ABELIAN - NON ABELIAN MIXING AND COSMIC INFLATION IN BORN-INFELD TYPE GRAVITY

A Thesis Submitted to
the Graduate School of Engineering and Sciences of
İzmir Institute of Technology
in Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

in Physics

by
ASLI ALTAŞ

December 2013
İZMİR

We approve the thesis of **ASLI ALTAŞ**

Examining Committee Members:

Prof. Dr. DURMUŞ ALİ DEMİR
Department of Physics,
İzmir Institute of Technology

Assoc. Prof. Dr. İSMAİL TURAN
Department of Physics,
Middle East Technical University

Assist. Prof. Dr. FATİH ERMAN
Department of Mathematics,
İzmir Institute of Technology

19 December 2013

Prof. Dr. Durmuş Ali Demir
Supervisor, Department of
İzmir Institute of Technology

Prof. Dr. Nejat Bulut
Head of the Department of
Physics

Prof. Dr. R. Tuğrul Senger
Dean of the Graduate School of
Engineering and Sciences

ACKNOWLEDGMENTS

While I was studying about my thesis, there are some people around me who help and support me and hence deserve my special thanks.

Firstly, I would like to express my endless respect and gratitude to my supervisor Prof. Dr. Durmuş Ali DEMİR. It's a great pleasure for me to work with him.

Besides my supervisor, I would like to thank warmly to the members of committee Assoc. Prof. Dr. İsmail TURAN and Assist. Prof. Dr. Fatih ERMAN for their contribution and giving suggestions.

All members of my family have supported me since when I decided to work on physics. I am not able to explain my indebtedness to them. Thank you for loving and promoting me.

Selin SOYSAL, who is my closest friend. I would like to thank her since she bears up to me always and in all situations. The most important thing which affects me to decide study on physics is to dream with you. I will never forget our dreams...

I would like to thank a lot Yaşar HİÇYILMAZ. He supports and encourages me always. It is not possible to forget his contribution.

Also Hale SERT and Barış ATEŞ, thank you for your endless support. I am fortunate for meeting with you.

I would like to thank Sıla Övgü KORKUT and Koray SEVİM . They are always with me.

Finally I would like to thank all of my friends from IZTECH for believing and supporting me.

ABSTRACT

ABELIAN - NON ABELIAN MIXING AND COSMIC INFLATION IN
BORN-INFELD TYPE GRAVITY

General Relativity (GR), which forms the basic framework for understanding cosmological and astrophysical phenomena, is based solely on general covariance. Therefore, the theory admits extensions regarding various phenomena related to inflation, massive gravity, dark energy etc.

In this thesis work we study Born-Infeld type extensions of the GR. There are mainly two parts in the thesis: The extension based on Ricci tensor (already known in literature) and a novel extension based on Riemann tensor. We call them respectively Born-Infeld-Einstein (BIE) and Born-Infeld-Riemann (BIR) extensions. The BIR formalism is being proposed and studied in this thesis work. In a comparative fashion, we study these two extensions for determining their implications for

1. Mixing between Abelian and Non-Abelian gauge fields, and
2. Inflationary phase of cosmic evolution.

As we prove explicitly, the two approaches yield distinct predictions for these phenomena. We emphasize that a slow-roll inflationary dynamics is naturally realized in BIR. The mixing between Abelian and Non-Abelian sectors enables cosmic photon production in inflationary phase.

ÖZET

BORN-INFELD KÜTLE ÇEKİM KURAMLARINDA
ABELYEN - ABELYEN OLMAYAN KARIŞIMI VE KOZMİK ŞİŞME

Kozmolojik ve astrofiziksel olguların anlaşılabilmesi için temel bir çerçeve olan Genel Görelilik (GR), genel kovaryans ilkesine bağlıdır. Buna göre, bu kuram kozmik şişme, karanlık enerji gibi değişik olguları açıklayabilecek genişletilmelere izin verir.

Bu tezde GR'ın Born-Infeld türü genişletilmiş formları üzerinde çalıştık. Esas olarak bu tez iki bölüme ayrılmıştır: Ricci tensörüne bağlı bir genişletme (bu tarz bir genişletme literatürde bulunmaktadır) ve Riemann tensörünün kendisine bağlı, alışılmışın dışında bir genişletme. Sırasıyla bunlar Born-Infeld-Einstein ve Born-Infeld-Riemann (BIE ve BIR) olarak adlandırılır. Bu tezde, bu iki yaklaşımın

1. Abelyen ve Abelyen olmayan ayar alanlarının karışımı
2. Kozmik evrimin şişme fazı

durumlarındaki etkileri üzerinde çalıştık. Açık bir şekilde gösterdiğimiz gibi, bu iki yaklaşım yukarıda verilen olgular için farklı öngörüler sağlar. Bu tezde kozmik şişmenin dinamiğinin doğal bir yolla BIR'dan çıktığını ve Abelyen ve Abelyen olmayan alanların karışımının, evrenin şişme fazında kozmik foton üretimine izin verdiğini gösterdik.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Understanding of Universe has been a giant puzzle since ancient times. Observations and experiments showed that it was born with Big Bang and then started to expand. This expansion caused to form of subatomic particles such as quarks, leptons and then fundamental interactions of nature. Now, we know that there are four fundamental interactions which are Electromagnetism, Weak Interaction, Strong Interaction and Gravitation. All theoretical physicist's dream is to unify these fundamental interactions. In the second half of the 20th century, Electromagnetism and Weak Interaction was unified in a theory which was called as Electroweak Theory. Moreover, Strong Interaction was added to them and this model was called as Standard Model.

So then, what about Gravitation?

Einstein gave a different perspective to gravity in the paper of General Relativity (GR) in 1916. He claimed that Gravity is the effects of curving of spacetime. Until now, it passed a numerous experimental tests. However, it still posses some problems at galactic and large scales. For example, it needs extra ingredients, called as Dark Matter and Dark Energy, to explain galaxy rotation curves. The another way to explain phenomena which could not explained by GR is to modify General Relativity. There are several types of modification and one of them is Born-Infeld Type Gravity (Born-Infeld, 1934). There are some papers which claim that Born-Infeld Type Gravity could explain galaxy rotation curves without Dark Matter (Banados, 2008).

In addition to the possibility of explaining phenomena which GR could not explain, Born- Infeld Type Gravity theories also give a chance to unify gravitation with other interactions. In this thesis, we worked on Born-Infeld-Einstein gravity which is obtained by using rank-2 tensor fields such as Ricci tensor and Metric tensor. Also we analyse Born-Infeld-Riemann Gravity which is obtained by using rank-4 tensor fields, especially Riemann tensor itself. Hence, to see their cosmological implications, we examine both theories in homogeneous and isotropic background, FRW background.

In this thesis we use metric theory of gravity and natural units. Detailed explanation is given in Appendix A.

The thesis was organised as follows. In Chapter 2, we gave a brief summary about gauge theories. Especially we examined Abelian Gauge Theories and Non Abelian Gauge

Theories to understand the importance of the mixing of these terms.

In Chapter 3, we explain Born-Infeld Theory and give their detailed calculations. As a result of this chapter we noticed that what the determinant notion give us.

In Chapter 4, we explain original Born-Infeld-Einstein Theory and then examine what happens if we add some new terms to the action and we realize that our modification brings in Abelian -Non Abelian Mixing naturally.

In Chapter 5, we analyse Born-Infeld-Riemann Gravity and its solutions due to our modification. This type of modification gave us Abelian-Non Abelian mixing terms naturally too.

In Chapter 6, firstly we introduced inflationary cosmology briefly, then analysed both Born- Infeld- Einstein and Born- Infeld- Riemann type modification in Friedmann-Robertson- Walker (FRW)background. From there, a question occurs: These modifications gave us inflation?

Finally, in Chapter 7 we concluded the thesis.

CHAPTER 2

GAUGE THEORIES

Symmetries are the most important concept of the physics. Noether Theorem says that, if a quantity remains invariant under appropriate changes, then there should be *symmetry*. We can classify symmetry in two parts: Discrete and Continuous symmetries. Lattice symmetries in condensed matter physics is an example of discrete symmetries while spacetime symmetries in quantum field theory is for continuous symmetries (Beisert, 2013).

In the language of mathematics, symmetries are described by group structure. In this thesis we deal with continuous symmetries and groups, which are also known as Lie Group. One of these symmetries is *gauge symmetry* which is continuous, local, internal symmetry (Enberg, 2013). If we want to make one more definition for gauge symmetry, we can say that any physical quantity has gauge symmetry if it remains unchanged under gauge transformation. So then, what is gauge transformation?

Gauge transformations are phase transformations. They can be global or local. Global means that, transformation does not depend on spacetime whereas local means that transformation depends on it. The set of gauge transformations constitutes gauge group. Moreover the theory which is invariant under gauge transformations is called Gauge Theory. From there, whole gauge theories have gauge symmetry. Gauge theory is one of the most important notion of physics since it is modern theory of fundamental interactions. The crucial role is played by phase factor which is associated with a parallel transport in an external gauge field (Makeenko, 1996).

The well-known example of gauge theory is Electromagnetism. Moreover, the Standard Model which is a unified description of all interactions of known particles except gravity is also gauge theory. Gauge theories could be Abelian which is commutative or non Abelian which is non commutative. Electromagnetism is an Abelian Gauge Theory and then its generalisation to non Abelian Gauge Theory, which is known as Yang-Mills theory, was given by Yang-Mills in 1950s . Standard model is also an example of non Abelian Gauge Theory.

In this thesis, we demand to obtain Abelian-non Abelian mixing terms which can have cosmological implications. Then, in the following sections we examine Abelian and non Abelian Gauge theories one by one in order to understand meaning of mixing.

2.1. Abelian Gauge Theory

Abelian gauge theory is a theory which remains invariant under Abelian gauge transformation. Under the roof of Particle Physics we deal with *unitary* transformations. In the set of unitary transformations, the only Abelian one is $U(1)$ gauge group which has one parameter and consists of all unitary 1×1 matrices. To understand how it remains invariant, let us examine briefly real scalar field Lagrangian, which is also called Klein-Gordon Lagrangian,

$$\mathcal{L}_{K-G} = \partial_\mu \Phi^* \partial^\mu \Phi + m^2 \Phi^* \Phi$$

where Φ is the real scalar field (we hold the complex conjugate since transformation contains complex components while the field itself does not). Under global $U(1)$ gauge transformations, which is the group of complex phase rotations, $\Phi(x)$ becomes

$$\Phi(x) \longrightarrow \Phi'(x) \equiv \exp(i\alpha)\Phi(x) \tag{2.1}$$

Since the transformation is global, phase, α, is independent of spacetime coordinate. Then the new Lagrangian which is a function of $\Phi'(x)$ is

$$\begin{aligned} \mathcal{L}'_{K-G} &= (\partial_\mu \Phi')^* \partial^\mu \Phi' + m^2 \Phi^{*\prime} \Phi' \\ &= \partial_\mu (\exp(-i\alpha)\Phi^*) \partial^\mu (\exp(i\alpha)\Phi) + m^2 \exp(-i\alpha)\Phi^* \exp(i\alpha)\Phi \end{aligned} \tag{2.2}$$

Hence due to independence of spacetime point of α, there is no contribution which comes from the exponential. Then the transformed Lagrangian takes the form as

$$\begin{aligned} \mathcal{L}'_{K-G} &= \partial_\mu \Phi^* \partial^\mu \Phi + m^2 \Phi^* \Phi \\ \mathcal{L}'_{K-G} &= \mathcal{L}_{K-G} \end{aligned} \tag{2.3}$$

It means that Lagrangian remains unchanged under global $U(1)$ transformation. This is the Gauge Symmetry!

Let us also examine local gauge transformation in which α depends on spacetime, i.e. $\alpha(x)$. Thus transformation can be written as

$$\Phi(x) \longrightarrow \Phi'(x) \equiv \exp(i\alpha(x))\Phi(x) \tag{2.4}$$

Then Lagrangian becomes,

$$\begin{aligned} \mathcal{L}'_{K-G} &= (\partial_\mu \Phi')^* \, \partial^\mu \Phi' + m^2 \Phi^{*\prime} \Phi' \\ &= \partial_\mu \left(\exp(-i\alpha(x))\Phi^*\right) \partial^\mu \left(\exp(i\alpha(x))\Phi\right) \\ &+ m^2 \exp(-i\alpha(x))\Phi^* \exp(i\alpha(x))\Phi \end{aligned} \tag{2.5}$$

Since phase factor depends on spacetime, the derivative operator acts on it also! Then Lagrangian is written as

$$\begin{aligned} \mathcal{L}'_{K-G} &= \left[-i \exp(-i\alpha(x))\Phi^* \partial_\mu \alpha(x) + \exp(-i\alpha(x))\partial_\mu \Phi^*\right] \\ &\times \left[i \exp(i\alpha(x))\Phi \partial^\mu \alpha(x) + \exp(i\alpha(x))\partial^\mu \Phi\right] \\ &+ m^2 \exp(-i\alpha(x))\Phi^* \exp(i\alpha(x))\Phi \\ &= \left[-i\Phi^* \partial_\mu \alpha(x) + \partial_\mu \Phi^*\right] \left[i\Phi \partial^\mu \alpha(x) + \partial^\mu \Phi\right] \\ &+ m^2 \Phi^* \Phi \end{aligned} \tag{2.6}$$

At first glance, the Lagrangian seems not to be invariant. However, if one changes derivative operator into gauge covariant derivative operator which is

$$D_\mu = \partial_\mu - igA_\mu$$

and applying gauge transformation to the *gauge field* A_μ as

$$A_\mu \longrightarrow A_\mu + \frac{1}{g}\partial_\mu \alpha(x)$$

it is straight forward to see that we have a gauge invariant Lagrangian under local gauge transformations (Lewis, 2009).

In this step we introduce a new field, gauge field, to the theory. In Particle Physics, gauge fields have a crucial role. They are force carriers of the gauge theories. In the light of gauge transformation of gauge field, we can deduce that the mass term of gauge field, $A_\mu A^\mu$ is not invariant under gauge transformation. Thus, to preserve gauge invariance, gauge fields should be massless. This explains why we need Higgs Mechanism (Griffiths, 1987).

Due to the force carrier properties of the gauge fields, they are called as gauge bosons. Each generator in gauge group corresponds to a gauge boson. In Abelian Gauge theories, which is in $U(1)$ gauge group, there is one generator and then one gauge boson. For Electromagnetism, this gauge boson is *photon*.

As we mention above, Electromagnetism is an Abelian Gauge Theory. So, to preserve its local gauge invariance we need gauge covariant derivative and gauge field. In the light of these, What is the Field Strength of Electromagnetism? It can be written as

$$[D_\mu, D_\nu] = -iqF_{\mu\nu} \tag{2.7}$$

From there, Field Strength Tensor of Electromagnetism, anymore we call it Abelian Field Strength Tensor, is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

2.2. Non-Abelian Gauge Theory

Non Abelian Gauge Theory is the theory which remains invariant under local transformations of non Abelian gauge group. Then, what is non Abelian gauge transformation? and what is the difference between Abelian and non Abelian gauge transformations? As we mentioned in previous section; in Abelian gauge transformation, the group elements are commutative although in non Abelian gauge transformation they are not. For non Abelian gauge group,

$$[T_a, T_b] = f_{abc}T_c$$

where T_a is the generator of the group and f_{abc} is the structure constant. In the class of unitary gauge groups, $U(n)$, all groups are non Abelian except $U(1)$. This is related to generator number of the group which is determined by n^2.

Our aim is to obtain invariant action under local gauge transformation which we call as G. However, in non Abelian gauge groups generators are represented in matrix form and so action contains $\det G$. This breaks the invariance of the action. Hence, to avoid this challenge, we use operator of non Abelian group whose determinant is 1. It is called *special unitary group* , in short $SU(n)$. The generator number of $SU(n)$ is given as $n^2 - 1$. From there $SU(2)$ and $SU(3)$, which are non Abelian gauge groups of the Standard Model, have 3 and 8 generators respectively.

Let us examine local transformation of gauge group $SU(2)$ on real scalar field $\Phi(x)$. Suppose that

$$\Phi \longrightarrow \Phi' = \exp\left(i\vec{L}.\vec{\alpha}(x)\right)\Phi$$

where $\vec{L}$ is the generator of the group and $\vec{\alpha}$ is phase factor. By using Einstein summation rule, scalar product can be avoid and transformation is written as

$$\Phi \longrightarrow \Phi' = \exp\left(iL_a\alpha^a(x)\right)\Phi$$

Here, since we work on $SU(2)$ gauge group (especially in §6) and it has 3 generator, a can take three different value as $a = 1, 2, 3$. In the following chapters of this thesis, we deal with $SU(2)$ for non Abelian gauge group, so our gauge indices take three different values. Then applying transformation to the Klein Gordon Lagrangian

$$\mathcal{L}_{K-G} = \partial_\mu\Phi^*\partial^\mu\Phi + m^2\Phi^*\Phi$$

$\mathcal{L}'_{K-G}$ becomes

$$\begin{aligned}\mathcal{L}'_{K-G} &= [-i\exp(-iL_a\alpha^a(x))\Phi^* L_a\partial_\mu\alpha^a(x) + \exp(-iL_a\alpha^a(x))\partial_\mu\Phi^*] \\ &\times\ [i\exp(iL_a\alpha^a(x))\Phi L_a\partial^\mu\alpha^a(x) + \exp(iL_a\alpha^a(x))\partial^\mu\Phi] \\ &+\ m^2\exp(-iL_a\alpha^a(x))\Phi^*\exp(iL_a\alpha^a(x))\Phi \\ &= [-i\Phi^* L_a\partial_\mu\alpha^a(x) + \partial_\mu\Phi^*]\,[i\Phi L_a\partial^\mu\alpha^a(x) + \partial^\mu\Phi] + m^2\Phi^*\Phi \qquad (2.8)\end{aligned}$$

In this step, one can deduce the covariant derivative as like as Abelian case.

$$D_\mu = \partial_\mu - igA^a{}_\mu L_a$$

where $A^a{}_\mu$ is non Abelian gauge field and it is transformation is given as

$$A^a{}_\mu L_a \longrightarrow A^a{}_\mu L_a + \frac{1}{g}\left(\partial_\mu \alpha^a\right) L_a + i[\alpha^a L_a, A^b{}_\mu L_b]$$

Under these transformations, Klein-Gordon Lagrangian has local non Abelian gauge invariance. In the previous section, we mentioned about gauge fields are force carriers of the theories. Then, since gauge index has three different values for $SU(2)$, there are three different force carriers, gauge bosons. They are $W^\pm$ and Z^0. These are gauge bosons of weak interactions. For $SU(3)$, there are 8 gauge bosons since this group has 8 generators and these are *gluons*. $SU(3)$ is the gauge group of strong interaction. Thus, field strength of non Abelian gauge theory is given as

$$[D_\mu, D_\nu] = -igF^a{}_{\mu\nu} L_a \tag{2.9}$$

From there, by using the commutation rule

$$[L_a, L_b] = f_{abc} L_c$$

Field Strength Tensor of the non Abelian fields, any more we call it as non Abelian Field Strength Tensor, takes form

$$F^a{}_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu - gf^a{}_{bc} A^b{}_\mu A^c{}_\nu$$

In the light of discussion given above, Standard Model which is the unified theory of fundamental interactions except gravity is non Abelian Gauge Theory. Gauge groups of this theory are (1) for electromagnetic interactions, $SU(2)$ for weak interactions (for electroweak interactions which is unification of electromagnetism and weak interaction, gauge group is given as $U(1) \times SU(2)$) and $SU(3)$ for strong interactions.

CHAPTER 3

BORN-INFELD THEORY

Born-Infeld Theory (Born-Infeld, 1934) was orginally proposed by Born and Infeld in 1934 to solve the problem of diverging Coulomb field and self energy of point particles in Maxwell's electrodynamics (Wohlfarth, 2004). This theory is non-linear extension of Maxwell's theory and reduces to Maxwell equations for small amplitudes (Nieto, 2004). Also, in 1980s, Born-Infeld Theory became of very much interest due to its relation with SUSY and String Theory. Thus it is a significant theory.

