IN-PLANE VIBRATIONS OF CURVED BEAMS HAVING VARIABLE CURVATURE AND CROSS-SECTION

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ABSTRACT

IN-PLANE VIBRATIONS OF CURVED BEAMS HAVING VARIABLE CURVATURE AND CROSS-SECTION

In this study, vibration characteristics of curved beams having variable curvatures and cross-sections are investigated. For convenience and progressive requirements, vibration characteristics of curved beams having; constant curvature and cross-section, variable curvature and constant cross-section, constant curvature and variable cross-section are also examined. The governing differential equations have derivatives with variable coefficients except for constant curvature and cross-sectioned case. Due to the fact that the solutions of differential equations with variable coefficients are analytically impossible except for special combinations of coefficients, in the investigation of eigenvalues of differential equations with variable coefficients usage of a numerical solution technique becomes necessary. At this point, the Finite Difference Method (FDM) is used to have the eigenvalues by converting continuous eigenvalue problem into discrete eigenvalue problem. Numerical solutions of the equations of motion with variable coefficients based on Finite Difference Method are carried out by using a symbolic program developed in Mathematica. The accuracy and numerical precisions of the developed program are evaluated by comparing the results with the analytical results given in literature. Good agreement is obtained in the comparisons of the present results with analytical results given in tabular form. Then, the effects of selected taper and curvature functions of beams on natural frequencies are found. The results are presented in tabular and graphical forms.

ÖZET

DEĞİŞKEN EĞRİLİK VE KESİT ALANLI EĞRİ ÇUBUKLARIN DÜZLEM İÇİ TİTREŞİMLERİ

Bu çalışmada, değişken eğrilik yarıçapına ve kesite sahip eğri çubukların titreşim özellikleri incelenmiştir. Uyum açısından ve incelemenin ilerleyen bir yapıda olması gerekliliğinden, sabit eğrilik yarıçapına ve kesit alanına, değişken eğrilik yarıçapına ve sabit kesit alanına, sabit eğrilik yarıçapına ve değişken kesit alanına sahip eğri çubukların titreşim özellikleri de incelenmiştir. Sabit eğrilik yarıçapı ve kesit alanı durumu hariç, hareket denklemi değişken katsayılı türevlere sahiptir. Bazı özel katsayı kombinasyonları hariç, değişken katsayılı diferansiyel denklemlerde özdeğer araştırmasının analitik olarak imkansız olduğu gerçeğinden dolayı, çalışmada özdeğer araştırması yapılırken bir sayısal çözüm yönteminin kullanımı zorunlu hale gelmiştir. Bu noktada, Sonlu Farklar Yöntemi (SFY), sürekli ortamdaki özdeğer problemini matris özdeğer problemine dönüştürerek özdeğerlere ulaşmada kullanılmıştır. Değişken katsayılı hareket denklemlerinin SFY tabanlı sayısal çözümü Mathematica'da geliştirilen sembolik program kullanılarak gerçekleştirilmiştir. Geliştirilen programın doğruluğu ve hassasiyeti sabit eğrilik yarıçapına ve kesit alanına sahip eğri çubuk durumunun literatürde verilen sonuçlarının, SFY programı kullanılarak elde edilen sonuçlarla karşılaştırılması ile değerlendirilmiştir. Bulunan sonuçlarla analitik sonuçlar arasında iyi bir uyum gözlenmiş olup, sonuçlar tablolar halinde verilmiştir. Daha sonra, kesit değişimi ve eğrilik yarıçapı fonksiyonlarının çubukların doğal frekansları üzerindeki etkileri bulunmuştur. Sonuçlar tablolar ve grafikler halinde verilmiştir.

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a_N , a_T	normal and tangential acceleration
A, $A(s)$, A_0, $A_1(s)$	constant cross-section, variable cross-section, cross-section at s=0, cross-section variation function
$b(s)$, b_0	width of the beam depending on s coordinate, width of the beam at s=0
$\boldsymbol{D}$	coefficient matrix
E	Young's modulus
$f(t)$	harmonic function
G'	internal moment about y- axis
$h(s)$, h_0	depth of the beam depending on s coordinate, depth of the beam at s=0
h	step-size
i	iterator
I_{YY}, $I_{YY}(s)$, I_0, $I_1(s)$	constant area moment of inertia about y-axis, variable area moment of inertia about y-axis, area moment of inertia about y-axis at s=0, variation function area moment of inertia about y-axis
m_P	mass per unit length
n	number of grids
s, $\bar{s}$, s_L	spatial variable (i.e., circumferential coordinate), dimensionless spatial variable, circumferential length of the beam
t	time
K	kinetic energy
N	internal normal force
T	internal tangential force
u	radial displacement
V	elastic strain energy
w, $\bar{w}$	tangential displacement, dimensionless tangential displacement
X	externally applied force along x- axis
Z	externally applied force along z- axis
(`)	derivative with respect to "s"

$(\dot{})$	derivative with respect to "t"
δ	curvature variation parameter
γ	width variation parameter
β	depth variation parameter
Φ_P	a constant function used in Richardson Extrapolation
ω, Ω	natural frequency, dimensionless natural frequency parameter
ρ	density of material of beam
ρ_0, $\rho_0(s)$	constant radius of curvature, variable radius of curvature
κ_0, $\kappa(s)$, k_0, $k(s)$	curvature; constant, variable, at s=0, variation function
κ_1'	curvature after displacement occurs
λ, λ_0	dimensionless curvature parameter; for constant curvature (opening angle), for variable curvature

CHAPTER 1

GENERAL INTRODUCTION

Vibration analysis of structural elements is one of the design steps which needs to be considered to avoid collapse of whole structure because of resonance, and affecting adjacent structures or machines. Preventing machines from spending more energy than actual need is also another cause which directs engineers and analyzers to investigate vibrations of structural elements, structures, and machines.

It is obvious that curved beams have been used widely in a lot of aerospace, machinery, and architectural applications. This is because their advantage in terms of storing mechanical energy, carrying loads better as a component of structures, and beauty of their appearance. Geometric and functional requirements also direct designers to employ such curved structures.

Curved beams can be utilized as parts of arch bridges, piping systems, aerospace structures, springs, watches, as well as smart structures. Besides, curved beams can be classified from different aspects. Namely, a curved beam can

- be a space curve or a plane curve,
- have constant (circular) or variable (such as catenary, elliptic, and parabolic etc.) curvature,
- have constant or variable (tapered etc.) cross section.

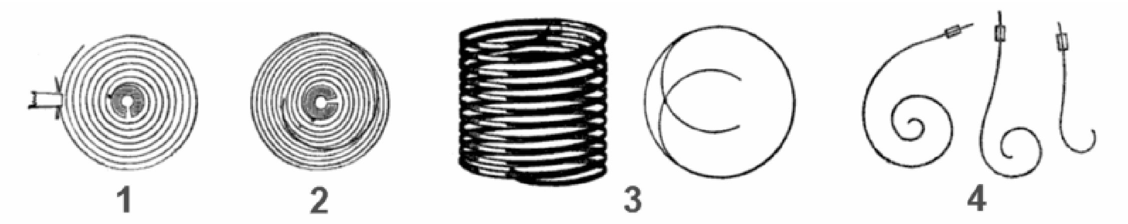

Figure 1.1. Balance springs: (1) flat spiral, (2) Breguet overcoil, (3) chronometer helix, showing curving ends, (4) early balance springs.

Because of the wide usage of curved beams, many investigators have studied curved beams, arches, and spirals. Researchers investigating vibration characteristics of curved beams dealt with problems of different points of view aforementioned. Some of those studies can be surveyed as in next paragraph.

An exact solution for in-plane vibration of arches was developed using Frobenius method combined with the dynamic stiffness method. The effects of rotary inertia and shear deformation were taken into account. The effects of rise to span length, slenderness ratio, and variation of cross-section on non-dimensional frequencies were also showed (Huang et al. 1998). A transfer matrix method was presented for predicting the natural frequencies of circular arcs with varying cross-section. The numerical results obtained by the method were compared with the experimental results (Irie et al. 1982). In-plane vibration of a free-clamped slender arc of varying cross section was analyzed using spline interpolation technique (Irie et al. 1980). In order to use this technique, the arc was divided into small elements. The in-plane displacement of each element was expressed by a spline function of seven degrees with unknown coefficients. An approximate method was presented to study both in-plane and out-of-plane free vibrations of horizontally curved beams with arbitrary shapes and variable cross sections. The characteristic equation was obtained by application of the Green function (Kawakami et al. 1995). The first two natural frequencies of vibration of symmetric circular arches with linearly varying thickness carrying concentrated masses were determined (Laura and Irassar 1988). The frequency equation was generated by means of the Ritz method and eigenvalues were optimized with respect to an exponential parameter. It was proposed that the Love strain form (Love 1944) of naturally curved and twisted rods are not valid for two dimensionally curved beams when shear deformation is considered (Leung and Chan 1997). The differential equations governing free, in-plane vibrations of non-circular arches with non-uniform cross-sections were derived (Oh et al. 1999). The effects of shear deformation, rotary inertia, and axial deformation were included. The governing equation was solved numerically to obtain frequencies and mode shapes. Numerical results showed agreement with results determined by means of finite element method. Besides, an experiment was conducted (Oh et al. 2000) and it was showed that the results obtained by use of theory are in agreement with those of experiments. The equations without making use of the assumption of no extension of the central line were derived (Philipsson 1956). It was showed that the forced vibration problem to require the possibility of extension in certain cases in which its neglecting in a free-vibration analysis is valid. An approximate method was presented to analyze the free vibration of any type of arches (Sakiyama 1985). The solutions of differential equations were obtained in discrete form, by translating the differential equations into integral equations and applying numerical integrations. The dynamic response of a plane curved bar with varying cross-section under a dynamic load was analyzed (Suzuki et al. 1985). The eigenfunction expansion method was employed to solve the equations of motion and also the time responses were compared with those for a static load to understand their characteristics. The in-plane vibrations of a uniform curved bar were investigated considering the bending, the extension, the shear deformation, and the rotary inertia of the bar (Suzuki and Takahashi 1979), (Suzuki and Takahashi 1982). The equations of vibration and boundary conditions were obtained from the stationarity condition of the Langrangian. As examples, calculations for elliptic arc bars with built-in ends were made. The equations of vibration were solved exactly by a series solution. The Rayleigh-Ritz method was used to find the lowest natural frequency of clamped parabolic arcs to understand the effect of the variation of depth and width on natural frequencies (Wang 1972). The in-plane and out-of-plane free vibration frequencies of Archimedes-type spirals were computed (Yıldırım 1997). The transfer matrix method was used and the shear deformation and rotary inertia were taken into account. To compute the overall dynamic transfer matrix, the complementary functions method was made use of. The coupled bending-bending vibration of pre-twisted tapered cantilever blades was dealt with (Carnegie and Thomas 1972). The domain was discretized using the finite difference method and a set of simultaneous algebraic equations was obtained. Solving this set of equations it was showed that taper has a considerable influence on both the frequencies of vibration and the associated modal shapes. Vibration characteristics of rectangular cross-sectioned pretwisted beams of which width and depth variations along the beam were expressed as power of axial coordinate of the beam were investigated by using the Finite Difference Method (Yardımoğlu and Kara 2009). The effects of width and depth power, multiplication, and pretwist parameters were showed.

