mall$ the class of short exact sequences 4.5, where $\text{Im}\, f \ll B$.

The κ-elements need not form a proper class in general. For instance, let $R = \mathbb{Z}$, and consider the composition $\beta \circ \alpha$ of the monomorphisms $\alpha : 2\mathbb{Z} \longrightarrow \mathbb{Z}$ and $\beta : \mathbb{Z} \longrightarrow \mathbb{Q}$, where α and β are the corresponding inclusions. Then we have $0 \longrightarrow 2\mathbb{Z} \xrightarrow{\beta\circ\alpha} \mathbb{Q} \longrightarrow \mathbb{Q}/2\mathbb{Z} \longrightarrow 0$ is a κ-element, but $0 \longrightarrow 2\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ is not a κ-element as $2\mathbb{Z}$ does not have a supplement in $\mathbb{Z}$.

There are some cases that enables $\mathcal{SB}$ to form a proper class. Furthermore, the following result shows that $\mathcal{SB}$ is coinjectively generated under the given condition.

Proposition 4.3 ((Demirci 2008), Proposition 4.3) *Let R be a Noetherian integral domain of Krull dimension 1. Then $\mathcal{SB} = \underline{k}(\mathcal{B})$. Hence $\mathcal{SB}$ is a proper class in this case.*

Definition 4.1 *A short exact sequence $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is said to be extend weak supplement if there is a short exact sequence $E' : 0 \longrightarrow A \xrightarrow{f} B' \longrightarrow C' \longrightarrow 0$ such that $\text{Im}\, f$ has (is) a weak supplement in B' and there is a homomorphism $g : C \to C'$ such that $E = g^*(E')$, i.e. there is commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 : E \\ & & \| & & \downarrow & & \downarrow g & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B' & \longrightarrow & C' & \longrightarrow & 0 : E'. \end{array} \tag{4.6}$$

The class of all extend weak supplement short exact sequences will be denoted by $\overline{\mathcal{WS}}$.

Over a hereditary ring, it is shown in (Alizade et al. 2012) that $\overline{\mathcal{WS}}$ forms a proper class. Moreover, the following result holds.

Corollary 4.2 ((Alizade et al. 2012), Corollary 3.13) $\langle \mathcal{S}mall \rangle = \langle \mathcal{S} \rangle = \langle \mathcal{WS} \rangle = \overline{\mathcal{WS}}$.

Projective, injective and coinjective objects of the class $\overline{\mathcal{WS}}$ are investigated in the same work, it is also shown that $\overline{\mathcal{WS}}$ is coinjectively generated. Let us remind that a module M is called small if it is a small submodule of some module N, or equivalently it is small in its injective hull (see (Leonard 1966) for more information about small modules).

Proposition 4.4 ((Alizade et al. 2012), Proposition 4.13) *Over a hereditary ring* $\overline{\mathcal{WS}} = \underline{k}(\mathcal{S}m)$*, where* $\mathcal{S}m$ *is the class of all small modules.*

4.3. Coneat-Homomorphisms

The main problem with the investigation of the κ-elements in $\mathrm{Ext}(C, A)$ is that they need not form a subgroup. The reason for this is the fact that, in general, for a homomorphism $g : C' \longrightarrow C$, the induced map $g^* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C', A)$ need not preserve κ-elements. The following results hold over the ring of integers.

A homomorphism $g : C' \longrightarrow C$ is called *coneat* if for every decomposition $g = \beta \circ \alpha$, where β is a small epimorphism, β is an isomorphism.

Lemma 4.1 ((Zöschinger 1978), Lemma 2.2)

(a) An epimorphism $g : C' \longrightarrow C$ *is coneat if and only if* $\mathrm{Ker}\, g$ *is coclosed in* C'*, i.e. for any submodule* X *of* $\mathrm{Ker}\, g$, $\mathrm{Ker}\, g/X \ll C'/X$ *implies* $X = \mathrm{Ker}\, g$.

(b) A splitting monomorphism $g : C' \longrightarrow C$ *is coneat if and only if* $\mathrm{Coker}\, g$ *has no small cover.*

(c) If $g = g_2 \circ g_1$ *is coneat, then* g_2 *is also coneat.*

Theorem 4.1 ((Zöschinger 1978), Theorem 2.3) *For a homomorphism* $g : C' \longrightarrow C$*, the following are equivalent:*

(i) g is coneat.

(ii) $\mathrm{Ker}\, g$ *is coclosed in* C' *and* $\mathrm{Im}\, g \supset \mathrm{Soc}\, C$.

(iii) $g(C'[p]) = C[p]$ *for all prime numbers p.*

(iv) *If the diagram below is a pullback diagram and* β *is a small epimorphism, then* β' *is also a small epimorphism.*

$$\begin{array}{ccc} B' & \xrightarrow{\beta'} & C' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[\beta]{} & C \end{array} \tag{4.7}$$

The following result establishes a connection between coneat homomorphisms and the κ-elements of $\mathrm{Ext}(C, A)$.

Corollary 4.3 ((Zöschinger 1978), Corollary 1 after Theorem 2.3) *If* $g : C' \longrightarrow C$ *is coneat, then* $g^* : \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C', A)$ *preserves* κ*-elements.*

CHAPTER 5

OPERATIONS ON PROPER CLASSES

In this chapter, we will define some operations defined on proper classes and give some results. More information can be found in (Pancar 1997).

In (Pancar 1997), the following operations are defined for the classes $\mathcal{R}$ and $\mathcal{L}$:

i) $\mathcal{R} + \mathcal{L} = \langle \mathcal{R} \cup \mathcal{L} \rangle$, the *sum* of the classes $\mathcal{R}$ and $\mathcal{L}$,

ii) $\mathcal{R}\underline{\circ}\mathcal{L}$, the class of short exact sequences whose monomorphisms are $\alpha \circ \beta$ where α is an $\mathcal{R}$-monomorphism and β is an $\mathcal{L}$-monomorphism,

iii $\mathcal{R}\overline{\circ}\mathcal{L}$, the class of short exact sequences whose epimorphisms are $\alpha \circ \beta$ where α is an $\mathcal{R}$-epimorphism and β is an $\mathcal{L}$-epimorphism,

iv) $\mathcal{R} * \mathcal{L}$, the class of short exact sequences defined by the formula

$$\mathrm{Ext}_{\mathcal{R}*\mathcal{L}}(C, A) = \mathrm{Ext}_{\mathcal{R}}(C, A) + \mathrm{Ext}_{\mathcal{L}}(C, A). \tag{5.1}$$

From the definitions we have,

1. $\mathcal{R} + \mathcal{L} = \mathcal{L} + \mathcal{R}$ is a proper class,
2. $\mathcal{R} * \mathcal{L} = \mathcal{L} * \mathcal{R}$,
3. $\mathcal{R}\underline{\circ}\mathcal{L} \subseteq \mathcal{R} + \mathcal{L}$, $\mathcal{R}\overline{\circ}\mathcal{L} \subseteq \mathcal{R} + \mathcal{L}$ and $\mathcal{R} * \mathcal{L} \subseteq \mathcal{R} + \mathcal{L}$.

In general, $\mathcal{R}\underline{\circ}\mathcal{L}$, $\mathcal{R}\overline{\circ}\mathcal{L}$ and $\mathcal{R} * \mathcal{L}$ need not be proper classes, but if $\mathcal{R}\underline{\circ}\mathcal{L}$ ($\mathcal{R}\overline{\circ}\mathcal{L}$ or $\mathcal{R} * \mathcal{L}$) is a proper class, then $\mathcal{R} + \mathcal{L} = \mathcal{R}\underline{\circ}\mathcal{L}$ ($\mathcal{R} + \mathcal{L} = \mathcal{R}\overline{\circ}\mathcal{L}$ or $\mathcal{R} + \mathcal{L} = \mathcal{R} * \mathcal{L}$).

The following theorem gives a condition under which $\mathcal{R} * \mathcal{L} \subseteq \mathcal{R}\underline{\circ}\mathcal{L}$.

Theorem 5.1 ((Pancar 1997), Theorem 3.1) *If the class $\mathcal{M}$ of modules is closed under extensions and submodules, then $\mathcal{R} * \mathcal{L} \subseteq \mathcal{R}\underline{\circ}\mathcal{L}$ for $\mathcal{R} = \overline{k}(\mathcal{M})$ and for every proper class $\mathcal{L}$.*

Using the following proposition, it is possible to write a coprojectively generated proper class as a composition of two classes one of which is the smallest proper class $\mathcal{S}plit$. Moreover, the dual statement for which the proof is included also holds.

Proposition 5.1 ((Pancar 1997), Proposition 3.1) *If the class of modules $\mathcal{M}$ is closed under submodules and extensions, then $\overline{k}(\mathcal{M}) = \mathcal{E}^{\mathcal{M}} \underline{\circ} \mathcal{S}plit$, $\mathcal{E}^{\mathcal{M}}$ being the class of all short exact sequences ending at modules from $\mathcal{M}$.*

Proposition 5.2 *If the class of modules $\mathcal{M}$ is closed under homomorphic images and extensions, then $\underline{k}(\mathcal{M}) = \mathcal{S}plit \,\overline{\circ}\, \varepsilon_{\mathcal{M}}$, $\varepsilon_{\mathcal{M}}$ being the class of all short exact sequences beginning with modules from $\mathcal{M}$ and $\mathcal{S}plit$ being the class of all splitting short exact sequences.*

Proof Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence in $\underline{k}(\mathcal{M})$. Since $\mathcal{M}$ is closed under extensions, by Theorem 3.3, there exist $M \in \mathcal{M}$, $E_1 \in \operatorname{Ext}(C, M)$ and a homomorphism $\alpha : M \longrightarrow A$ such that $E = \alpha_*(E_1)$. Since $\mathcal{M}$ is closed under homomorphic images, $M/\operatorname{Ker}\alpha \in \mathcal{M}$. Therefore, α can be taken as a monomorphism. From the cohomology sequence

$$\cdots \longrightarrow \operatorname{Ext}(C, M) \xrightarrow{\alpha_*} \operatorname{Ext}(C, A) \xrightarrow{\beta_*} \operatorname{Ext}(C, X) \longrightarrow \cdots \tag{5.2}$$

for $0 \longrightarrow M \xrightarrow{\alpha} A \xrightarrow{\beta} X \longrightarrow 0$, $X = \operatorname{Coker}\alpha$, we have $E \in \operatorname{Im}\alpha_* = \operatorname{Ker}\beta_*$. Then in the following commutative diagram with exact rows and columns, $\beta_*(E) \in \mathcal{S}plit$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & M & = & M & & & & \\
 & & & & \downarrow{\scriptstyle\alpha} & & \downarrow & & & & \\
E : & & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0. \\
 & & & & \downarrow{\scriptstyle\beta} & & \downarrow & & \| & & \\
\alpha_*(E) : & & 0 & \longrightarrow & X & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array} \tag{5.3}$$

On the other hand, $0 \longrightarrow M \longrightarrow B \longrightarrow B' \longrightarrow 0 \in \varepsilon_{\mathcal{M}}$ by the definition of $\varepsilon_{\mathcal{M}}$, therefore $E \in \mathcal{S}plit \,\overline{\circ}\, \varepsilon_{\mathcal{M}}$.

Conversely, since $\varepsilon_{\mathcal{M}} \subseteq \langle \varepsilon_{\mathcal{M}} \rangle$ and $\mathcal{S}plit \subseteq \langle \varepsilon_{\mathcal{M}} \rangle$, we have $\mathcal{S}plit \,\overline{\circ}\, \varepsilon_{\mathcal{M}} \subseteq \langle \varepsilon_{\mathcal{M}} \rangle \subseteq \underline{k}(\mathcal{M})$. □

We can relate the conclusion of this proposition with the class $\overline{\mathcal{WS}}$ using following results.

Corollary 5.1 *Over a hereditary ring R,* $\overline{\mathcal{WS}} = \mathcal{S}plit \,\overline{\circ}\, \varepsilon_{\mathcal{S}m}$*, where* $\mathcal{S}m$ *is the class of small modules and* $\varepsilon_{\mathcal{S}m}$ *is the class of all short exact sequences beginning with small modules.*

Proof It is easy to show that $\mathcal{S}m$ is closed under homomorphic images and extensions. By Proposition 5.2, $\underline{k}(\mathcal{S}m) = \mathcal{S}plit \,\overline{\circ}\, \varepsilon_{\mathcal{S}m}$. Proposition 4.4 completes the proof. □

The sum of proper classes is investigated in (Alizade et al. 1997) for the case of abelian groups. The direct sum $\mathcal{P}\oplus\mathcal{L}$ of proper classes $\mathcal{P}$ and $\mathcal{L}$ is defined, and an example for a direct sum is given in the same work.