Born and Infeld used determinantal form given below.

$$S_{B-I} = \int d^4x M^4 \left[\det(A_{\mu\nu})\right]^{1/2} \tag{3.1}$$

$A_{\mu\nu}$ can be expanded as

$$A_{\mu\nu} = g_{\mu\nu} + af_{\mu\nu}$$

where $g_{\mu\nu}$ is symmetric and $f_{\mu\nu}$ is antisymmetric parts of $A_{\mu\nu}$. Hence the action becomes;

$$S_{B-I} = \int d^4x M^4 \left[\det(g_{\mu\nu} + af_{\mu\nu})\right]^{1/2} \tag{3.2}$$

For an exact physical theory, action

$$S = \int d^4x \mathcal{L}$$

must be dimensionless.

- In D dimensional spacetime, $[d^Dx] = M^{-D}$, for $D = 4 \longrightarrow [d^4x] = M^{-4}$.
- $D = 4 \longrightarrow [\mathcal{L}] = M^4$

By using this fact, let us examine dimensions of constituents term by term.

- Metric tensor is dimensionless $[g_{\mu\nu}] = M^0$.

- Dimension of antisymmetric tensor (Assume that it is electromagnetic tensor)

  $[f_{\mu\nu}] = M^2$,

- $[a] = M^{-2}$ to cancel out dimension of $f_{\mu\nu}$

Here we should examine Born-Infeld action in order to understand Born-Infeld procedure. Let us start with considering $[\det(g_{\mu\nu} + af_{\mu\nu})]^{1/2}$. By bracketing of $g_{\mu\alpha}$

$$\det(g_{\mu\nu} + af_{\mu\nu}) \quad = \quad \det[g_{\mu\alpha}(\delta^{\alpha}{}_{\nu} + af^{\alpha}{}_{\nu})] \tag{3.3}$$

and by using the property of determinant given below

$$\det(A.B) = \det(A).\det(B) \tag{3.4}$$

it becomes

$$\det(g_{\mu\nu} + af_{\mu\nu}) \quad = \quad \det\left(g_{\mu\alpha}\right)\det\left(\delta^{\alpha}{}_{\nu} + af^{\alpha}{}_{\nu}\right) \tag{3.5}$$

Then Eq.(3.2) becomes,

$$S_{B-I} \quad = \quad \int d^4xM^4\left(\det\left(g_{\mu\alpha}\right)\right)^{1/2}\left[\det\left(\delta^{\alpha}{}_{\nu} + af^{\alpha}{}_{\nu}\right)\right]^{1/2} \tag{3.6}$$

Here we have to find expansion of $\det\left(\delta^{\alpha}{}_{\nu} + af^{\alpha}{}_{\nu}\right)$. The series expansion is given as

$$\det\left(\delta^{\alpha}{}_{\nu} + af^{\alpha}{}_{\nu}\right) = \sum_{k=0}^{\infty}\frac{1}{k!}\left[-\sum_{j=1}^{\infty}\frac{(-1)^j}{j}Tr\left[(af^{\alpha}{}_{\nu})^j\right]\right]^k \tag{3.7}$$

If we expand this series up to third order and use the trace property

$$Tr(aX) = aTr(X)$$

where a is a constant;

$$\begin{aligned}
\det\left(\delta^{\alpha}{}_{\nu} + af^{\alpha}{}_{\nu}\right) &= \sum_{k=0}^{\infty}\frac{1}{k!}\left[Tr(af^{\alpha}{}_{\nu}) - \frac{1}{2}Tr[(af^{\alpha}{}_{\nu})^2] + \mathcal{O}(A^3)\right]^k \\
&= \left[1 + aTr(f^{\alpha}{}_{\nu}) - \frac{1}{2}a^2Tr[(f^{\alpha}{}_{\nu})^2] + \frac{1}{2}a^2[Tr(f^{\alpha}{}_{\nu})]^2\right. \\
&+ \left.\mathcal{O}(A^3)\right] \qquad (3.8)
\end{aligned}$$

To find

$$\begin{aligned}
[det(g_{\mu\nu} + af_{\mu\nu})]^{1/2} &= \left[1 + aTr(f^{\alpha}{}_{\nu}) - \frac{1}{2}a^2Tr[(f^{\alpha}{}_{\nu})^2] + \frac{1}{2}a^2[Tr(f^{\alpha}{}_{\nu})]^2\right. \\
&+ \left.\mathcal{O}(A^3)\right]^{1/2} \qquad (3.9)
\end{aligned}$$

we should also use binomial expansion which is

$$(1+x)^{\epsilon} = \sum_{k=0}^{\infty}\binom{\epsilon}{k}x^k = \sum_{k=0}^{\infty}\frac{\epsilon!}{k!(\epsilon-k)!}x^k \qquad (3.10)$$

For $\epsilon = \frac{1}{2}$;

$$\begin{aligned}
(1+x)^{\frac{1}{2}} &= \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}x^k \\
&= 1 + \frac{\frac{1}{2}!}{(\frac{1}{2}-1)!}x + \frac{\frac{1}{2}!}{2!(\frac{1}{2}-2)!}x^2 + \mathcal{O}(A^3) \\
&= 1 + \frac{\frac{1}{2}(\frac{1}{2}-1)!}{(\frac{1}{2}-1)!}x + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)!}{2!(\frac{1}{2}-2)!}x^2 + \mathcal{O}(A^3) \\
&= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}(A^3) \qquad (3.11)
\end{aligned}$$

Hence, if we choose x as

$$x = aTr(f^{\alpha}{}_{\nu}) - \frac{1}{2}a^2Tr[(f^{\alpha}{}_{\nu})^2] + \frac{1}{2}a^2[Tr(f^{\alpha}{}_{\nu})]^2 \qquad (3.12)$$

Then x^2 takes the form as

$$\begin{aligned} x^2 &= \left[aTr(f^{\alpha}{}_{\nu}) - \frac{1}{2}a^2Tr[(f^{\alpha}{}_{\nu})^2] + \frac{1}{2}a^2[Tr(f^{\alpha}{}_{\nu})]^2\right]^2 \\ &= a^2Tr[(f^{\alpha}{}_{\nu})^2 + \mathcal{O}(A^3) \end{aligned} \tag{3.13}$$

By inserting (3.12) and(3.13) in (3.9), $[\det(g_{\mu\nu} + af_{\mu\nu})]^{1/2}$ becomes;

$$\begin{aligned} [\det(g_{\mu\nu} + af_{\mu\nu})]^{1/2} &= [\det(g_{\mu\alpha})]^{1/2}\Big\{1 + \frac{1}{2}\Big[aTr(f^{\alpha}{}_{\nu}) - \frac{1}{2}a^2Tr[(f^{\alpha}{}_{\nu})^2] \\ &+ \frac{1}{2}a^2[Tr(f^{\alpha}{}_{\nu})]^2\Big] - \frac{1}{8}\Big[a^2Tr[(f^{\alpha}{}_{\nu})^2\Big] + \mathcal{O}(A^3)\Big\} \\ &= [\det(g_{\mu\alpha})]^{1/2}\Big[1 + \frac{1}{2}aTr(f^{\alpha}{}_{\nu}) - \frac{1}{4}a^2Tr[(f^{\alpha}{}_{\nu})^2] \\ &+ \frac{1}{8}a^2[Tr(f^{\alpha}{}_{\nu})]^2 + \mathcal{O}(A^3)\Big] \end{aligned} \tag{3.14}$$

This is the expansion of determinant given in Eq. (3.6). By substituting the result given in Eq.(3.14)in the Eq. (3.6), the action becomes

$$\begin{aligned} S_{B-I} &= \int d^4xM^4[\det(g_{\mu\alpha})]^{1/2}\Big(1 + \frac{1}{2}aTr(f^{\alpha}{}_{\nu}) - \frac{1}{4}a^2Tr[(f^{\alpha}{}_{\nu})^2] \\ &+ \frac{1}{8}a^2[Tr(f^{\alpha}{}_{\nu})]^2\Big) \end{aligned} \tag{3.15}$$

From now, we have an action containing *trace of tensors* instead of determinantal form. Trace of nth order any rank-2 tensor is given as

$$TrA^n = A_{\mu_1\mu_2}A^{\mu_2\mu_3}\dots A_{\mu_{n-1}\mu_n}A^{\mu_n\mu_1} \tag{3.16}$$

(Koerber, 2004). Thus,

$$Tr(f^{\alpha}{}_{\nu}) = f^{\alpha}{}_{\alpha} \tag{3.17}$$

Since we assign $f_{\mu\nu}$ as electromagnetic tensor which is

$$f_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & B_y & B_x & 0 \end{bmatrix}$$

$$Tr(f^{\alpha}{}_{\nu}) = 0 \tag{3.18}$$

and so by using the equation given in Eq.(3.18)

$$[Tr(f^{\alpha}{}_{\nu})]^2 = 0 \tag{3.19}$$

too. There is no contributions from trace of $f_{\mu\nu}$ or square of it. The only term giving non-vanishing contribution to the action is $Tr[(f^{\alpha}{}_{\nu})^2]$ and it is expanded as

$$\begin{aligned} Tr[(f^{\alpha}{}_{\nu})^2] &= f^{\alpha}{}_{\nu} f^{\nu}{}_{\alpha} \\ &= f_{\mu\nu} g^{\alpha\mu} f^{\nu}{}_{\alpha} \\ &= f_{\mu\nu} f^{\nu\mu} \\ &= -f_{\mu\nu} f^{\mu\nu} \end{aligned} \tag{3.20}$$

From there, since $[\det(g_{\mu\alpha})]^{1/2} \equiv \sqrt{-g}$,

$$S_{B-I} = \int d^4x M^4 \sqrt{-g}\Big(1 + \frac{1}{4} a^2 f_{\mu\nu} f^{\mu\nu}\Big) \tag{3.21}$$

By arranging;

$$S_{B-I} = \int d^4x \sqrt{-g}\Big(M^4 + \frac{1}{4} M^4 a^2 f_{\mu\nu} f^{\mu\nu}\Big) \tag{3.22}$$

Here, M^4 corresponds to accumulated energy in the universe.$f_{\mu\nu}f^{\mu\nu}$ is just Lagrangian term of electromagnetic field. Therefore, Born-Infeld theory provides us cosmological constant and electromagnetism. One of the remarkable result of this theory is that electromagnetism is embedded in geometry since we use *determinant* notion. Moreover, in Born-Infeld theory, metric is not a dynamical variable. It means that we have no variation with respect to metric tensor. The only role of metric tensor is to supply invariance of action in the case of coordinate transformation.

As a consequence; Born-Infeld Theory yields to Electromagnetism.

CHAPTER 4

BORN-INFELD-EINSTEIN GRAVITY

Eddigton introduced the action (Eddington, 1924),

$$S_{EDD} = \int d^4x \left[\det(R_{\mu\nu}(\Gamma))\right] \tag{4.1}$$

and inspiring from Eddington; Deser and Gibbons proposed that Born-Infeld-Einstein Gravity (Deser, 1998) and their action is given below:

$$S_{D-G} = \int d^4x M_{Pl}^2 \left[-\det\left(ag_{\mu\nu} + bR_{\mu\nu} + X_{\mu\nu}\right)\right]^{1/2} \tag{4.2}$$

Dimensional analysis:

- $[R_{\mu\nu}] = M^2$
- $[g_{\mu\nu}] = M^0$
- $[a], [X_{\mu\nu}] = M^1$
- $[b] = M^{-1}$

If one choose $Q_{\mu\nu} = bR_{\mu\nu} + X_{\mu\nu}$, then S_{D-G} becomes;

$$S_{D-G} = \int d^4x M_{Pl}^2 \left[-\det\left(ag_{\mu\nu} + Q_{\mu\nu}\right)\right]^{1/2} \tag{4.3}$$

In this step let us examine expansion of determinant and to do this let us follow the same procedure given in previous chapter. Thus, by bracketing of $ag_{\mu\alpha}$

$$-\det\left(ag_{\mu\nu} + Q_{\mu\nu}\right) = -\det\left\{ag_{\mu\alpha}\left(\delta^{\alpha}{}_{\nu} + \frac{1}{a}Q^{\alpha}{}_{\nu}\right)\right\} \tag{4.4}$$

and also by using (3.4),

$$-\det\left(ag_{\mu\nu}+Q_{\mu\nu}\right)=\left[-\det\left(ag_{\mu\alpha}\right)\right]\det\left(\delta^{\alpha}{}_{\nu}+\frac{1}{a}Q^{\alpha}{}_{\nu}\right) \tag{4.5}$$

The one other property of determinant;

$$\det(cA)=c^{n}\det(A) \tag{4.6}$$

for $n\times n$ matrix. Then,

$$\det\left(ag_{\mu\alpha}\right)=a^{4}\det\left(g_{\mu\alpha}\right) \tag{4.7}$$

Since $g_{\mu\alpha}$ is 4×4 matrix for $D=4$. For $\det\left(\delta^{\alpha}{}_{\nu}+\frac{1}{a}Q^{\alpha}{}_{\nu}\right)$, we can use the expansion given in Eq.(3.8),

$$\begin{aligned}\det\left(\delta^{\alpha}{}_{\nu}+\frac{1}{a}Q^{\alpha}{}_{\nu}\right) &= 1+\frac{1}{a}Tr\left(Q^{\alpha}{}_{\nu}\right)-\frac{1}{2a^{2}}Tr\left[\left(Q^{\alpha}{}_{\nu}\right)^{2}\right]\\ &+ \frac{1}{2a^{2}}\left[Tr\left(Q^{\alpha}{}_{\nu}\right)\right]^{2}+\mathcal{O}(A^{3})\end{aligned} \tag{4.8}$$

and also using binomial expansion given (3.11)

$$\begin{aligned}\left[-\det(ag_{\mu\nu}+Q_{\mu\nu})\right]^{1/2} &= a^{2}\left[-\det(g_{\mu\alpha})\right]^{1/2}\left\{1+\frac{1}{2}\left[\frac{1}{a}Tr(Q^{\alpha}{}_{\nu})-\frac{1}{2a^{2}}Tr[(Q^{\alpha}{}_{\nu})^{2}]\right.\right.\\ &+ \left.\frac{1}{2a^{2}}[Tr(Q^{\alpha}{}_{\nu})]^{2}\right]-\frac{1}{8}\left[\frac{1}{a^{2}}Tr[(Q^{\alpha}{}_{\nu})^{2}\right]+\left.\mathcal{O}(A^{3})\right\}\\ &= a^{2}[-\det(g_{\mu\alpha})]^{1/2}\Big[1+\frac{1}{2a}Tr(Q^{\alpha}{}_{\nu})-\frac{1}{4a^{2}}Tr[(Q^{\alpha}{}_{\nu})^{2}]\\ &+ \frac{1}{8a^{2}}[Tr(Q^{\alpha}{}_{\nu})]^{2}+\mathcal{O}(A^{3})\Big]\end{aligned} \tag{4.9}$$

Then action becomes,

$$S_{D-G} = \int d^4x M_{Pl}^2 a^2 [-\det(g_{\mu\alpha})]^{1/2} \Big[1 + \frac{1}{2a} Tr(Q^{\alpha}{}_{\nu}) - \frac{1}{4a^2} Tr[(Q^{\alpha}{}_{\nu})^2] + \frac{1}{8a^2}[Tr(Q^{\alpha}{}_{\nu})]^2 + \mathcal{O}(A^3)\Big] \quad (4.10)$$

Let us find traces being in S_{D-G}. To do this, we use (3.16):

$$\begin{aligned} Tr(Q^{\alpha}{}_{\nu}) &= Q^{\alpha}{}_{\alpha} \\ &= bR^{\alpha}{}_{\alpha} + X^{\alpha}{}_{\alpha} \end{aligned} \quad (4.11)$$

and it can be rewritten as

$$Tr(Q^{\alpha}{}_{\nu}) = bR + X \quad (4.12)$$

and from(4.12),

$$[Tr(Q^{\alpha}{}_{\nu})]^2 = b^2R^2 + 2RX + X^2 \quad (4.13)$$

For $Tr[(Q^{\alpha}{}_{\nu})^2]$, we use again Eq.(3.16)

$$\begin{aligned} Tr[(Q^{\alpha}{}_{\nu})^2] &= Q^{\alpha}{}_{\nu} Q^{\nu}{}_{\alpha} \\ &= Q_{\mu\nu} g^{\mu\alpha} Q^{\nu}{}_{\alpha} \\ &= Q_{\mu\nu} Q^{\nu\mu} \end{aligned} \quad (4.14)$$

Since $Q_{\mu\nu} = bR_{\mu\nu} + X_{\mu\nu}$ and $Q^{\nu\mu} = bR^{\nu\mu} + X^{\nu\mu}$

$$Tr[(Q^{\alpha}{}_{\nu})^2] = Q_{\mu\nu} Q^{\nu\mu} = R_{\mu\nu} R^{\nu\mu} + R_{\mu\nu} X^{\nu\mu} + X_{\mu\nu} R^{\nu\mu} + X_{\mu\nu} X^{\nu\mu} \quad (4.15)$$

Here $R_{\mu\nu}$ and $X_{\mu\nu}$ are symmetric tensors. Therefore, $R_{\mu\nu} = R_{\nu\mu}$ and $X_{\mu\nu} = X_{\nu\mu}$. Moreover, $R_{\mu\nu}X^{\nu\mu} = R^{\rho\sigma}g_{\mu\rho}g_{\nu\sigma}X^{\nu\mu} = R^{\rho\sigma}X_{\sigma\rho}$. By changing indices as $\rho \to \nu$ and $\sigma \to \mu$,$R_{\mu\nu}X^{\nu\mu} = R^{\nu\mu}X_{\mu\nu}$. Thus

$$Tr\left[(Q^{\alpha}{}_{\nu})^2\right] \quad = \quad R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu}X^{\mu\nu} + X_{\mu\nu}X^{\mu\nu} \tag{4.16}$$

Hence, by subsituting (4.12),(4.13) and (4.16) in (4.10) and changing representation as $[-\det(g_{\mu\alpha})]^{1/2} \equiv \sqrt{-g}$

$$\begin{aligned} S_{D-G} = \int d^4x M_{Pl}^2 a^2 \sqrt{-g}\Big[1 + \frac{1}{2a}\left(bR + X\right) - \frac{1}{4a^2}\Big(R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu}X^{\mu\nu} \\ + X_{\mu\nu}X^{\mu\nu}\Big) + \frac{1}{8a^2}\left(b^2R^2 + 2RX + X^2\right) + \mathcal{O}(A^3)\Big] \end{aligned} \tag{4.17}$$

If we want to obtain natural theory, we should avoid non physical components from the theory. Here, $R^{\mu\nu}R_{\mu\nu}$ term results in *ghost* when it is expand perturbatively around Minkowski background. Ghost- sometimes called as bad ghost- means that wrong signed kinetic energy of any scalar field. For instance ,If metric tensor is Diag[-,+,+,+] and kinetic energy of scalar field positive signed, then it is called as *GHOST* (Karahan, 2010). Since ghosts allows probabilities to be negative, it violates unitarity. Higher derivative curvature theories always contains ghosts (Stelle, 1978). Therefore, to have a natural theory, we cancel out non-linear curvature terms besides ghosts by using $X_{\mu\nu}$.

Suppose that

$$\frac{1}{D}g_{\mu\nu}X^{\alpha}{}_{\alpha} = X_{\mu\nu} \tag{4.18}$$

To test this assumption let us multiply both sides with $g^{\mu\nu}$

$$\frac{1}{D}\underbrace{g_{\mu\nu}g^{\mu\nu}}_{\delta^{\mu}{}_{\mu}=D}X^{\alpha}{}_{\alpha} \quad = \quad \underbrace{X_{\mu\nu}g^{\mu\nu}}_{X^{\nu}{}_{\nu}} \tag{4.19}$$

$$\begin{aligned} \frac{1}{D} D X^{\alpha}{}_{\alpha} &= X^{\nu}{}_{\nu} \\ \implies X &= X \end{aligned} \tag{4.20}$$

Then our assumption is hold! To cancel out ghosts, let us define $X^{\alpha}{}_{\alpha}$

$$X^{\alpha}{}_{\alpha} = \frac{1}{2a} b^2 \left(R_{\rho\sigma} R^{\rho\sigma} - \frac{1}{2} R^2 \right) \tag{4.21}$$

In light of our assumption, multiply both sides with $\frac{1}{4} g_{\mu\nu}$ since we work on 4-dimensional spacetime.

$$\underbrace{\frac{1}{4} g_{\mu\nu} X^{\alpha}{}_{\alpha}}_{X_{\mu\nu}} = \frac{1}{4} g_{\mu\nu} \frac{1}{2a} b^2 \left(R_{\rho\sigma} R^{\rho\sigma} - \frac{1}{2} R^2 \right) \tag{4.22}$$

$$X_{\mu\nu} = \frac{b^2}{8a} g_{\mu\nu} \left(R_{\rho\sigma} R^{\rho\sigma} - \frac{1}{2} R^2 \right) \tag{4.23}$$

By substituting (4.23) in (4.17)

$$S_{D-G} = \int d^4 x M_{Pl}^2 a^2 \sqrt{-g} \Big(1 + \frac{1}{2} \frac{b}{a} R + \mathcal{O}(A^3) \Big) \tag{4.24}$$

All surviving terms without $\left(1 + \frac{b}{2a} R \right)$ are higher order terms because X is defined by using square of Ricci tensor.