Although, the problem to define the effects of parameters affecting vibration characteristics of curved beams is much investigated and a considerable amount of publications have been published so far, it still holds attraction because of wide usage. Therefore, in this study, the effects of variable radius of curvature and cross-section on vibration characteristics of curved beams are determined. The method employed to solve the eigenvalue problem is chosen as the Finite Difference Method, because the method has not been used for such curved structures yet.

A symbolic program is developed in Mathematica environment based on the Finite Difference Method. Using the program developed, dimensionless natural frequencies are obtained and the effects of parameters those defining variable radius of curvature and cross-section are determined.

The accuracy and numerical precision of the developed program are evaluated by using the analytical results given in literature for a curved beam of constant curvature and cross-section. Good agreement is reached in the comparisons of the present results with analytical results. As novel examples of the study, curved beams having polynomial, sinusoidal, and exponential curvature and/or having linearly varying width and depth are investigated. Besides the effects of parameters defining the shape of cross-section and curvature, the effects of parameters concerning opening angle (or dimensionless curvature parameter) are examined. The effects of cross-section and curvature variation parameters are given in tabular and graphical form.

CHAPTER 2

THEORETICAL VIBRATION ANALYSIS

2.1. Introduction

In this chapter, statement of the investigated problem is given. Geometry, curvature expression and energies which are needed to derive equations are given. Two methods which may be used to derive the equations of motions governing the stated problem are introduced. The derived equations of motions governing free vibrations can be found in following sections.

Before employing a method to find eigenvalues, equations of motions which are derived need to be non-dimensionalized. Non-dimensional parameters need to be introduced to make investigation general and based on those given parameters.

At the end of this chapter, the logic behind the Finite Difference Method is explained and determination of natural frequencies using the Finite Difference Method is mentioned. Finally the Richardson Extrapolation Method, which is a method utilized to have better approximations, is introduced.

2.2 Statement of the Problem

Curved beams having variable curvature and cross-section are considered. The material of the beam is thought to be linearly isotropic. For the cases of constant radius of curvature and cross section, variable radius of curvature and constant cross-section, constant curvature and variable cross-section, and variable radius of curvature and cross-section, equations which govern free vibrations are obtained in a manner that is dependent from specific geometry. If desired equations for radius of curvature and cross-sectional area are substituted in derived equations, specific governing equations for that geometry can be obtained. In Figure 2.1., a representative curved beam with both variable radius of curvature and cross-section can be seen.

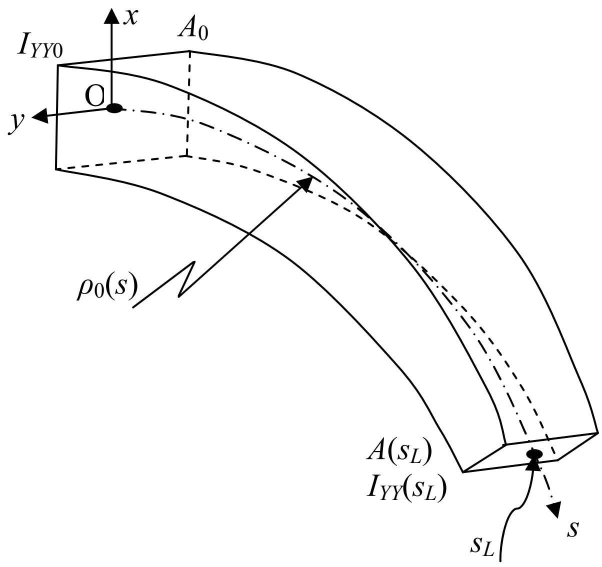

Figure 2.1. A representative curved beam with variable curvature and cross section

2.3 Derivation of the Equation of Motion

In this section methods which may be employed to derive equations of motion of continuous systems are presented. Equations of motion of continuous systems can be obtained by using different methods; such as; "Dynamic Equilibrium Approach", "Variational Method", and "Integral Equation Approach". In this study, Dynamic Equilibrium Approach is applied using Newtonian Method and Variational Method is applied using Hamilton's Principle. On the other hand, Integral Equation Approach is not mentioned, because it is based on the derivation of governing differential equations using other methods. Besides, equations of motions and boundary conditions which are derived by hand and by using Mathematica are introduced for all of the cases investigated.

2.3.1 Newtonian Method

Dynamic Equilibrium approach can be applied by using either Newtonian Method or D'Alembert's Principle. Due to the fact that the D'Alembert's Principle is just the restatement of the Newtonian Method, it is not used to derive the equations of motion.

Newtonian method is based on dynamic equilibrium of internal and external forces and moments. Therefore, in Newtonian method what needs to be done is to equalize the external forces and internal forces, and external moments and internal moments in same direction. Newtonian method can be given using following equations (Rao 2007):

$$\sum_i \vec{F}_i = \frac{d}{dt}(m\vec{v}) = m\vec{a} \tag{2.1}$$

$$\sum_i \vec{M}_i = \frac{d}{dt}(I\vec{\omega}) = I\vec{\alpha} \tag{2.2}$$

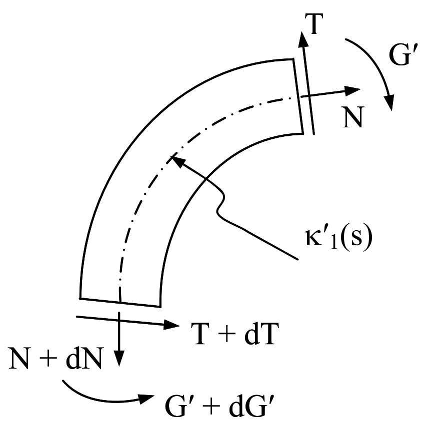

Figure 2.2. A differential curved beam with internal forces and bending moment

If forces and moments which can be seen in Figure 2.2 are substituted in Equations 2.1 and 2.2, force and moment equilibrium equations can be obtained as follows (Love 1944):

$$\frac{dN}{ds} + T\kappa_1' + X = m_P\, a_N \tag{2.3.a}$$

$$\frac{dT}{ds} - N\kappa_1' + Z = m_P\, a_T \tag{2.3.b}$$

$$\frac{dG'}{ds} + N + K' = 0 \tag{2.3.c}$$

where $m_P\, a_N$ and $m_P\, a_T$ are inertial forces. Bending moment and curvature can be written as follows:

$$G' = EI_{yy}(s)\frac{d}{ds}\left(\frac{du}{ds} + \kappa_0 w\right) \qquad (2.4.a)$$

$$\kappa_1' = \kappa_0 + \frac{d}{ds}\left(\frac{du}{ds} + w\kappa_0\right) \qquad (2.4.b)$$

If central line is assumed as unextended, the inextensionality condition is wrıtten in following equation:

$$\frac{dw}{ds} = u\kappa_0 \qquad (2.5)$$

Internal force along z-axis "T" vanishes. Also, knowing that there is no externally applied moment, "K'"; equilibrium equations becomes:

$$-\frac{d^2 G'}{ds^2} + X = m_P\, a_N \qquad (2.6.a)$$

$$\kappa_1' \frac{dG'}{ds} + Z = m_P\, a_T \qquad (2.6.b)$$

After substituting the expressions of G' and κ_1' and inextensionality condition given by Equation 2.5 the equation of motion of curved beam is obtained. One of the disadvantages of using Newtonian Method is that it does not yield boundary conditions associated with differential equations unlike Hamilton's method. Besides, while derivation of equations of motions using Newtonian method, it is needed to neglect small quantities of higher order unlike Hamilton's Principle.

2.3.2 Hamilton's Method

The variational principle that can be used for dynamic problems is called Hamilton's principle. Hamiltonian method is a more powerful approach than Newtonian method, because it also yields boundary conditions. Variation of a function can be depicted as shown in Figure 2.3.

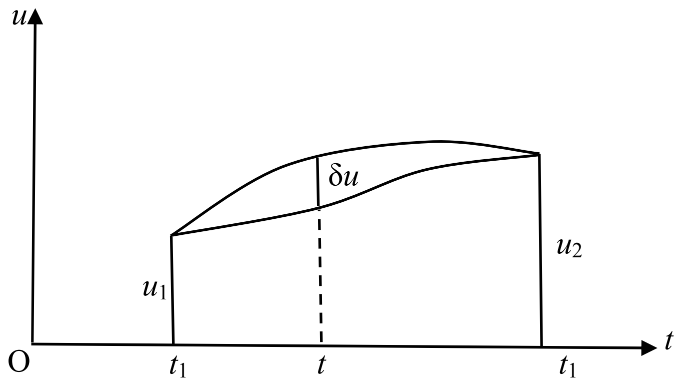

Figure 2.3. Variation of $u(t)$

The principle can be stated as follows; "Of all possible time histories of displacement states that satisfy the compatibility equations and the constraints or the kinematic boundary conditions and that also satisfy the conditions at initial and final times t_1 and t_2, the history to the actual solution makes the Langrangian a minimum" (Meirovitch 1967). The principle can be defined mathematically as follows;

$$\delta\int_{t_1}^{t_2} L = \delta\int_{t_1}^{t_2} (K - V) = 0 \tag{2.7}$$

Where K is the kinetic energy and V is the elastic strain energy- i.e. potential energy of the curved beam. Kinetic energy and elastic strain energy of the curved beam examined are as follows;

$$K = \frac{1}{2}\int_{0}^{s_L} m(\dot{u}^2 + \dot{w}^2)ds \tag{2.8.a}$$

$$V = \int_0^{s_L} (\frac{1}{2} G'(\kappa_1' - \kappa_0))ds \tag{2.8.b}$$

If G' and κ_1' given in Equations 2.4 are substituted in Equations 2.8, and employing inextensionality condition given in Equation 2.5 energies given in Equation 2.8 becomes:

$$K = \frac{1}{2} \int_0^{s_L} \rho A (\frac{\dot{w}'^2}{\kappa_0} + \dot{w}^2) ds \tag{2.9.a}$$

$$V = \frac{1}{2} \int_0^{s_L} E I_{YY} ((\frac{w''}{\kappa_0})' + \kappa_0 w')^2 ds \tag{2.9.b}$$

Using energies given in Equations 2.9, differential equations governing free vibrations of curved beams with constant radius of curvature and cross-section can be obtained with associated boundary conditions as follows:

$$\frac{EI_{yy}}{\kappa_0^{\ 2}} w^{vi} + 2EI_{yy} w^{iv} + \kappa_0^{\ 2} EI_{yy} w'' = \rho A (\ddot{w} - \frac{\ddot{w}''}{\kappa_0^{\ 2}}) \tag{2.10}$$

$$(\frac{w'''}{\kappa_0} + w')\delta w''\Big|_0^{s_L} = 0 \tag{2.11.a}$$

$$(\frac{w'''}{\kappa_0} + w')'\delta w'\Big|_0^{s_L} = 0 \tag{2.11.b}$$

$$(\frac{w'''}{\kappa_0} + w')\delta w\Big|_0^{s_L} = 0 \tag{2.11.c}$$

Physical interpretations for boundary conditions corresponding to Equations 2.11.a-c as follows:

a) Either bending moment is zero(pinned or free), or slope is zero(clamped).

b) Either shear force is zero(free), or displacement is zero(pinned or clamped).

c) Either bending moment is zero(pinned or free), or displacement is zero(pinned or clamped).