Theorem 5.2 ((Alizade et al. 1997), Theorem 1) *If cl. is a class of groups closed under direct sums such that* $\pi^{-1}(cl.)$ *is projective, then*

$$\pi^{-1}(cl.) + \overline{k}(cl.) = \mathcal{A}bs. \tag{5.4}$$

If cl. is closed under direct products such that $\iota^{-1}(cl.)$ *is injective, then*

$$\iota^{-1}(cl.) + \underline{k}(cl.) = \mathcal{A}bs \tag{5.5}$$

Remark 5.1 ((Alizade et al. 1997), Remark after Theorem 1) *Let* $\mathcal{TF}$ *be the class of all torsion free groups. Theorem 5.2 implies that*

$$\pi^{-1}(\mathcal{TF}) + \mathcal{D} = \mathcal{A}bs \tag{5.6}$$

Definition 5.1 *Proper Classes* $\mathcal{P}$ *and* $\mathcal{L}$ *are said to be disjoint if* $\mathcal{P} \cap \mathcal{L} = \mathcal{S}plit$*. A proper class* C *is called a direct sum of proper classes* $\mathcal{P}$ *and* $\mathcal{L}$ *if* C *is a sum of disjoint classes* $\mathcal{P}$ *and* $\mathcal{L}$*, that is if* $C = \mathcal{P} + \mathcal{L}$ *and* $\mathcal{P} \cap \mathcal{L} = \mathcal{S}plit$*.*

Theorem 5.3 ((Alizade et al. 1997), Theorem 4) $\mathcal{A}bs = \pi^{-1}(red.) \oplus \overline{k}(red.)$*, where red. is the class of all reduced groups.*

The following theorem shows an example for which the operator $*$ does not give a proper class.

Theorem 5.4 ((Alizade et al. 1997), Theorem 5) *$\pi^{-1}(red.) * \overline{k}(red.) \neq \mathcal{A}bs$, and therefore $\pi^{-1}(red.) * \overline{k}(red.)$ is not a proper class.*

CHAPTER 6

CLOSURE OF PROPER CLASSES

In this chapter, we will review some results on closure of proper classes and define new ones.

In the first section, we will work on integral domains and in the second one the rings we consider will be associative with an identity element unless otherwise stated.

6.1. The Classes $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_r$

Let R be an integral domain throughout this section. For a class $\mathcal{P}$ of short exact sequences of R-modules, we denote by $\hat{\mathcal{P}}$ the class of short exact sequences $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules such that $kE \in \mathcal{P}$ for some $0 \neq k \in R$ where k also denotes the multiplication homomorphism by $k \in R$. Thus

$$\hat{\mathcal{P}} = \{E \mid kE \in \mathcal{P} \text{ for some } 0 \neq k \in R\}. \tag{6.1}$$

For $E \in \mathcal{P}$, we have $1 \cdot E = E \in \mathcal{P}$, therefore $\mathcal{P} \subseteq \hat{\mathcal{P}}$ for every class $\mathcal{P}$ of short exact sequences.

In case of abelian groups, the class $\hat{\mathcal{P}}$ was studied in (Walker 1964, Alizade 1986) for $\mathcal{P} = \mathcal{S}plit$, in (Hart 1974) for $\mathcal{P} = \mathcal{P}ure$ and $\mathcal{P} = \mathcal{D}$, torsion splitting short exact sequences and in (Alizade 1986) for $\mathcal{P} = \mathcal{S}plit$, where it was denoted by *Text* since $\text{Ext}_{\hat{\mathcal{S}plit}}(C, A) = T(\text{Ext}(C, A))$ the torsion part of $\text{Ext}(C, A)$ and for every proper class $\mathcal{P}$. The following result gives a general answer when R is an integral domain.

Theorem 6.1 ((Alizade et al. 2004), Theorem 3.1) *For every proper class $\mathcal{P}$ of short exact sequences of R-modules, the class $\hat{\mathcal{P}}$ is proper.*

The following result is an example that shows applying the operations +, * and $\underline{\circ}$ can give the same class.

Theorem 6.2 ((Pancar 1997), Theorem 4.2) *Let R be the ring of integers. Then*

$$\mathcal{D} * \hat{\mathcal{S}plit} = \mathcal{D} \underline{\circ} \hat{\mathcal{S}plit} = \mathcal{D} + \hat{\mathcal{S}plit} = \hat{\mathcal{D}}. \tag{6.2}$$

So, every $\hat{\mathcal{D}}$-monomorphism is a composition of a $\hat{\mathcal{S}plit}$-monomorphism and a $\mathcal{D}$-monomorphism.

We can also consider $\hat{M}$ for R-modules $M \leq N$ with the definition

$$\hat{M} = \{x \in N \mid kx \in M \text{ for some } 0 \neq k \in R\}. \tag{6.3}$$

The following result is shown in (Pancar 1997) for the case of abelian groups.

Proposition 6.1 ((Pancar 1997), Lemma 4.1) *Let B_d be the maximal divisible subgroup and $T(B)$ be the torsion part of a group B. Then*

$$\hat{B_d} = B_d + T(B) = B_d + \hat{0}. \tag{6.4}$$

The submodule $\hat{M}$ of the R-module N can be used in finding closures. Let us first define the closure of an R-module.

Definition 6.1 *A submodule X of an R-module Z is closed in Z if $X \leq Y \leq Z$ and $X \trianglelefteq Y$ implies $Y = X$. For R-modules $M \leq X \leq N$, X is called a closure of M in N if $M \trianglelefteq X$ and X is closed in N.*

Closure of a module need not be unique in general as the following example shows.

Example 6.1 *Let R be the ring of integers and $S = \left\{\frac{a}{b} \in \mathbb{Q} \mid (b, p) = (b, q) = 1\right\}$ for prime numbers p and q. Then pS and qS are both closures of the submodule pqS in S.*

Proposition 6.2 *Let $M \leq N$ be R-modules, then $\hat{M}$ is the unique closure of $M + T(N)$ in N, where $T(N)$ is the torsion part of N.*

Proof For $x \in \hat{M}$, $rx \in M \leq M + T(N)$ for some $0 \neq r \in R$, therefore $M + T(N) \trianglelefteq \hat{M}$. If $\hat{M} \trianglelefteq M'$ for some $M' \leq N$, then for $y \in M'$, $ky \in \hat{M}$ for some $0 \neq k \in R$ which implies

$l(ky) \in M$ for some $0 \neq l \in R$. Since $(lk)y = l(ky) \in M$ for $0 \neq lk \in R$, $y \in \hat{M}$. Therefore, $\hat{M}$ is closed.

Let N' be another closure of $M + T(N)$ in N. If $z \in N'$, then $sz \in M + T(N)$ for some $0 \neq s \in R$ which shows that $N' \leq \hat{M}$. Since $M + T(N) \trianglelefteq N' \leq \hat{M}$ and N' is closed in N, $N' = \hat{M}$. □

Corollary 6.1 *Let N be a torsion-free R-module. Then for every submodule M of N, $\hat{M}$ is the unique closure of M in N.*

We have seen that $\hat{\mathcal{P}}$ is a closure of the class $\mathcal{P}$ of short exact sequences and is indeed a proper class when the underlying class $\mathcal{P}$ is proper. It is possible to define another closure of the class $\mathcal{P}$ that happens to be in between $\mathcal{P}$ and $\hat{\mathcal{P}}$.

For a class $\mathcal{P}$ of short exact sequences of R-modules and $0 \neq r \in R$, we denote by $\hat{\mathcal{P}}_r$ the class of short exact sequences $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules such that $r^t E \in \mathcal{P}$ for some for some nonnegative integer t where r^t also denotes the multiplication homomorphism by $r^t \in R$. Thus

$$\hat{\mathcal{P}}_r = \{E |\, r^t E \in \mathcal{P} \text{ for some nonnegative integer } t\}. \tag{6.5}$$

For $E \in \mathcal{P}$, we have $r^0 \cdot E = 1 \cdot E = E \in \mathcal{P}$ and clearly $\hat{\mathcal{P}}_r \subseteq \hat{\mathcal{P}}$, therefore $\mathcal{P} \subseteq \hat{\mathcal{P}}_r \subseteq \hat{\mathcal{P}}$ for every class $\mathcal{P}$ of short exact sequences.

For a given proper class $\mathcal{P}$, the following result gives us proper classes that are contained in $\hat{\mathcal{P}}$.

Proposition 6.3 *$\hat{\mathcal{P}}_r$ is a proper class for every proper class $\mathcal{P}$ and every* $0 \neq r \in R$.

The proof of this result uses similar ideas used in the proof of (Alizade 1986, Theorem 1).

Proof For any $r \in R$, let r also denote the homomorphism multiplication by r.

Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \hat{\mathcal{P}}_r$ with $r^t E \in \mathcal{P}$.

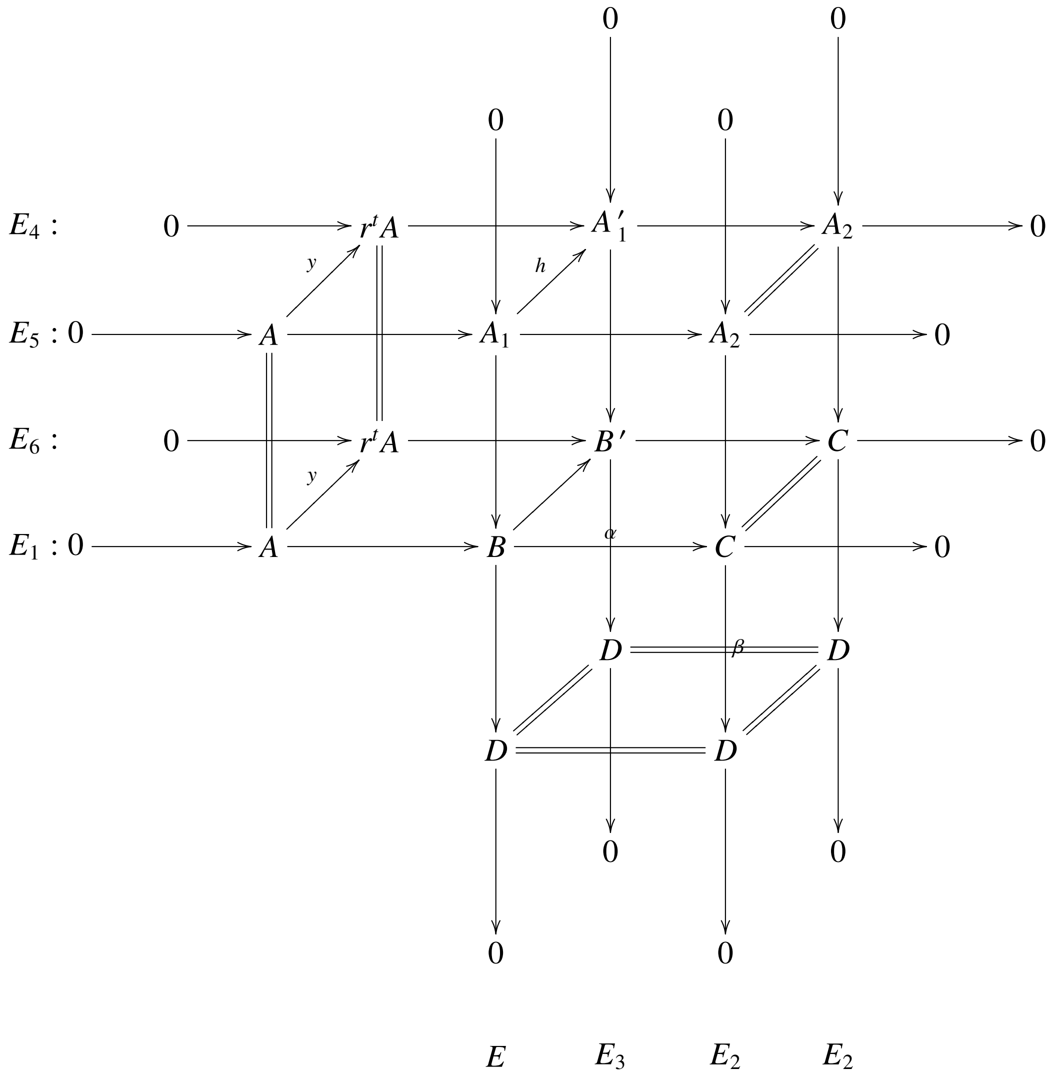
If $f : A \longrightarrow A'$, then $r^t f_*(E) = (r^{t*} \circ f_*)(E) = (f_* \circ r^{t*})(E) = f_*(r^{t*}(E)) \in \mathcal{P}$ since $\text{Ext}_{\mathcal{P}}$ is a subfunctor of Ext. Then $f_*(E) \in \hat{\mathcal{P}}_r$.

If $g : C' \longrightarrow C$, then $r^t g^*(E) = (r^t_* \circ g^*)(E) = (g^* \circ r^t_*)(E) = g^*(r^t_*(E)) \in \mathcal{P}$ since $\text{Ext}_{\mathcal{P}}$ is a subfunctor of Ext. Then $g^*(E) \in \hat{\mathcal{P}}_r$.