Here, we can explicitly see that Deser and Gibbons theory consist of cosmological constant and general relativity. This theory reduces to Einstein Equations in the limit of small curvature. Moreover, in case of Deser and Gibbons theory, metric is not only provide invariance of the action, but also has dynamical role. Einstein equation can be obtain by taking variation of action with respect to metric tensor.

As a result, Born-Infeld-Einstein Gravity yields to Cosmological Constant and General Relativity

4.1. Born-Infeld-Einstein Gravity with Abelian and Non-Abelian Fields

In this section, we claim that BIE theory can be extended by adding new fields to the action. Our suggestion is

$$S_{BIE} \;\; = \;\; \int d^4x M_{Pl}^2 \Big[- \det(\tilde{a} g_{\mu\nu} + bR_{\mu\nu} + cF_{\mu\nu} + dF^a{}_{\mu\alpha} F^a{}_{\beta\nu} g^{\alpha\beta} + X_{\mu\nu}) \Big]^{1/2} \quad (4.25)$$

Here $F_{\mu\nu}$ is *Abelian* Field strength tensor and defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (4.26)$$

and $F^a_{\mu\alpha}$ is *non-Abelian* Field strength tensor, it is called also as Yang-Mills Field.

$$F^a{}_{\mu\alpha} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu - g\epsilon^a{}_{bc} A^b{}_\mu A^c{}_\alpha \quad (4.27)$$

where g is coupling constant and $\epsilon^a{}_{bc}$ is antisymmetric Levi-Civita symbol. Let us examine dimensional analysis:

- $[R_{\mu\nu}], [F_{\mu\nu}], [F^a{}_{\mu\nu}] = M^2$
- $[g_{\mu\nu}] = M^0$
- $[\tilde{a}], [X_{\mu\nu}] = M^1$
- $[b], [c] = M^{-1}$
- $[d] = M^{-3}$

By bracketing of $\tilde{a} g_{\mu\sigma}$

$$\tilde{a} g_{\mu\nu} + bR_{\mu\nu} + cF_{\mu\nu} + dF^a{}_{\mu\alpha} F^a{}_{\beta\nu} g^{\alpha\beta} + X_{\mu\nu} \equiv \tilde{a} g_{\mu\sigma} \Big(\delta^\sigma{}_\nu + \frac{b}{\tilde{a}} R^\sigma{}_\nu + \frac{c}{\tilde{a}} F^\sigma{}_\nu + \frac{d}{\tilde{a}} F^{a\sigma}{}_\alpha F^a{}_{\beta\nu} g^{\alpha\beta} + \frac{1}{\tilde{a}} X^\sigma{}_\nu \Big) \quad (4.28)$$

if one makes definition given below,

$$\frac{b}{\tilde{a}}R^{\sigma}{}_{\nu} + \frac{c}{\tilde{a}}F^{\sigma}{}_{\nu} + \frac{d}{\tilde{a}}F^{a\sigma}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta} + \frac{1}{\tilde{a}}X^{\sigma}{}_{\nu} = A^{\sigma}{}_{\nu} \tag{4.29}$$

then S_{BIE} takes form as

$$\begin{aligned} S_{BIE} &= \int d^4x M_{Pl}^2 \left[-\det(\tilde{a}g_{\mu\sigma})\det(\delta^{\sigma}{}_{\nu} + A^{\sigma}{}_{\nu})\right]^{1/2} \\ &= \int d^4x M_{Pl}^2 \left[-\det(\tilde{a}g_{\mu\sigma})\right]^{1/2}\left[\det(\delta^{\sigma}{}_{\nu} + A^{\sigma}{}_{\nu})\right]^{1/2} \end{aligned} \tag{4.30}$$

By using (4.6) and (4.7)

$$\begin{aligned} \left[-\det(\tilde{a}g_{\mu\sigma})\right]^{1/2} &= \left[\tilde{a}^4[-\det(g_{\mu\sigma})]\right]^{1/2} \\ &= \tilde{a}^2\sqrt{-g} \end{aligned} \tag{4.31}$$

In this theory, to obtain a reasonable results, we should expand the series up to fourth order. Therefore, let us rewrite expansions: By using (3.7)

$$\begin{aligned} \det(\delta^{\sigma}{}_{\nu} + A^{\sigma}{}_{\nu}) &= \sum_{k=o}^{\infty}\frac{1}{k!}\left\{Tr\,(A^{\sigma}{}_{\nu}) - \frac{1}{2}Tr\left[(A^{\sigma}{}_{\nu})^2\right] + \frac{1}{3}Tr\left[(A^{\sigma}{}_{\nu})^3\right] + \mathcal{O}(A^4)\right\}^k \\ &= 1 + \left[Tr\,(A^{\sigma}{}_{\nu}) - \frac{1}{2}Tr\left[(A^{\sigma}{}_{\nu})^2\right] + \frac{1}{3}Tr\left[(A^{\sigma}{}_{\nu})^3\right]\right] \\ &+ \frac{1}{2}\left[Tr\,(A^{\sigma}{}_{\nu}) - \frac{1}{2}Tr\left[(A^{\sigma}{}_{\nu})^2\right] + \frac{1}{3}Tr\left[(A^{\sigma}{}_{\nu})^3\right]\right]^2 \\ &+ \frac{1}{6}\left[Tr\,(A^{\sigma}{}_{\nu}) - \frac{1}{2}Tr\left[(A^{\sigma}{}_{\nu})^2\right] + \frac{1}{3}Tr\left[(A^{\sigma}{}_{\nu})^3\right]\right]^3 + \mathcal{O}(A^4) \end{aligned} \tag{4.32}$$

$$\begin{aligned} \det(\delta^{\sigma}{}_{\nu} + A^{\sigma}{}_{\nu}) &= 1 + Tr\,(A^{\sigma}{}_{\nu}) - \frac{1}{2}Tr\left[(A^{\sigma}{}_{\nu})^2\right] + \frac{1}{3}Tr\left[(A^{\sigma}{}_{\nu})^3\right] + \frac{1}{2}\left[Tr\,(A^{\sigma}{}_{\nu})\right]^2 \\ &- \frac{1}{2}Tr\left[(A^{\sigma}{}_{\nu})\right]Tr\left[(A^{\sigma}{}_{\nu})^2\right] + \frac{1}{6}\left[Tr\,(A^{\sigma}{}_{\nu})\right]^3 + \mathcal{O}(A^4) \end{aligned} \tag{4.33}$$

If we expand binomial series up to 4^{th} order, by using (3.10),

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{3}{96}x^3 + \mathcal{O}(A^4) \tag{4.34}$$

and by choosing

$$\begin{aligned} x &= Tr\,(A^\sigma{}_\nu) - \frac{1}{2}Tr\left[(A^\sigma{}_\nu)^2\right] + \frac{1}{3}Tr\left[(A^\sigma{}_\nu)^3\right] + \frac{1}{2}\left[Tr\,(A^\sigma{}_\nu)\right]^2 \\ &- \frac{1}{2}Tr\left[(A^\sigma{}_\nu)\right]Tr\left[(A^\sigma{}_\nu)^2\right] + \frac{1}{6}\left[Tr\,(A^\sigma{}_\nu)\right]^3 + \mathcal{O}(A^4) \end{aligned} \tag{4.35}$$

and so

$$x^2 = \left[Tr\,(A^\sigma{}_\nu)\right]^2 - Tr\,(A^\sigma{}_\nu)\,Tr\left[(A^\sigma{}_\nu)^2\right] + \left[Tr\,(A^\sigma{}_\nu)\right]^3 + \mathcal{O}(A^4) \tag{4.36}$$

$$x^3 = \left[Tr\,(A^\sigma{}_\nu)\right]^3 + \mathcal{O}(A^4) \tag{4.37}$$

substituting (4.35),(4.36) and (4.37) in (4.34)

$$\begin{aligned} \left[\det\,(\delta^\sigma{}_\nu + A^\sigma{}_\nu)\right]^{1/2} &= 1 + \frac{1}{2}Tr\,(A^\sigma{}_\nu) - \frac{1}{4}Tr\left[(A^\sigma{}_\nu)^2\right] + \frac{1}{6}Tr\left[(A^\sigma{}_\nu)^3\right] \\ &+ \frac{1}{8}\left[Tr\,(A^\sigma{}_\nu)\right]^2 - \frac{1}{8}Tr\,(A^\sigma{}_\nu)\,Tr\left[(A^\sigma{}_\nu)^2\right] \\ &- \frac{1}{96}\left[Tr\,(A^\sigma{}_\nu)\right]^3 + \mathcal{O}(A^4) \end{aligned} \tag{4.38}$$

Then,

$$\begin{aligned} S_{BIE} &= \int d^4x M_{Pl}^2 \tilde{a}^2 \sqrt{-g}\Big\{1 + \frac{1}{2}Tr(A^\sigma{}_\nu) - \frac{1}{4}Tr[(A^\sigma{}_\nu)^2] + \frac{1}{6}Tr[(A^\sigma{}_\nu)^3] \\ &+ \frac{1}{8}[Tr(A^\sigma{}_\nu)]^2 - \frac{1}{8}Tr(A^\sigma{}_\nu)Tr[(A^\sigma{}_\nu)^2] - \frac{1}{96}[Tr(A^\sigma{}_\nu)]^3 + \mathcal{O}(A^4)\Big\} \end{aligned} \tag{4.39}$$

Lt us calculate traces one by one:

$$
\begin{aligned}
Tr(A^{\sigma}{}_{\nu}) &= A^{\nu}{}_{\nu} \\
&= \frac{b}{\tilde{a}}R + \frac{c}{\tilde{a}}F + \frac{d}{\tilde{a}}F^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta} + \frac{1}{\tilde{a}}X
\end{aligned} \tag{4.40}
$$

Since $F = 0$ due to antisymmetric nature of $F_{\mu\nu}$,

$$
Tr(A^{\sigma}{}_{\nu}) = \frac{b}{\tilde{a}}R + \frac{d}{\tilde{a}}F^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta} + \frac{1}{\tilde{a}}X \tag{4.41}
$$

Then,

$$
\begin{aligned}
[Tr(A^{\sigma}{}_{\nu})]^2 &= \frac{b^2}{\tilde{a}^2}R^2 + \frac{2bd}{\tilde{a}^2}RF^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta} + \frac{2b}{\tilde{a}^2}RX \\
&+ \frac{d^2}{\tilde{a}^2}F^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}F^{b\lambda}{}_{\varepsilon}F^{b}{}_{\theta\lambda}g^{\varepsilon\theta} \\
&+ \frac{2d}{\tilde{a}^2}XF^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta} + \frac{1}{\tilde{a}^2}X^2
\end{aligned} \tag{4.42}
$$

$$
\begin{aligned}
[Tr(A^{\sigma}_{\nu})]^3 &= \frac{b^3}{\tilde{a}^3}R^3 + \frac{d^3}{\tilde{a}^3}F^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}F^{b\lambda}{}_{\varepsilon}F^{b}{}_{\theta\lambda}g^{\varepsilon\theta}F^{c\eta}{}_{\omega}F^{c}{}_{\tau\eta}g^{\omega\tau} + \frac{1}{\tilde{a}^3}X^3 \\
&+ \frac{3b^2d}{\tilde{a}^3}R^2F^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta} + \frac{3b^2}{\tilde{a}^3}R^2X \\
&+ \frac{3bd^2}{\tilde{a}^3}RF^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}F^{b\lambda}{}_{\varepsilon}F^{b}{}_{\theta\lambda}g^{\varepsilon\theta} + \frac{6bd}{\tilde{a}^3}RXF^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta} \\
&+ \frac{3b}{\tilde{a}^3}RX^2 + \frac{3d^2}{\tilde{a}^3}XF^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}F^{b\lambda}{}_{\varepsilon}F^{b}{}_{\theta\lambda}g^{\varepsilon\theta} \\
&+ \frac{3d}{\tilde{a}^3}X^2F^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}
\end{aligned} \tag{4.43}
$$

i.e.

$$
[Tr(A^{\sigma}_{\nu})]^3 \supset \frac{b^3}{\tilde{a}^3}R^3 + \frac{3b^2d}{\tilde{a}^3}R^2F^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta} \tag{4.44}
$$

To find $Tr[(A^{\sigma}{}_{\nu})^2]$ and $Tr[(A^{\sigma}{}_{\nu})^3]$ we should use (3.16)

$$
\begin{aligned}
Tr[(A^{\sigma}{}_{\nu})^2] &= A_{\mu\nu}A^{\mu\nu} \\
&= \frac{b^2}{\tilde{a}^2}R_{\mu\nu}R^{\mu\nu} + \frac{2bd}{\tilde{a}^2}F^{a}{}_{\mu\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}R^{\mu\nu} + \frac{2b}{\tilde{a}^2}R_{\mu\nu}X^{\mu\nu} + \frac{c^2}{\tilde{a}^2}F_{\mu\nu}F^{\mu\nu} \\
&+ \frac{2c}{\tilde{a}^2}F_{\mu\nu}X^{\mu\nu} + \frac{d^2}{\tilde{a}^2}F^{a}{}_{\mu\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}F^{b\mu}{}_{\varepsilon}F^{b}{}_{\theta}^{\nu}g^{\varepsilon\theta} \\
&+ \frac{2d}{\tilde{a}^2}F^{a}{}_{\mu\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}X_{\mu\nu} + \frac{1}{\tilde{a}^2}X_{\mu\nu}X^{\mu\nu} \qquad (4.45)
\end{aligned}
$$

and

$$
\begin{aligned}
Tr[(A^{\sigma}{}_{\nu})^3] &= A_{\mu\nu}A^{\nu\kappa}A_{\kappa}{}^{\mu} \\
&\supset \frac{b^3}{\tilde{a}^3}R^{\mu\kappa}R_{\kappa}{}^{\nu}R_{\mu\nu} + \frac{b^2 d}{\tilde{a}^3}R^{\kappa}{}_{\nu}R_{\mu\kappa}F^{a\nu}{}_{\lambda}F^{a\lambda\mu} \\
&+ \frac{b^2 d}{\tilde{a}^3}R^{\nu\lambda}R_{\lambda\kappa}F^{a\kappa\mu}F^{a}{}_{\mu\nu} + \frac{c^2 d}{\tilde{a}^3}F_{\kappa}{}^{\nu}F_{\mu\nu}F^{a\mu}{}_{\alpha}F^{a\kappa}{}_{\beta}g^{\alpha\beta} \\
&+ \frac{c^2 d}{\tilde{a}^3}F^{a\kappa\mu}F^{a}{}_{\mu\nu}F^{\nu\lambda}F_{\lambda\kappa} + \frac{cd}{\tilde{a}^3}F^{a\kappa\mu}F^{a}{}_{\mu\nu}F_{\lambda\kappa}X^{\nu\lambda} \\
&+ \frac{cd}{\tilde{a}^3}F^{a\nu\mu}F^{a}{}_{\mu\lambda}g^{\lambda\alpha}F_{\alpha\kappa}X^{\kappa}{}_{\nu} + \frac{b^2 d}{\tilde{a}^3}R^{\kappa}{}_{\mu}R^{\mu\nu}F^{a}{}_{\nu\lambda}F^{a\lambda\sigma}g_{\sigma\kappa} \\
&+ \frac{c^2 d}{\tilde{a}^3}F^{\kappa}{}_{\nu}F^{\nu\lambda}F^{a}{}_{\lambda\sigma}F^{a\sigma}{}_{\kappa} + \frac{cd}{\tilde{a}^3}F^{\kappa}{}_{\nu}F^{a}{}_{\lambda\sigma}F^{a\sigma}{}_{\kappa}X^{\nu\lambda} \\
&+ \frac{cd}{\tilde{a}^3}F^{\nu\lambda}F^{a}{}_{\lambda\sigma}F^{a\sigma}{}_{\kappa}X^{\kappa}{}_{\nu} \qquad (4.46)
\end{aligned}
$$

By using (4.41) and (4.45)

$$
\begin{aligned}
Tr(A^{\sigma}{}_{\nu})Tr[(A^{\sigma}{}_{\nu})^2] &\supset \frac{b^3}{\tilde{a}^3}RR^{\mu\nu}R_{\mu\nu} + \frac{b^2 d}{\tilde{a}^3}R^{\mu\nu}R_{\mu\nu}F^{a\lambda}{}_{\alpha}F^{a}{}_{\beta\lambda}g^{\alpha\beta} \\
&- \frac{c^2 d}{\tilde{a}^3}F^{a\lambda}{}_{\alpha}F^{a}{}_{\beta\lambda}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu} \\
&+ \frac{2cd}{\tilde{a}^3}F^{a\lambda}{}_{\alpha}F^{a}{}_{\beta\lambda}g^{\alpha\beta}F_{\mu\nu}X^{\mu\nu} \qquad (4.47)
\end{aligned}
$$