Boundary conditions given in Equations 2.11.a-c remain the same for all of the cases.

Differential equation governing free vibrations of curved beam with variable curvature and constant cross-section can be obtained by choosing a curvature variation function and substituting it into Equations 2.8 as follows:

$$
\begin{aligned}
&\frac{EI_{YY}}{\kappa_0(s)^2}\frac{\partial^6 w}{\partial s^6}-6\frac{EI_{YY}}{\kappa_0(s)^2}\frac{\partial \kappa_0}{\partial s}\frac{\partial^5 w}{\partial s^5}+(2EI_{YY}+24\frac{EI_{YY}}{\kappa_0(s)^4}\frac{\partial \kappa_0}{\partial s}^{2}-10\frac{EI_{YY}}{\kappa_0(s)^3}\frac{\partial^2 \kappa_0}{\partial s^2})\frac{\partial^4 w}{\partial s^4}\\
&+(-72\frac{EI_{YY}}{\kappa_0(s)^5}\frac{\partial \kappa_0}{\partial s}^{3}+66\frac{EI_{YY}}{\kappa_0(s)^4}\frac{\partial \kappa_0}{\partial s}\frac{\partial^2 \kappa_0}{\partial s^2}-10\frac{EI_{YY}}{\kappa_0(s)^3}\frac{\partial^3 \kappa_0}{\partial s^3})\frac{\partial^3 w}{\partial s^3}+(EI_{YY}\kappa_0(s)^2\\
&+2\frac{EI_{YY}}{\kappa_0(s)^2}\frac{\partial \kappa_0}{\partial s}^{2}+144\frac{EI_{YY}}{\kappa_0(s)^6}\frac{\partial \kappa_0}{\partial s}^{4}-204\frac{EI_{YY}}{\kappa_0(s)^5}\frac{\partial \kappa_0}{\partial s}^{2}\frac{\partial^2 \kappa_0}{\partial s^2}+30\frac{EI_{YY}}{\kappa_0(s)^4}\frac{\partial^2 \kappa_0}{\partial s^2}^{2}\\
&+44\frac{EI_{YY}}{\kappa_0(s)^4}\frac{\partial \kappa_0}{\partial s}\frac{\partial^3 \kappa_0}{\partial s^3}-5\frac{EI_{YY}}{\kappa_0(s)^3}\frac{\partial^4 \kappa_0}{\partial s^4})\frac{\partial^2 w}{\partial s^2}+(2EI_{YY}\kappa_0(s)\frac{\partial \kappa_0}{\partial s}-4\frac{EI_{YY}}{\kappa_0(s)^3}\frac{\partial \kappa_0}{\partial s}^{3}\\
&-144\frac{EI_{YY}}{\kappa_0(s)^7}\frac{\partial \kappa_0}{\partial s}^{5}+4\frac{EI_{YY}}{\kappa_0(s)^2}\frac{\partial \kappa_0}{\partial s}\frac{\partial^2 \kappa_0}{\partial s^2}+276\frac{EI_{YY}}{\kappa_0(s)^6}\frac{\partial \kappa_0}{\partial s}^{3}\frac{\partial^2 \kappa_0}{\partial s^2}-96\frac{EI_{YY}}{\kappa_0(s)^5}\frac{\partial \kappa_0}{\partial s}\frac{\partial^2 \kappa_0}{\partial s^2}^{2}\\
&-68\frac{EI_{YY}}{\kappa_0(s)^5}\frac{\partial \kappa_0}{\partial s}^{2}\frac{\partial^3 \kappa_0}{\partial s^3}+20\frac{EI_{YY}}{\kappa_0(s)^4}\frac{\partial^2 \kappa_0}{\partial s^2}\frac{\partial^3 \kappa_0}{\partial s^3}+11\frac{EI_{YY}}{\kappa_0(s)^4}\frac{\partial \kappa_0}{\partial s}\frac{\partial^4 \kappa_0}{\partial s^4}-\frac{EI_{YY}}{\kappa_0(s)^3}\frac{\partial^5 \kappa_0}{\partial s^5})\frac{\partial w}{\partial s}\\
&-\rho A\frac{\partial^2 w}{\partial t^2}-\frac{2\rho A}{\kappa_0(s)^3}\frac{\partial \kappa_0}{\partial s}\frac{\partial^3 w}{\partial s\partial t^2}+\frac{\rho A}{\kappa_0(s)^2}\frac{\partial^4 w}{\partial s^2\partial t^2}=0
\end{aligned}
$$

(2.12)

Differential equation governing free vibrations of curved beam with constant curvature and variable cross-section can be obtained by choosing a cross-section variation function and substituting it into Equations 2.8 as follows:

$$
\begin{aligned}
&\frac{EI_{YY}(s)}{\kappa_0^{\,2}}\frac{\partial^6 w}{\partial s^6}+3\frac{E}{\kappa_0^{\,2}}\frac{\partial I}{\partial s}\frac{\partial^5 w}{\partial s^5}+(2EI_{YY}(s)+3\frac{E}{\kappa_0^{\,2}}\frac{\partial^2 I_{YY}}{\partial s^2})\frac{\partial^4 w}{\partial s^4}+(4E\frac{\partial I_{YY}}{\partial s}+\frac{E}{\kappa_0^{\,2}}\frac{\partial^3 I_{YY}}{\partial s^3})\frac{\partial^3 w}{\partial s^3}\\
&(E\kappa_0^{\,2}I_{YY}(s)+3E\frac{\partial^2 I_{YY}}{\partial s^2})\frac{\partial^2 w}{\partial s^2}-\rho A(s)\frac{\partial^2 w}{\partial t^2}+\frac{\rho}{\kappa_0^{\,2}}\frac{\partial A}{\partial s}\frac{\partial^3 w}{\partial s\partial t^2}+\frac{\rho A(s)}{\kappa_0^{\,2}}\frac{\partial^4 w}{\partial s^2\partial t^2}=0
\end{aligned}
$$

(2.13)

Differential equation governing free vibrations of curved beam with variable curvature and cross-section can be obtained by choosing variation functions of both curvature and cross-section and substituting them into Equations 2.8 as follows:

$$
\begin{aligned}
&\frac{EI_{YY}(s)}{\kappa_0(s)^2}\frac{\partial^6 w}{\partial s^6}+(3\frac{E}{\kappa_0(s)^2}\frac{dI_{YY}}{ds}-6\frac{EI_{YY}(s)}{\kappa_0(s)^3}\frac{d\kappa_0}{ds})\frac{\partial^5 w}{\partial s^5}+(2EI_{YY}(s)-14\frac{E}{\kappa_0(s)^3}\frac{dI_{YY}}{ds}\frac{d\kappa_0}{ds}\\
&+24\frac{EI_{YY}(s)}{\kappa_0(s)^4}\frac{d\kappa_0}{ds}^2+3\frac{E}{\kappa_0(s)^2}\frac{d^2I_{YY}}{ds^2}-10\frac{EI_{YY}(s)}{\kappa_0(s)^3}\frac{d^2\kappa_0}{ds^2})\frac{\partial^4 w}{\partial s^4}+(4E\frac{dI_{YY}}{ds}+42\frac{E}{\kappa_0(s)^4}\frac{dI_{YY}}{ds}\frac{d\kappa_0}{ds}^2\\
&-72\frac{EI_{YY}(s)}{\kappa_0(s)^5}\frac{d\kappa_0}{ds}^3-10\frac{E}{\kappa_0(s)^3}\frac{d^2I_{YY}}{ds^2}\frac{d\kappa_0}{ds}-18\frac{E}{\kappa_0(s)^3}\frac{dI_{YY}}{ds}\frac{d^2\kappa_0}{ds^2}+66\frac{EI_{YY}(s)}{\kappa_0(s)^4}\frac{d\kappa_0}{ds}\frac{d^2\kappa_0}{ds^2}\\
&+\frac{E}{\kappa_0(s)^2}\frac{d^3I_{YY}}{ds^3}-10\frac{EI_{YY}(s)}{\kappa_0(s)^3}\frac{d^3\kappa_0}{ds^3})\frac{\partial^3 w}{\partial s^3}+(EI_{YY}(s)\kappa_0(s)^2+2\frac{E}{\kappa_0(s)}\frac{dI_{YY}}{ds}\frac{d\kappa_0}{ds}+2\frac{EI_{YY}(s)}{\kappa_0(s)^2}\frac{d\kappa_0}{ds}^2\\
&-84\frac{E}{\kappa_0(s)^5}\frac{dI_{YY}}{ds}\frac{d\kappa_0}{ds}^3+144\frac{EI_{YY}(s)}{\kappa_0(s)^6}\frac{d\kappa_0}{ds}^4+3E\frac{d^2I_{YY}}{ds^2}+20\frac{E}{\kappa_0(s)^4}\frac{d^2I_{YY}}{ds^2}\frac{d\kappa_0}{ds}^2\\
&+78\frac{E}{\kappa_0(s)^4}\frac{dI_{YY}}{ds}\frac{d\kappa_0}{ds}\frac{d^2\kappa_0}{ds^2}-204\frac{EI_{YY}(s)}{\kappa_0(s)^5}\frac{d\kappa_0}{ds}^2\frac{d^2\kappa_0}{ds^2}-9\frac{E}{\kappa_0(s)^3}\frac{d^2I_{YY}}{ds^2}\frac{d^2\kappa_0}{ds^2}+30\frac{EI_{YY}(s)}{\kappa_0(s)^4}\frac{d^2\kappa_0}{ds^2}^2\\
&-2\frac{E}{\kappa_0(s)^3}\frac{d^3I_{YY}}{ds^3}\frac{d\kappa_0}{ds}-12\frac{E}{\kappa_0(s)^3}\frac{dI_{YY}}{ds}\frac{d^3\kappa_0}{ds^3}+44\frac{EI_{YY}(s)}{\kappa_0(s)^4}\frac{d\kappa_0}{ds}\frac{d^3\kappa_0}{ds^3}-5\frac{EI_{YY}(s)}{\kappa_0(s)^3}\frac{d^4\kappa_0}{ds^4})\frac{\partial^2 w}{\partial s^2}\\
&+(E\frac{dI_{YY}}{ds}\kappa_0(s)^2+2EI_{YY}(s)\kappa_0(s)\frac{d\kappa_0}{ds}-4\frac{EI_{YY}(s)}{\kappa_0(s)^3}\frac{d\kappa_0}{ds}^3+84\frac{E}{\kappa_0(s)^6}\frac{dI_{YY}}{ds}\frac{d^2\kappa_0}{ds^2}^4\\
&-144\frac{EI_{YY}(s)}{\kappa_0(s)^7}\frac{d\kappa_0}{ds}^5+2\frac{E}{\kappa_0(s)}\frac{d^2I_{YY}}{ds^2}\frac{d\kappa_0}{ds}-20\frac{E}{\kappa_0(s)^5}\frac{d^2I_{YY}}{ds^2}\frac{d^2\kappa_0}{ds^2}^3+2\frac{E}{\kappa_0(s)}\frac{dI_{YY}}{ds}\frac{d^2\kappa_0}{ds^2}\\
&+4\frac{EI_{YY}(s)}{\kappa_0(s)^2}\frac{d\kappa_0}{ds}\frac{d^2\kappa_0}{ds^2}-120\frac{E}{\kappa_0(s)^5}\frac{dI_{YY}}{ds}\frac{d\kappa_0}{ds}^2\frac{d^2\kappa_0}{ds^2}+276\frac{EI_{YY}(s)}{\kappa_0(s)^6}\frac{d^2\kappa_0}{ds^2}^3\frac{d^2\kappa_0}{ds^2}\\
&+19\frac{E}{\kappa_0(s)^4}\frac{d^2I_{YY}}{ds^2}\frac{d\kappa_0}{ds}\frac{d^2\kappa_0}{ds^2}+18\frac{E}{\kappa_0(s)^4}\frac{dI_{YY}}{ds}\frac{d^2\kappa_0}{ds^2}^2-96\frac{EI_{YY}(s)}{\kappa_0(s)^5}\frac{d\kappa_0}{ds}\frac{d^2\kappa_0}{ds^2}^2+E\frac{d^3I_{YY}}{ds^3}\\
&+2\frac{E}{\kappa_0(s)^4}\frac{d^3I_{YY}}{ds^3}\frac{d\kappa_0}{ds}^2-\frac{E}{\kappa_0(s)^3}\frac{d^3I_{YY}}{ds^3}\frac{d^2\kappa_0}{ds^2}+26\frac{E}{\kappa_0(s)^4}\frac{dI_{YY}}{ds}\frac{d\kappa_0}{ds}\frac{d^3\kappa_0}{ds^3}-68\frac{EI_{YY}(s)}{\kappa_0(s)^5}\frac{d\kappa_0}{ds}^2\frac{d^3\kappa_0}{ds^3}\\
&-3\frac{E}{\kappa_0(s)^3}\frac{d^2I_{YY}}{ds^2}\frac{d^3\kappa_0}{ds^3}+20\frac{EI_{YY}(s)}{\kappa_0(s)^4}\frac{d^2\kappa_0}{ds^2}\frac{d^3\kappa_0}{ds^3}-3\frac{E}{\kappa_0(s)^3}\frac{dI_{YY}}{ds}\frac{d^4\kappa_0}{ds^4}+11\frac{EI_{YY}(s)}{\kappa_0(s)^4}\frac{d\kappa_0}{ds}\frac{d^4\kappa_0}{ds^4}\\
&-\frac{EI_{YY}(s)}{\kappa_0(s)^3}\frac{d^5\kappa_0}{ds^5})\frac{\partial w}{\partial s}-\rho A(s)\frac{\partial^2 w}{\partial t^2}+(\frac{\rho}{\kappa_0(s)^2}\frac{\partial A}{\partial s}-2\frac{\rho A(s)}{\kappa_0(s)^3}\frac{d\kappa_0}{ds})\frac{\partial^3 w}{\partial s\partial t^2}+(\frac{\rho A(s)}{\kappa_0(s)^2})\frac{\partial^4 w}{\partial s^2\partial t^2}=0
\end{aligned}
\tag{2.14}
$$

After obtaining governing equations of the problem, the separation of variables technique is used to convert partial differential equation into an ordinary differential equation. This technique can be introduced as follows:

$$w(s,t)=w(s)f(t) \tag{2.15}$$

Because of the fact that a conservative system has constant total energy, the function $f(t)$ in Equation 2.15 needs to be a harmonic function (Meirovitch 1967). Only for the differential equation governing free vibrations of curved beams with constant radius of curvature and cross-section is separated using Equations 2.15 as a representative example. It is obvious that this step should be performed for other cases as well. Therefore, substituting Equation 2.15 into Equation 2.10, the partial differential equation reduces to the following ordinary differential equation form:

$$\frac{EI_{yy}}{{\kappa_0}^2} w^{vi} + 2EI_{yy} w^{iv} + {\kappa_0}^2 EI_{yy} w'' = -\rho A \omega^2 (w - \frac{w''}{{\kappa_0}^2}) \tag{2.16}$$

2.4. Non-dimensionalization of Equation of Motions

Non-dimensionalization is one of the most important steps in the analysis of a system of differential equations. It comprises scaling each variable (dependent and independent) by a typical or representative value, providing a nondimensional variable. Non-dimensionalization is important to analyze such problems because;

1. It identifies the dimensionless groups (ratios of dimensional parameters) which control the solution behavior.
2. Terms in the equations are now dimensionless and so allow comparison of their sizes.
3. It permits estimations of the effects of extra features to the original model through the new dimensional group(s) connected to the extra term(s).
4. Finally, it can decrease the amount of parameters appearing in the problem by creating the nondimensional parameters or dimensionless groups.

Because of those four reasons aforementioned, in our analysis, problem should be non-dimensionalized, and dimensionless variables and parameters have to be introduced. Dimensionless parameters, variables, and differential equation governing free vibrations of curved beams having constant curvature and cross-section can be written as follows:

$$\bar{s} = \frac{s}{s_L}, \qquad \bar{w} = \frac{w}{s_L}, \qquad \lambda = \kappa_0 s_L, \qquad \Omega = \frac{\rho A}{E I_{YY} {\kappa_0}^4} \omega^2 \tag{2.17}$$

$$\frac{\overline{w}^{vi}}{\lambda^6}+\frac{2\overline{w}^{iv}}{\lambda^4}+\frac{\overline{w''}}{\lambda^2}=\Omega(\frac{\overline{w''}}{\lambda^2}-\overline{w}) \tag{2.18}$$

Dimensionless parameters given in Equation 2.17 can be described as; $\overline{w}$ is dimensionless displacement, λ is dimensionless curvature parameter (i.e. κ_0 has the unit of m^{-1}), and Ω is dimensionless frequency parameter. It is clear that, for such a problem, to identify only the dimensionless curvature parameter λ is enough to have a frequency parameter that is independent from curvature itself, arc length, cross-section, and material of the beam.

For other three cases, dimensionless parameters are going to be written knowing following cross-section and curvature variation definitions:

$$\kappa(s)=k_0 k(s) \tag{2.19.a}$$

$$I_{YY}(s)=I_0 I_1(s) \tag{2.19.b}$$

$$A(s)=A_0 A_1(s) \tag{2.19.c}$$

where, k_0 is curvature, I_0 is second moment of area, and A_0 is cross-sectional area at s=0. $k(s)$, $I_1(s)$, and $A_1(s)$ are predefined functions to determine the variations of curvature, second moment of area, and cross-sectional area, respectively. Combinations of Equations 2.19.a-c must be chosen for determination of cross-sectional area and curvature expressions. For the cases of variable curvature and constant cross-section, constant curvature and variable cross-section and variable curvature and cross-section, dimensionless parameters and variables can be written as follows:

$$\overline{s}=\frac{s}{s_L}, \qquad \overline{w}=\frac{w}{s_L}, \qquad \lambda_0=k_0 s_L, \qquad \Omega=\frac{\rho A_0}{E I_0 k_0{}^4}\omega^2 \tag{2.20}$$

Substituting proper dimensionless parameters given in Equations 2.17 and 2.20 into Equations 2.10, 2.12, 2.13, or 2.14, the equations are transformed into dimensionless form like given in Equation 2.18 for the simplest case.

2.5. Finite Difference Method to Determine Natural Frequencies

In mathematics, the Finite Difference Methods are numerical methods for approximating the solutions to differential equations using finite difference equations to approximate derivatives (Hildebrand 1987).

Because differential equations having variable coefficients are analytically unsolvable except for equations having special combinations of coefficients, the Finite Difference Method is used. In addition to that, as it is mentioned before, the Finite Difference Method has not been used to solve such curved beam vibration problems, although it is used to solve pretwisted beam problems (Carnegie and Thomas 1972).

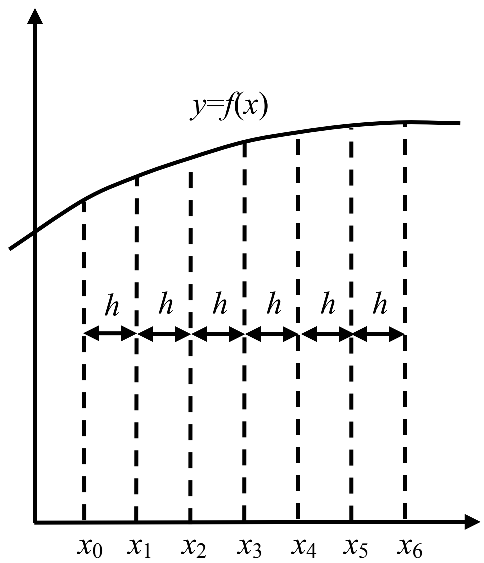

Figure 2.4. A domain divided into six subdomains for approximation

Following two assumptions are used to employ the Finite Difference Method;

1. The derivatives of dependent variable (for instance y) in the differential equation(s) are replaced by the finite difference approximations.
2. The differential equation(s) is/are enforced only at mesh points. As a result, the differential equations are replaced by n simultaneous algebraic equations, the unknowns being y_i , i=1,2,….,n.

In the problem, the Finite Difference Method can be applied by;

a) Dividing the range $0 \leq \bar{s} \leq 1$ into n number of equal parts of step length $h(=1/n)$ (as shown in Figure 2.4.),

b) Replacing the derivatives of $\overline{w}$ appearing in equations by their corresponding central difference relationships, given in Table 2.1.

c) Replacing the derivatives of $\overline{w}$ appearing in left and right boundaries by their corresponding central difference relationships, given in Table 2.1.

d) Grouping like terms. After grouping like terms n simultaneous algebraic equations are obtained.