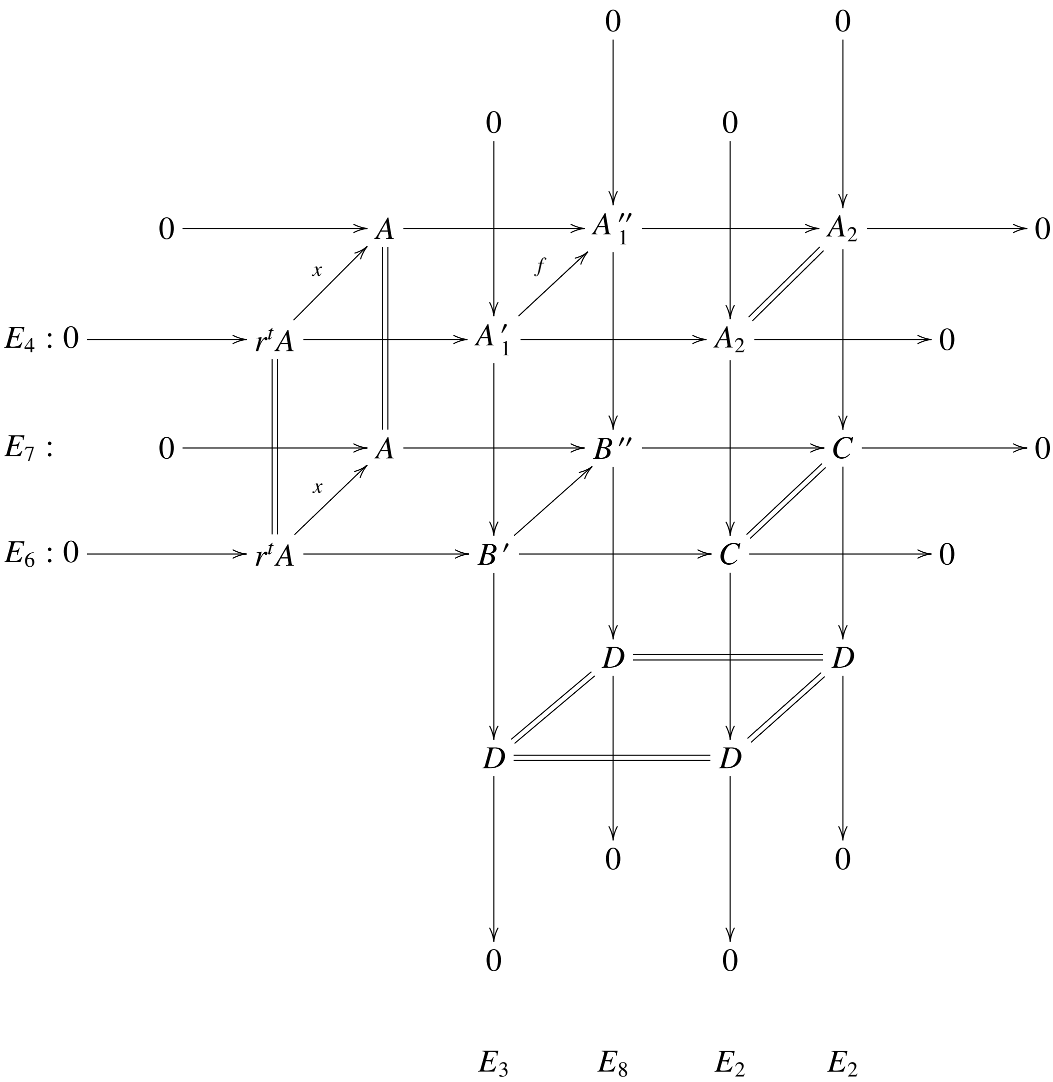
Let $E' : 0 \longrightarrow A \longrightarrow B' \longrightarrow C \longrightarrow 0 \in \hat{\mathcal{P}}_r$ with $r^s(E') \in \mathcal{P}$. Then $r^{t+s}(E - E') = r^s(r^t E) - r^t(r^s E') \in \mathcal{P}$ since $\mathcal{P}$ is a proper class.

These arguments show that $\hat{\mathcal{P}}_r$ gives an e-functor. Using (Nunke 1963, Theorem 1.1), in order to show that $\hat{\mathcal{P}}_r$ is a proper class, it is enough to show that the composition of two $\hat{\mathcal{P}}_r$-epimorphisms is a $\hat{\mathcal{P}}_r$-epimorphism.

Let $\alpha : B \longrightarrow C$ and $\beta : C \longrightarrow D$ be $\hat{\mathcal{P}}_r$-epimorphisms. Since $E_1 : 0 \longrightarrow A \longrightarrow B \xrightarrow{\alpha} C \longrightarrow 0 \ \in \hat{\mathcal{P}}_r$, there is a nonnegative integer t such that $r^t_*(E_1) : 0 \longrightarrow A \longrightarrow B' \longrightarrow C \longrightarrow 0 \ \in \mathcal{P}$. We can write the homomorphism r^t as $r^t = x \circ y$, where $x : r^tA \longrightarrow A$ is the inclusion and $y : A \longrightarrow r^tA$ is the standard epimorphism. Then we obtain the following commutative diagrams with exact rows and columns:



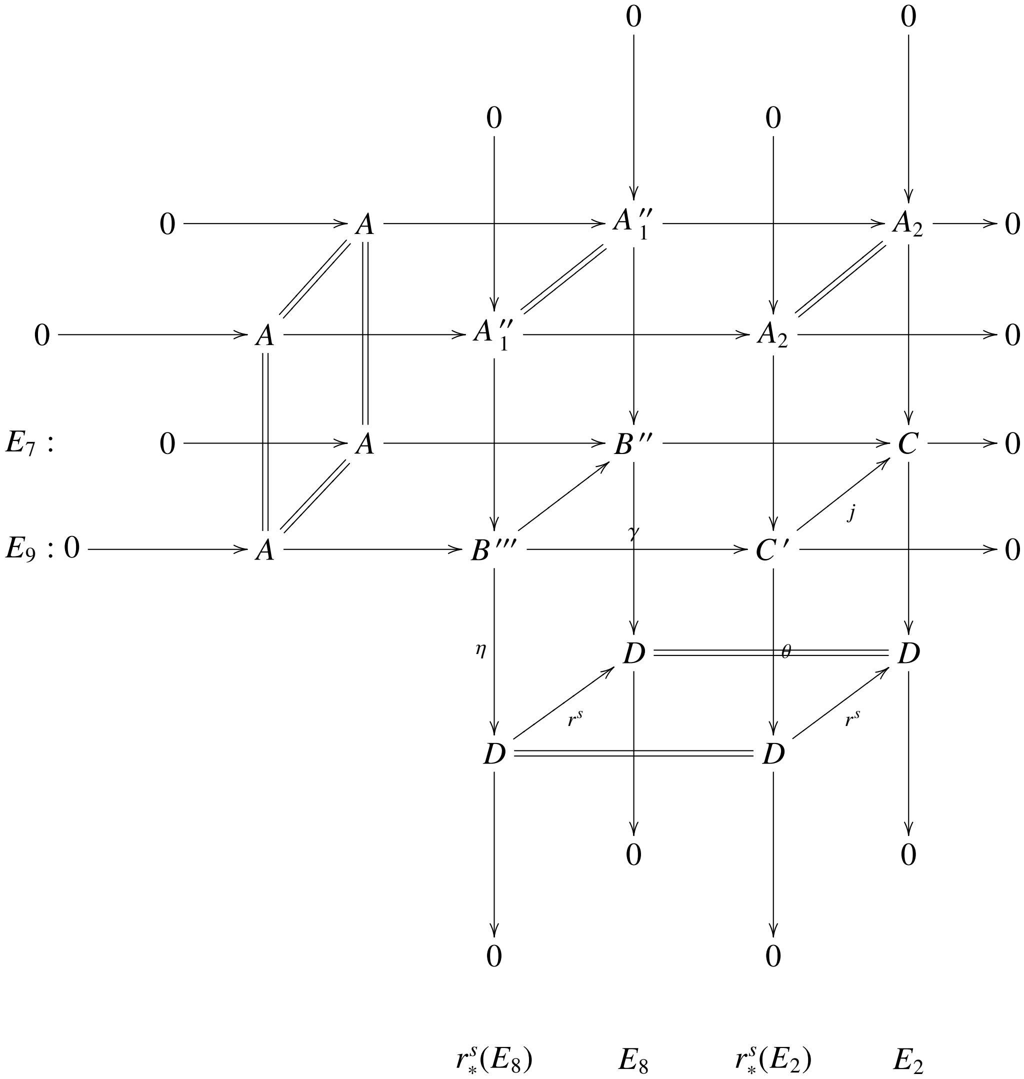
(6.6)



(6.7)

In the diagram (6.6), $E_4 = y_*(E_5)$, $E_6 = y_*(E_1)$, h is constructed using pushout, and $B \longrightarrow B'$ and $B' \longrightarrow D$ are epimorphisms. The diagram (6.7) is constructed similarly.

Since $E_2 \in \hat{\mathcal{P}}_r$, there is a nonnegative integer s such that $r^{s*}(E_2) \in \mathcal{P}$. Applying the homomorphism r^s to some part of the diagram (6.7), we obtain the following commutative diagram:

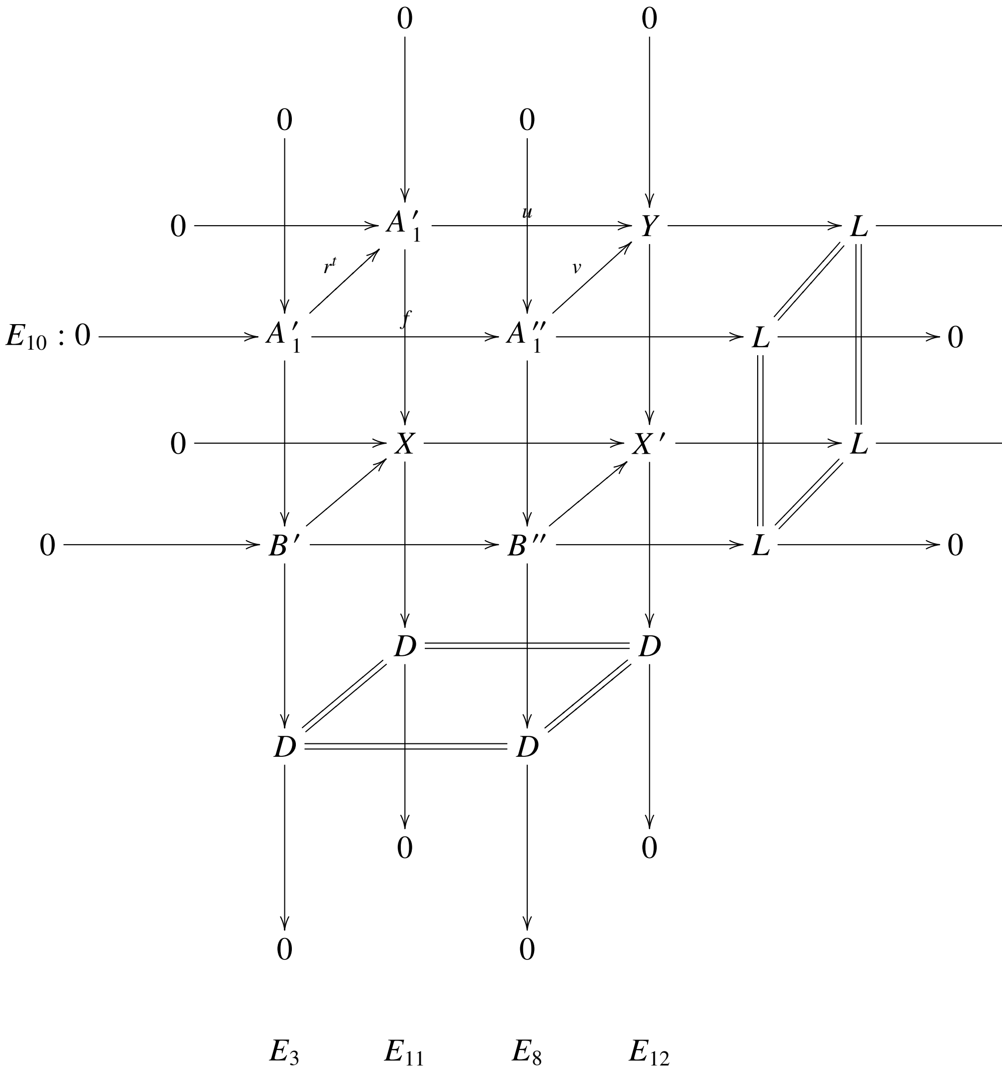


(6.8)

In the diagram (6.8), the homomorphisms η, θ and j are constructed using pull-back.

Since $E_7 = x_*(E_6) = x_*(y_*(E_1)) = (x \circ y)_*(E_1) = r^t_*(E_1) \in \mathcal{P}$ and $\mathrm{Ext}_{\mathcal{P}}$ is a subfunctor of Ext, $E_9 = j^*(E_7) \in \mathcal{P}$, that is $\gamma : B''' \longrightarrow C'$ is a $\mathcal{P}$-epimorphism. $\theta : C' \longrightarrow D$ is also a $\mathcal{P}$-epimorphism. Then $\eta = \theta \circ \gamma$ is a $\mathcal{P}$-epimorphism, and $r^{s*}(E_8) \in \mathcal{P}$. Therefore $E_8 \in \hat{\mathcal{P}}_r$.