If we consider all these terms given in (4.41),(4.42),(4.43),(4.45),(4.46)and (4.47), then the action of BIE with Abelian and Non Abelian Fields becomes,

$$
\begin{aligned}
S_{BIE} &= \int d^4x M_{Pl}^2 \tilde{a}^2 \sqrt{-g} \Big\{ 1 + \frac{1}{2}\Big[\frac{b}{\tilde{a}} R + \frac{d}{\tilde{a}} F^{a\nu}{}_{\alpha} F^{a}{}_{\beta\nu} g^{\alpha\beta} + \frac{1}{\tilde{a}} X \Big] \\
&- \frac{1}{4}\Big[\frac{b^2}{\tilde{a}^2} R_{\mu\nu} R^{\mu\nu} + \frac{2bc}{\tilde{a}^2} F^{a}{}_{\mu\alpha} F^{a}{}_{\beta\nu} g^{\alpha\beta} R^{\mu\nu} + \frac{2b}{\tilde{a}^2} R_{\mu\nu} X^{\mu\nu} + \frac{c^2}{\tilde{a}^2} F_{\mu\nu} F^{\mu\nu} \\
&+ \frac{2c}{\tilde{a}^2} F_{\mu\nu} X^{\mu\nu} + \frac{d^2}{\tilde{a}^2} F^{a}{}_{\mu\alpha} F^{a}{}_{\beta\nu} g^{\alpha\beta} F^{b\mu}{}_{\varepsilon} F^{b}{}_{\theta}^{\nu} g^{\varepsilon\theta} + \frac{2d}{\tilde{a}^2} F^{a}{}_{\mu\alpha} F^{a}{}_{\beta\nu} g^{\alpha\beta} X_{\mu\nu} \\
&+ \frac{1}{\tilde{a}^2} X_{\mu\nu} X^{\mu\nu} \Big] + \frac{1}{6}\Big[\frac{b^3}{\tilde{a}^3} R R^{\mu\nu} R_{\mu\nu} + \frac{b^2 d}{\tilde{a}^3} R^{\mu\nu} R_{\mu\nu} F^{a\lambda}{}_{\alpha} F^{a}{}_{\beta\lambda} g^{\alpha\beta} \\
&- \frac{c^2 d}{\tilde{a}^3} F^{a\lambda}{}_{\alpha} F^{a}{}_{\beta\lambda} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} + \frac{2cd}{\tilde{a}^3} F^{a\lambda}{}_{\alpha} F^{a}{}_{\beta\lambda} g^{\alpha\beta} F_{\mu\nu} X^{\mu\nu} \Big] \\
&+ \frac{1}{8}\Big[\frac{b^2}{\tilde{a}^2} R^2 + \frac{2bd}{\tilde{a}^2} R F^{a\nu}{}_{\alpha} F^{a}{}_{\beta\nu} g^{\alpha\beta} + \frac{2b}{\tilde{a}^2} R X \\
&+ \frac{d^2}{\tilde{a}^2} F^{a\nu}{}_{\alpha} F^{a}{}_{\beta\nu} g^{\alpha\beta} F^{b\lambda}{}_{\varepsilon} F^{b}{}_{\theta\lambda} g^{\varepsilon\theta} + \frac{2d}{\tilde{a}^2} X F^{a\nu}{}_{\alpha} F^{a}{}_{\beta\nu} g^{\alpha\beta} + \frac{1}{\tilde{a}^2} X^2 \Big] \\
&- \frac{1}{8}\Big[\frac{b^3}{\tilde{a}^3} R^{\mu\kappa} R_{\kappa}{}^{\nu} R_{\mu\nu} + \frac{b^2 d}{\tilde{a}^3} R^{\kappa}{}_{\nu} R_{\mu\kappa} F^{a\nu}{}_{\lambda} F^{a\lambda\mu} \\
&+ \frac{b^2 d}{\tilde{a}^3} R^{\nu\lambda} R_{\lambda\kappa} F^{a\kappa\mu} F^{a}{}_{\mu\nu} + \frac{c^2 d}{\tilde{a}^3} F_{\kappa}{}^{\nu} F_{\mu\nu} F^{a\mu}{}_{\alpha} F^{a\kappa}{}_{\beta} g^{\alpha\beta} \\
&+ \frac{c^2 d}{\tilde{a}^3} F^{a\kappa\mu} F^{a}{}_{\mu\nu} F^{\nu\lambda} F_{\lambda\kappa} + \frac{cd}{\tilde{a}^3} F^{a\kappa\mu} F^{a}{}_{\mu\nu} F_{\lambda\kappa} X^{\nu\lambda} \\
&+ \frac{cd}{\tilde{a}^3} F^{a\nu\mu} F^{a}{}_{\mu\lambda} g^{\lambda\alpha} F_{\alpha\kappa} X^{\kappa}{}_{\nu} + \frac{b^2 d}{\tilde{a}^3} R^{\kappa}{}_{\mu} R^{\mu\nu} F^{a}{}_{\nu\lambda} F^{a\lambda\sigma} g_{\sigma\kappa} \\
&+ \frac{c^2 d}{\tilde{a}^3} F^{\kappa}{}_{\nu} F^{\nu\lambda} F^{a}{}_{\lambda\sigma} F^{a\sigma}{}_{\kappa} + \frac{cd}{\tilde{a}^3} F^{\kappa}{}_{\nu} F^{a}{}_{\lambda\sigma} F^{a\sigma}{}_{\kappa} X^{\nu\lambda} \\
&+ \frac{cd}{\tilde{a}^3} F^{\nu\lambda} F^{a}{}_{\lambda\sigma} F^{a\sigma}{}_{\kappa} X^{\kappa}{}_{\nu} \Big] \\
&+ \frac{1}{48}\Big[\frac{b^3}{\tilde{a}^3} R^3 + \frac{3b^2 d}{\tilde{a}^3} R^2 F^{a\nu\beta} F^{a}{}_{\beta\nu} \Big] + \mathcal{O}(A^4) \Big\}
\end{aligned}
\tag{4.48}
$$

Let us define $X_{\mu\nu}$ in light of Eq.(4.18),

$$
\begin{aligned}
X_{\mu\nu} = \frac{1}{8}\frac{b^2}{\tilde{a}} \quad \Big(& \frac{b}{2\tilde{a}} R R_{\alpha\beta} R^{\alpha\beta} - \frac{2b}{3\tilde{a}} R^{\alpha\kappa} R_{\kappa}{}^{\beta} R_{\alpha\beta} + R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{2} R^2 \\
- & \frac{b}{12\tilde{a}} R^3 + \frac{d}{2\tilde{a}} R_{\alpha\beta} R^{\alpha\beta} F^{a\kappa\sigma} F^{a}{}_{\sigma\kappa} - \frac{2d}{\tilde{a}} R^{\nu\delta} R_{\delta\sigma} F^{a\beta\sigma} F^{a}{}_{\beta\nu} \\
- & \frac{d}{8\tilde{a}} R^2 F^{a\kappa\sigma} F^{a}{}_{\sigma\kappa} + \frac{d}{\tilde{a}} R R^{\nu\sigma} F^{a\sigma\alpha} F^{a\alpha}{}_{\nu} \Big) g_{\mu\nu}
\end{aligned}
\tag{4.49}
$$

Thus, if one substitute $X_{\mu\nu}$ in the action, R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R^{\mu\kappa}R^{\nu}{}_{\kappa}R_{\mu\nu}$ terms cancel out. Furthermore, all terms which contain $X_{\mu\nu}$ or X are higher order. Considering this situation and making some arrangement, action takes the form as

$$
\begin{aligned}
S_{BIE} &= \int d^4x M_{Pl}^2 \sqrt{-g} \Big\{ \tilde{a}^2 + \frac{\tilde{a}}{2} \Big[bR - \frac{c^2}{2\tilde{a}} F_{\mu\nu} F^{\mu\nu} + dF^{a\nu}{}_{\alpha} F^{a}{}_{\beta\nu} g^{\alpha\beta} \Big] \\
&+ \frac{1}{8} \Big[-4bdR^{\mu\nu} F^{a}{}_{\mu\alpha} F^{a}{}_{\beta\nu} g^{\alpha\beta} + 2bdRF^{a\nu}{}_{\alpha} F^{a}{}_{\beta\nu} g^{\alpha\beta} \\
&- 2d^2 F^{a}{}_{\mu\alpha} F^{a}{}_{\beta\nu} g^{\alpha\beta} F^{b\mu}_{\ \ \varepsilon} F^{b\nu}{}_{\theta} g^{\varepsilon\theta} + d^2 F^{a\nu}{}_{\alpha} F^{a}{}_{\beta\nu} g^{\alpha\beta} F^{b\lambda}{}_{\varepsilon} F^{b}{}_{\theta\lambda} g^{\varepsilon\theta} \Big] \\
&+ \frac{1}{48\tilde{a}} \Big[d^3 F^{a\sigma\beta} F^{a}{}_{\beta\sigma} F^{b\lambda\rho} F^{b}{}_{\rho\lambda} F^{c\varepsilon\eta} F^{c}{}_{\eta\varepsilon} + 6d^3 F^{a}{}_{\mu\alpha} F^{a\alpha}{}_{\nu} F^{b\nu\rho} F^{b\mu}{}_{\rho} F^{c\varepsilon\eta} F^{c}{}_{\eta\varepsilon} \\
&- 8d^3 F^{a\sigma\beta} F^{a}{}_{\beta\nu} F^{b\nu\rho} F^{b\mu}{}_{\rho} F^{c}{}_{\mu\eta} F^{c\eta}{}_{\sigma} + bd^2 RF^{a\sigma\beta} F^{a}{}_{\beta\sigma} F^{b\lambda\rho} F^{b}{}_{\rho\lambda} \\
&+ 6bd^2 RF^{a}{}_{\mu\alpha} F^{a\alpha}{}_{\nu} F^{b\nu\rho} F^{b\mu}{}_{\rho} - 12bd^2 R^{\nu\mu} F^{a}{}_{\mu\alpha} F^{a\alpha}{}_{\nu} F^{b\sigma\eta} F^{b}{}_{\eta\sigma} \\
&+ 24bd^2 R^{\nu\mu} F^{a\sigma\beta} F^{a}{}_{\beta\nu} F^{b}{}_{\mu\eta} F^{b\eta}{}_{\sigma} - 6bc^2 RF^{\mu\nu} F_{\nu\mu} + 24bc^2 R^{\nu\mu} F^{\sigma}{}_{\nu} F_{\mu\sigma} \\
&- 6c^2 dF^{\mu\nu} F_{\nu\mu} F^{a\sigma\beta} F^{a}{}_{\beta\sigma} + 24c^2 dF^{\sigma}{}_{\nu} F^{\nu\mu} F^{a}{}_{\mu\eta} F^{a\eta}{}_{\sigma} \Big] \Big\} \qquad (4.50)
\end{aligned}
$$

4.2. Abelian - Non Abelian Mixing in BIE

In the end of very long calculations, we obtain an action given in Eq. (4.50). Let us rewrite it and analyse term by term.

$$\begin{aligned}
S_{BIE} &= \int d^4x M_{Pl}^2\sqrt{-g}\Big\{\tilde{a}^2 + \frac{\tilde{a}}{2}\Big[b\underbrace{R}_{1} - \frac{c^2}{2\tilde{a}}\underbrace{F_{\mu\nu}F^{\mu\nu}}_{2} + d\underbrace{F^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}}_{3}\Big] \\
&+ \frac{1}{8}\Big[-4bd\underbrace{R^{\mu\nu}F^{a}{}_{\mu\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}}_{4} + 2bd\underbrace{RF^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}}_{5} \\
&- 2d^2\underbrace{F^{a}{}_{\mu\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}F^{b\mu}{}_{\varepsilon}F^{b\nu}{}_{\theta}g^{\varepsilon\theta}}_{6} + d^2\underbrace{F^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}F^{b\lambda}{}_{\varepsilon}F^{b}{}_{\theta\lambda}g^{\varepsilon\theta}}_{7}\Big] \\
&+ \frac{1}{48\tilde{a}}\Big[d^3\underbrace{F^{a\sigma\beta}F^{a}{}_{\beta\sigma}F^{b\lambda\rho}F^{b}{}_{\rho\lambda}F^{c\varepsilon\eta}F^{c}{}_{\eta\varepsilon}}_{8} + 6d^3\underbrace{F^{a}{}_{\mu\alpha}F^{a\alpha}{}_{\nu}F^{b\nu\rho}F^{b\mu}{}_{\rho}F^{c\varepsilon\eta}F^{c}{}_{\eta\varepsilon}}_{9} \\
&- 8d^3\underbrace{F^{a\sigma\beta}F^{a}{}_{\beta\nu}F^{b\nu\rho}F^{b\mu}{}_{\rho}F^{c}{}_{\mu\eta}F^{c\eta}{}_{\sigma}}_{10} + bd^2\underbrace{RF^{a\sigma\beta}F^{a}{}_{\beta\sigma}F^{b\lambda\rho}F^{b}{}_{\rho\lambda}}_{11} \\
&+ 6bd^2\underbrace{RF^{a}{}_{\mu\alpha}F^{a\alpha}{}_{\nu}F^{b\nu\rho}F^{b\mu}{}_{\rho}}_{12} - 12bd^2\underbrace{R^{\nu\mu}F^{a}{}_{\mu\alpha}F^{a\alpha}{}_{\nu}F^{b\sigma\eta}F^{b}{}_{\eta\sigma}}_{13} \\
&+ 24bd^2\underbrace{R^{\nu\mu}F^{a}{}_{\beta\nu}F^{a\sigma\beta}F^{b}{}_{\mu\eta}F^{b\eta}{}_{\sigma}}_{14} - 6bc^2\underbrace{RF^{\mu\nu}F_{\nu\mu}}_{15} + 24bc^2\underbrace{R^{\nu\mu}F^{\sigma}{}_{\nu}F_{\mu\sigma}}_{16} \\
&- 6c^2d\underbrace{F^{\mu\nu}F_{\nu\mu}F^{a\sigma\beta}F^{a}{}_{\beta\sigma}}_{17} + 24c^2d\underbrace{F^{\sigma}{}_{\nu}F^{\nu\mu}F^{a}{}_{\mu\eta}F^{a\eta}{}_{\sigma}}_{18}\Big]\Big\}
\end{aligned} \tag{4.51}$$

Here, the term labelled as 1 is just the Einstein-Hilbert term. It gives us General Relativity in small curvature limit.

The term labelled as 2 is just Lagrangian of Field Strength. This term supplies us to Electromagnetism.

3 is called as Yang-Mills term. It carries Non-Abelian part of the theory.

The terms $1, 2, 3$ are not mixed terms. All of them are responsible for their own fields. From there let us examine mixed terms.

The term labelled as $4, 5, 11, 12, 13$ and 14 have mixing of Non-Abelian Field Strength tensor and Curvature. It can be interpreted as graviton-gluon coupling and can be detected by means of High Luminosity experiments.

$6, 7, 8, 9$ and 10 are coupling of two Non Abelian fields such as gluon-gluon coupling.

The terms 15 and 16 are the coupling of Curvature and Abelian Field Strength

Tensor. One can deduce that it is photon graviton coupling.

17 and 18 are Abelian-Non Abelian Coupling. This is the main purpose of our thesis. Born-Infeld-Einstein Gravity allows the mixing of Abelian and Non-Abelian fields. This term can be responsible for photon-gluon coupling and it can be observed by High Luminosity Experiments.

CHAPTER 5

BORN-INFELD-RIEMANN GRAVITY (BIR)

Born-Infeld Rieman Gravity was firstly proposed in (Soysal, 2012). Main idea of this theory is using rank-4 tensor fields to obtain an invariant action. In this thesis we suggest a new form of Born-Infeld -Riemann Gravity which contains both Abelian and Non Abelian fields.

5.1. Born-Infeld-Riemann Gravity with Abelian and Non-Abelian Fields

Our suggestion is

$$S_{BIR} = \int d^4x M_{Pl}^2 \Big[DDet\Big(\kappa^2 \tilde{g}_{\mu\alpha\nu\beta} + \lambda_R R_{\mu\alpha\nu\beta} + \lambda_F \hat{F}_{\mu\alpha\nu\beta} + \tilde{\lambda}_F \tilde{F}_{\mu\alpha\nu\beta} + X_{\mu\alpha\nu\beta} \Big) \Big]^{1/4} \tag{5.1}$$

where

$$\begin{aligned} \tilde{g}_{\mu\alpha\nu\beta} &= g_{\mu\nu} g_{\alpha\beta} - g_{\mu\beta} g_{\alpha\nu} \\ \hat{F}_{\mu\alpha\nu\beta} &= F_{\mu\alpha} F_{\nu\beta} \\ \tilde{F}_{\mu\alpha\nu\beta} &= F^a{}_{\mu\alpha} F^a{}_{\nu\beta} \end{aligned} \tag{5.2}$$

and dimensions of constants are

- $[R_{\mu\alpha\nu\beta}] = M^2$
- $[\hat{F}_{\mu\alpha\nu\beta}] = M^4$
- $[\tilde{F}_{\mu\alpha\nu\beta}] = M^4$
- $[\tilde{g}_{\mu\alpha\nu\beta}] = M^0$

- $[\kappa] = M^1$
- $[\lambda_R] = M^0$
- $[\lambda_F] = M^{-2}$
- $[\tilde{\lambda}_F] = M^{-2}$

Thus, substituting Eq.(5.2) in (6.89)

$$S_{BIR} = \int d^4x M_{Pl}^2 \Big[DDet\Big(\kappa^2 (g_{\mu\nu} g_{\alpha\beta} - g_{\mu\beta} g_{\alpha\nu}) + \lambda_R R_{\mu\alpha\nu\beta} + \lambda_F F_{\mu\alpha} F_{\nu\beta} + \tilde{\lambda}_F F^a{}_{\mu\alpha} F^a{}_{\nu\beta} + X_{\mu\alpha\nu\beta} \Big) \Big]^{1/4} \tag{5.3}$$

By bracketing $\big(\kappa^2 (g_{\mu\nu'} g_{\alpha\beta'} - g_{\mu\beta'} g_{\alpha\nu'})\big)$ and using Eq.(3.4)

$$\begin{aligned} S_{BIR} &= \int d^4x M_{Pl}^2 \Big[DDet\Big(\kappa^2 (g_{\mu\nu'} g_{\alpha\beta'} - g_{\mu\beta'} g_{\alpha\nu'}) \Big) \Big]^{1/4} \\ &\times \Big[DDet\Big(I^{\nu'\beta'}{}_{\nu\beta} + \frac{\lambda_R}{\kappa^2} Inv^{\mu'\alpha'\nu'\beta'} R_{\mu'\alpha'\nu\beta} + \frac{\lambda_F}{\kappa^2} Inv^{\mu'\alpha'\nu'\beta'} F_{\mu'\alpha'} F_{\nu\beta} \\ &+ \frac{\tilde{\lambda}_F}{\kappa^2} Inv^{\mu'\alpha'\nu'\beta'} F^a{}_{\mu'\alpha'} F^a{}_{\nu\beta} + \frac{1}{\kappa^2} Inv^{\mu'\alpha'\nu'\beta'} X_{\mu'\alpha'\nu\beta} \Big) \Big]^{1/4} \end{aligned} \tag{5.4}$$

If one use expansion of determinant given in Eq.(3.8) and Binomial expansion given in Eq.(3.10)

$$\begin{aligned} S_{BIR} &= \int d^4x M_{Pl}^2 \Big[DDet\Big(\kappa^2 (g_{\mu\nu'} g_{\alpha\beta'} - g_{\mu\beta'} g_{\alpha\nu'}) \Big) \Big]^{1/4} \\ &\times \Big[1 + \frac{1}{4} Tr(A^{\nu'\beta'}{}_{\nu\beta}) - \frac{1}{8} Tr[(A^{\nu'\beta'}{}_{\nu\beta})^2] + \frac{1}{32} [Tr(A^{\nu'\beta'}{}_{\nu\beta})]^2 \\ &+ \mathcal{O}(A^3) \Big] \end{aligned} \tag{5.5}$$

$$\begin{aligned} S_{BIR} &= \int d^4x M_{Pl}^2 \Big[DDet\Big(\kappa^2 (g_{\mu\nu'} g_{\alpha\beta'} - g_{\mu\beta'} g_{\alpha\nu'}) \Big) \Big]^{1/4} \\ &\times \Big[DDet\Big(I^{\nu'\beta'}{}_{\nu\beta} + A^{\nu'\beta'}{}_{\nu\beta} \Big) \Big]^{1/4} \end{aligned} \tag{5.6}$$

$$A^{\nu'\beta'}{}_{\nu\beta} = \frac{\lambda_R}{\kappa^2} R^{\nu'\beta'}{}_{\nu\beta} + \frac{\lambda_F}{\kappa^2} F^{\nu'\beta'} F_{\nu\beta} + \frac{\tilde{\lambda}_F}{\kappa^2} F^{a\nu'\beta'} F^{a}{}_{\nu\beta} + \frac{1}{\kappa^2} X^{\nu'\beta'}{}_{\nu\beta} \tag{5.7}$$

where

$$I^{\nu'\beta'}{}_{\nu\beta} = \frac{1}{2}\left(\delta^{\nu'}{}_{\nu}\delta^{\beta'}{}_{\beta} - \delta^{\beta'}{}_{\nu}\delta^{\nu'}{}_{\beta}\right) \tag{5.8}$$

and

$$Inv^{\mu'\alpha'\nu'\beta'} = \frac{1}{2}\left(g^{\mu'\nu'}g^{\alpha'\beta'} - g^{\mu'\beta'}g^{\alpha'\nu'}\right) \tag{5.9}$$

Firstly, let us examine $DDet\Big(\kappa^2\left(g_{\mu\nu'}g_{\alpha\beta'} - g_{\mu\beta'}g_{\alpha\nu'}\right)\Big)$:

$$DDet\Big(\kappa^2\left(g_{\mu\nu'}g_{\alpha\beta'} - g_{\mu\beta'}g_{\alpha\nu'}\right)\Big) = \kappa^8 DDet\Big(g_{\mu\nu'}g_{\alpha\beta'} - g_{\mu\beta'}g_{\alpha\nu'}\Big) = \kappa^8 DDet\Big(Q_{\mu\alpha\nu'\beta'}\Big)$$

for $D = 4$ where

$$Q_{\mu\alpha\nu'\beta'} = g_{\mu\nu'}g_{\alpha\beta'} - g_{\mu\beta'}g_{\alpha\nu'}$$