After these steps, problem defined in Equation 2.18 for a curved beam having constant cross-section and curvature becomes the form of a matrix eigenvalue problem as given in following equation:

$$[D]\{w_i\} = \Omega\{w_i\} \tag{2.21}$$

Table 2.1. Differences approximations for derivatives

Term	Central Difference Expressions
$\frac{dw}{ds}$	$\frac{w(i+1)-w(i-1)}{2h}$
$\frac{d^2w}{ds^2}$	$\frac{w(i+1)-2w(i)+w(i-1)}{h^2}$
$\frac{d^3w}{ds^3}$	$\frac{w(i+2)-2w(i+1)+2w(i-1)-w(i-2)}{2h^3}$
$\frac{d^4w}{ds^4}$	$\frac{w(i+2)-4w(i+1)+6w(i)-4w(i-1)+w(i-2)}{h^4}$
$\frac{d^5w}{ds^5}$	$\frac{w(i+3)-4w(i+2)+5w(i+1)-5w(i-1)+4w(i-2)-w(i-3)}{h^5}$
$\frac{d^6w}{ds^6}$	$\frac{w(i+3)-6w(i+2)+15w(i+1)-20w(i)+15w(i-1)-6w(i-2)+w(i-3)}{h^6}$

After substituting boundary conditions given in Equation 2.18 into Equation 2.11, the problem reduces to finding the eigenvalues of coefficient matrix ***D.***

For the cases having variable cross-sections or/and variable curvature, coefficient matrix ***D*** has variable terms as well.

i appearing in Equation 2.21 should be defined considering boundary conditions. The values that *i* must be assigned are tabulated (0 is left boundary and n is right boundary) in Table 2.2.

Table 2.2. *i* values considering boundary conditions

Boundary Conditions	*i* values
Cantilever	$1 \leq i \leq n$
Fixed-Fixed	$1 \leq i \leq n-1$
Fixed-Simple	$1 \leq i \leq n-1$
Simple-Free	$1 \leq i \leq n$
Simple-Fixed	$1 \leq i \leq n-1$
Simple-Simple	$1 \leq i \leq n-1$
Free-Free	$0 \leq i \leq n$
Free-Simple	$0 \leq i \leq n-1$

2.6. Richardson Extrapolation Method

Richardson extrapolation is a method for improving the accuracy of some numerical procedures, including finite difference approximations and numerical integration. The method is named after English mathematician, physicist, meteorologist, psychologist, and pacifist Lewis Fry Richardson.

Let us suppose that we have an approximate means of computing the dimensionless natural frequency *Ω* of a curved beam. It is known that, the result depends on the parameter *h,* the stepsize. Richardson extrapolation formula then can be written as follows (Richardson and Gaunt 1927):

$$\Omega(h) = \Omega + \sum_{p=1} h^{2p} \phi_p \tag{2.22}$$

where ϕ_p is a constant function and the upper bound of *p* needs to be chosen considering how many approximate values are used to find an extrapolated value. It is also worth to mention that the value assigned for the upper bound of p affects the elimination of the order of error. It means, if p=2 is chosen (i.e. there are three approximated dimensionless frequency parameter values), an error of O(h^2) is eliminated.

CHAPTER 3

NUMERICAL RESULTS AND DISCUSSION

3.1. Introduction

In this chapter, numerical applications for four cases are done. These are:

a) a curved beam with constant curvature and cross-section,

b) a curved beam with variable curvature and constant cross-section,

c) a curved beam with constant curvature and variable cross-section,

d) a curved beam with variable curvature and cross-section.

The numerical results are found and compared with those existing in the literature. Observations concerning tendency of results given in tabular and graphical forms are discussed. The accuracy of obtained values is also improved by using the Richardson Extrapolation Method which is explained before.

3.2. Applications for Constant Curvature and Cross-Section

Vibration problem of a curved beam having constant cross-section and curvature is the simplest among the cases considered in this study. For this case, governing sixth order differential equation and boundary conditions in dimensionless form may be written as follows:

$$\frac{\overline{w}^{vi}}{\lambda^6}+\frac{2\overline{w}^{iv}}{\lambda^4}+\frac{\overline{w''}}{\lambda^2}=\Omega\left(\frac{\overline{w''}}{\lambda^2}-\overline{w}\right) \quad (3.1)$$

$$\left.\left(\frac{\overline{w'''}}{\kappa_0}+\overline{w'}\right)\delta\overline{w''}\right|_0^{sL}=0 \quad (3.2.a)$$

$$\left.\left(\frac{\overline{w'''}}{\kappa_0}+\overline{w'}\right)'\delta\overline{w'}\right|_0^{sL}=0 \quad (3.2.b)$$

$$\left.\left(\frac{\overline{w'''}}{\kappa_0}+\overline{w'}\right)\delta\overline{w}\right|_0^{sL}=0 \quad (3.2.c)$$

Although the problem can be solved analytically, it is also solved by using the Finite Difference Method to show accuracy and precision of the developed symbolic Mathematica program. Applying finite difference scheme to governing equation and also boundary conditions to approximate the derivatives, an aforementioned matrix eigenvalue problem can be obtained.

3.2.1 Comparison of Results with Those Existing in the Literature

The results obtained by using Finite Difference Method and extrapolated by using Richardson extrapolation method given in Equation 2.23 are compared with the analytical results available in the literature (Archer 1960).

Table 3.1. Comparison of dimensionless frequency parameters (Ω) for a fixed-fixed constant curvature and cross-sectioned curved beam with analytical results (Archer 1960) for different λ values

Mode	λ	**Ω (Archer 1960)**	**Finite Difference Method**			
			Ω(n =20)	**Ω**(n =50)	**Ω**(n =100)	Extrapolated
1	π	19.22	19.153	19.2128	19.2207	19.2233
2		93.15	91.4226	92.8919	93.0927	93.1591
3		321.5	306.428	318.9	320.621	321.191
4		756.3	698.581	748.321	755.262	757.567
1	1.5π	1.946	1.92795	1.94304	1.94513	1.94582
2		12.85	12.5576	12.8075	12.8421	12.8536
3		49.58	47.2065	49.2117	49.4893	49.5813
4		126.6	118.154	126.924	128.151	128.559
1	2π	0.3208	0.313559	0.319682	0.320546	0.320833
2		2.545	2.45968	2.53129	2.54134	2.54468
3		11.46	10.8293	11.3572	11.4309	11.4554
4		33.06	30.2361	32.6646	33.0058	33.1192

3.2.2 Discussion of Results

It is obvious from the Table 3.1 that present results are in a good agreement with analytical results (Archer 1960). It can be seen that for all λ values, errors are larger at higher modes than those at lower modes.

Table 3.1 also gives the convergence patterns and extrapolated values to improve the accuracy of results. If the first natural frequency of the curved beam with $\lambda = 2\pi$ is investigated, it can be concluded that, the extrapolated values from n= 20, 50, 100 are more reasonable than those values themselves. There is one exception of this fact, which is fourth vibration mode of $\lambda = 1.5\pi$, where frequency parameter found by n= 50 is closer to analytical value than extrapolated result.

Analyzing Table 3.1, it can also be seen that, if λ value increases, natural frequencies decrease. It is also reasonable, because the meaning of increase of λ-the dimensionless curvature parameter (which corresponds to the opening angle of the beam for constant curvature case) is that curved beam gets longer.

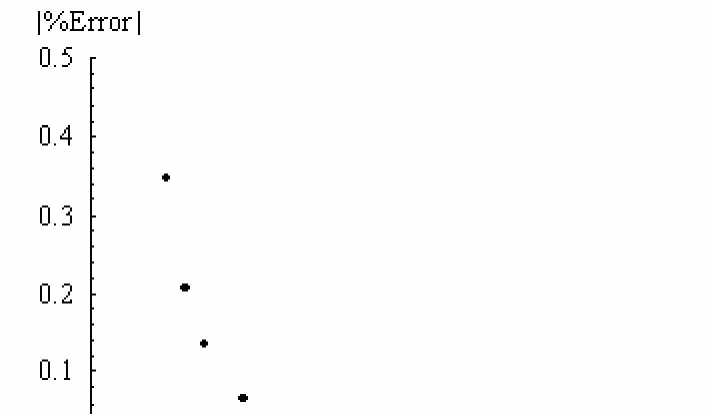

Figure 3.1. Absolute percentage error of dimensionless first natural frequency Ω for a fixed-fixed curved beam of $\lambda=\pi$ vs. n-number of grids

In Figure 3.1, absolute percentage error versus number of grids for first natural frequency of a curved beam having $\lambda= \pi$ is depicted. It is seen that, while number of grids increase, absolute percentage error decreases. At about n =75 absolute percentage error is almost zero. Although this problem is a simple one, maximum absolute percentage error is about 1.85% at n= 10. But still, as seen in Figure 3.1, the smallest value of absolute percentage error is about 0.0003% and it can be said that this is a good approximation to the exact value.

3.3. Applications for Variable Curvature and Constant Cross-Section

In this section, numerical applications for the case of variable curvature and constant cross-section are done for various functions of curvature. The challenge while solving this problem is differential equation which is tried to be solved to yield eigenvalues is order of six and has variable coefficients of differential terms. Due to the fact that differential equations having variable coefficients cannot be solved except for special cases, a numerical method becomes necessary to utilize.

The results obtained by using Finite Difference Method are tabulated for curved beam having variable curvature and constant cross-section. As descriptive examples: curved beams having neutral axis of which shape is a linearly, quadraticly, cubically, exponentially, and sinusoidally varying curve are chosen and results are given in tabular form.

For linearly varying curvature case, the chosen curvature variation function can be written as:

$$\kappa(s) = k_0 (1 + \delta \frac{s}{s_L}) \tag{3.3}$$

If differential equation is derived using the curvature form given in Equation 3.3., and non-dimensionalized employing proper parameters given in Equation 2.16 and Equation 2.19, equations are ready to solve using finite difference scheme.

Table 3.2. Dimensionless frequency parameters for fixed-fixed linearly varying curvature and constant cross-sectioned curved beam ($\lambda_o = \pi$, δ=0.1)

Mode	$n = 20$	$n = 50$	$n = 100$	Extrapolated
1.	17.9369	17.9971	18.005	18.0076
2.	88.4424	89.8868	90.0843	90.1496
3.	299.476	311.678	313.361	313.919

For quadraticly varying curvature case, the chosen curvature variation function can be written as:

$$\kappa(s) = k_0(1 + \delta(\frac{s}{s_L})^2) \tag{3.4}$$

If differential equation is derived using the curvature form given in Equation 3.4., and non-dimensionalized employing proper parameters given in Equation 2.16 and Equation 2.19, equations are ready to solve using finite difference scheme.