In the diagram (6.7), we have $L = \mathrm{Coker}\, x \cong \mathrm{Coker}\, f$. Consider the short exact sequence $E_{10} : 0 \longrightarrow A'_1 \longrightarrow A''_1 \longrightarrow L \longrightarrow 0$. Since $r^t L = 0$, the short exact sequence $r^t_*(E_{10}) : 0 \longrightarrow A'_1 \longrightarrow Y \longrightarrow L \longrightarrow 0$ is splitting. Consider the following commutative diagram:



(6.9)

In the diagram (6.9), we have $E_{11} = r^t_*(E_3)$ and $E_{12} = (u \circ r^t)_*(E_3) = (v \circ f)_*(E_3) = v_*(E_8) \in \hat{\mathcal{P}}_r$, since $\mathrm{Ext}_{\hat{\mathcal{P}}_r}$ is a subfunctor of Ext. $r^t E_{10}$ is splitting, therefore $E_{11} \in \hat{\mathcal{P}}_r$. Then there is a nonnegative integer q such that $r^q E_{11} \in \mathcal{P}$. We also know that $E_{11} = r^t E_3$, therefore $r^{q+t} E_3 \in \mathcal{P}$ and $E_3 \in \hat{\mathcal{P}}_r$.

Consider the diagram (6.6). We can write the homomorphism $r^t : A_1 \longrightarrow A_1$ as $r^t = \psi \circ \phi$ where $\phi : A_1 \longrightarrow r^t A_1$ is the standard epimorphism and $\psi : r^t A_1 \longrightarrow A_1$ is the inclusion. Then $r^t_* = \psi_* \circ \phi_*$. Let $i : A'_1 \longrightarrow r^t A_1$ be the epimorphism (note that h is also an epimorphism). Then we have $\phi_*(E) = (i_* \circ h_*)(E) = i_*(E_3)$. Since $E_3 \in \hat{\mathcal{P}}_r$ and $\mathrm{Ext}_{\hat{\mathcal{P}}_r}$ is a subfunctor of Ext, $r^t E = r^t_*(E) = (\psi_* \circ \phi_*)(E) \in \hat{\mathcal{P}}_r$. Then there is a nonnegative integer p such that $r^{p+t} E \in \mathcal{P}$, hence $E \in \hat{\mathcal{P}}_r$. □

We have seen the direct sum of two proper classes in the previous chapter. We say that the sum $\sum_{i\in I}\mathcal{L}_i$ of proper classes is direct if $\mathcal{L}_j\bigcap\sum_{\substack{i\neq j\\ i\in I}}\mathcal{L}_i = \mathcal{S}plit$ for every $j\in I$. Since we are dealing with finite number of operations in the last equality, we can replace this equality by $\mathcal{L}_j\bigcap\sum_{\substack{i\neq j\\ i\in F}}\mathcal{L}_i = \mathcal{S}plit$ for every finite index set $F\subseteq I$ with $j\notin F$.

Over the ring of integers, for every group A we have a decomposition of $T(A)$ into primary parts as $T(A) = \bigoplus_{p \text{ prime}} T_p(A)$. The following result holds when $R=\mathbb{Z}$, and it gives a direct sum decomposition for the class $\mathcal{S}plit$ of splitting short exact sequences in terms of proper classes.

Proposition 6.4 *Over the ring of integers,*

$$\hat{\mathcal{S}plit} = \bigoplus_{p} \hat{\mathcal{S}plit}_p, \tag{6.10}$$

where p ranges over all prime numbers.

The following theorem gives the relation between the class $\hat{\mathcal{P}}$ and the operation $+$. A similar result holds for the class $\hat{\mathcal{P}}_r$.

Theorem 6.3 ((Alizade et al. 2004), Theorem 3.2) *Let $\mathcal{P}$ be a proper class. Then*

$$\mathcal{P} + \hat{\mathcal{S}plit} = \hat{\mathcal{P}}. \tag{6.11}$$

Theorem 6.4 *Let $\mathcal{P}$ be a proper class. Then for every $0\neq r\in R$,*

$$\mathcal{P} + \hat{\mathcal{S}plit}_r = \hat{\mathcal{P}}_r. \tag{6.12}$$

The proof of this theorem uses similar ideas used in proving the previous one.

Proof Let $E\in\mathcal{P}$, then $E = 1\cdot E = r^0\cdot E\in\hat{\mathcal{P}}_r$, therefore $\mathcal{P}\subseteq\hat{\mathcal{P}}_r$.Since $\mathcal{S}plit\subseteq\mathcal{P}$ for every proper class, $\hat{\mathcal{S}plit}\subseteq\hat{\mathcal{P}}$. Combining these inclusions we obtain $\mathcal{P}+\hat{\mathcal{S}plit}_r\subseteq\hat{\mathcal{P}}_r$.

Let us write $\mathcal{P}+\hat{\mathcal{S}plit}_r = \mathcal{L}$. We will show that $\hat{\mathcal{P}}_r\subseteq\mathcal{L}$.

Let $E: 0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0 \in\hat{\mathcal{P}}_r$. Then there is a nonnegative integer t such that $r^tE\in\mathcal{P}$. Let us write the endomorphism of multiplication by r^t on A as

$r^t = \alpha \circ \beta$, where $\beta : A \longrightarrow r^tA$ is the standard epimorphism and $\alpha : r^tA \longrightarrow A$ is the inclusion.

Using the homomorphisms α and β we obtain the following commutative diagrams with exact rows and columns (note that $A[r^t] = \{a \in A|\, r^ta = 0\}$):

$$\begin{CD}
@. 0 @. 0 \\
@. @VVV @VVV \\
@. A[r^t] @= A[r^t] \\
@. @VVV @VVV \\
E : 0 @>>> A @>>> B @>\mu>> C @>>> 0 \\
@. @V\beta VV @V\theta VV @| \\
E' : 0 @>>> r^tA @>\delta>> B' @>\sigma>> C @>>> 0 \\
@. @VVV @VVV \\
@. 0 @. 0
\end{CD} \tag{6.13}$$

$$\begin{CD}
@. 0 @. 0 \\
@. @VVV @VVV \\
E' : 0 @>>> r^tA @>\delta>> B' @>\sigma>> C @>>> 0 \\
@. @V\alpha VV @V\mu VV @| \\
r^tE : 0 @>>> A @>\gamma>> B'' @>>> C @>>> 0 \\
@. @VVV @VVV \\
@. A/r^tA @= A/r^tA \\
@. @VVV @VVV \\
@. 0 @. 0
\end{CD} \tag{6.14}$$

A/r^tA is annihilated by r^t, therefore it is $\hat{\mathcal{S}plit}_r$-coprojective, and α is an $\mathcal{L}$-monomorphism since $\hat{\mathcal{S}plit}_r \subseteq \mathcal{L}$. Since $r^tE \in \mathcal{P} \subseteq \mathcal{L}$, γ is an $\mathcal{L}$-monomorphism. Since $\mathcal{L}$ is a proper class, $\mu \circ \delta = \gamma \circ \alpha$ is an $\mathcal{L}$-monomorphism, and since $\mathrm{Ext}_{\mathcal{L}}$ is a subfunctor of Ext, δ is an $\mathcal{L}$-monomorphism, hence $E' \in \mathcal{L}$.

$A[r^t]$ is annihilated by r^t, therefore it is $\hat{\mathcal{S}plit}_r$-coinjective, and θ is an $\mathcal{L}$-epimorphism since $\hat{\mathcal{S}plit}_r \subseteq \mathcal{L}$. Then $\mu = \sigma \circ \theta$ is an $\mathcal{L}$-epimorphism since $\mathcal{L}$ is a proper class. Hence $E \in \mathcal{L}$ and $\hat{\mathcal{P}}_r \subseteq \mathcal{L}$. □

In order to show the relation of the classes defined so far with the proper classes related to supplements, let us introduce Ivanov classes and give some results over the ring of integers (see (Ivanov 1978), (Alizade 1986)).

Definition 6.2 *For the classes $\mathcal{M}$ and $\mathcal{J}$ of modules, the class $i(\mathcal{M}, \mathcal{J})$ of short exact sequences is the least proper class for which every module from $\mathcal{M}$ is coprojective and every module from $\mathcal{J}$ is coinjective.*

Let $\mathcal{M}$ and $\mathcal{J}$ be classes of R-modules. The class $i(\mathcal{M}, \mathcal{J})$ is described in (Ivanov 1978) when $\mathcal{M}$ and $\mathcal{J}$ are closed under extensions with $\mathcal{M}$ closed under submodules, $\mathcal{J}$ closed under factor modules and satisfy the properties

(i) If $A \leq B$, $A \in \mathcal{J}$ and $B \in \mathcal{M}$, then $B/A \in \mathcal{M}$,

(ii) If $A \leq B$, $B \in \mathcal{J}$ and $B/A \in \mathcal{M}$, then $A \in \mathcal{J}$.

In (Alizade 1986) these conditions are called *Ivanov conditions*, and there is given a result related to the operation * defined in the previous chapter.

Proposition 6.5 ((Alizade 1986), Lemma 5) *Let R be hereditary and the classes $\mathcal{M}$ and $\mathcal{J}$ of R-modules satisfy the Ivanov conditions. Then*

$$i(\mathcal{M}, \mathcal{J}) = \bar{k}(\mathcal{M}) * \underline{k}(\mathcal{J}). \tag{6.15}$$

Theorem 6.5 ((Alizade 1986), Theorem 2) *For all classes $\mathcal{M}$ and $\mathcal{J}$ of abelian groups*

$$\widehat{i(\mathcal{M}, \mathcal{J})} = i(\mathcal{M} \cup \mathcal{B}, \mathcal{J} \cup \mathcal{B}), \tag{6.16}$$

where $\mathcal{B}$ denotes the class of bounded abelian groups.

This result is also true for modules over an integral domain since its proof can easily be modified. It is also possible to use the proof of this result and obtain another closure of the class $i(\mathcal{M}, \mathcal{J})$.

Theorem 6.6 *Let R be an integral domain, $\mathcal{M}$ and $\mathcal{J}$ classes of R-modules. Then for every* $0 \neq r \in R$*,*

$$i\widehat{(\mathcal{M},\mathcal{J})}_r = i(\mathcal{M} \cup \mathcal{B}_r, \mathcal{J} \cup \mathcal{B}_r), \tag{6.17}$$

where $\mathcal{B}_r$ denotes the class of R-modules bounded by a power of r.

Proof Let $i(\mathcal{M} \cup \mathcal{B}_r, \mathcal{J} \cup \mathcal{B}_r) = \mathcal{L}$. It is clear that $i(\mathcal{M},\mathcal{J}) \subseteq \mathcal{L}$ and $\mathcal{L} \subseteq i\widehat{(\mathcal{M},\mathcal{J})}_r$ since every module from $\mathcal{B}_r$ is $i\widehat{(\mathcal{M},\mathcal{J})}_r$-coinjective and $i\widehat{(\mathcal{M},\mathcal{J})}_r$-coprojective. To show that $i\widehat{(\mathcal{M},\mathcal{J})}_r \subseteq \mathcal{L}$, let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in i\widehat{(\mathcal{M},\mathcal{J})}_r$. Then there is a nonnegative integer t such that $r^t E \in i(\mathcal{M},\mathcal{J})$. We can write the homomorphism $r^t : C \longrightarrow C$ as $r^t = \alpha \circ \beta$, where $\alpha : r^t C \longrightarrow C$ is the inclusion and $\beta : C \longrightarrow r^t C$ is the standard epimorphism. Applying these homomorphisms we obtain the following commutative diagrams with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
E' : 0 & \longrightarrow & A & \xrightarrow{\delta} & B' & \xrightarrow{\sigma} & r^t C & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\alpha} & & \\
E : 0 & \longrightarrow & A & \xrightarrow{\gamma} & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C/r^t C & = & C/r^t C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{6.18}$$

$$
\begin{CD}
@. @. 0 @. 0 @. \\
@. @. @VVV @VVV @. \\
@. @. C/r^tC @= C/r^tC @. \\
@. @. @VVV @VVV @. \\
r^{t*}(E):0 @>>> A @>>> B'' @>\xi>> C @>>> 0 \\
@. @| @V\eta VV @V\beta VV @. \\
E':0 @>>> A @>\delta>> B' @>\sigma>> r^tC @>>> 0 \\
@. @. @VVV @VVV @. \\
@. @. 0 @. 0 @.
\end{CD}
\tag{6.19}
$$

In the diagram (6.19), β is an $\mathcal{L}$-epimorphism since $C[r^t] \in \mathcal{B}_r$ and ξ is also an $\mathcal{L}$-epimorphism. Then $\sigma \circ \eta = \beta \circ \xi$ is an $\mathcal{L}$-epimorphism since $\mathcal{L}$ is a proper class and σ is an $\mathcal{L}$-epimorphism since $\mathrm{Ext}_{\mathcal{L}}$ is a subfunctor of Ext. Therefore, $E' \in \mathcal{L}$ and δ is an $\mathcal{L}$-monomorphism.

In the diagram (6.18), θ is an $\mathcal{L}$-monomorphism since $C/r^tC \in \mathcal{B}_r$. Since $\mathcal{L}$ is a proper class, $\alpha = \theta \circ \sigma$ is an $\mathcal{L}$-monomorphism and $E \in \mathcal{L}$. □

Corollary 6.2 *Let R be a Noetherian integral domain of Krull dimension 1, and* $0 \neq r \in R$. *Then*

$$
\begin{aligned}
\widehat{\mathcal{S}\mathcal{B}} &= i(\mathcal{B}, \mathcal{B}) \\
\widehat{\mathcal{S}\mathcal{B}_r} &= i(\mathcal{B}_r, \mathcal{B}),
\end{aligned}
\tag{6.20}
$$

where $\mathcal{B}$ is the class of bounded R-modules and $\mathcal{B}_r$ is the class of R-modules bounded by a power of r.