Then,

$$DDet\Big(Q_{\mu\alpha\nu'\beta'}\Big) = \left(\sqrt{-g}\right)^4 c_{DD}^4 \tag{5.10}$$

and since we need $DDet\Big(Q_{\mu\alpha\nu'\beta'}\Big)^{1/4}$

$$DDet\Big(Q_{\mu\alpha\nu'\beta'}\Big)^{1/4} = \sqrt{-g}\,c_{DD} \tag{5.11}$$

Expansion of $DDet\left(I^{\nu'\beta'}{}_{\nu\beta} + A^{\nu'\beta'}{}_{\nu\beta}\right)$ is given also as in (3.7). From there

$$\Big[DDet(I^{\nu'\beta'}{}_{\nu\beta} + A^{\nu'\beta'}{}_{\nu\beta}\Big]^{1/4} = 1 + \frac{1}{4}Tr(A^{\nu'\beta'}{}_{\nu\beta}) - \frac{1}{8}Tr[(A^{\nu'\beta'}{}_{\nu\beta})^2] + \frac{1}{32}[Tr(A^{\nu'\beta'}{}_{\nu\beta})]^2 + \mathcal{O}(A^3) \tag{5.12}$$

$$Tr(A^{\nu'\beta'}{}_{\nu\beta}) = A^{\nu'\beta'}{}_{\nu\beta} I^{\nu'\beta'}{}_{\nu\beta} \tag{5.13}$$

Therefore

$$Tr(A^{\nu'\beta'}{}_{\nu\beta}) = \frac{\lambda_R}{\kappa^2} R + \frac{\lambda_F}{\kappa^2} F^{\nu\beta} F_{\nu\beta} + \frac{\tilde{\lambda}_F}{\kappa^2} F^{a\nu\beta} F^a{}_{\nu\beta} + \frac{1}{\kappa^2} X \tag{5.14}$$

$$\begin{aligned}
[Tr(A^{\nu'\beta'}{}_{\nu\beta})]^2 &= \frac{\lambda_R^2}{\kappa^4} R^2 + \frac{2\lambda_R \lambda_F}{\kappa^4} R F^{\nu\beta} F_{\nu\beta} + \frac{2\lambda_R \tilde{\lambda}_F}{\kappa^4} R F^{a\nu\beta} F^a{}_{\nu\beta} + \frac{2\lambda_R}{\kappa^4} R X \\
&+ \frac{\lambda_F^2}{\kappa^4} F^{\nu\beta} F_{\nu\beta} F^{\sigma\rho} F_{\sigma\rho} + \frac{2\lambda_F \tilde{\lambda}_F}{\kappa^4} F^{a\nu\beta} F^a{}_{\nu\beta} F^{\sigma\rho} F_{\sigma\rho} + \frac{2\lambda_F}{\kappa^4} F^{\nu\beta} F_{\nu\beta} X \\
&+ \frac{\tilde{\lambda}_F^2}{\kappa^4} F^{a\nu\beta} F^a{}_{\nu\beta} F^{b\sigma\rho} F^b{}_{\sigma\rho} + \frac{2\tilde{\lambda}_F}{\kappa^4} F^{a\nu\beta} F^a{}_{\nu\beta} X + \frac{1}{\kappa^4} X^2
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
Tr[(A^{\nu'\beta'}{}_{\nu\beta})^2] &= \frac{\lambda_R^2}{\kappa^4} R^{\nu'\beta'}{}_{\nu\beta} R^{\nu\beta}{}_{\nu'\beta'} + \frac{2\lambda_R \lambda_F}{\kappa^4} R^{\nu'\beta'}{}_{\nu\beta} F^{\nu\beta} F_{\nu'\beta'} \\
&+ \frac{2\lambda_R \tilde{\lambda}_F}{\kappa^4} R^{\nu'\beta'}{}_{\nu\beta} F^{a\nu\beta} F^a{}_{\nu'\beta'} + \frac{2\lambda_R}{\kappa^4} R^{\nu'\beta'}{}_{\nu\beta} X^{\nu\beta}{}_{\nu'\beta'} \\
&+ \frac{\lambda_F^2}{\kappa^4} F^{\nu\beta} F_{\nu'\beta'} F_{\nu\beta} F^{\nu'\beta'} + \frac{2\lambda_F \tilde{\lambda}_F}{\kappa^4} F^{\nu'\beta'} F_{\nu\beta} F^{a\nu\beta} F^a{}_{\nu'\beta'} \\
&+ \frac{2\lambda_F}{\kappa^4} F^{\nu'\beta'} F_{\nu\beta} X^{\nu\beta}{}_{\nu'\beta'} + \frac{\tilde{\lambda}_F^2}{\kappa^4} F^{a\nu\beta} F^a{}_{\nu'\beta'} F^b{}_{\nu\beta} F^{b\nu'\beta'} \\
&+ \frac{2\tilde{\lambda}_F}{\kappa^4} F^{a\nu'\beta'} F^a{}_{\nu\beta} X^{\nu\beta}{}_{\nu'\beta'} + \frac{1}{\kappa^4} X^{\nu'\beta'}{}_{\nu\beta} X^{\nu\beta}{}_{\nu'\beta'}
\end{aligned} \tag{5.16}$$

In the light of these traces, we have also some terms which contain *ghost*. To cancel out these term let us rewrite $X^{\nu'\beta'}{}_{\nu\beta}$ by using Eq.(4.18)

$$X^{\nu'\beta'}{}_{\nu\beta} = \frac{1}{24\kappa^2} \lambda_R^2 \left[R^{\lambda\rho}{}_{\mu\tau} R^{\mu\tau}{}_{\lambda\rho} - \frac{1}{4} R^2 \right] \left(\delta^{\nu'}{}_{\nu} \delta^{\beta'}{}_{\beta} - \delta^{\beta'}{}_{\nu} \delta^{\nu'}{}_{\beta} \right) \tag{5.17}$$

By inserting Eq.(5.17) and all of the traces from the begining of the this chapter into the action given in Eq.(5.6) we obtain,

$$
\begin{aligned}
S_{BIR} &= \int d^4x M_{Pl}^2 \kappa^2 c_{DD} \sqrt{-g} \Big[1 + \frac{1}{4}\Big(\frac{\lambda_R}{\kappa^2} R + \frac{\lambda_F}{\kappa^2} F^{\nu\beta} F_{\nu\beta} + \frac{\tilde{\lambda}_F}{\kappa^2} F^{a\nu\beta} F^{a}{}_{\nu\beta} \Big) \\
&- \frac{1}{8}\Big(\frac{2\lambda_R\lambda_F}{\kappa^4} R^{\nu'\beta'}{}_{\nu\beta} F^{\nu\beta} F_{\nu'\beta'} + \frac{2\lambda_R\tilde{\lambda}_F}{\kappa^4} R^{\nu'\beta'}{}_{\nu\beta} F^{a\nu\beta} F^{a}{}_{\nu'\beta'} \\
&+ \frac{\lambda_F^2}{\kappa^4} F^{\nu\beta} F_{\nu'\beta'} F_{\nu\beta} F^{\nu'\beta'} + \frac{2\lambda_F\tilde{\lambda}_F}{\kappa^4} F^{\nu'\beta'} F_{\nu\beta} F^{a\nu\beta} F^{a}{}_{\nu'\beta'} \\
&+ \frac{\tilde{\lambda}_F^2}{\kappa^4} F^{a\nu\beta} F^{a}{}_{\nu'\beta'} F^{b}{}_{\nu\beta} F^{b\nu'\beta'} \Big) \\
&+ \frac{1}{32}\Big(\frac{2\lambda_R\lambda_F}{\kappa^4} R F^{\nu\beta} F_{\nu\beta} + \frac{2\lambda_R\tilde{\lambda}_F}{\kappa^4} R F^{a\nu\beta} F^{a}{}_{\nu\beta} + \frac{\lambda_F^2}{\kappa^4} F^{\nu\beta} F_{\nu\beta} F^{\sigma\rho} F_{\sigma\rho} \\
&+ \frac{2\lambda_F\tilde{\lambda}_F}{\kappa^4} F^{a\nu\beta} F^{a}{}_{\nu\beta} F^{\sigma\rho} F_{\sigma\rho} + \frac{\tilde{\lambda}_F^2}{\kappa^4} F^{a\nu\beta} F^{a}{}_{\nu\beta} F^{b\sigma\rho} F^{b}{}_{\sigma\rho} \Big) + \mathcal{O}(A^3) \Big]
\end{aligned}
\tag{5.18}
$$

For simplicity, let us define two new tensors

$$
K_R{}^{\nu'\beta'}{}_{\nu\beta} \equiv R^{\nu'\beta'}{}_{\nu\beta} - \frac{1}{4} R \delta^{\nu'}{}_{\nu} \delta^{\beta'}{}_{\beta} \tag{5.19}
$$

$$
K_F{}^{\nu'\beta'}{}_{\nu\beta} \equiv F^{a\nu'\beta'} F^{a}{}_{\nu\beta} - \frac{1}{4} F^{a\rho\sigma} F^{a}{}_{\rho\sigma} \delta^{\nu'}{}_{\nu} \delta^{\beta'}{}_{\beta} \tag{5.20}
$$

and with the new tensors, the last form of BIR action is

$$
\begin{aligned}
S_{BIR} &= \int d^4x M_{Pl}^2 \kappa^2 c_{DD} \sqrt{-g} \Big[1 + \frac{1}{4}\Big(\frac{\lambda_R}{\kappa^2} R + \frac{\lambda_F}{\kappa^2} F^{\nu\beta} F_{\nu\beta} + \frac{\tilde{\lambda}_F}{\kappa^2} F^{a\nu\beta} F^{a}{}_{\nu\beta} \Big) \\
&- \frac{1}{8}\Big(\frac{2\lambda_R\lambda_F}{\kappa^4} K_R^{\nu'\beta'}{}_{\nu\beta} F^{\nu\beta} F_{\nu^{prime}\beta'} + \frac{2\lambda_R\tilde{\lambda}_F}{\kappa^4} K_R^{\nu'\beta'}{}_{\nu\beta} F^{a\nu\beta} F^{a}{}_{\nu'\beta'} \\
&+ \frac{3\lambda_F^2}{4\kappa^4} F^{\nu\beta} F_{\nu'\beta'} F_{\nu\beta} F^{\nu'\beta'} + \frac{2\lambda_F\tilde{\lambda}_F}{\kappa^4} K_F^{\nu'\beta'}{}_{\nu\beta} F^{\nu\beta} F_{\nu'\beta'} \\
&+ \frac{\tilde{\lambda}_F^2}{\kappa^4} K_F^{\nu'\beta'}{}_{\nu\beta} F^{a\nu\beta} F^{a}{}_{\nu\beta} + \mathcal{O}(A^3) \Big]
\end{aligned}
\tag{5.21}
$$

5.2. Abelian - Non Abelian Mixing in BIR

After long calculations we obtain the action given in Eq.(5.21). let us rewrite it and examine term by term.

$$
\begin{aligned}
S_{BIR} &= \int d^4x M_{Pl}^2 \kappa^2 c_{DD} \sqrt{-g}\Big[1 + \frac{1}{4}\Big(\underbrace{\frac{\lambda_R}{\kappa^2}R}_{1} + \underbrace{\frac{\lambda_F}{\kappa^2}F^{\nu\beta}F_{\nu\beta}}_{2} + \underbrace{\frac{\tilde{\lambda}_F}{\kappa^2}F^{a\nu\beta}F^{a}{}_{\nu\beta}}_{3}\Big) \\
&- \frac{1}{8}\Big(\underbrace{\frac{2\lambda_R\lambda_F}{\kappa^4}K_R^{\nu'\beta'}{}_{\nu\beta}F^{\nu\beta}F_{\nu'\beta'}}_{4} + \underbrace{\frac{2\lambda_R\tilde{\lambda}_F}{\kappa^4}K_R^{\nu'\beta'}{}_{\nu\beta}F^{a\nu\beta}F^{a}{}_{\nu'\beta'}}_{5} \\
&+ \underbrace{\frac{3\lambda_F^2}{4\kappa^4}F^{\nu\beta}F_{\nu'\beta'}F_{\nu\beta}F^{\nu'\beta'}}_{6} + \underbrace{\frac{2\lambda_F\tilde{\lambda}_F}{\kappa^4}K_F^{\nu'\beta'}{}_{\nu\beta}F^{\nu\beta}F_{\nu'\beta'}}_{7} \\
&+ \underbrace{\frac{\tilde{\lambda}_F^2}{\kappa^4}K_F^{\nu'\beta'}{}_{\nu\beta}F^{a\nu\beta}F^{a}{}_{\nu\beta}}_{8} + \mathcal{O}(A^3)\Big]
\end{aligned}
\tag{5.22}
$$

Here, the term labelled as 1 is just the Einstein-Hilbert term. It gives us General Relativity in small curvature limit.

The term labelled as 2 is just Lagrangian of Field Strength. This term supplies us to Electromagnetism.

3 is called as Yang-Mills term. It carries Non-Abelian part of the theory.

The terms $1, 2, 3$ are not mixed terms. All of them responsible for their own field. From there let us examine mixed terms.

The term labelled as 4 has mixing of Abelian Field Strength tensor and Curvature. It can be interpreted as graviton-photon coupling and can be detected by means of High Luminosity experiments.

5 is the Curvature- Non Abelian Field Strength Tensor coupling and it can be interpreted graviton-gluon coupling due to High Energy experiments.

The term 6 is coupling of Abelian Field Strength Tensors, such as photon-photon coupling.

7 is Abelian-Non Abelian Coupling. This is the main purpose of our thesis.

8 is coupling of two Non Abelian fields such as gluon-gluon coupling.

As a result, Born-Infeld-Riemann Gravity allows the mixing of Abelian and Non-Abelian fields. This term can be responsible for photon-gluon coupling and it can be observed by High Luminosity Experiments.

CHAPTER 6

APPLICATION TO COSMOLOGY

A large part of modern cosmological theories are based on cosmological principle. Cosmological principle is the hypothesis that the universe is spatially flat and isotropic (Weinberg, 1972). Thus the standard cosmology is based on maximally spatially symmetric FRW line element which is

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \quad \text{for} \quad k = 0 \tag{6.1}$$

(Riotto, 2002).In this chapter we examine our theories in terms of cosmology and what they give us about cosmology. Let us start with brief explanation for inflationary cosmology.

6.1. Inflationary Cosmology

Although Einstein claimed that universe must be static, Hubble's observation showed that universe was expanding with an accelerating rate (Hubble, 1929). In the light of this observation and the others which were made after Hubble's discovery inspire the scientist to search a theory which explains accelerated universe. One of the explanation is given by *inflation*. This theory was originally proposed by Alan Guth in 1980 (Guth, 1980). Inflation is the period of the accelerated expansion of the universe. Scalar and vector fields can be source of the inflation. The expansion rate is related to the scale factor which is originated from FRW metric (Appendix A). The evolution of the scale factor is governed by Einstein Field Equations.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{6.2}$$

In the case of perfect fluid, energy momentum tensor becomes

$$T_{\mu\nu} = Diag\left[\rho, a^2P, a^2P, a^2P\right]$$

Let us examine Einstein Equation term by term. In case of $\mu = 0$ and $\nu = 0$

$$R_{00} - \frac{1}{2}Rg_{00} = 8\pi G T_{00} \tag{6.3}$$

In light of Appendix A, this term becomes

$$\frac{3\dot{a}^2}{a^2} = 8\pi G\rho \tag{6.4}$$

Here $\dfrac{\dot{a}}{a}$ has a special meaning. It is called Hubble parameter and defined as

$$H = \frac{\dot{a}}{a} \tag{6.5}$$

From there, Einstein Field Equations for $\mu = 0$ and $\nu = 0$ take the form as

$$H^2 = \frac{8\pi G}{3}\rho \tag{6.6}$$

Moreover, in case of $\mu = i$ and $\nu = j$

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi G T_{ij} \tag{6.7}$$

Using curvature tensor in FRW background (see Appendix A),

$$\left(\ddot{a}a + 2\dot{a}^2\right)\delta_{ij} - \frac{1}{2}6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)a^2\delta_{ij} = 8\pi G\left(a^2P\delta_{ij}\right) \tag{6.8}$$

In this equation we have second derivative of scale factor. We find the second and third derivative of the scale factor by taking derivative on Eq. (6.5) with respect to time. Then we obtain

$$\ddot{a} = \left(\dot{H} + H^2\right) a \qquad \& \qquad \dddot{a} = \left(\ddot{H} + 3\dot{H}H + H^3\right) a \tag{6.9}$$

By inserting this result in Eq.(6.8) and in case of $i = j$ Eq.(6.8) becomes,

$$\dot{H} = -4\pi G\left(\rho + P\right) \tag{6.10}$$

As a result, Einstein equation for perfect fluid becomes

$$\dot{H} = -4\pi G\left(\rho + P\right) \qquad \& \qquad H^2 = \frac{8\pi G}{3}\rho \tag{6.11}$$

Since $M_{pl}^2 = \dfrac{1}{8\pi G}$,

$$\dot{H} = \frac{-1}{2M_{pl}^2}\left(\rho + P\right) \qquad \& \qquad H^2 = \frac{1}{3M_{pl}^2}\rho \tag{6.12}$$

By using equations given in Eq.(6.12), we can write density and pressure in terms of Hubble parameter. Thus

$$\rho = 3M_{pl}^2 H^2 \qquad \& \qquad P = -M_{pl}^2\left(2\dot{H} + 3H^2\right) \tag{6.13}$$

On the other hand, in terms of any scalar field- called as inflaton- H^2 and $\dot{H}$ can be written as

$$\begin{aligned} H^2 &= \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \\ \dot{H} &= \frac{-1}{2}\dot{\phi}^2 \end{aligned} \tag{6.14}$$

Then equation of motion for the scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V(\phi) = 0 \tag{6.15}$$

Here, the term which contains $\dot{\phi}$ is just the friction term. To get inflation from scalar field let us assume that we have a very flat potential, i.e the field vary slowly with respect to time (Vaudrevange, 2010). In the light of this assumption, we can avoid terms which contains derivative of the fields. From there let us define slow roll parameter:

$$\varepsilon = \frac{-\dot{H}}{H^2} \tag{6.16}$$

Slow roll inflation is achieved where $\varepsilon << 1$. Moreover, to obtain a slow roll inflation, we apply a constraint on the field ϕ such as

$$\eta << 1 \qquad \text{where} \qquad \eta = \frac{\dot{\phi}}{H\phi} \tag{6.17}$$

In this thesis we obtain inflation field by means of Non Abelian field strength tensor. In (§4) and (§) 5 we showed that Non Abelian field strength tensor embedded into geometry. Thus, by the definition given below, we embedded not only Non-Abelian fields but also inflaton field to the geometry. We obtain a scalar field such as inflaton, by applying gauge fixing to the Non Abelian Field Strength tensor in FRW background. Gauge fixing is given as

$$A^a{}_\mu = \begin{cases} \mu = 0 \rightarrow 0 \\ \mu = i \rightarrow \phi(t)\delta^a{}_i \end{cases} \tag{6.18}$$

Non-Abelian Field Strength tensor is

$$F^a{}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon^a_{bc} A^b{}_\mu A^c{}_\nu$$

By applying gauge fixing in FRW background, it takes the form as

$$\begin{aligned} F^{a}{}_{0i} &= \dot{\phi}\delta^{a}{}_{i} \quad \& \quad F^{a0i} = a^{-2}\dot{\phi}\delta^{ai} \\ F^{a}{}_{ij} &= -g\phi^{2}\epsilon^{a}_{ij} \quad \& \quad F^{aij} = -a^{-4}g\phi^{2}\epsilon^{aij} \end{aligned} \tag{6.19}$$

and also notice that for Abelian part we should use metric tensor also for raising and lowering indices.

$$F^{0i} = -a^{-2}F_{0i} \text{ and } F^{ij} = a^{-4}F_{ij} \tag{6.20}$$

In the following sections, we use these forms given in Eq. (6.19) and Eq. (6.20) for both Abelian and Non Abelian Field Strength tensor.