Table 3.3. Dimensionless frequency parameters for fixed-fixed quadraticly varying curvature and constant cross-sectioned curved beam ($\lambda_o = \pi$, δ=0.1)

Mode	$n = 20$	$n = 50$	$n = 100$	Extrapolated
1.	19.221	19.2791	19.2866	19.2891
2.	91.4524	92.9208	93.1215	93.1879
3.	305.666	318.073	319.784	320.351

For cubically varying curvature case, the chosen curvature variation function can be written as:

$$\kappa(s) = k_0(1 + \delta(\frac{s}{s_L})^3) \tag{3.5}$$

If differential equation is derived using the curvature form given in equation 3.5., and non-dimensionalized employing proper parameters given in Equation 2.16 and Equation 2.19, equations are ready to solve using finite difference scheme.

Table 3.4. Dimensionless frequency parameters for fixed-fixed cubically varying curvature and constant cross-sectioned curved beam ($\lambda_o = \pi$, δ=0.1)

Mode	$n = 20$	$n = 50$	$n = 100$	Extrapolated
1.	19.9072	19.9644	19.9718	19.9742
2.	93.0434	94.5262	94.7289	94.796
3.	308.788	321.3	323.026	323.598

For exponentially varying curvature case, the chosen curvature variation function can be written as:

$$\kappa(s) = k_0\, e^{\delta \frac{s}{s_L}} \tag{3.6}$$

If differential equation is derived using the curvature form given in Equation 3.6., and non-dimensionalized employing proper parameters given in Equation 2.16 and Equation 2.19, equations are ready to solve using finite difference scheme.

Table 3.5 Dimensionless frequency parameters for fixed-fixed exponentially varying curvature and constant cross-sectioned curved beam ($\lambda_0 = \pi$, δ=0.1)

Mode	$n = 20$	$n = 50$	$n = 100$	Extrapolated
1.	17.938	17.9981	18.006	18.0086
2.	88.4334	89.8776	90.0752	90.1406
3.	299.42	311.618	313.302	313.86

For sinusoidally varying curvature case, the chosen curvature variation function can be written as:

$$\kappa(s) = k_0\,(1 + Sin(\delta \frac{s}{s_L})) \tag{3.7}$$

If differential equation is derived using the curvature form given in Equation 3.7., and non-dimensionalized employing proper parameters given in Equation 2.16 and Equation 2.19, equations are ready to solve using finite difference scheme.

Table 3.6. Dimensionless frequency parameters for fixed-fixed sinusoidally varying curvature and constant cross-sectioned curved beam ($\lambda_0 = \pi$, δ=0.1)

Mode	$n = 20$	$n = 50$	$n = 100$	Extrapolated
1.	17.9358	17.996	18.0039	18.0065
2.	88.4402	89.8845	90.0821	90.1475
3.	299.473	311.674	313.358	313.916

Tables 3.2-3.4 need to be considered together because they all have the same polynomial function form. As it can be seen from these tables, if the order of the polynomial increases, dimensionless frequency parameters also increase, for the same dimensionless curvature parameter. Besides, it can be seen from Tables 3.5 and 3.6 that dimensionless frequency parameters for curved beams having exponentially and sinusoidally varying curvature are almost equal to each other for the same dimensionless curvature parameter.

3.3.1. The Effects of Parameters Defining Variation of Curvature

In this section, the effects of parameters defining of variation of curvature (δ) and dimensionless curvature parameter are investigated. Results are given in graphical form for various cases of curvature variation.

If curvature variation is chosen considering Equation 3.3 (i.e. linearly varying curvature), the effect of parameters can be depicted in the following figure:

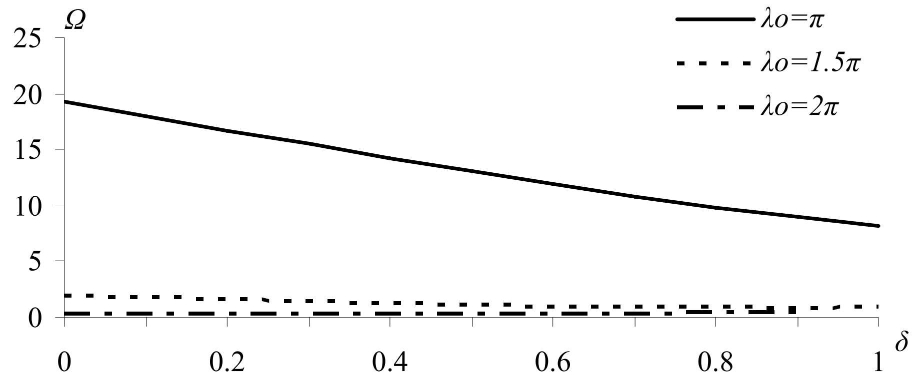

Figure 3.2. Dimensionless first natural frequency for a fixed-fixed curved beam of different λ_0's vs. δ-linear variation parameter (n=100)

In Figure 3.2, dimensionless first natural frequency parameter of a curved beam having linearly varying curvature versus δ-variation parameter can be seen for various λ_0-dimensionless curvature parameters. First thing to deduce from Figure 3.2 is that if dimensionless curvature parameter λ_0 increases, dimensionless frequency parameter decreases for a curved beam having linearly varying curvature. When λ_0=π, dimensionless frequency parameter has the biggest form then, and while δ-variation parameter gets bigger, it decreases.

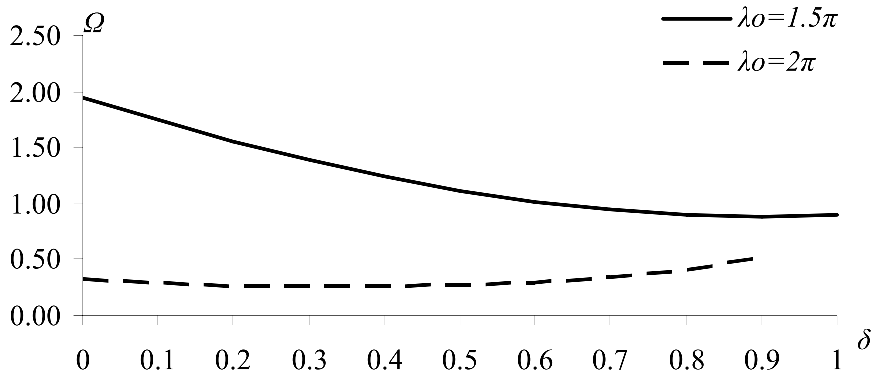

Figure 3.3. Dimensionless first natural frequency for a fixed-fixed curved beam of two different λ_0's vs. δ-linear variation parameter (n=100)

For other values of dimensionless curvature parameters tendency of change of dimensionless natural frequency parameters is different as it is seen in Figure 3.3. When λ_0=1.5π dimensionless frequency parameter increases as δ-linear variation parameter reaches the value of almost 0.8. After that value, Ω almost remains constant. On the other hand, different than other λ_0 values, when λ_0=2π dimensionless frequency parameter decreases as δ-linear variation parameter reaches the value of almost 0.4. After that value, Ω starts increasing.

If curvature variation is chosen considering Equation 3.4 (i.e. quadraticly varying curvature), the effect of parameters can be depicted in the following figure,

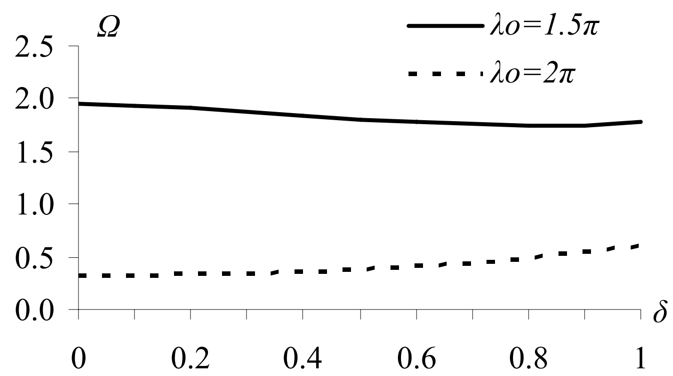

Figure 3.4. Dimensionless first natural frequency for a fixed-fixed curved beam of two different λ_0's vs. δ-quadratic variation parameter (n=100)

In Figure 3.4, dimensionless first natural frequency parameter of a curved beam having quadraticly varying curvature versus δ-variation parameter can be seen for two different λ_0-dimensionless curvature parameters. First thing to deduce from Figure 3.4 is that if dimensionless curvature parameter λ_0 increases, dimensionless frequency parameter decreases for a curved beam having quadraticly varying curvature. When $\lambda_0=1.5\pi$ dimensionless frequency parameter shows a small decrease as δ-quadratic variation parameter increases. On the other hand, different than other λ_0 values, when $\lambda_0=2\pi$ dimensionless frequency parameter increases.

3.3.2 Discussion of Results

In this section, curved beams having constant cross-section and variable radius of curvature are investigated. Results are given in tabular and graphical form. It is noticed that when variation parameters are constant, for all cases, dimensionless natural frequencies decrease as dimensionless curvature parameters increase. On the other hand, the tendency of change of dimensionless frequency parameter when dimensionless curvature parameters are constant does not have a standard increase or decrease behavior. Comments on specific cases having non-standard tendency are mentioned in previous subsection.

3.4. Applications for Constant Curvature and Variable Cross-Section

In this section, numerical application for the case of constant curvature and variable cross-section is done for linear functions of width and depth. The challenge while solving this problem is differential equation which is tried to be solved to yield eigenvalues is order of six and has variable coefficients of differential terms. Due to the fact that differential equations having variable coefficients cannot be solved except for special cases, a numerical method becomes necessary to utilize.

The results obtained by using Finite Difference Method are tabulated for curved beam having constant curvature and variable cross-section. As descriptive examples, curved beams of which width and depth are linear functions of independent variable are chosen.

For the case of linearly varying width and depth, the chosen width and depth variation functions can be written as in following equations, respectively:

$$b(s) = b_0(1+\gamma\frac{s}{s_L}) \tag{3.8.a}$$

$$h(s) = h_0(1+\beta\frac{s}{s_L}) \tag{3.8.b}$$

where b_0 and h_0 defines width and depth at s=0. If differential equation is derived using width ($b(s)$) and depth ($h(s)$) variation forms given in Equation 3.8., and non-dimensionalized employing proper parameters given in Equations 2.16 and 2.19, equations are ready to solve using finite difference scheme.

Dimensionless frequency parameters for curved beams having linearly varying width and depth with two different dimensionless curvature parameters are tabulated in Table 3.7. As it can be seen from table, if λ increases dimensionless frequency parameters tend to decrease. If Table 3.7 is studied, it is seen that, for given grid numbers, results are converged well. Again using Richardson Extrapolation Method, results are corrected.