Proof By Proposition 4.3, $\mathcal{S}\mathcal{B} = \underline{k}(\mathcal{B}) = i(\emptyset, \mathcal{B})$. Theorem 6.5 and Theorem 6.6 complete the proof. □

Corollary 6.3 *Let R be a Dedekind domain and* $0 \neq r \in R$. *Then*

$$
\begin{aligned}
\overline{\widehat{\mathcal{W}\mathcal{S}}} &= i(\mathcal{B}, \mathcal{S}m) \\
\overline{\widehat{\mathcal{W}\mathcal{S}_r}} &= i(\mathcal{B}_r, \mathcal{S}m),
\end{aligned}
\tag{6.21}
$$

where $\mathcal{B}$ is the class of bounded R-modules and $\mathcal{B}_r$ is the class of R-modules bounded by a power of r.

Proof By Proposition 4.4, $\overline{\mathcal{WS}} = \underline{k}(\mathcal{S}m) = i(\emptyset, \mathcal{S}m)$. By Theorem 6.5 and Theorem 6.6, we have $\hat{\overline{\mathcal{WS}}} = i(\mathcal{B}, \mathcal{S}m \cup \mathcal{B})$ and $\hat{\overline{\mathcal{WS}}}_r = i(\mathcal{B}_r, \mathcal{S}m \cup \mathcal{B}_r)$. Since over a Dedekind domain R, $\mathcal{B}_r \subseteq \mathcal{B} \subseteq \mathcal{S}m$ for every $0 \neq r \in R$, the result follows. □

Until now, we have dealt with the classes that include the given class $\mathcal{P}$. In (Alizade et al. 2004), the existence of a class that is contained in the given class $\mathcal{P}$ is given using the class $\hat{\mathcal{P}}$ over a principal ideal domain. It is also possible to obtain a similar result using the class $\hat{\mathcal{P}}_r$.

Before we give the results, let us remind the following definition. Note also that for a class $\mathcal{P}$, by $r\mathcal{P}$ we mean the class $r\mathcal{P} = \{E : E = rE' \text{ for some } E' \in \mathcal{P}\}$ defined in (Alizade et al. 2004).

Definition 6.3 *Let R be a principal deal domain. An element r of R is said to be divisible by $s \in R$ (denoted as $s|r$) if $r = r_i s$ for some $r_i \in R$.*

Theorem 6.7 ((Alizade et al. 2004), Theorem 4.2) *Let R be a principal ideal domain. A short exact sequence* $E : 0 \longrightarrow A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \longrightarrow 0$ *is divisible by a nonzero element $r \in R$ if and only if $\alpha(sA) = \alpha(A) \cap sB$ for every $s \in R$ dividing r.*

This result is used to prove that for a proper class $\mathcal{P}$, the class $r\mathcal{P}$ is proper under an extra condition.

Theorem 6.8 ((Alizade et al. 2004), Theorem 4.4) *Let R be a principal ideal domain. Then for every proper class $\mathcal{P}$ with $\hat{\mathcal{P}} = \mathcal{P}$, the class $k\mathcal{P}$ is proper for every $k \in R$, .*

With a slight change on the condition given, we obtain the following result for the class $\hat{\mathcal{P}}_r$.

Theorem 6.9 *Let R be a principal ideal domain. Then for every proper class $\mathcal{P}$ with $\hat{\mathcal{P}}_r = \mathcal{P}$, the class $r^t\mathcal{P}$ is proper for every $0 \neq r \in R$ and every nonnegative integer t.*

Proof Let $r^tE, r^tE' \in \text{Ext}_{r^t\mathcal{P}}(C, A)$, $E, E' \in \text{Ext}_{\mathcal{P}}(C, A)$ and $f : A \longrightarrow A'$, $g : C' \longrightarrow C$ be homomorphisms.

Since $\text{Ext}_{\mathcal{P}}$ is a subfunctor of Ext, $f_*(g^*(E)) \in \text{Ext}_{\mathcal{P}}(C', A')$, and then $f_*(g^*(r^tE)) = r^tf_*(g^*(E)) \in \text{Ext}_{r^t\mathcal{P}}$.

Since $\text{Ext}_{\mathcal{P}}(C, A)$ is a subgroup of $\text{Ext}(C, A)$, $E - E' \in \text{Ext}_{\mathcal{P}}(C, A)$. Then $r^tE - r^tE' = r^t(E - E') \in \text{Ext}_{r^t\mathcal{P}}(C, A)$.

These arguments show that $\text{Ext}_{r^t\mathcal{P}}$ gives an e-functor. Using (Nunke 1963, Theorem 1.1), to show that $r^t\mathcal{P}$ is a proper class, it is enough to show that the composition of two $r^t\mathcal{P}$-monomorphisms is a $r^t\mathcal{P}$-monomorphism.

Let $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ be $r^t\mathcal{P}$-monomorphisms. Let $s \in R$ and $s|r^t$. The sequences $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow X \longrightarrow 0$ and $0 \longrightarrow B \xrightarrow{\beta} C \longrightarrow Y \longrightarrow 0$ are divisible by r^t by definition of $r^t\mathcal{P}$. By Theorem 4.2, $\alpha(sA) = \alpha(A) \cap sB$ and $\beta(sB) = \beta(B) \cap sC$. Then $(\beta \circ \alpha)(sA) = \beta(\alpha(sA)) = \beta(\alpha(A) \cap B) = (\beta \circ \alpha)(A) \cap \beta(sB) = (\beta \circ \alpha)(A) \cap \beta(B) \cap sC = (\beta \circ \alpha)(A) \cap sC$ since β is a monomorphism. By Theorem 4.2, $E : 0 \longrightarrow A \xrightarrow{\beta\circ\alpha} C \longrightarrow Z \longrightarrow 0$ is divisible by r^t, that is $E = r^tE'$ for some $E' \in \text{Ext}(C, A)$.

The monomorphisms α and β are also $\mathcal{P}$-monomorphisms, then the composition $\beta \circ \alpha$ is a $\mathcal{P}$-monomorphism, therefore $E \in \mathcal{P}$. Since $\mathcal{P} = \hat{\mathcal{P}}_r$ and $r^tE' \in \mathcal{P}$, we have $E' \in \hat{\mathcal{P}}_r = \mathcal{P}$. Hence $E = r^tE' \in r^t\mathcal{P}$. □

6.2. The Class $\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$

In the previous section, the classes $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_r$ for the given class $\mathcal{P}$ of short exact sequences are defined over an integral domain using the homomorphisms multiplication by elements of R since multiplication by elements of R give homomorphisms when R is an integral domain. We cannot use elements of R to obtain homomorphisms in order to find a closure for the class $\mathcal{P}$ when R is an associative ring with an identity element since the multiplication by an element of R does not give a homomorphism when R is not commutative. Therefore, we turn our attention to the classes of R-module homomorphisms. Throughout this section, we will study modules over an associative ring R with an identity element.

Definition 6.4 *Let $\mathcal{F}$ and $\mathcal{G}$ be families of homomorphisms of R-modules. We say that $\mathcal{F}$ is closed under pushout diagrams if for all homomorphisms $f : A \longrightarrow B$, $f \in \mathcal{F}$ and $\alpha : A \longrightarrow A'$, we have $f' \in \mathcal{F}$ in the pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ A' & \xrightarrow[f']{} & P \end{array} \quad . \tag{6.22}$$

We say that $\mathcal{G}$ is closed under pullback diagrams if for all homomorphisms $g : C \longrightarrow D$, $g \in \mathcal{G}$ and $\beta : D' \longrightarrow D$, we have $g' \in \mathcal{G}$ in the pullback diagram

$$\begin{array}{ccc} C & \xrightarrow{\;g\;} & D \\ \uparrow & & \uparrow{\scriptstyle \beta} \\ P' & \xrightarrow[\;g'\;]{} & D' \end{array} . \tag{6.23}$$

Lemma 6.1 *Let $\mathcal{F}$ and $\mathcal{G}$ be families of homomorphisms of R-modules. Let $\mathcal{F}$ be closed under pushout diagrams and $\mathcal{G}$ closed under pullback diagrams. Then the following hold.*

(i) If $f : A \longrightarrow A_1$ belongs to $\mathcal{F}$, then the homomorphisms $f \oplus 1_A : A \oplus A \longrightarrow A_1 \oplus A$ and $1_A \oplus f : A \oplus A \longrightarrow A \oplus A_1$ are also in $\mathcal{F}$.

(ii) If $f : A \longrightarrow B$ belongs to $\mathcal{F}$, then the inclusion $i : \operatorname{Im} f \longrightarrow B$ (or the monomorphism $\bar{i} : A/\operatorname{Ker} f \longrightarrow B$ induced by f) is also in $\mathcal{F}$.

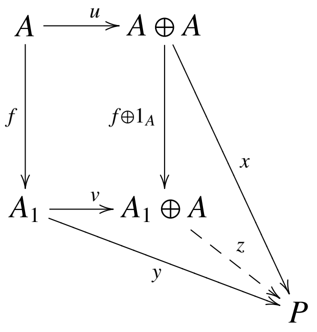
(iii) If $g : C_1 \longrightarrow C$ belongs to $\mathcal{G}$, then the homomorphisms $g \oplus 1_C : C_1 \oplus C \longrightarrow C \oplus C$ and $1_C \oplus g : C \oplus C_1 \longrightarrow C \oplus C$ are also in $\mathcal{G}$.

(iv) If $g : C \longrightarrow D$ belongs to $\mathcal{G}$, then the canonical epimorphism $\pi : C \longrightarrow C/\operatorname{Ker} g$ induced by g (or the epimorphism $\bar{\pi} : C \longrightarrow \operatorname{Im} g$ induced by g) is also in $\mathcal{G}$.

Proof (i) The diagram

$$\begin{array}{ccc} A & \xrightarrow{\;u\;} & A \oplus A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\oplus 1_A} \\ A_1 & \xrightarrow[\;v\;]{} & A_1 \oplus A \end{array} \tag{6.24}$$

with $u(a) = (a, 0)$ and $v(a_1) = (a_1, 0)$ for $a, a' \in A$, $a_1 \in A_1$ is commutative. Suppose that there are homomorphisms $x : A \oplus A \longrightarrow P$ and $y : A_1 \longrightarrow P$ for some R-module P such that $x \circ u = y \circ f$. Define the homomorphism $z : A_1 \oplus A \longrightarrow P$ by $z((a_1, a)) = y(a_1) + x((0, a))$ for $(a_1, a) \in A_1 \oplus A$. Then in the diagram

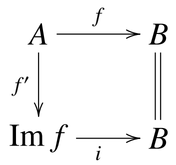


(6.25)

we have $x = z \circ (f \oplus 1_A)$ and $y = z \circ v$. Uniqueness of z follows from the commutativity of the diagram (6.25), therefore the diagram (6.24) is a pushout diagram. Hence $f \oplus 1_A \in \mathcal{F}$ by given condition on $\mathcal{F}$.

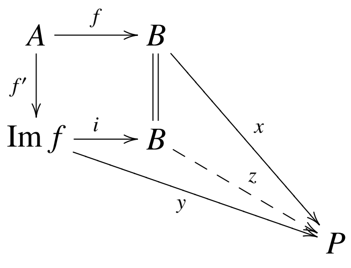
Similarly, one can show that $1_A \oplus f \in \mathcal{F}$.

(*ii*) The diagram



(6.26)

with $f'(a) = f(a)$ and i being the inclusion is commutative. Suppose that there are homomorphisms $x : B \longrightarrow P$ and $y : \operatorname{Im} f \longrightarrow P$ for some R-module P such that $x \circ f = y \circ f'$. Define the homomorphism $z : B \longrightarrow P$ by $z(b) = x(b)$ for $b \in B$. Then in the diagram



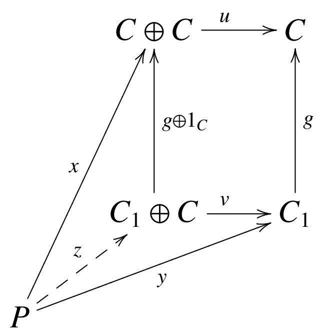
(6.27)

we have $x = z \circ 1_B$ and $y = z \circ f'$. Uniqueness of z follows from the commutativity of the diagram (6.27), therefore the diagram (6.26) is a pushout diagram. Hence $i : \operatorname{Im} f \longrightarrow B$ is in $\mathcal{F}$ by given condition on $\mathcal{F}$.