6.2. BIE in FRW Background

In this section we consider Born-Infeld-Einstein gravity in FRW Background to understand its effect on cosmology. The action of BIE with Abelian and Non-Abelian Fields was found in Eq. (4.50) and it is

$$
\begin{aligned}
S_{BIE} \;=\; & \int d^4x M_{Pl}^2 \sqrt{-g}\Big\{\tilde{a}^2 + \frac{\tilde{a}}{2}\Big[b \underbrace{R}_{0} - \frac{c^2}{2\tilde{a}} \underbrace{F_{\mu\nu}F^{\mu\nu}}_{1} + d \underbrace{F^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}}_{2}\Big] \\
+ & \frac{1}{8}\Big[-4bd \underbrace{R^{\mu\nu}F^{a}{}_{\mu\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}}_{3} + 2bd \underbrace{RF^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}}_{4} \\
- & 2d^2 \underbrace{F^{a}{}_{\mu\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}F^{b\mu}{}_{\varepsilon}F^{b\nu}{}_{\theta}g^{\varepsilon\theta}}_{5} + d^2 \underbrace{F^{a\nu}{}_{\alpha}F^{a}{}_{\beta\nu}g^{\alpha\beta}F^{b\lambda}{}_{\varepsilon}F^{b}{}_{\theta\lambda}g^{\varepsilon\theta}}_{6}\Big] \\
+ & \frac{1}{48\tilde{a}}\Big[d^3 \underbrace{F^{a\sigma\beta}F^{a}{}_{\beta\sigma}F^{b\lambda\rho}F^{b}{}_{\rho\lambda}F^{c\varepsilon\eta}F^{c}{}_{\eta\varepsilon}}_{7} + 6d^3 \underbrace{F^{a}{}_{\mu\alpha}F^{a\alpha}{}_{\nu}F^{b\nu\rho}F^{b\mu}{}_{\rho}F^{c\varepsilon\eta}F^{c}{}_{\eta\varepsilon}}_{8} \\
- & 8d^3 \underbrace{F^{a\sigma\beta}F^{a}{}_{\beta\nu}F^{b\nu\rho}F^{b\mu}{}_{\rho}F^{c}{}_{\mu\eta}F^{c\eta}{}_{\sigma}}_{9} + bd^2 \underbrace{RF^{a\sigma\beta}F^{a}{}_{\beta\sigma}F^{b\lambda\rho}F^{b}{}_{\rho\lambda}}_{10} \\
+ & 6bd^2 \underbrace{RF^{a}{}_{\mu\alpha}F^{a\alpha}{}_{\nu}F^{b\nu\rho}F^{b\mu}{}_{\rho}}_{11} - 12bd^2 \underbrace{R^{\nu\mu}F^{a}{}_{\mu\alpha}F^{a\alpha}{}_{\nu}F^{b\sigma\eta}F^{b}{}_{\eta\sigma}}_{12} \\
+ & 24bd^2 \underbrace{R^{\nu\mu}F^{a\sigma\beta}F^{a}{}_{\beta\nu}F^{b}{}_{\mu\eta}F^{b\eta}{}_{\sigma}}_{13} - 6bc^2 \underbrace{RF^{\mu\nu}F_{\nu\mu}}_{14} + 24bc^2 \underbrace{R^{\nu\mu}F^{\sigma}{}_{\nu}F_{\mu\sigma}}_{15} \\
- & 6c^2 d \underbrace{F^{\mu\nu}F_{\nu\mu}F^{a\sigma\beta}F^{a}{}_{\beta\sigma}}_{16} + 24c^2 d \underbrace{F^{\sigma}{}_{\nu}F^{\nu\mu}F^{a}{}_{\mu\eta}F^{a\eta}{}_{\sigma}}_{17}\Big]\Big\}
\end{aligned}
\tag{6.21}
$$

Here we should expand action given above in FRW background. Curvature terms give us scalar factor and its derivative. Non Abelian fields give us also scalar field which inflates the universe. Let us examine terms in the action one by one: The term labelled as 0 is the just Curvature scalar. Expansion of this is given in Appendix A.

The term labelled as 1 is Abelian Field Strength Tensor. Since it is Abelian, it can not be expand in FRW backgorund. It can be written just as

$$1 \rightarrow 2F_{0i}F^{i0} + F_{ij}F^{ji} \tag{6.22}$$

Non Abelian Field Strength Tensor given in 2 in FRW background it takes the form as

$$2 \to 6a^{-2}\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right) \tag{6.23}$$

3 is the mixing of Ricci Tensor and Non Abelian Field Strength tensor. It is given as

$$3 \to 12a^{-4}\left(\ddot{a}a + \dot{a}^2\right)\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right) \tag{6.24}$$

4 is combination of curvature scalar and Non Abelian Field Strength Tensor and expansion of that term is

$$4 \to 36a^{-4}\left(\ddot{a}a + \dot{a}^2\right)\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right) \tag{6.25}$$

Non Abelian Field Strength tensors are combined in 5

$$5 \to 36a^{-4}\left(\dot{\phi}^4 - 2a^{-2}g^2\dot{\phi}^2\phi^4 + a^{-4}g^4\phi^8\right) \tag{6.26}$$

and also the other type of mixing of Non Abelian Fields is given in 6,

$$6 \to 6a^{-4}\left(2\dot{\phi}^4 - a^{-2}g^2\dot{\phi}^2\phi^4\right) \tag{6.27}$$

In 7 we also see mixing of Non Abelian Fields

$$7 \to 216a^{-6}\left(\dot{\phi}^6 - 3a^{-2}g^2\dot{\phi}^4\phi^4 + 3a^{-4}g^4\dot{\phi}^2\phi^8 - a^{-6}g^6\phi^{12}\right) \tag{6.28}$$

Then the expansion of 8,

$$8 \to 36a^{-6}\left(2\dot{\phi}^6 - 3a^{-2}g^2\dot{\phi}^4\phi^4 + a^{-4}g^4\dot{\phi}^2\phi^8\right) \tag{6.29}$$

and 9 takes the form as

$$9 \to 6a^{-6}\left(5\dot{\phi}^6 - a^{-2}g^2\dot{\phi}^4\phi^4 - 2a^{-4}g^4\dot{\phi}^2\phi^8 + 4a^{-6}g^6\phi^{12}\right) \tag{6.30}$$

Coupling of curvature terms and four Non Abelian fields are given in 10

$$10 \to 216a^{-6}\left(\ddot{a}a + \dot{a}^2\right)\left(\dot{\phi}^4 - 2a^{-2}g^2\dot{\phi}^2\phi^4 + a^{-4}g^4\phi^8\right) \tag{6.31}$$

for 11,

$$11 \to 36a^{-6}\left(\ddot{a}a + \dot{a}^2\right)\left(2\dot{\phi}^4 - a^{-2}g^2\dot{\phi}^2\phi^4\right) \tag{6.32}$$

for 12,

$$\begin{aligned} 12 \to 36a^{-6}\Big[&\left(2\ddot{a}a + \dot{a}^2\right)\dot{\phi}^4 - 3\left(\ddot{a}a + \dot{a}^2\right)a^{-2}g^2\dot{\phi}^2\phi^4 \\ &+ \left(\ddot{a}a + 2\dot{a}^2\right)a^{-4}g^4\phi^8\Big] \end{aligned} \tag{6.33}$$

and 13

$$\begin{aligned} 13 \to 6a^{-6}\left(5\ddot{a}a + \dot{a}^2\right)\dot{\phi}^4 - 42a^{-8}\left(\ddot{a}a + 2\dot{a}^2\right)g^2\dot{\phi}^2\phi^4 \\ +72a^{-10}\left(\ddot{a}a + 2\dot{a}^2\right)g^4\phi^8 \end{aligned} \tag{6.34}$$

Abelian fields and curvature couplings signed as 14 and 15 are combined like as

$$14 + 15 \to 2a^{-2}\left(14\ddot{a}aF_{0i}F^{i0} + 5\dot{a}^2F_{ij}F^{ji}\right) \tag{6.35}$$

Expansion of Abelian-Non Abelian coupling are given in 16

$$16 \to 12a^{-2}\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right)F_{0i}F^{i0} + 6a^{-2}\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right)F_{ij}F^{ji} \tag{6.36}$$

and 17

$$17 \to 8a^{-2}\left(2\dot{\phi}^2 + a^{-2}g^2\phi^4\right)F_{0i}F^{i0} + 4a^{-2}\left(\dot{\phi}^2 + 2a^{-2}g^2\phi^4\right)F_{ij}F^{ji} \tag{6.37}$$

The next step we should follow to substitute these expansions into the action. However there are coefficients in action which are unknown. We have four coefficients in the action which are $\tilde{a}$, b, c, d. To determine these coefficients, let us use well-known actions-Einstein-Hilbert, Electromagnetism and Yang-Mills. Because the first way to understand we have a consistent theory or not is to seek Einstein-Hilbert action in the limit of small curvature (Escobar, 2012). Thus curvature terms in the action, should be the same as Einstein-Hilbert term (Carroll, 2004).

$$S_{EH} = \frac{1}{2}\int d^4x\sqrt{-g}M_{pl}^2 R \tag{6.38}$$

and Yang-Mills action (Quigg, 1983)

$$S_{YM} = \frac{-1}{4}\int d^4x\sqrt{-g}F^{a\alpha\beta}F^{a}{}_{\alpha\beta} \tag{6.39}$$

Electromagnetism Action (Quigg, 1983)

$$S_{EM} = \frac{-1}{4}\int d^4x\sqrt{-g}F^{\alpha\beta}F_{\alpha\beta} \tag{6.40}$$

Then by comparing these well-known actions to the our action, coefficients can be determined as

$$\tilde{a}b = 1 \ , \ c^2 = \frac{1}{M_{pl}^2} \ , \ d = \frac{1}{2\tilde{a}M_{pl}^2}$$

$$
\begin{aligned}
S &= \int d^4x\sqrt{-g}\Big\{ \Big[M_{pl}^2\tilde{a}^2 + 3M_{pl}^2 a^{-2}\left(\ddot{a}a + \dot{a}^2\right) - \frac{1}{4}F_{\gamma\nu}F^{\gamma\nu} + \frac{3}{2}a^2\left(\dot{\phi}^2 - a^2g^2\phi^4\right)\Big] \\
&+ \frac{3a^{-4}}{2\tilde{a}^2}\Big[3\Big(\ddot{a}a + \dot{a}^2\Big)\Big(\dot{\phi}^2 - a^{-2}g^2\phi^4\Big) - \Big(\Big(2\ddot{a}a + \dot{a}^2\Big)\dot{\phi}^2 - \Big(\ddot{a}a + 2\dot{a}^2\Big)a^{-2}g^2\phi^4\Big) \\
&+ \frac{a^2}{6}\left(12\ddot{a}aF_{0i}F^{i0} + 5\dot{a}^2F_{ij}F^{ji}\right)\Big] + \frac{3a^{-4}}{\tilde{a}^2M_{pl}^2}\Big[3\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right)^2 \\
&- \frac{1}{2}\left(\dot{\phi}^4 - 2a^{-2}g^2\dot{\phi}^2\phi^4\right) + a^2\Big(\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right)\left(2F_{0i}F^{i0} + F_{ij}F^{ji}\right)\Big) \\
&+ \frac{8a^2}{3}\Big(\left(4\dot{\phi}^2 + 2a^{-2}g^2\phi^4\right)F_{0i}F^{i0} + \left(\dot{\phi}^2 + 2a^{-2}g^2\phi^4\right)F_{ij}F^{ji}\Big)\Big] \\
&+ \frac{3a^{-6}}{4\tilde{a}^4M_{pl}^2}\Big[\frac{-9}{4}\left(\ddot{a}a + \dot{a}^2\right)\left(2\dot{\phi}^4 - a^{-2}g^2\dot{\phi}^2\phi^4\right) - 3\Big(\left(2\ddot{a}a + \dot{a}^2\right)\dot{\phi}^4 \\
&- 3a^{-2}\left(\ddot{a}a + \dot{a}^2\right)g^2\dot{\phi}^2\phi^4 + a^{-4}\left(\ddot{a}a + 2\dot{a}^2\right)g^4\phi^8\Big) \\
&+ \Big(\left(5\ddot{a}a + \dot{a}^2\right)\dot{\phi}^4 - 7a^{-2}\left(\ddot{a}a + \dot{a}^2\right)g^2\dot{\phi}^2\phi^4 + 12a^{-4}\left(\ddot{a}a + 2\dot{a}^2\right)g^4\phi^8\Big)\Big] \\
&+ \frac{a^{-6}}{16\tilde{a}^4M_{pl}^4}\Big[\Big(\dot{\phi}^6 - 2a^{-2}g^2\dot{\phi}^4\phi^4 - 12a^{-4}g^4\dot{\phi}^2\phi^8 + 17a^{-6}g^6\phi^{12}\Big) \\
&+ 27\left(\ddot{a}a + \dot{a}^2\right)\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right)^2\Big]\Big\} \qquad (6.41)
\end{aligned}
$$

Let us define

$$\tilde{a}^2 \equiv \epsilon M^2$$

where M denotes mass (which is approximately in the order of M_{pl}, but not equal to it) and

$$\epsilon \equiv \pm 1$$

. Then the action becomes,

$$
\begin{aligned}
S &= \int d^4x\sqrt{-g}\Big\{\epsilon M^2M_{pl}^2 + 3M_{pl}^2a^{-2}\left(\ddot{a}a + \dot{a}^2\right) - \frac{1}{4}F_{\gamma\nu}F^{\gamma\nu} \\
&+ \frac{3a^{-2}}{2}\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right) + \frac{3a^{-4}}{2\epsilon M^2}\Big[3\left(\ddot{a}a + \dot{a}^2\right)\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right) \\
&- \Big(\left(2\ddot{a}a + \dot{a}^2\right)\dot{\phi}^2 - a^{-2}\left(\ddot{a}a + 2\dot{a}^2\right)g^2\phi^4\Big) + \frac{a^2}{6}\left(12\ddot{a}aF_{0i}F^{i0} + 5\dot{a}^2F_{ij}F^{ji}\right)\Big] \\
&+ \frac{3a^{-4}}{8\epsilon M^2M_{pl}^2}\Big[3\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right)^2 - \frac{1}{2}\left(\dot{\phi}^4 - 2a^{-2}g^2\dot{\phi}^2\phi^4\right) \\
&+ a^2\Big(\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right)\left(2F_{0i}F^{i0} + F_{ij}F^{ji}\right)\Big) \\
&+ \left(4\dot{\phi}^2 + 2a^{-2}g^2\phi^4\right)F_{0i}F^{i0} + \left(\dot{\phi}^2 + 2a^{-2}g^2\phi^4\right)F_{ij}F^{ji}\Big]\Big\} \qquad (6.42)
\end{aligned}
$$

we neglect $\dfrac{1}{\epsilon^2 M^4 M_{pl}^2}$ and $\dfrac{1}{\epsilon^2 M^4 M_{pl}^4}$ terms since they give very small contribution to the action. Let us examine FRW background briefly:

One of the solutions of the Einstein Equations is FRW solution. FRW solution defines universe as isotropic and homogeneous. Metric tensor for FRW solution is given as

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$$

where $a(t)$ denotes *Scale Factor* which is responsible from expanding of the universe. Scale factor is also related to radius of the universe. Moreover scale factor is a *dimensionless* quantity since metric tensor is also dimensionless. In matrix form, metric tensor for FRW background is written as

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{pmatrix} \tag{6.43}$$

Bu using this matrix form, we calculate determinant of metric tensor $g_{\mu\nu}$

$$\det\left(g_{\mu\nu}\right) = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{vmatrix} \tag{6.44}$$

$$\det g_{\mu\nu} = -1 \begin{vmatrix} a(t)^2 & 0 & 0 \\ 0 & a(t)^2 & 0 \\ 0 & 0 & a(t)^2 \end{vmatrix} + 0 + 0 + 0 \tag{6.45}$$

$$\det g_{\mu\nu} = -a^6 \tag{6.46}$$

Substituting $\det(g_{\mu\nu})$ in

$$\sqrt{-g} \equiv \sqrt{-\det(g_{\mu\nu})}$$

, $\sqrt{-g}$ becomes,

$$\sqrt{-g} = a^3 \tag{6.47}$$

As we mention before, scale factor ($a(t)$) correlates with radius of the universe (R). Thus let us make a definition:

$$R \equiv \xi a \tag{6.48}$$

where ξ is a constant that does not depend on time and $[\xi] = M^{-1}$. Then,

$$\int d^4x = \int dt \int_0^R d^3x = \int dt R^3 = \int dt \xi^3 a^3 \tag{6.49}$$

Therefore,

$$\int d^4x \sqrt{-g} = \int dt \xi^3 a^6 \tag{6.50}$$

Let us substitute (6.50) to the action

$$\begin{aligned} S &= \int dt \Big\{ \epsilon \xi^3 a^6 M^2 M_{pl}^2 + 3\xi^3 M_{pl}^2 a^4 \left(\ddot{a}a + \dot{a}^2\right) - \frac{\xi^3 a^6}{4} F_{\gamma\nu} F^{\gamma\nu} \\ &+ \frac{3\xi^3 a^4}{2} \left(\dot{\phi}^2 - a^{-2} g^2 \phi^4\right) + \frac{3\xi^3 a^2}{2\epsilon M^2} \Big[\left(\ddot{a}a + 2\dot{a}^2\right) \dot{\phi}^2 - \left(2\ddot{a}a + \dot{a}^2\right) a^{-2} g^2 \phi^4 \\ &+ \frac{a^2}{6} \left(12\ddot{a}a F_{0i} F^{i0} + 5\dot{a}^2 F_{ij} F^{ji}\right) \Big] + \frac{3\xi^3 a^2}{8\epsilon M^2 M_{pl}^2} \Big[\frac{5}{2} \dot{\phi}^4 - 5a^{-2} g^2 \dot{\phi}^2 \phi^4 \\ &+ 3a^{-4} g^4 \phi^8 + \frac{a^2}{3} \Big(\left(38\dot{\phi}^2 + 10a^{-2} g^2 \phi^4\right) F_{0i} F^{i0} \\ &+ \left(11\dot{\phi}^2 + 13a^{-2} g^2 \phi^4\right) F_{ij} F^{ji} \Big) \Big] \Big\} \end{aligned} \tag{6.51}$$

We obtain a theory which consists of two scalar field, $a(t)$ and $\phi(t)$. Both fields only depend on a parameter, time. Also,

$$\epsilon\xi^3 a^6 M^2 M_{pl}^2 \tag{6.52}$$

denotes vacuum energy.

6.2.1. Inflation from BIE?

To examine inflationary setup, we choose $c = 0$ and so all contributions coming from Abelian Field Strength Tensor disappear in action. Hence,

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\Big\{\epsilon M^2 M_{pl}^2 + 3M_{pl}^2 a^{-2}\left(\ddot{a}a + \dot{a}^2\right) + \frac{3a^{-2}}{2}\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right) \\ &+ \frac{3a^{-4}}{2\epsilon M^2}\left[\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi}^2 - \left(2\ddot{a}a + \dot{a}^2\right)a^{-2}g^2\phi^4\right] \\ &+ \frac{3a^{-4}}{8\epsilon M^2 M_{pl}^2}\left[\frac{5}{2}\dot{\phi}^4 - 5a^{-2}g^2\dot{\phi}^2\phi^4 + 3a^{-4}g^4\phi^8\right]\Big\} \end{aligned} \tag{6.53}$$

and inserting Eq.(6.50)

$$\begin{aligned} S = \int dt\Big\{ &\left[\epsilon\xi^3 a^6 M^2 M_{pl}^2 + 3\xi^3 M_{pl}^2 a^4\left(\ddot{a}a + \dot{a}^2\right) + \frac{3\xi^3 a^4}{2}\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right)\right] \\ &+ \frac{3\xi^3 a^2}{2\epsilon M^2}\left[\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi}^2 - \left(2\ddot{a}a + \dot{a}^2\right)a^{-2}g^2\phi^4\right] \\ &+ \frac{3\xi^3 a^2}{8\epsilon M^2 M_{pl}^2}\left[\frac{5}{2}\dot{\phi}^4 - 5a^{-2}g^2\dot{\phi}^2\phi^4 + 3a^{-4}g^4\phi^8\right]\Big\} \end{aligned} \tag{6.54}$$

Eq.(6.54) is the action without Abelian contributions. Since the action does not contain second derivative of ϕ, i.e. $\ddot{\phi}$, the theory does not contain ghosts.