Table 3.7. Dimensionless frequency parameters for fixed-fixed curved beam having constant curvature and linearly varying width and depth (γ=0.1, β=0.1)

Mode	n = 20	n = 50	n = 100	Extrapolated
$\lambda = 1.5\pi$				
1.	2.13028	2.14736	2.14972	2.1505
2.	13.7896	14.0659	14.1041	14.1168
3.	51.9200	54.1319	54.4381	54.5396
$\lambda = 2\pi$				
1.	0.3511	0.3578	0.3588	0.3591
2.	2.7034	2.7828	2.7939	2.7975
3.	11.9084	12.4907	12.5720	12.5990

3.4.1. The Effects of Parameters Defining Variation of Cross-Section

In this section, the effects of parameters defining of variation of width (γ) and depth (β) and dimensionless curvature parameter on dimensionless frequency parameters are investigated. Results are given in graphical form If cross-section variation is chosen considering Equation 3.8 (i.e. linearly varying width and depth), the tendency of change of dimensionless frequency parameters can be depicted as follows;

In Figure 3.5.a, dimensionless first natural frequency parameter of a curved beam having constant curvature versus γ-linear width variation parameter can be seen for β=0.1. It is seen in the figure that, firstly frequency parameter decreases and after the γ reaches the value of 0.2, tendency changes and parameter starts to increase.

In Figure 3.5.b, dimensionless first natural frequency parameter of a curved beam having constant curvature versus γ-linear width variation parameter can be seen β=0.2. It is seen in the figure that, firstly frequency parameter decreases and after the γ reaches the value of 0.3, tendency changes and parameter starts to increase.

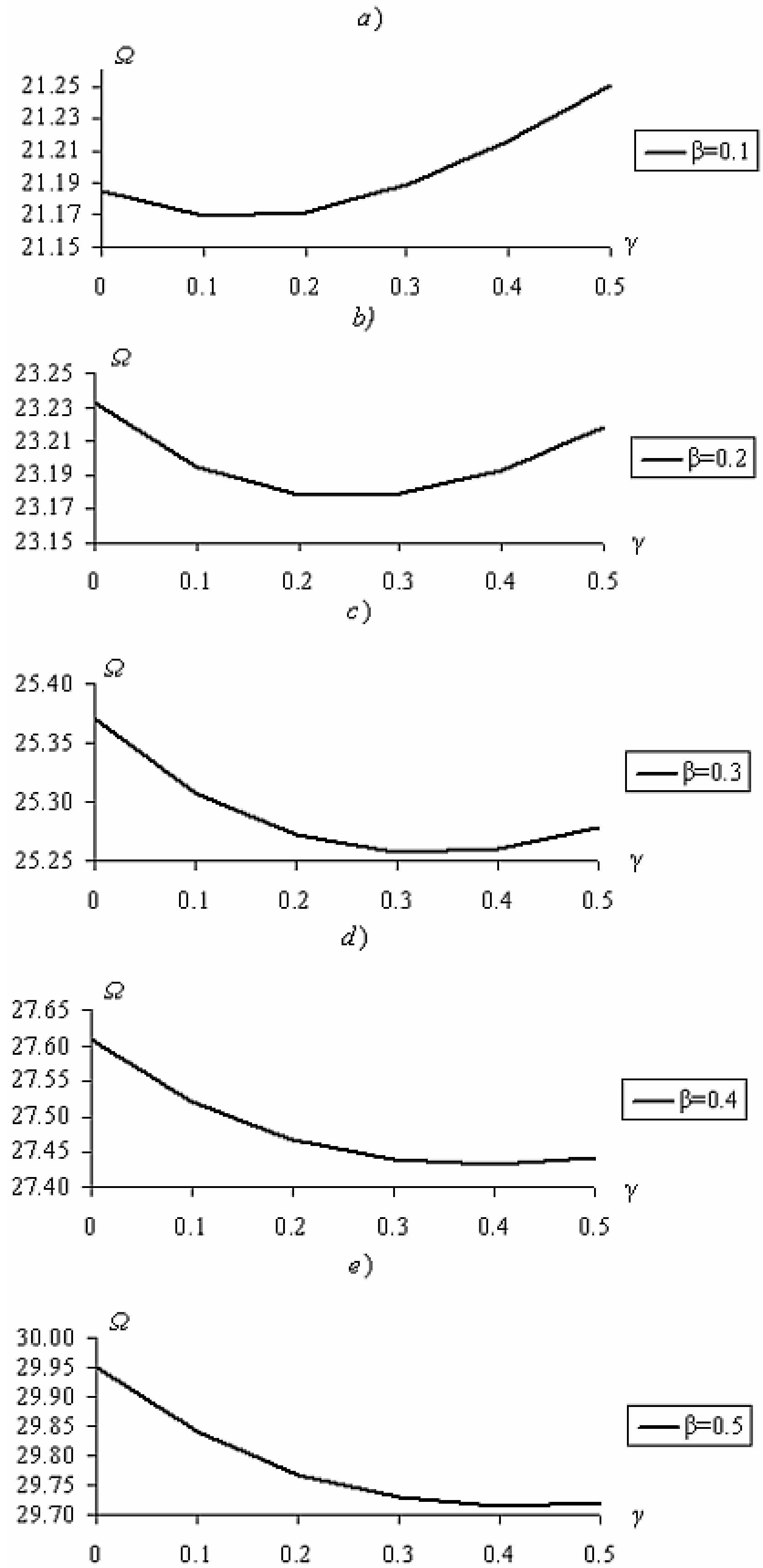

Figure 3.5. Dimensionless first natural frequency for a fixed-fixed curved beam of λ=π vs. γ-width variation parameter for different β values (n=100)

In Figure 3.5.c., system starts to show a different behavior. Although again frequency parameter firstly decreases, increase of the results is not as much as in previous graphics.

In Figure 3.5.d. and e., dimensionless frequency parameters decrease as γ reaches the value of almost 0.35. After that value of γ frequency parameter remains almost constant.

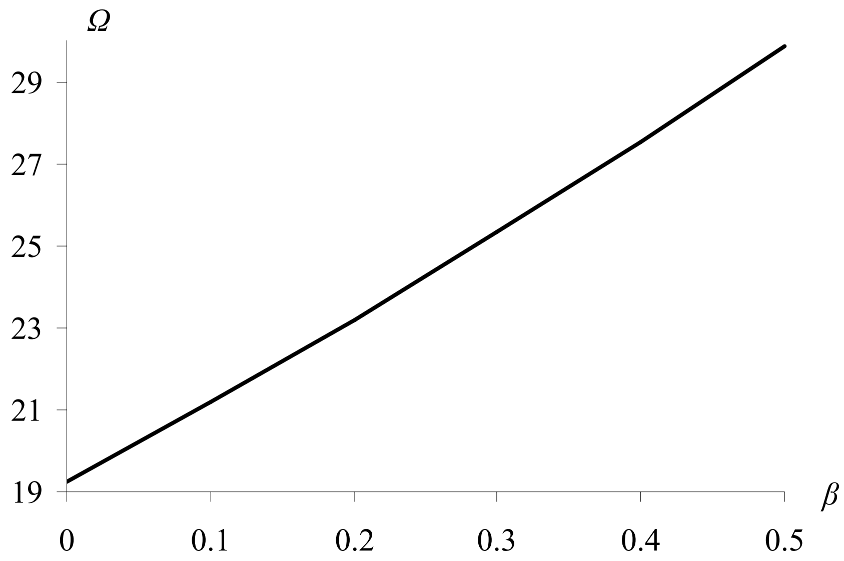

Figure 3.6. Dimensionless first natural frequency for a fixed-fixed curved beam of $\lambda=\pi$ and $\gamma=0.1$ vs. β-linear cross-section variation parameter ($n=100$)

In Figure 3.6., dimensionless frequency parameter increase linearly as β increases. If Figures 3.5 and 3.6 are compared, it can be said, system is more delicate according to change of β. Because, when β increases to 0.5 from 0(γ is constant); rate of change of dimensionless frequency parameter is about 45%, but when γ increases to 0.5 from 0(β is constant); rate of change of frequency parameter is about -0.67%.

3.4.2 Discussion of Results

In this section, curved beams having constant curvature and linearly varying width and depth are investigated. Results are given in tabular and graphical form. It is noticed that when variation parameters are constant, for all cases, dimensionless natural frequencies decrease as dimensionless curvature parameters increase.

The tendency of change of dimensionless frequency parameter when dimensionless curvature parameters are constant does not have a standard increase or decrease behavior. Comments on specific cases having non-standard tendency are mentioned in previous subsections.

As mentioned before dimensionless frequency parameter increase linearly as β-linear depth variation parameter increases. On the other hand, when β and λ are constant, dimensionless frequency parameter shows different behaviors as γ-linear width variation increases. In those cases, when β=0.4 and 0.5 the tendency of curves can be said to be nearly the same.

3.5. Applications for Variable Curvature and Cross-Section

In this section, numerical application for the case of variable curvature and variable cross-section is done for linear functions of curvature, width, and depth. The challenge while solving this problem is differential equation which is tried to be solved to yield eigenvalues is order of six and has variable coefficients of differential terms. Due to the fact that differential equations having variable coefficients cannot be solved except for special cases, a numerical method becomes necessary to utilize.

The results obtained by using Finite Difference Method are tabulated for curved beam having variable curvature and cross-section. As descriptive examples, curved beams of which curvature, width, and depth are linear functions of independent variable are chosen.

For the case of linearly varying curvature, width, and depth, the chosen curvature, width, and depth variation functions function can be written as:

$$\kappa(s) = k_0(1+\delta\frac{s}{s_L}) \tag{3.9.a}$$

$$b(s) = b_0(1+\gamma\frac{s}{s_L}) \tag{3.9.b}$$

$$h(s) = h_0(1+\beta\frac{s}{s_L}) \tag{3.9.c}$$

where k_0, b_0 and h_0 values of curvature, width, and depth at s=0 respectively. If differential equation is derived using curvature ($\kappa(s)$), width ($b(s)$), and depth ($h(s)$) variation forms given in Equation 3.9., and non-dimensionalized employing proper parameters given in Equations 2.16 and 2.19, equations are ready to solve using finite difference scheme.

Table 3.8. Dimensionless frequency parameters for fixed-fixed curved beam having constant curvature and linearly varying width and depth (δ=0.1, γ=0.1, β=0.1)

Mode	n = 20	n = 50	n = 100	Extrapolated
λ_0 = 1.5π				
1.	1.9186	1.9362	1.9386	1.9394
2.	12.9675	13.2378	13.2753	13.2877
3.	49.7862	51.9321	52.2295	52.3281
λ_0 = 2π				
1.	0.3167	0.3234	0.3244	0.3247
2.	2.4379	2.5155	2.5264	2.5300
3.	11.0855	11.6470	11.7256	11.7517

Dimensionless frequency parameters for curved beams having linearly varying curvature, width, and depth with two different dimensionless curvature parameters are tabulated in Table 3.8. As it can be seen from table, if λ_0 increases dimensionless frequency parameters tend to decrease. If Table 3.8 is studied, it is seen that, for given grid numbers, results are converged well. Again using Richardson Extrapolation Method, results are corrected.

3.5.1. The Effects of Parameters Defining Variation of Curvature and Cross-Section

In this section, the effects of parameters defining of variation of width (γ) and depth (β), curvature (δ), and of λ_0 on Ω are investigated.