(iii) The diagram

$$\begin{array}{ccc} C \oplus C & \xrightarrow{u} & C \\ {\scriptstyle g\oplus 1_C}\uparrow & & \uparrow{\scriptstyle g} \\ C_1 \oplus C & \xrightarrow[v]{} & C_1 \end{array} \tag{6.28}$$

with $u((c, d)) = c$ and $v((c_1, e)) = c_1$ for $c, d, e \in C$, $c_1 \in C_1$ is commutative. Suppose that there are homomorphisms $x : P \longrightarrow C \oplus C$ and $y : P \longrightarrow C_1$ for some R-module P such that $u \circ x = g \circ y$. Let $u' : C \oplus C \longrightarrow C$, $u'((c, d)) = d$ for $c, d \in C$, be the standard epimorphism and define the homomorphism $z : P \longrightarrow C_1 \oplus C$ by $z(p) = (y(p), (u' \circ x)(p))$ for $p \in P$. Then in the diagram



(6.29)

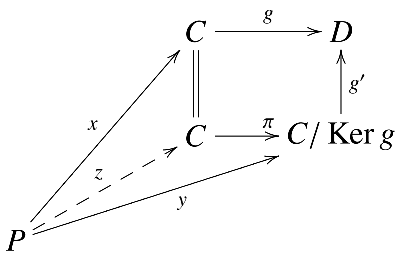
we have $x = (g \oplus 1_C) \circ z$ and $y = v \circ z$. Uniqueness of z follows from the commutativity of the diagram (6.29), therefore the diagram (6.28) is a pullback diagram. Hence $(g \oplus 1_C) \in \mathcal{G}$ by given condition on $\mathcal{G}$.

Similarly, one can show that $(1_C \oplus g) \in \mathcal{G}$.

(iv) The diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ \| & & \uparrow{\scriptstyle g'} \\ C & \xrightarrow[\pi]{} & C/\operatorname{Ker} g \end{array} \tag{6.30}$$

with $g'(c + \operatorname{Ker} g) = g(c)$ and π being the canonical epimorphism is commutative. Suppose that there are homomorphisms $x : P \longrightarrow C$ and $y : P \longrightarrow C/\operatorname{Ker} g$ for some R-module P such that $g \circ x = g' \circ y$. Define the homomorphism $z : P \longrightarrow C$ by $z(p) = x(p)$ for $p \in P$. Then in the diagram



(6.31)

we have $x = 1_C \circ z$ and $y = \pi \circ z$. Uniqueness of z follows from the commutativity of the diagram (6.31), therefore the diagram (6.30) is a pullback diagram. Hence $\pi : C \longrightarrow C/\operatorname{Ker} g$ is in $\mathcal{G}$ by given condition on $\mathcal{G}$. □

Definition 6.5 *Let R be an associative ring with identity. Let $\mathcal{F}$ and $\mathcal{G}$ be nonempty families of homomorphisms of R-modules, and $\mathcal{P}$ a class of short exact sequences. We say that the pair $(\mathcal{F}, \mathcal{G})$ is compatible for the class $\mathcal{P}$ if for every short exact sequence E, there is $f \in \mathcal{F}$ such that $f_*(E) \in \mathcal{P}$ if and only if there is $g \in \mathcal{G}$ such that $g^*(E) \in \mathcal{P}$ with one (or both) of the following conditions satisfied:*

(i) $\mathcal{F}$ is closed under compositions and pushout diagrams,

(ii) $\mathcal{G}$ is closed under compositions and pullback diagrams.

For a class $\mathcal{P}$ of short exact sequences and a compatible pair $(\mathcal{F}, \mathcal{G})$ for $\mathcal{P}$, we define the class $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ as

$$\begin{aligned} \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}} &= \{E \mid f_*(E) \in \mathcal{P} \text{ for some } f \in \mathcal{F}\} \\ &= \{E \mid g^*(E) \in \mathcal{P} \text{ for some } g \in \mathcal{G}\}. \end{aligned} \tag{6.32}$$

Theorem 6.10 *For every proper class $\mathcal{P}$ of short exact sequences and every compatible pair $(\mathcal{F}, \mathcal{G})$ for $\mathcal{P}$, the class $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is proper.*

Proof Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ with $f_{1*}(E)$, $g_1^*(E) \in \mathcal{P}$ for $f_1 : A \longrightarrow A_1$, $f_1 \in \mathcal{F}$ and $g_1 : C_1 \longrightarrow C$, $g_1 \in \mathcal{G}$.

If $f : A \longrightarrow A'$, then $g_1^*(f_*(E)) = (g_1^* \circ f_*)(E) = (f_* \circ g_1^*)(E) = f_*(g_1^*(E)) \in \mathcal{P}$ since $\operatorname{Ext}_{\mathcal{P}}$ is a subfunctor of Ext. Then $f_*(E) \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ since $g_1 \in \mathcal{G}$.

If $g : C' \longrightarrow C$, then $f_{1*}(g^*(E)) = (f_{1*} \circ g^*)(E) = (g^* \circ f_{1*})(E) = g^*(f_{1*}(E)) \in \mathcal{P}$ since $\operatorname{Ext}_{\mathcal{P}}$ is a subfunctor of Ext. Then $g^*(E) \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ since $f_1 \in \mathcal{F}$.

These arguments show that $\mathrm{Ext}_{\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}}$ is a subfunctor of Ext.

At this point, we separate the proof into two parts because of the conditions on $\mathcal{F}$ and $\mathcal{G}$.

In the first case, let $\mathcal{F}$ be closed under compositions and pushout diagrams. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, $E' : 0 \longrightarrow A \longrightarrow B' \longrightarrow C \longrightarrow 0 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ with $f_{1*}(E) \in \mathcal{P}$ for $f_1 : A \longrightarrow A_1$, $f_1 \in \mathcal{F}$ and $f_{2*}(E') \in \mathcal{P}$ for $f_2 : A \longrightarrow A_2$, $f_2 \in \mathcal{F}$, say $f_{1*}(E) : 0 \longrightarrow A_1 \longrightarrow B_1 \longrightarrow C \longrightarrow 0$ and $f_{2*}(E') : 0 \longrightarrow A_2 \longrightarrow B_2 \longrightarrow C \longrightarrow 0$. We have the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \oplus A & \longrightarrow & B \oplus B' & \longrightarrow & C \oplus C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1 \oplus 1_A} & & \downarrow & & \| & & \\ 0 & \longrightarrow & A_1 \oplus A & \longrightarrow & B_1 \oplus B' & \longrightarrow & C \oplus C & \longrightarrow & 0 \end{array} \tag{6.33}$$

and

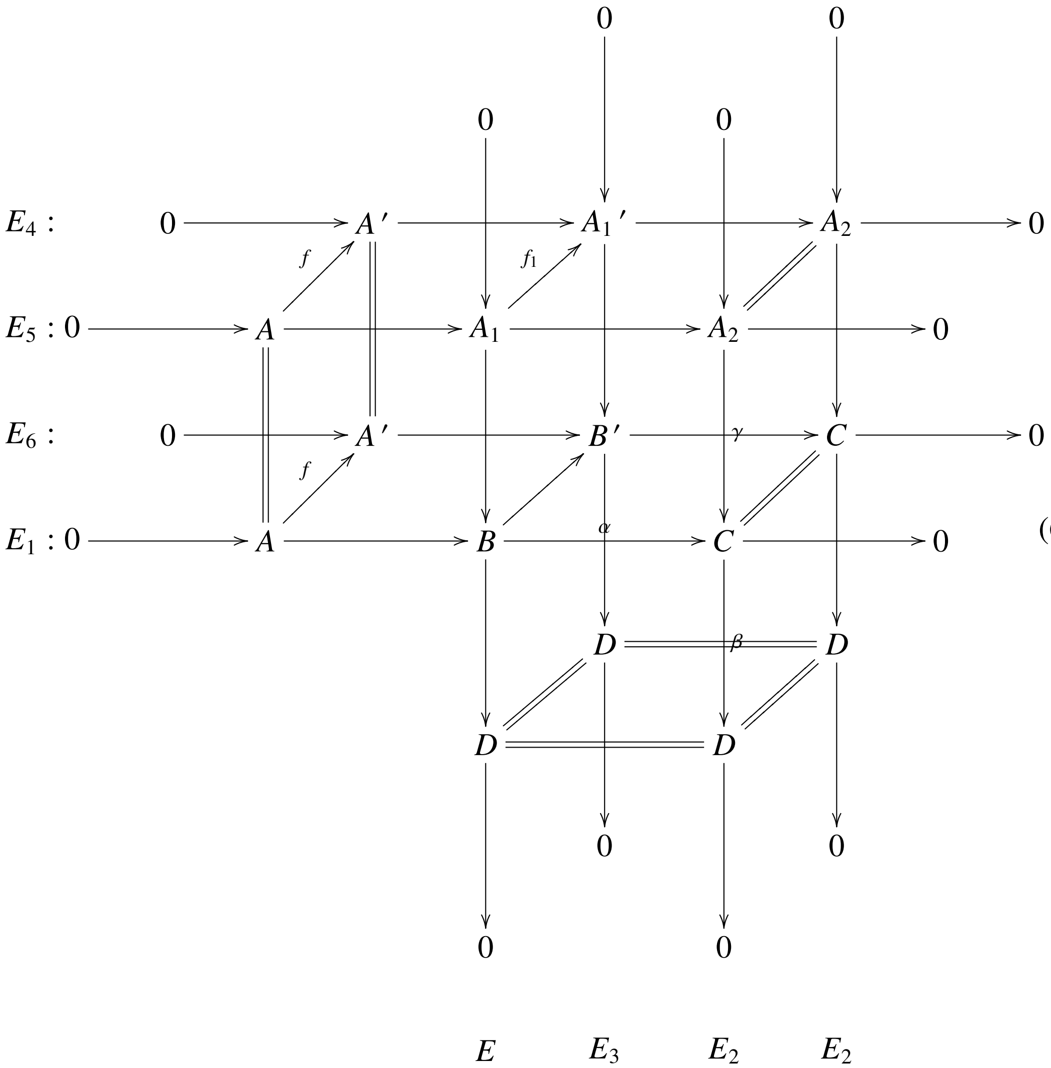
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 \oplus A & \longrightarrow & B_1 \oplus B' & \longrightarrow & C \oplus C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_{A_1} \oplus f_2} & & \downarrow & & \| & & \\ 0 & \longrightarrow & A_1 \oplus A_2 & \longrightarrow & B_1 \oplus B_2 & \longrightarrow & C \oplus C & \longrightarrow & 0 \end{array} \tag{6.34}$$

with exact rows.

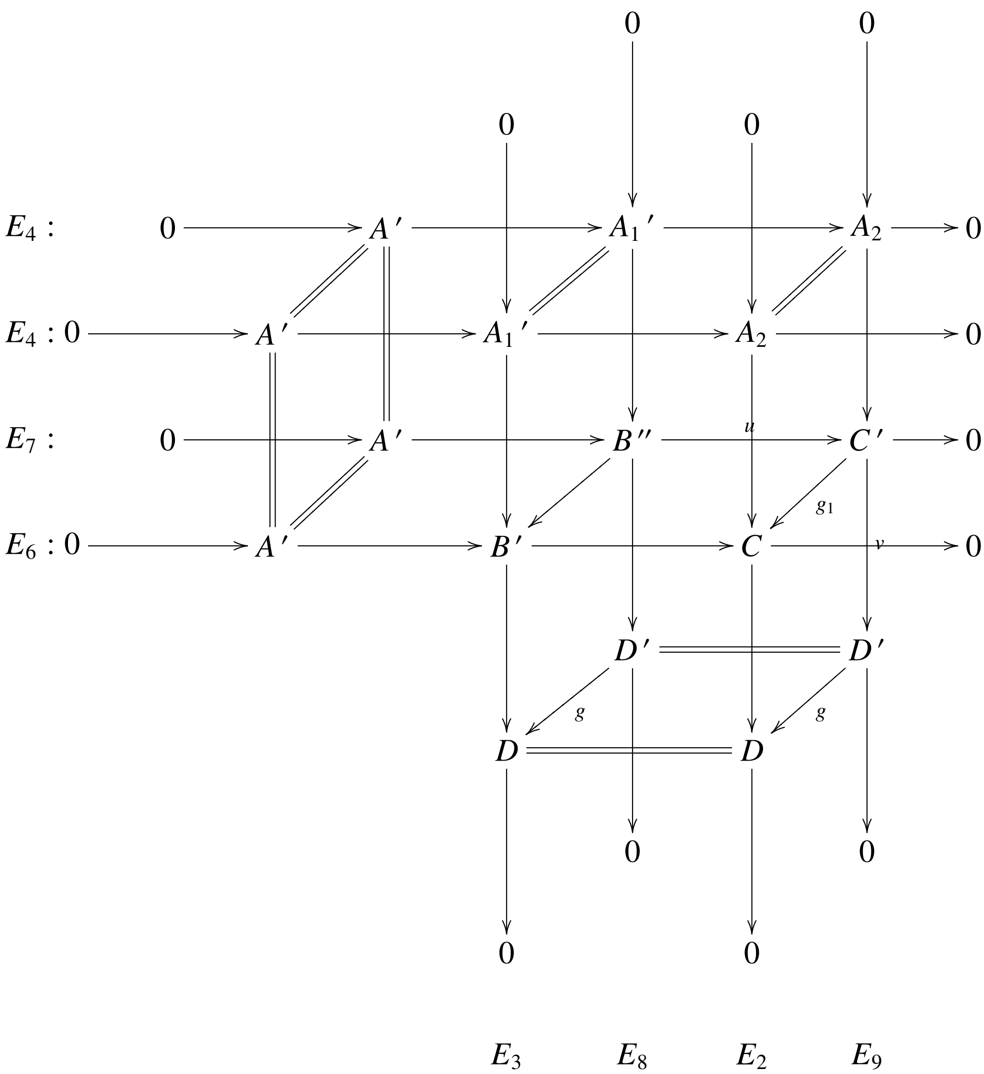
By Lemma 6.1, $f_1 \oplus 1_A$ and $1_{A_1} \oplus f_2$ are in $\mathcal{F}$. Since $\mathcal{F}$ is closed under compositions, we have $(1_{A_1} \oplus f_2) \circ (f_1 \oplus 1_A) \in \mathcal{F}$. Since $\mathcal{P}$ is a proper class, $[(1_{A_1} \oplus f_2) \circ (f_1 \oplus 1_A)]_*(E \oplus E') \in \mathcal{P}$ with $(1_{A_1} \oplus f_2) \circ (f_1 \oplus 1_A) \in \mathcal{F}$. Therefore, $(E \oplus E') \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$.