Let us find Equations of motion. Since we have two dynamical variables, we should have e.o.m for both of them. Let us start with $\phi(t)$. To obtain equations of motions, we can use Euler-Lagrange equation which is (see Appendix B)

$$\frac{d}{dt}\left(\frac{\partial L_{eff}}{\partial\dot{\phi}}\right) - \left(\frac{\partial L_{eff}}{\partial\phi}\right) = 0 \tag{6.55}$$

Our Lagrange is

$$L_{eff} = \Big\{ \Big[\epsilon\xi^3 a^6 M^2 M_{pl}^2 + 3\xi^3 M_{pl}^2 a^4 \left(\ddot{a}a + \dot{a}^2\right) + \frac{3\xi^3 a^4}{2}\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right)\Big] + \frac{3\xi^3 a^2}{2\epsilon M^2}\Big[\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi}^2 - \left(2\ddot{a}a + \dot{a}^2\right)a^{-2}g^2\phi^4\Big] + \frac{3\xi^3 a^2}{8\epsilon M^2 M_{pl}^2}\Big[\frac{5}{2}\dot{\phi}^4 - 5a^{-2}g^2\dot{\phi}^2\phi^4 + 3a^{-4}g^4\phi^8\Big]\Big\} \quad (6.56)$$

From there inserting Eq. (6.56) in (6.55)

$$0 = \frac{d}{dt}\left[3\xi^3 a^4\dot{\phi} + \frac{3\xi^3 a^2}{\epsilon M^2}\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi} + \frac{15\xi^3 a^2}{4\epsilon M^2 M_{pl}^2}\dot{\phi}^3 - \frac{15\xi^3}{4\epsilon M^2 M_{pl}^2}g^2\dot{\phi}\phi^4\right] - \left[-6\xi^3 a^2 g^2\phi^3 - \frac{6\xi^3}{\epsilon M^2}\left(2\ddot{a}a + \dot{a}^2\right)g^2\phi 3 - \frac{30\xi^3}{4\epsilon M^2 M_{pl}^2}g^2\dot{\phi}^2\phi^3 + \frac{9\xi^3 a^{-2}}{\epsilon M^2 M_{pl}^2}g^4\phi^7\right] \quad (6.57)$$

By taking derivative with respect to time,

$$0 = \Big[12\xi^3 a^3\dot{a}\dot{\phi} + 3\xi^3 a^4\ddot{\phi} + \frac{3\xi^3}{\epsilon M^2}\left(3a^2\dot{a}\ddot{a} + a^3\dddot{a} + 4a\dot{a}^3 + 4a^2\dot{a}\ddot{a}\right)\dot{\phi} + \frac{3\xi^3 a^2}{\epsilon M^2}\left(\ddot{a}a + 2\dot{a}^2\right)\ddot{\phi} + \frac{30\xi^3}{4\epsilon M^2 M_{pl}^2}a\dot{a}\dot{\phi}^3 + \frac{45\xi^3}{4\epsilon M^2 M_{pl}^2}a^2\dot{\phi}^2\ddot{\phi} - \frac{15\xi^3}{4\epsilon M^2 M_{pl}^2}g^2\ddot{\phi}\phi^4 - \frac{15\xi^3 g^2}{4\epsilon M^2 M_{pl}^2}\dot{\phi}^2\phi^3\Big] - \Big[-6\xi^3 a^2 g^2\phi^3 - \frac{6\xi^3}{\epsilon M^2}\left(2\ddot{a}a + \dot{a}^2\right)g^2\phi^3 - \frac{30\xi^3}{4\epsilon M^2 M_{pl}^2}g^2\dot{\phi}^2\phi^3 + \frac{9\xi^3 a^{-2}}{\epsilon M^2 M_{pl}^2}g^4\phi^7\Big] \quad (6.58)$$

If we divide both sides with $3\xi^3$ and arrange a little bit,

$$\begin{aligned} 0 &= \Big[a^4 + \frac{a^2}{\epsilon M^2}\left(\ddot{a}a + 2\dot{a}^2\right) + \frac{15}{4\epsilon M^2 M_{pl}^2}a^2\dot{\phi}^2 - \frac{5}{4\epsilon M^2 M_{pl}^2}g^2\phi^4\Big]\ddot{\phi} \\ &+ \frac{10}{4\epsilon M^2 M_{pl}^2}a\dot{a}\dot{\phi}^3 + \frac{5}{4\epsilon M^2 M_{pl}^2}g^2\phi^3\dot{\phi}^2 \\ &+ \Big[4a^3\dot{a} + \frac{1}{\epsilon M^2}\left(7a^2\dot{a}\ddot{a} + a^3\dddot{a} + 4a\dot{a}^3\right)\Big]\dot{\phi} \\ &+ \Big[2a^2 g^2 + \frac{2}{\epsilon M^2}\left(2\ddot{a}a + \dot{a}^2\right)g^2\Big]\phi^3 + \frac{3a^{-2}}{\epsilon M^2 M_{pl}^2}g^4\phi^7 \end{aligned} \quad (6.59)$$

and replacing derivatives of scale factor with Hubble parameter by using Eq. (6.12)

$$\begin{aligned}
0 &= \left[a^4 + \frac{a^4}{\epsilon M^2}\left(\dot{H} + H^2 + 2H^2\right) + \frac{15}{4\epsilon M^2 M_{pl}^2} a^2 \dot{\phi}^2 - \frac{5}{4\epsilon M^2 M_{pl}^2} g^2 \phi^4\right] \ddot{\phi} \\
&+ \frac{10}{4\epsilon M^2 M_{pl}^2} a^2 H \dot{\phi}^3 + \frac{5}{4\epsilon M^2 M_{pl}^2} g^2 \phi^3 \dot{\phi}^2 \\
&+ \left[4a^4 H + \frac{1}{\epsilon M^2}\left(7a^4\left(\dot{H}H + H^3\right) + a^4\left(\ddot{H} + 3\dot{H}H + H^3\right) + 4a^4 H^3\right)\right]\dot{\phi} \\
&+ \left[2a^2 g^2 + \frac{2a^2}{\epsilon M^2}\left(2\dot{H} + 3H^2\right) g^2\right]\phi^3 + \frac{3a^{-2}}{\epsilon M^2 M_{pl}^2} g^4 \phi^7 \qquad (6.60)
\end{aligned}$$

and finally equation for motion for $\phi(t)$ in terms of Hubble Parameter is

$$\begin{aligned}
0 &= \left[a^4 + \frac{a^4}{\epsilon M^2}\left(\dot{H} + 3H^2\right) + \frac{15}{4\epsilon M^2 M_{pl}^2} a^2 \dot{\phi}^2 - \frac{5}{4\epsilon M^2 M_{pl}^2} g^2 \phi^4\right] \ddot{\phi} \\
&+ \frac{10}{4\epsilon M^2 M_{pl}^2} a^2 H \dot{\phi}^3 + \frac{5}{4\epsilon M^2 M_{pl}^2} g^2 \phi^3 \dot{\phi}^2 \\
&+ \left[4a^4 H + \frac{1}{\epsilon M^2} a^4\left(\ddot{H} + 10\dot{H}H + 12H^3\right)\right]\dot{\phi} \\
&+ \left[2a^2 g^2 + \frac{2a^2}{\epsilon M^2}\left(2\dot{H} + 3H^2\right) g^2\right]\phi^3 + \frac{3a^{-2}}{\epsilon M^2 M_{pl}^2} g^4 \phi^7 \qquad (6.61)
\end{aligned}$$

In the case of inflation, since variation on ϕ should be negligible, we can avoid terms in equation of motion for $\phi(t)$ which consist of derivative of ϕ. Then only surviving terms are

$$0 = \frac{3a^{-2}}{\epsilon M^2 M_{pl}^2} g^4 \phi^7 + \left\{2a^2 g^2 \left(1 + \frac{\left(2\dot{H} + 3H^2\right)}{\epsilon M^2}\right)\right\}\phi^3 \qquad (6.62)$$

Then we can also neglect the term which contains $\frac{\left(2\dot{H}+3H^2\right)}{\epsilon M^2}$ because of M^2. Then it becomes,

$$0 = \frac{3a^{-2}}{\epsilon M^2 M_{pl}^2} g^4 \phi^7 + 2a^2 g^2 \phi^3 \qquad (6.63)$$

As a result, it takes the form as

$$\left(\frac{\phi}{a}\right)^4 = -\frac{2\epsilon M^2 M_{pl}^2}{3g^2}. \tag{6.64}$$

This is the equation of ϕ in case of inflation.

The next step we should do is to find density and pressure for BIE gravity. Density and pressure are found by using

$$\begin{aligned} \rho &= \frac{\partial \mathcal{L}_{eff}}{\partial \dot{\phi}}\dot{\phi} - \mathcal{L}_{eff} \\ P &= \frac{\partial (a^3 \mathcal{L}_{eff})}{\partial a^3} \end{aligned} \tag{6.65}$$

(Maleknejad, 2011). Lagrangian of the system given is

$$\begin{aligned} \mathcal{L}_{eff} &= \epsilon M^2 M_{pl}^2 + 3M_{pl}^2 a^{-2}\left(\ddot{a}a + \dot{a}^2\right) + \frac{3a^{-2}}{2}\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right) \\ &+ \frac{3a^{-4}}{2\epsilon M^2}\left[\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi}^2 - \left(2\ddot{a}a + \dot{a}^2\right)a^{-2}g^2\phi^4\right] \\ &+ \frac{3a^{-4}}{8\epsilon M^2 M_{pl}^2}\left[\frac{5}{2}\dot{\phi}^4 - 5a^{-2}g^2\dot{\phi}^2\phi^4 + 3a^{-4}g^4\phi^8\right] \end{aligned} \tag{6.66}$$

By using $\mathcal{L}_{eff}$,

$$\frac{\partial \mathcal{L}_{eff}}{\partial \dot{\phi}} = 3a^{-2}\dot{\phi} + \frac{3a^{-4}}{\epsilon M^2}\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi} + \frac{15a^{-4}}{4\epsilon M^2 M_{pl}^2}\dot{\phi}^3 - \frac{15a^{-2}}{4\epsilon M^2 M_{pl}^2}g^2\dot{\phi}\phi^4 \tag{6.67}$$

Substituting this result in Eq. (6.65),

$$\begin{aligned} \rho &= -\epsilon M^2 M_{pl}^2 - 3M_{pl}^2 a^{-2}\left(\ddot{a}a + \dot{a}^2\right) + \frac{3a^{-2}}{2}\left(\dot{\phi}^2 + a^{-2}g^2\phi^4\right) \\ &+ \frac{3a^{-4}}{2\epsilon M^2}\left[\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi}^2 + \left(2\ddot{a}a + \dot{a}^2\right)a^{-2}g^2\phi^4\right] \\ &+ \frac{3a^{-4}}{8\epsilon M^2 M_{pl}^2}\left[\frac{15}{2}\dot{\phi}^4 - 5a^{-2}g^2\dot{\phi}^2\phi^4 - 3a^{-4}g^4\phi^8\right] \end{aligned} \tag{6.68}$$

By replacing $\dot{a}$ and $\ddot{a}$ by $\dot{H}$ and H^2 using Eq. (6.12),

$$\begin{aligned}\rho = -\epsilon M^2 M_{pl}^2 - 3M_{pl}^2\left(\dot{H}+2H^2\right)+\frac{3}{2}a^{-2}\left(\dot{\phi}^2+a^{-2}g^2\phi^4\right)\\ +\frac{3a^{-2}}{2\epsilon M^2}\left[\left(\dot{H}+3H^2\right)\dot{\phi}^2+\left(2\dot{H}+3H^2\right)a^{-2}g^2\phi^4\right]\\ +\frac{3a^{-4}}{8\epsilon M^2 M_{pl}^2}\left[\frac{15}{2}\dot{\phi}^4-5a^{-2}g^2\dot{\phi}^2\phi^4-3a^{-4}g^4\phi^3\right]\end{aligned} \tag{6.69}$$

This is density of the system in terms of Hubble parameter. To find pressure,

$$\begin{aligned}a^3\mathcal{L}_{eff} &= a^3\epsilon M^2 M_{pl}^2 + 3M_{pl}^2 a\left(\ddot{a}a+\dot{a}^2\right)+\frac{3a}{2}\left(\dot{\phi}^2-a^{-2}g^2\phi^4\right)\\ &+ \frac{3a^{-1}}{2\epsilon M^2}\left[\left(\ddot{a}a+2\dot{a}^2\right)\dot{\phi}^2-\left(2\ddot{a}a+\dot{a}^2\right)a^{-2}g^2\phi^4\right]\\ &+ \frac{3a^{-1}}{8\epsilon M^2 M_{pl}^2}\left[\frac{5}{2}\dot{\phi}^4-5a^{-2}g^2\dot{\phi}^2\phi^4+3a^{-4}g^4\phi^8\right]\end{aligned} \tag{6.70}$$

Thus inserting Eq. (6.70) in (6.65),

$$\begin{aligned}P &= \epsilon M^2 M_{pl}^2 + M_{pl}^2 a^{-2}\left(2\ddot{a}a+\dot{a}^2\right)+\frac{a^{-2}}{2}\left(\dot{\phi}^2+a^{-2}g^2\phi^4\right)\\ &+ \frac{a^{-4}}{\epsilon M^2}\left[-2\dot{a}^2\dot{\phi}^2+a^{-2}\left(4\ddot{a}a+3\dot{a}^2\right)g^2\phi^4\right]\\ &+ \frac{a^{-4}}{8\epsilon M^2 M_{pl}^2}\left[-\frac{5}{2}\dot{\phi}^4+15a^{-2}g^2\dot{\phi}^2\phi^4-15a^{-4}g^4\phi^8\right]\end{aligned} \tag{6.71}$$

and by using Eq. (6.12), it becomes

$$\begin{aligned}P &= \epsilon M^2 M_{pl}^2 + M_{pl}^2 a^{-2}\left(2\dot{H}+3H^2\right)+\frac{a^{-2}}{2}\left(\dot{\phi}^2+a^{-2}g^2\phi^4\right)\\ &+ \frac{a^{-2}}{\epsilon M^2}\left[-2H^2\dot{\phi}^2+a^{-2}\left(4\dot{H}a+7H^2\right)g^2\phi^4\right]\\ &+ \frac{a^{-4}}{8\epsilon M^2 M_{pl}^2}\left[-\frac{5}{2}\dot{\phi}^4+15a^{-2}g^2\dot{\phi}^2\phi^4-15a^{-4}g^4\phi^8\right]\end{aligned} \tag{6.72}$$

From now we have density and pressure equations given by BIE with Abelian and Non Abelian Fields. Einstein's Field Equations in FRW background also gives us density and pressure equations. Let us equate density equations, one of them is calculated by us and the other is given in Eq. (6.13).

$$\begin{aligned}3M_{pl}^2H^2 = -\epsilon M^2M_{pl}^2 - 3M_{pl}^2\dot{H} - 6M_{pl}^2H^2 + \frac{3}{2}a^{-2}\left(\dot{\phi}^2 + a^{-2}g^2\phi^4\right)\\ +\frac{3a^{-2}\dot{\phi}^2}{2\epsilon M^2}\dot{H} + \frac{9a^{-2}\dot{\phi}^2}{2\epsilon M^2}H^2 + \frac{3a^{-4}g^2\phi^4}{\epsilon M^2}\dot{H} + \frac{9a^{-4}g^2\phi^4}{2\epsilon M^2}H^2\\ +\frac{3a^{-4}}{8\epsilon M^2M_{pl}^2}\left\{\frac{15}{2}\dot{\phi}^4 - 5a^{-2}g^2\dot{\phi}^2\phi^4 - 3a^{-4}g^4\phi^8\right\} \qquad (6.73)\end{aligned}$$

If we arrange it,

$$\begin{aligned}&\left(9M_{pl}^2 - \frac{9a^{-2}}{2\epsilon M^2}\left(\dot{\phi}^2 + a^{-2}g^2\phi^4\right)\right)H^2 + \left(3M_{pl}^2 - \frac{3a^{-2}}{2\epsilon M^2}\left(\dot{\phi}^2 + 2a^{-2}g^2\phi^4\right)\right)\dot{H}\\ = &-\epsilon M^2M_{pl}^2 + \frac{3}{2}a^{-2}\left(\dot{\phi}^2 + a^{-2}g^2\phi^4\right)\\ + &\frac{3a^{-4}}{8\epsilon M^2M_{pl}^2}\left[\frac{15}{2}\dot{\phi}^4 - 5a^{-2}g^2\dot{\phi}^2\phi^4 - 3a^{-4}g^4\phi^8\right] \qquad (6.74)\end{aligned}$$

By applying slow roll condition given in Eq. (6.17)

$$\begin{aligned}&\left(9M_{pl}^2 - \frac{9a^{-2}}{2\epsilon M^2}\left(\eta^2H^2\phi^2 + a^{-2}g^2\phi^4\right)\right)H^2\\ + &\left(3M_{pl}^2 - \frac{3a^{-2}}{2\epsilon M^2}\left(\eta^2H^2\phi^2 + 2a^{-2}g^2\phi^4\right)\right)\dot{H}\\ = &-\epsilon M^2M_{pl}^2 + \frac{3}{2}a^{-2}\left(\eta^2H^2\phi^2 + a^{-2}g^2\phi^4\right)\\ + &\frac{3a^{-4}}{8\epsilon M^2M_{pl}^2}\left[\frac{15}{2}\eta^4H^4\phi^4 - 5a^{-2}g^2\eta^2H^2\phi^6 - 3a^{-4}g^4\phi^8\right] \qquad (6.75)\end{aligned}$$

All terms which consist of η negligible because of $\eta << 1$. Then equation given above becomes,

$$\begin{aligned}\left(9M_{pl}^2 - \frac{9a^{-4}g^2\phi^4}{2\epsilon M^2}\right)H^2 + \left(3M_{pl}^2 - \frac{3a^{-4}g^2\phi^4}{\epsilon M^2}\right)\dot{H}\\ = -\epsilon M^2M_{pl}^2 + \frac{3}{2}a^{-4}g^2\phi^4 - \frac{-9a^{-3}}{8\epsilon M^2M_{pl}^2}g^4\phi^8 \qquad (6.76)\end{aligned}$$

Finally, inserting Eq.(6.64)

$$\left(9M_{pl}^2 - 3M_{pl}^2\right)H^2 + \left(3M_{pl}^2 - 2M_{pl}^2\right)\dot{H} = -\epsilon M^2 M_{pl}^2 + \epsilon M^2 M_{pl}^2 - \frac{\epsilon M^2 M_{pl}^2}{2} \quad (6.77)$$

Combining terms which are related to each other and dividing both sides with M_{pl}^2

$$6H^2 + \dot{H} = -\frac{\epsilon M^2}{2} \quad (6.78)$$

Let us follow same procedure for pressure. By using Eq. (6.13),

$$\begin{aligned} -M_{pl}^2\left(2\dot{H} + 3H^2\right) &= \epsilon M^2 M_{pl}^2 + M_{pl}^2\left(2\dot{H} + 3H^2\right) + \frac{a^{-2}}{2}\left(\dot{\phi}^2 + a^{-2}g^2\phi^4\right) \\ &+ \frac{a^{-2}}{\epsilon M^2}\left(-2H^2\dot{\phi}^2 + a^{-2}\left(4\dot{H} + 7H^2\right)g^2\phi^4\right) \\ &+ \frac{a^{-4}}{8\epsilon M^2 M_{pl}^2}\left(\frac{-5}{2}\dot{\phi}^4 + 15a^{-2}g^2\dot{\phi}^2\phi^4 - 15a^{-4}g^4\phi^8\right) \quad (6.79) \end{aligned}$$

and then

$$\begin{aligned} &\left(-4M_{pl}^2 - \frac{4a^{-4}}{\epsilon M^2}g^2\phi^4\right)\dot{H} + \left(-6M_{pl}^2 - \frac{2a^{-2}}{\epsilon M^2}\dot{\phi}^2 - \frac{7a^{-4}}{\epsilon M^2}g'2\phi^4\right)H^2 \\ &= \epsilon M^2 M_{pl}^2 + \frac{a^{-2}}{2}\left(\dot{\phi}^2 + a^{-2}g^2\phi^4\right) \\ &+ \frac{a^{-4}}{8\epsilon M^2 M_{pl}^2}\left(\frac{-5}{2}\dot{\phi}^4 + 15a^{-2}g^2\dot{\phi}^2\phi^4 - 15a^{-4}g^4\phi^8\right) \quad (6.80) \end{aligned}$$