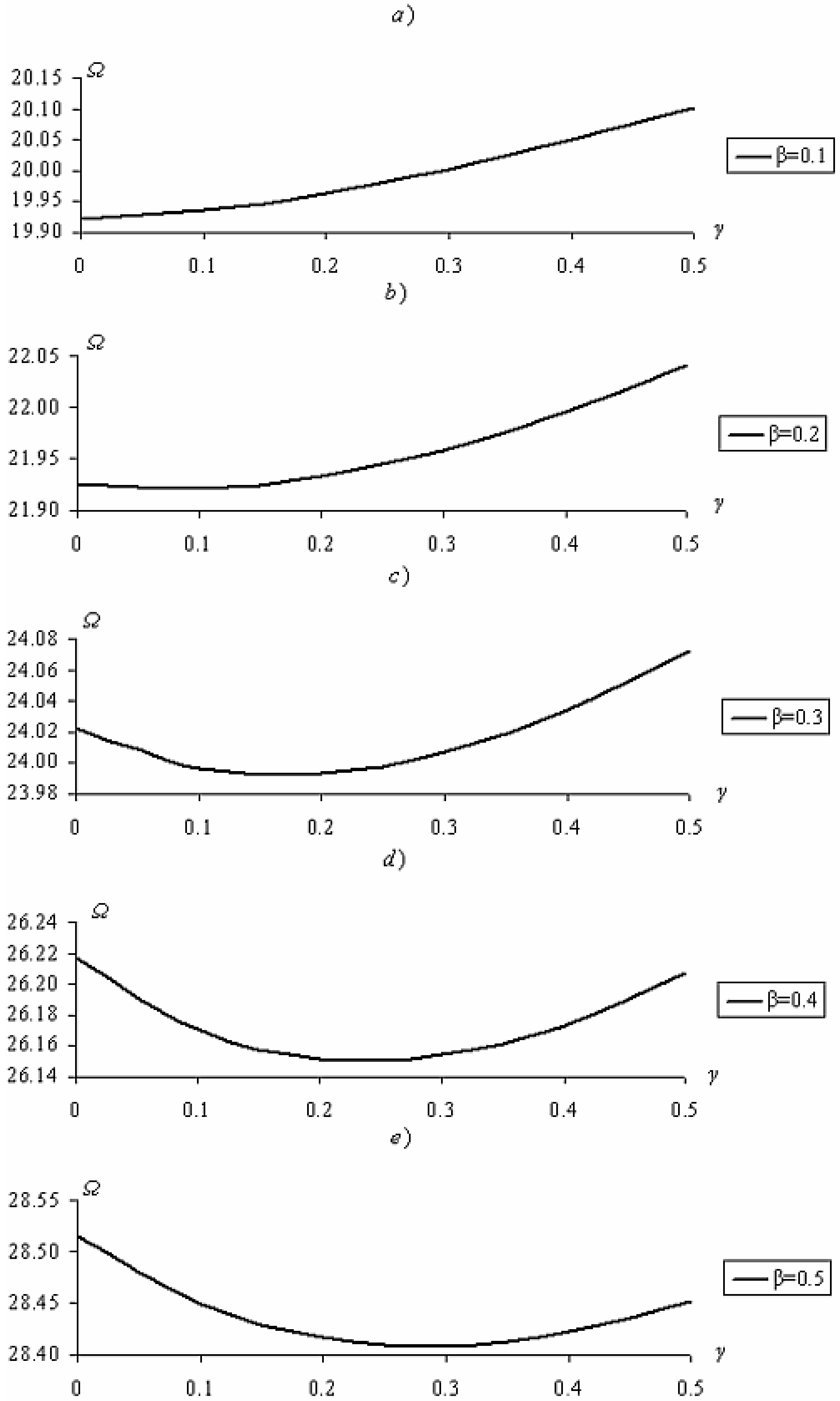

Figure 3.7. Dimensionless first natural frequency for a fixed-fixed curved beam of $\lambda_0=\pi$ and $\delta=0.1$ vs. γ-width variation parameter for different β values ($n=100$)

In Figure 3.7.a, dimensionless first natural frequency parameter of a curved beam having variable curvature versus γ-linear width variation parameter can be seen for β=0.1, δ=0.1. It is seen in the figure that dimensionless frequency parameters shows an increase. It is notified that, this tendency is different from Figure 3.5.a.

In Figure 3.7.b, dimensionless first natural frequency parameter of a curved beam having variable curvature versus γ-linear width variation parameter can be seen β=0.2, δ=0.1. It is seen in the figure that, firstly frequency parameter decreases and after the γ reaches the value of 0.1, tendency changes and parameter starts to increase.

In Figure 3.7.c, system starts to show a different behavior. Although again frequency parameter firstly decreases, increase of the results is not as much as in previous graphics.

In Figure 3.7.d and e, dimensionless frequency parameters decrease as γ reaches the value of almost 0.2 and 0.3 respectively. After that value of γ frequency parameter starts to increase slightly.

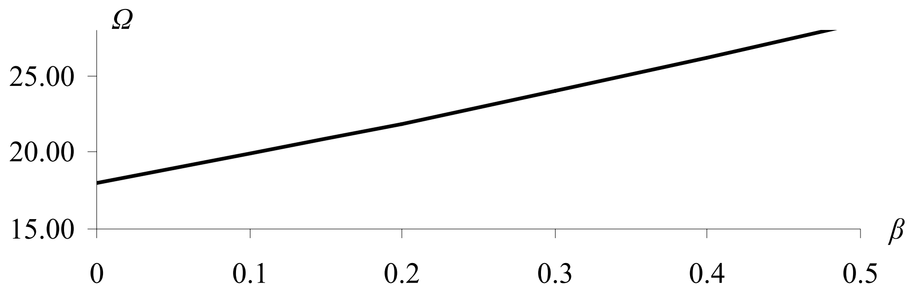

Figure 3.8. Dimensionless first natural frequency for a fixed-fixed curved beam of λ_0=π, δ=0.1 and, γ=0.1 vs. β-linear depth variation parameter (n=100)

In Figure 3.8., dimensionless frequency parameter increase linearly as β increases. If Figures 3.7 and 3.8 are compared, it can be said, system is more delicate according to change of β. Because, when β increases to 0.5 from 0(γ is constant); rate of change of dimensionless frequency parameter is about 36%, but when γ increases to 0.5 from 0(β is constant); rate of change of frequency parameter is about -0.21%.

In order to understand the effect of parameter defining variation of curvature of a curved beam having variable cross-section, parameter of linear width variation γ is kept constant, and for different linear depth variation parameters (β) change in the frequency is plotted in Figure 3.9.

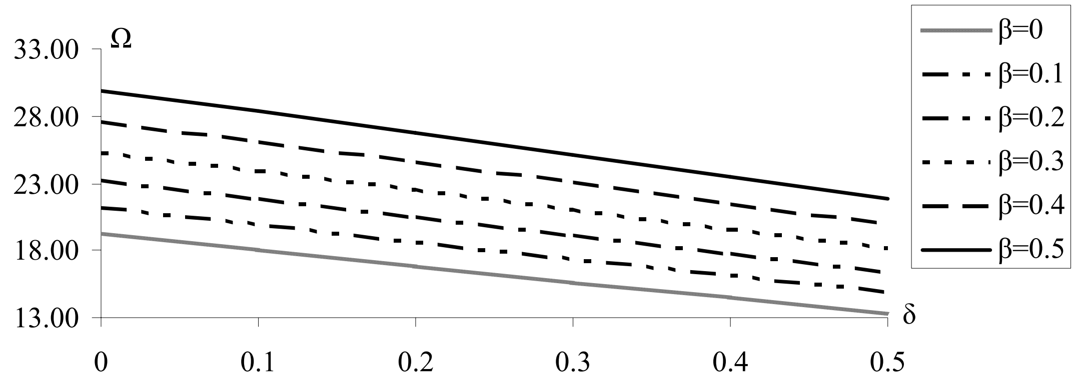

Figure 3.9. Dimensionless first natural frequency for a fixed-fixed curved beam of λ_0=π, γ=0.1 vs. δ-linear curvature variation parameter for different β values (n=100)

In Figure 3.9. dimensionless first natural frequency linearly decreases as δ - linear depth variation increases, as other two parameters kept constant. Besides, it is notified that while linear width variation parameter increases, dimensionless frequency parameter increases.

3.5.2 Discussion of Results

In this section, curved beams having linearly varying curvature, width, and depth are investigated. Results are given in tabular and graphical form. It is noticed that when variation parameters are constant, for all cases, dimensionless natural frequencies decrease as dimensionless curvature parameters increase.

The tendency of change of dimensionless frequency parameter when dimensionless curvature parameters are constant does not have a standard increase or decrease behavior. Comments on specific cases having non-standard tendency are mentioned in previous subsections.

As mentioned before dimensionless frequency parameter increases linearly as β-linear depth variation parameter increases. On the other hand, when β, δ, and λ_0 are constant, dimensionless frequency parameter shows different behaviors as γ-linear width variation increases.

In order to understand the effect of variation of curvature, Tables 3.7 and 3.8 can be compared with each other. It is obvious that dimensionless first natural frequency of a curved beam having linearly varying curvature and cross-section is smaller than the beam of constant curvature and linearly varying cross-section.

CHAPTER 4

CONCLUSIONS

In this study, in-plane vibrations of curved beams having variable curvature and cross-section are investigated. For convenience and progressive requirements, vibration characteristics of curved beams having constant/variable curvature and cross-section are also examined.

Derivations of equations governing free vibrations are presented by using Newtonian and Hamiltonian approaches. After this step, non-dimensionalization process is carried out and dimensionless parameters are introduced. Derivatives in dimensionless governing equations are replaced by respective central difference equivalents. Therefore, the problem is converted to matrix eigenvalue problem and then eigenvalues are obtained. The obtained eigenvalues are corrected also by using the Richardson Extrapolation Method.

Applicability of the FDM is showed for such vibration analyses of curved beams. A Mathematica code is developed and used to obtain eigenvalues. In order to test the accuracy and precision of the program developed, analysis of vibrations of curved beams having constant curvature and cross-section is done. Results are compared with given in literature and a good agreement is achieved.

It is understood that if dimensionless curvature parameter (for cases of constant curvature; opening angle) increases, dimensionless frequency parameter decreases. The decrease of frequency parameter when curvature parameter decreases from 2π to 1.5π is more dramatical than when curvature parameter decreases from 1.5π to π.

The results show that, for curved beams of variable curvature and constant cross-section, the effects of curvature function variation parameter depend on dimensionless curvature parameter. The results show different behaviours for different curvature variation parameters.

The tendency of change of dimensionless frequency parameter depending on width variation parameter does not show a standard increase or decrease behavior depending on depth variation parameter. Dimensionless frequency parameter increases linearly as linear depth variation parameter increases.

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