These arguments show that $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is an e-functor. Using (Nunke 1963, Theorem 1.1), in order to show that $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is a proper class, it is enough to show that the composition of two $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$-epimorphisms is a $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$-epimorphism.

Let $E_1 : 0 \longrightarrow A \longrightarrow B \xrightarrow{\alpha} C \longrightarrow 0$ and $E_2 : 0 \longrightarrow A_2 \xrightarrow{\gamma} C \xrightarrow{\beta} D \longrightarrow 0 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. Then there are homomorphisms $f : A \longrightarrow A'$, $f \in \mathcal{F}$ and $g : D' \longrightarrow D$, $g \in \mathcal{G}$ such that $f_*(E_1) \in \mathcal{P}$ and $g^*(E_2) \in \mathcal{P}$. We have the following commutative diagrams with exact rows and columns:



(6.35)



(6.36)

In the diagram (6.35), $E_6 = f_*(E_1) \in \mathcal{P}$, $E_4 = f_*(E_5) = f_*(\gamma^*(E_1)) = \gamma^*(f_*(E_1))$ since $\mathrm{Ext}_{\mathcal{P}}$ is a subfunctor of Ext, and $f_1 \in \mathcal{F}$ since f_1 is constructed using pushout.

In the diagram (6.36), $E_9 = g^*(E_2)$, $E_7 = g_1^*(E_6) \in \mathcal{P}$. Then $u : B'' \longrightarrow C'$ and $v : C' \longrightarrow D'$ are $\mathcal{P}$-epimorphisms. Since $\mathcal{P}$ is a proper class, $v \circ u$ is a $\mathcal{P}$-epimorphism and $E_8 \in \mathcal{P}$. $g^*(E_3) = E_8$ and $g \in \mathcal{G}$ implies $E_3 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$, so there is $f_2 : A_1{}' \longrightarrow A_1{}''$, $f_2 \in \mathcal{F}$ for some R-module $A_1{}''$ such that $f_{2*}(E_3) \in \mathcal{P}$.

Then $(f_2 \circ f_1)_*(E) = f_{2*}(f_{1*}(E)) = f_{2*}(E_3) \in \mathcal{P}$ and $(f_2 \circ f_1) \in \mathcal{F}$ since $f_1, f_2 \in \mathcal{F}$, and $\mathcal{F}$ is closed under compositions. Hence $E \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$.

For the second case, let $\mathcal{G}$ be closed under compositions and pullback diagrams. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, $E' : 0 \longrightarrow A \longrightarrow B' \longrightarrow C \longrightarrow 0 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ with $g_1^*(E) \in \mathcal{P}$ for $g_1 : C_1 \longrightarrow C$, $g_1 \in \mathcal{G}$ and $g_2^*(E') \in \mathcal{P}$ for $g_2 : C_2 \longrightarrow C$, $g_2 \in \mathcal{G}$, say $g_1^*(E) : 0 \longrightarrow A \longrightarrow B_1 \longrightarrow C_1 \longrightarrow 0$ and

$g_2^*(E') : 0 \longrightarrow A \longrightarrow B_2 \longrightarrow C_2 \longrightarrow 0$. We have the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \oplus A & \longrightarrow & B \oplus B' & \longrightarrow & C \oplus C & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow {\scriptstyle g_1 \oplus 1_C} & & \\ 0 & \longrightarrow & A \oplus A & \longrightarrow & B_1 \oplus B' & \longrightarrow & C_1 \oplus C & \longrightarrow & 0 \end{array} \tag{6.37}$$

and

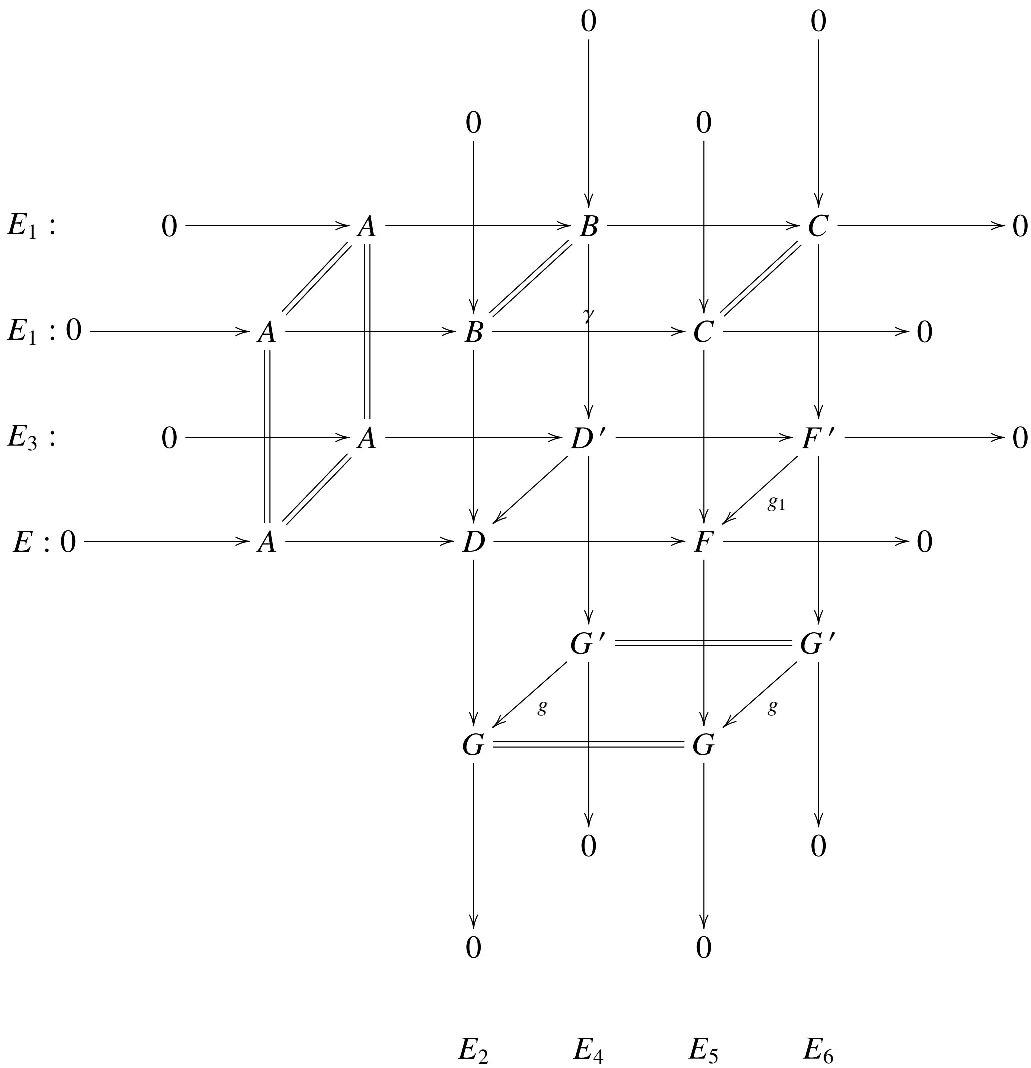
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \oplus A & \longrightarrow & B_1 \oplus B' & \longrightarrow & C_1 \oplus C & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow {\scriptstyle 1_{C_1} \oplus g_2} & & \\ 0 & \longrightarrow & A \oplus A & \longrightarrow & B_1 \oplus B_2 & \longrightarrow & C_1 \oplus C_2 & \longrightarrow & 0 \end{array} \tag{6.38}$$

with exact rows.

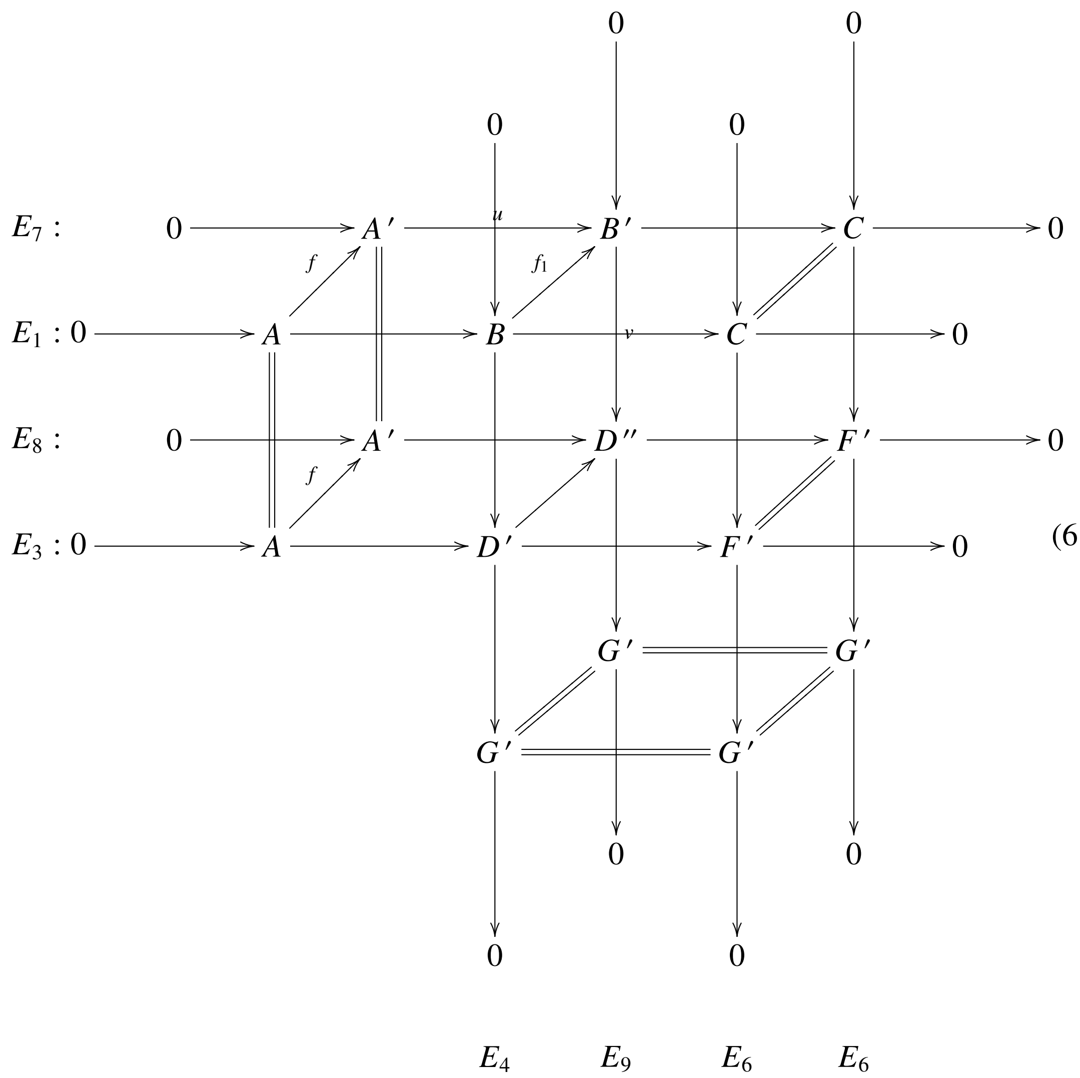
By Lemma 6.1, $g_1 \oplus 1_C$ and $1_{C_1} \oplus g_2$ are in $\mathcal{G}$. Since $\mathcal{G}$ is closed under compositions, $(g_1 \oplus 1_C) \circ (1_{C_1} \oplus g_2) \in \mathcal{G}$. Since $\mathcal{P}$ is a proper class, $[(g_1 \oplus 1_C) \circ (1_{C_1} \oplus g_2)]^*(E \oplus E') \in \mathcal{P}$ with $(g_1 \oplus 1_C) \circ (1_{C_1} \oplus g_2) \in \mathcal{G}$. Therefore, $(E \oplus E') \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$.

These arguments show that $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is an e-functor. Using (Nunke 1963, Theorem 1.1), in order to show that $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ is a proper class, it is enough to show that the composition of two $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$-monomorphisms is a $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$-monomorphism.

Let $E_1 : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\gamma} C \longrightarrow 0$ and $E_2 : 0 \longrightarrow B \xrightarrow{\beta} D \longrightarrow G \longrightarrow 0 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. Then there are homomorphisms $f : A \longrightarrow A'$, $f \in \mathcal{F}$ and $g : G' \longrightarrow G$, $g \in \mathcal{G}$ such that $f_*(E_1) \in \mathcal{P}$ and $g^*(E_2) \in \mathcal{P}$. We have the following commutative diagrams with exact rows and columns:



(6.39)



(6.40)

In the diagram (6.39), $E_4 = g^*(E_2) \in \mathcal{P}$, $E_6 = g^*(E_5) = g^*(\gamma_*(E_2)) = \gamma_*(g^*(E_2))$ since $\operatorname{Ext}_{\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}}$ is a subfunctor of Ext, and $g_1 \in \mathcal{G}$ since g_1 is constructed using pullback.

In the diagram (6.40), $E_7 = f_*(E_1)$, $E_9 = f_{1*}(E_4) \in \mathcal{P}$. Then $u : A' \longrightarrow B'$ and $v : B' \longrightarrow D''$ are $\mathcal{P}$-monomorphisms. Since $\mathcal{P}$ is a proper class, $v \circ u$ is a $\mathcal{P}$-monomorphism and $E_8 \in \mathcal{P}$. $f_*(E_3) = E_8$ and $f \in \mathcal{F}$ implies $E_3 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$, so there is $g_2 : C' \longrightarrow F'$, $g_2 \in \mathcal{G}$ for some R-module C' such that $g_2^*(E_3) \in \mathcal{P}$.

Then $(g_1 \circ g_2)^*(E) = g_2^*(g_1^*(E)) = g_2^*(E_3) \in \mathcal{P}$ and $(g_1 \circ g_2) \in \mathcal{G}$ since $g_1, g_2 \in \mathcal{G}$ and $\mathcal{G}$ is closed under compositions. Hence $E \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. □

A similar result obtained for the classes $\mathcal{P}$ and $\hat{\mathcal{P}}_r$ in the previous section also holds for the class $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ under some extra conditions on the classes $\mathcal{F}$ and $\mathcal{G}$.

Theorem 6.11 *Let $\mathcal{P}$ be a proper class and $(\mathcal{F}, \mathcal{G})$ a compatible pair for $\mathcal{P}$. Moreover, let the inclusions $i_f : \operatorname{Ker} f \longrightarrow A$ belong to $\mathcal{F}$ for all $f : A \longrightarrow A' \in \mathcal{F}$ for R-modules A, A' if $\mathcal{F}$ is closed under pushout diagrams, and the epimorphisms $\pi_g : C \longrightarrow C/\operatorname{Im} g$*

belong to $\mathcal{G}$ for all $g : C' \longrightarrow C \in \mathcal{G}$ for R-modules C, C' if $\mathcal{G}$ is closed under pullback diagrams. Then we have

$$\mathcal{P} + \hat{\mathcal{S}plit}_{\mathcal{F}}^{\mathcal{G}} = \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}. \tag{6.41}$$

Proof First we will show that $\mathcal{P} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \mathcal{P}$. For the first case, let $\mathcal{F}$ be closed under pushout diagrams. Since $\mathcal{F}$ is nonempty, there is a homomorphisms $h : X \longrightarrow X'$ in $\mathcal{F}$ for some R-modules X and X'. Then in the pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ {\scriptstyle 0}\downarrow & & \downarrow \\ A & \xrightarrow[h']{} & P \end{array} \tag{6.42}$$

we have $h' \in \mathcal{F}$ since $\mathcal{F}$ is closed under pushout diagrams. Since $\mathrm{Ext}_{\mathcal{P}}$ is a subfunctor of Ext, $h'_*(E) \in \mathcal{P}$, therefore $E \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ and $\mathcal{P} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$.
For the second case, let $\mathcal{G}$ be closed under pullback diagrams. Since $\mathcal{G}$ is nonempty, there is a homomorphisms $w : Y' \longrightarrow Y$ in $\mathcal{G}$ for some R-modules Y and Y'. Then in the pullback diagram

$$\begin{array}{ccc} Y' & \xrightarrow{w} & Y \\ \uparrow & & \uparrow{\scriptstyle 0} \\ P' & \xrightarrow[w']{} & C \end{array} \tag{6.43}$$

we have $w' \in \mathcal{G}$ since $\mathcal{G}$ is closed under pullback diagrams. Since $\mathrm{Ext}_{\mathcal{P}}$ is a subfunctor of Ext, $w'_*(E) \in \mathcal{P}$, therefore $E \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$ and $\mathcal{P} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. These arguments show that $\mathcal{P} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. Since $\mathcal{S}plit \subseteq \mathcal{P}$ for every proper class $\mathcal{P}$, $\hat{\mathcal{S}plit}_{\mathcal{F}}^{\mathcal{G}} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. Combining these inclusions we obtain $\mathcal{P} + \hat{\mathcal{S}plit}_{\mathcal{F}}^{\mathcal{G}} \subseteq \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$.

Let us write $\mathcal{P} + \hat{\mathcal{S}plit}_{\mathcal{F}}^{\mathcal{G}} = \mathcal{L}$. We will show that $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}} \subseteq \mathcal{L}$.

At this point, we separate the proof into two parts because of the conditions on $\mathcal{F}$ and $\mathcal{G}$.

In the first case, let $\mathcal{F}$ be closed under compositions and pushout diagrams. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \in \hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}}$. Then there is a homomorphism $f : A \longrightarrow$

A', $f \in \mathcal{F}$ for some R-module A' such that $f_*(E) \in \mathcal{P}$. We can write the homomorphism f as $f = i \circ f'$, where $f' : A \longrightarrow \operatorname{Im} f$ is the epimorphism induced by f and $i : \operatorname{Im} f \longrightarrow A'$ is the inclusion.

Using the homomorphisms f' and i, we obtain the following commutative diagrams with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Ker} f & = & \operatorname{Ker} f & & & & \\
 & & \downarrow & & \downarrow & & & & \\
E : 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\gamma} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \alpha} & & \| & & \\
E' : 0 & \longrightarrow & \operatorname{Im} f & \xrightarrow{\mu} & B' & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{6.44}$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
E' : 0 & \longrightarrow & \operatorname{Im} f & \xrightarrow{\mu} & B' & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \lambda} & & \| & & \\
f_*(E) : 0 & \longrightarrow & A' & \xrightarrow{\nu} & B'' & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & A'/\operatorname{Im} f & = & A'/\operatorname{Im} f & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & & \\
 & & E_1 & & & & & &
\end{array} \tag{6.45}$$

We have $i_*(E') = f_*(E) \in \mathcal{P}$, then ν is a $\mathcal{P}$-monomorphism. Since $\mathcal{P} \subseteq \mathcal{L}$, ν is an $\mathcal{L}$-monomorphism. Since $i \in \mathcal{F}$ by Lemma 6.1 and $i_*(E_1) \in \mathcal{S}plit$ by (Mac Lane 1963, Ch. 3, Proposition 1.7), i is a $\hat{\mathcal{S}plit}^{\mathcal{G}}_{\mathcal{F}}$-monomorphism. Since $\hat{\mathcal{S}plit}^{\mathcal{G}}_{\mathcal{F}} \subseteq \mathcal{L}$, i is an $\mathcal{L}$-monomorphism. Then $\lambda \circ \mu = \nu \circ i$ is an $\mathcal{L}$-monomorphism since $\mathcal{L}$ is a proper class by definition, and μ is an $\mathcal{L}$-monomorphism since $\mathrm{Ext}_{\mathcal{L}}$ is a subfunctor of Ext. Therefore β is an $\mathcal{L}$-epimorphism.

Since $i_f : \mathrm{Ker}\, f \longrightarrow A$ and $f : A \longrightarrow A'$ belong to $\mathcal{F}$, $f \circ i_f$ is in $\mathcal{F}$ since $\mathcal{F}$ is closed under compositions. Therefore, $\mathrm{Ker}\, f$ is $\hat{\mathcal{S}plit}^{\mathcal{G}}_{\mathcal{F}}$-coinjective and α is a $\hat{\mathcal{S}plit}^{\mathcal{G}}_{\mathcal{F}}$-epimorphism, and α is an $\mathcal{L}$-epimorphism since $\hat{\mathcal{S}plit}^{\mathcal{G}}_{\mathcal{F}} \subseteq \mathcal{L}$. Then $\gamma = \beta \circ \alpha$ is an $\mathcal{L}$-epimorphism since $\mathcal{L}$ is a proper class by definition. Hence $E \in \mathcal{L}$ and $\hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}} \subseteq \mathcal{L}$.

For the second case, let $\mathcal{G}$ be closed under compositions and pullback diagrams. Let $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \;\in \hat{\mathcal{P}}^{\mathcal{G}}_{\mathcal{F}}$. Then there is a homomorphism $g : C' \longrightarrow C$, $g \in \mathcal{G}$ for some R-module C' such that $g^*(E) \in \mathcal{P}$. We can write the homomorphism g as $g = g' \circ j$, where $g' : \mathrm{Im}\, g \longrightarrow C$ is the inclusion and $j : C' \longrightarrow \mathrm{Im}\, g$ is the epimorphism induced by g.

Using the homomorphisms j and g', we obtain the following commutative diagrams with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
E' : 0 & \longrightarrow & A & \xrightarrow{\theta} & B' & \xrightarrow{\sigma} & \mathrm{Im}\, g & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle g'} & & \\
E : 0 & \longrightarrow & A & \xrightarrow{\eta} & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C/\mathrm{Im}\, g & = & C/\mathrm{Im}\, g & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\qquad (6.46)
$$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X & = & X & & \\
 & & & & \downarrow & & \downarrow & & \\
g^*(E): 0 & \longrightarrow & A & \longrightarrow & B'' & \xrightarrow{\delta} & C' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle j} & & \\
E': 0 & \longrightarrow & A & \xrightarrow{\theta} & B' & \xrightarrow{\sigma} & \operatorname{Im} g & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{6.47}
$$

E_2

We have $g'^*(E') = g^*(E) \in \mathcal{P}$, then δ is a $\mathcal{P}$-epimorphism. Since $\mathcal{P} \subseteq \mathcal{L}$, δ is an $\mathcal{L}$-epimorphism. Since $j \in \mathcal{G}$ by Lemma 6.1 and $j^*(E_2) \in \mathcal{S}plit$ by (Mac Lane 1963, Ch. 3, Proposition 1.7), j is a $\hat{\mathcal{S}plit}_{\mathcal{F}}^{\mathcal{G}}$-epimorphism. Since $\hat{\mathcal{S}plit}_{\mathcal{F}}^{\mathcal{G}} \subseteq \mathcal{L}$, j is an $\mathcal{L}$-epimorphism. Then $\sigma \circ \omega = j \circ \delta$ is an $\mathcal{L}$-epimorphism since $\mathcal{L}$ is a proper class by definition, and σ is an $\mathcal{L}$-epimorphism since $\operatorname{Ext}_{\mathcal{L}}$ is a subfunctor of Ext. Therefore θ is an $\mathcal{L}$-monomorphism.

Since $\pi_g : C \longrightarrow C/\operatorname{Im} g$ and $g : C' \longrightarrow C$ belong to $\mathcal{G}$, $\pi_g \circ g$ is in $\mathcal{G}$ since $\mathcal{G}$ is closed under compositions. Therefore, $C/\operatorname{Im} g$ is $\hat{\mathcal{S}plit}_{\mathcal{F}}^{\mathcal{G}}$-coprojective and ξ is a $\hat{\mathcal{S}plit}_{\mathcal{F}}^{\mathcal{G}}$-monomorphism, and ξ is an $\mathcal{L}$-monomorphism since $\hat{\mathcal{S}plit}_{\mathcal{F}}^{\mathcal{G}} \subseteq \mathcal{L}$. Then $\eta = \xi \circ \theta$ is an $\mathcal{L}$-monomorphism since $\mathcal{L}$ is a proper class by definition. Hence $E \in \mathcal{L}$ and $\hat{\mathcal{P}}_{\mathcal{F}}^{\mathcal{G}} \subseteq \mathcal{L}$. $\square$

CHAPTER 7

CONCLUSIONS

In this thesis we applied the operations defined on classes of short exact sequences to the classes related to supplements. We showed that the class of extended weak supplements is the result of the operation $\overline{\circ}$ applied to two classes one of which is the class of splitting short exact sequences.

We have introduced closures for a class of short exact sequences, which are proved to be proper when the underlying class is proper, and investigated their relation with the proper classes related to supplements and the operations mentioned. We have given a direct sum decomposition for the class $\mathcal{S}\hat{plit}$ of quasi-splitting short exact sequences over the ring of integers in terms of proper classes. We have found a relation between a proper class and its closure using the operation +. We have introduced Ivanov classes and described the closures of proper classes $\mathcal{SB}$ and $\overline{\mathcal{WS}}$ in terms of Ivanov classes under some restrictions on the ring concerned. For a given class $\mathcal{P}$ of short exact sequences over an integral domain, we have also defined other classes which are proper and included in the class $\mathcal{P}$ when $\mathcal{P}$ is proper.

A closure of submodules of a module over an integral domain is defined using a similar definition for modules over an integral domain (see (Pancar 1997)). Using this definition, we have proved the uniqueness of closures for submodules of a torsion-free module over an integral domain.

We have defined a closure for a class of short exact sequences over an associative ring with an identity element using classes of homomorphisms, and proved that this closure is proper when the underlying class is proper. We have also proved a result that relates the closure to the underlying class, under some conditions on the classes of homomorphisms used in defining the closure.

Results we have proved shows that the operation + and closures of splitting short exact sequences plays an important role on the closures of proper classes defined in this work.

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