Applying slow roll condition given in Eq. (6.17),

$$\begin{aligned} &\left(-4M_{pl}^2 - \frac{4a^{-4}}{\epsilon M^2}g^2\phi^4\right)\dot{H} + \left(-6M_{pl}^2 - \frac{2a^{-2}}{\epsilon M^2}\eta^2 H^2\phi^2 - \frac{7a^{-4}}{\epsilon M^2}g'2\phi^4\right)H^2 \\ &= \epsilon M^2 M_{pl}^2 + \frac{a^{-2}}{2}\left(\eta^2 H^2\phi^2 + a^{-2}g^2\phi^4\right) \\ &+ \frac{a^{-4}}{8\epsilon M^2 M_{pl}^2}\left(\frac{-5}{2}\eta^4 H^4\phi^4 + 15a^{-2}g^2\eta^2 H^2\phi^6 - 15a^{-4}g^4\phi^8\right) \quad (6.81) \end{aligned}$$

omitting terms which contain η,

$$\left(-4M_{pl}^2 - \frac{4a^{-4}}{\epsilon M^2} g^2 \phi^4\right) \dot{H} + \left(-6M_{pl}^2 - \frac{7a^{-4}}{\epsilon M^2} g' 2 \phi^4\right) H^2$$
$$= \epsilon M^2 M_{pl}^2 + \frac{a^{-4}}{2} g^2 \phi^4 - \frac{15a^{-8}}{8\epsilon M^2 M_{pl}^2} g^4 \phi^8 \qquad (6.82)$$

By inserting Eq.(6.64)

$$\left\{4M_{pl}^2 - \frac{4g^2}{\epsilon M^2}\left(\frac{2\epsilon M^2 M_{pl}^2}{3g^2}\right)\right\} \dot{H} + \left\{6M_{pl}^2 - \frac{7g^2}{\epsilon M^2}\left(\frac{2\epsilon M^2 M_{pl}^2}{3g^2}\right)\right\} H^2$$
$$= \epsilon M^2 M_{pl}^2 + \frac{g^2}{2}\left(\frac{2\epsilon M^2 M_{pl}^2}{3g^2}\right) - \frac{15g^4}{8\epsilon M^2 M_{pl}^2}\left(\frac{4\epsilon^2 M^4 M_{pl}^4}{9g^4}\right) \qquad (6.83)$$

and hence combining terms

$$\frac{-20M_{pl}^2}{3}\dot{H} - \frac{32M_{pl}^2}{3}H^2 = \frac{-37\epsilon M^2 M_{pl}^2}{6} \qquad (6.84)$$

Dividing both sides with M_{pl}^2

$$\frac{-20}{3}\dot{H} - \frac{32}{3}H^2 = \frac{-37\epsilon M^2}{6} \qquad (6.85)$$

In the end, the last step to understand whether Born-Infeld-Einstein Gravity with Non Abelian Fields gives us inflation or not is to solve Eq.(6.78) and (6.85) together. Then the result is

$$H^2 = \frac{-57}{176}\epsilon M^2 \qquad (6.86)$$

and

$$\dot{H} = \frac{127}{88}\epsilon M^2 \qquad (6.87)$$

Finally let us calculate slow roll parameter ε given in Eq.(6.16). For a successful inflationary model, this parameter should supply the condition $\varepsilon << 1$

$$\begin{aligned}\varepsilon &= \frac{-\dot{H}}{H^2} \\ &= \frac{\left(\frac{-127\epsilon M^2}{88}\right)}{\left(\frac{-57\epsilon M^2}{176}\right)} \\ &\simeq 4,46\end{aligned} \tag{6.88}$$

As a result Born-Infeld-Einstein Gravity with Non Abelian Fields does not explain Inflation

6.3. BIR in FRW Background

We examine Born-Infeld-Riemann Gravity in detail in (§5.1). In this section we explain how BIR with Abelian - Non Abelian Fields gives us inflation.

$$\begin{aligned} S &= \int d^4x M_{pl}^2 \kappa^2 c_{DD} \sqrt{-g} \Big\{ 1 + \frac{1}{4\kappa^2} \Big(\lambda_R R + \lambda_F F^{\nu\beta} F_{\nu\beta} + \lambda_{\tilde{F}} F^{a\nu\beta} F^a{}_{\nu\beta} \Big) \\ &- \frac{1}{8\kappa^4} \Big[2\lambda_R \lambda_F K_R{}^{\nu\beta}{}_{\nu'\beta'} F^{\nu'\beta'} F_{\nu\beta} + 2\lambda_R \lambda_{\tilde{F}} K_R{}^{\nu\beta}{}_{\nu'\beta'} F^{a\nu'\beta'} F^a{}_{\nu\beta} \\ &+ \frac{3\lambda_F^2}{4} F^{\nu\beta} F_{\nu\beta} F^{\nu'\beta'} F_{\nu'\beta'} + 2\lambda_F \lambda_{\tilde{F}} K_F{}^{\nu\beta}{}_{\nu'\beta'} F^{\nu'\beta'} F_{\nu\beta} + \lambda_{\tilde{F}}^2 K_F{}^{\nu\beta}{}_{\nu'\beta'} F^{b\nu'\beta'} F^b{}_{\nu\beta} \Big] \\ &+ \mathcal{O}(A^3) \Big\} \end{aligned} \tag{6.89}$$

Here Eq.(6.89) is the last BIR action of the theory where $K_R{}^{\nu\beta}{}_{\nu'\beta'}$ and $K_F{}^{\nu\beta}{}_{\nu'\beta'}$ defined in Eq.(5.19) and Eq. (5.20) respectively. Let us expand these tensors one by one and see what they gives in the case of FRW background and gauge fixing.

In light of Appendix A,

$$
\begin{aligned}
K_{R\ 0j}^{\ 0i} &= R^{0i}_{0j} - \frac{1}{4} R \underbrace{\delta^0{}_0}_{1} \delta^i{}_j \\
&= g^{i\mu} R^0{}_{\mu 0 j} - \frac{1}{4} R \delta^i{}_j \\
&= \left(\underbrace{g^{i0} R^0{}_{00j}}_{0} + \underbrace{g^{ik}}_{a^{-2}\delta^{ik}} \underbrace{R^0{}_{k0j}}_{R_{kj}} \right) - \frac{1}{4} R \delta^i{}_j \\
&= \left(a^{-2} R_{kj} - \frac{1}{4} R \delta_{kj} \right) \delta^{ki} \\
&= \left(a^{-2} \left(\ddot{a} a + 2\dot{a}^2 \right) \delta_{kj} - \frac{6}{4} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \delta_{kj} \right) \delta^{ki} \\
&= \frac{-1}{2} \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) \underbrace{\delta_{kj} \delta^{ki}}_{\delta^i{}_j} \\
&= \frac{-1}{2} \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) \delta^i{}_j \qquad (6.90)
\end{aligned}
$$

$$
\begin{aligned}
K_{R\ j0}^{\ 0i} &= \underbrace{R^{0i}_{j0}}_{-R^{0i}_{0j}} - \frac{1}{4} R \underbrace{\delta^0{}_j \delta^i{}_0}_{\delta^i{}_j} \\
&= -g^{i\mu} R^0{}_{\mu 0 j} - \frac{1}{4} R \delta^i{}_j \\
&= -\left(\underbrace{g^{i0} R^0{}_{00j}}_{0} + \underbrace{g^{ik}}_{a^{-2}\delta^{ik}} \underbrace{R^0{}_{k0j}}_{R_{kj}} \right) - \frac{1}{4} R \delta^i{}_j \\
&= \left(-a^{-2} R_{kj} - \frac{1}{4} R \delta_{kj} \right) \delta^{ki} \\
&= \left(-a^{-2} \left(\ddot{a} a + 2\dot{a}^2 \right) \delta_{kj} - \frac{6}{4} \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \delta_{kj} \right) \delta^{ki} \\
&= \frac{-1}{2} \left(\frac{5\ddot{a}}{a} + \frac{7\dot{a}^2}{a^2} \right) \underbrace{\delta_{kj} \delta^{ki}}_{\delta^i{}_j} \\
&= \frac{-1}{2} \left(\frac{5\ddot{a}}{a} + \frac{7\dot{a}^2}{a^2} \right) \delta^i{}_j \qquad (6.91)
\end{aligned}
$$

$$
\begin{aligned}
K_{R\ \ jk}^{\ 0i} &= R^{0i}_{\ \ jk} - \frac{1}{4} R \underbrace{\delta^0_{\ j}\delta^i_{\ k}}_{0} \\
&= g^{i\mu} R^0_{\ \mu jk} \\
&= \underbrace{g^{i0} R^0_{\ 0jk}}_{0} + \underbrace{g^{il}}_{a^{-2}\delta^{il}} R^0_{\ ljk} \\
&= a^{-2} R^0_{\ ijk} \\
&= a^{-2} R_{ik} \underbrace{\delta^0_{\ j}}_{0} \\
&= 0
\end{aligned}
\qquad (6.92)
$$

$$
\begin{aligned}
K_{R\ \ 0k}^{\ ij} &= R^{ij}_{\ \ 0k} - \frac{1}{4} R \underbrace{\delta^i_{\ 0}\delta^j_{\ k}}_{0} \\
&= g^{j\mu} R^i_{\ \mu 0k} \\
&= \underbrace{g^{j0} R^i_{\ 00k}}_{0} + \underbrace{g^{jl}}_{a^{-2}\delta^{jl}} R^i_{\ l0k} \\
&= a^{-2} R^i_{\ j0k} \\
&= a^{-2} R_{jk} \underbrace{\delta^i_{\ 0}}_{0} \\
&= 0
\end{aligned}
\qquad (6.93)
$$

$$
\begin{aligned}
K_R{}^{ij}{}_{kl} &= R^{ij}{}_{kl} - \frac{1}{4}R\delta^i{}_k\delta^j{}_l \\
&= g^{j\mu}R^i{}_{\mu kl} - \frac{1}{4}R\delta^i{}_k\delta^j{}_l \\
&= \Bigg(\underbrace{g^{j0}R^i{}_{0kl}}_{0} + \underbrace{g^{jm}}_{a^{-2}\delta^{jm}} R^i{}_{mkl}\Bigg) - \frac{1}{4}R\delta^i{}_k\delta^j{}_l \\
&= a^{-2}\delta^{jm}R_{ml}\delta^i{}_k - \frac{1}{4}R\delta^i{}_k\delta^j{}_l \\
&= \left(a^{-2}R_{ml} - \frac{1}{4}R\delta_{ml}\delta^j{}_l\right)\delta^{jm}\delta^i{}_k \\
&= \left[a^{-2}\left(\ddot{a}a + 2\dot{a}^2\right)\delta_{ml} - \frac{6}{4}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\delta_{ml}\right]\delta^{jm}\delta^i{}_k \\
&= \frac{-1}{2}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\underbrace{\delta_{ml}\delta^{jm}}_{\delta^j{}_l}\delta^i{}_k \\
&= \frac{-1}{2}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\delta^j{}_l\delta^i{}_k
\end{aligned}
\tag{6.94}
$$

Due to symmetry conditions of Riemann Tensor,

$$
\begin{aligned}
K_R{}^{0i}{}_{j0} &= K_R{}^{0i}{}_{0j} \\
K_R{}^{i0}{}_{j0} &= K_R{}^{0i}{}_{0j} \\
K_R{}^{0i}{}_{jk} &= -K_R{}^{i0}{}_{jk} \\
K_R{}^{ij}{}_{0k} &= -K_R{}^{ij}{}_{k0}
\end{aligned}
\tag{6.95}
$$

Let us expand $K_{R\ \ \nu'\beta'}^{\ \nu\beta}F^{\nu'\beta'}F_{\nu\beta}$. We expand indices ν, β, ν', β' respectively.

$$\begin{aligned}
K_{R\ \ \nu'\beta'}^{\ \nu\beta}F^{\nu'\beta'}F_{\nu\beta} &= K_{R\ \ \nu'\beta'}^{\ 0\beta}F^{\nu'\beta'}F_{0\beta} + K_{R\ \ \nu'\beta'}^{\ i\beta}F^{\nu'\beta'}F_{i\beta} \\
&= \Big(K_{R\ \nu'\beta'}^{\ 00}F^{\nu'\beta'}\underbrace{F_{00}}_{0} + K_{R\ \nu'\beta'}^{\ 0i}F^{\nu'\beta'}F_{0i}\Big) \\
&+ \Big(K_{R\ \nu'\beta'}^{\ i0}F^{\nu'\beta'}F_{i0} + K_{R\ \nu'\beta'}^{\ ij}F^{\nu'\beta'}F_{ij}\Big) \\
&= \Big(K_{R\ 0\beta'}^{\ 0i}F^{0\beta'}F_{0i} + K_{R\ j\beta'}^{\ 0i}F^{j\beta'}F_{0i}\Big) \\
&+ \Big(K_{R\ 0\beta'}^{\ i0}F^{0\beta'}F_{i0} + K_{R\ j\beta'}^{\ i0}F^{j\beta'}F_{i0}\Big) \\
&+ \Big(K_{R\ 0\beta'}^{\ ij}F^{0\beta'}F_{ij} + K_{R\ k\beta'}^{\ ij}F^{k\beta'}F_{ij}\Big) \\
&= \Big(K_{R\ 00}^{\ 0i}\underbrace{F^{00}}_{0}F_{0i} + K_{R\ 0j}^{\ 0i}F^{0j}F_{0i}\Big) \\
&+ \big(K_{R\ j0}^{\ 0i}F^{j0}F_{0i} + K_{R\ jk}^{\ 0i}F^{jk}F_{0i}\big) \\
&+ \Big(K_{R\ 00}^{\ i0}\underbrace{F^{00}}_{0}F_{i0} + K_{R\ 0j}^{\ i0}F^{0j}F_{i0}\Big) \\
&+ \big(K_{R\ j0}^{\ i0}F^{j0}F_{i0} + K_{R\ jk}^{\ i0}F^{jk}F_{i0}\big) \\
&+ \Big(K_{R\ 00}^{\ ij}\underbrace{F^{00}}_{0}F_{ij} + K_{R\ 0k}^{\ ij}F^{0k}F_{ij}\Big) \\
&+ \Big(K_{R\ k0}^{\ ij}F^{k0}F_{ij} + K_{R\ kl}^{\ ij}F^{kl}F_{ij}\Big) \\
&= K_{R\ 0j}^{\ 0i}F^{0j}F_{0i} + K_{R\ j0}^{\ 0i}F^{j0}F_{0i} + K_{R\ jk}^{\ 0i}F^{jk}F_{0i} + K_{R\ 0j}^{\ i0}F^{0j}F_{i0} \\
&+ K_{R\ j0}^{\ i0}F^{j0}F_{i0} + K_{R\ jk}^{\ i0}F^{jk}F_{i0} + K_{R\ 0k}^{\ ij}F^{0k}F_{ij} + K_{R\ k0}^{\ ij}F^{k0}F_{ij} \\
&+ K_{R\ kl}^{\ ij}F^{kl}F_{ij} \qquad (6.96)
\end{aligned}$$

In this step we can use Eq. (6.90), (6.91), (6.92), (6.93), (6.94) and (6.95). Also by using antisymmetric property of Abelian Field Strength tensor

$$F_{\mu\nu} = -F_{\nu\mu} \ \text{ or } \ F^{\mu\nu} = -F^{\nu\mu}$$

Eq.(6.96) becomes

$$K_{R\ \ \nu'\beta'}^{\ \nu\beta}F^{\nu'\beta'}F_{\nu\beta} = 4\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right)F^{0i}F_{0i} - \frac{1}{2}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)F^{ij}F_{ij} \qquad (6.97)$$

Let us expand $K_{R\ \ \nu'\beta'}^{\ \nu\beta}F^{a\nu'\beta'}F^{a}_{\ \nu\beta}$. This term has the same structure with $K_{R\ \ \nu'\beta'}^{\ \nu\beta}F^{\nu'\beta'}F_{\nu\beta}$

since both Abelian and Non Abelian Field Strength tensors are antisymmetric. Thus all diagonal elements of Non-Abelian Field Strength tensor are also zero. Then, we can use the expansion given in Eq.(6.96) for this term too.

$$\begin{aligned} K_{R\ \nu'\beta'}^{\ \nu\beta} F^{a\nu'\beta'} F^{a}_{\ \nu\beta} &= K_{R\ 0j}^{\ 0i} F^{a0j} F^{a}_{\ 0i} + K_{R\ j0}^{\ 0i} F^{aj0} F^{a}_{\ 0i} + K_{R\ jk}^{\ 0i} F^{ajk} F^{a}_{\ 0i} \\ &+ K_{R\ 0j}^{\ i0} F^{a0j} F^{a}_{\ i0} + K_{R\ j0}^{\ i0} F^{aj0} F^{a}_{\ i0} + K_{R\ jk}^{\ i0} F^{ajk} F^{a}_{\ i0} \\ &+ K_{R\ 0k}^{\ ij} F^{a0k} F^{a}_{\ ij} + K_{R\ k0}^{\ ij} F^{ak0} F^{a}_{\ ij} + K_{R\ kl}^{\ ij} F^{akl} F^{a}_{\ ij} \end{aligned} \quad (6.98)$$

By using Eq. (6.19), (6.90), (6.91,)(6.92), (6.93), (6.94) and (6.95)

$$K_{R\ \nu'\beta'}^{\ \nu\beta} F^{a\nu'\beta'} F^{a}_{\ \nu\beta} = 3a^{-2}\left[4\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right)\dot{\phi}^2 - a^{-2}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) g^2\phi^4\right] \quad (6.99)$$

We calculated terms which are related to curvature until now. The next step to obtain a theory based on FRW background is to find contributions which comes from Non Abelian Field Strength Tensor. We can see this part in the action as $K_{F\ \nu'\beta'}^{\ \nu\beta} F^{\nu'\beta'} F_{\ \nu\beta}$ and $K_{F\ \nu'\beta'}^{\ \nu\beta} F^{a\nu'\beta'} F^{a}_{\ \nu\beta}$. To find these terms, firstly let us calculate $K_{F\ \nu'\beta'}^{\ \nu\beta}$ which is given in Eq. (5.20).

$$K_{F\ 0j}^{\ 0i} = F^{a0i} F^{a}_{\ 0j} - \frac{1}{4} \underbrace{F^{a\rho\sigma} F^{a}_{\ \rho\sigma}}_{A} \underbrace{\delta^{0}_{\ 0}}_{1} \delta^{i}_{\ j} \quad (6.100)$$

The term called as A is in general form and we use it for all of $K_{F\ \nu'\beta'}^{\ \nu\beta}$. Then we should calculate it firstly.

$$\begin{aligned} F^{a\rho\sigma} F^{a}_{\ \rho\sigma} &= F^{a0\sigma} F^{a}_{\ 0\sigma} + F^{ai\sigma} F^{a}_{\ i\sigma} \\ &= \Big(\underbrace{F^{a00} F^{a}_{\ 00}}_{0} + F^{a0i} F^{a}_{\ 0i}\Big) + \left(F^{ai0} F^{a}_{\ i0} + F^{aij} F^{a}_{\ ij}\right) \\ &= \left(-a^{-2}\dot{\phi}\delta^{ai}\dot{\phi}\delta^{a}_{\ i}\right) + \left[\left(a^{-2}\dot{\phi}\delta^{ai}\right)\left(\dot{\phi}\delta^{a}_{\ i}\right) + \left(-a^{-4} g\phi^2 \epsilon^{aij}\right)\left(-g\phi^2 \epsilon^{a}_{\ ij}\right)\right] \\ &= 6a^{-2}\left(a^{-2} g^2 \phi^4 - \dot{\phi}^2\right) \end{aligned} \quad (6.101)$$

From there,

$$
\begin{aligned}
K_F{}^{0i}_{\ 0j} &= \left[\left(-a^{-2}\dot{\phi}\delta^{ai}\right)\left(\dot{\phi}\delta^{a}{}_{j}\right)\right] - \frac{6}{4}\left(a^{-4}g^2\phi^4 - a^{-2}\dot{\phi}^2\right)\delta^{i}{}_{j} \\
&= \frac{a^{-2}}{2}\left(\dot{\phi}^2 - 3a^{-2}g^2\phi^4\right)\delta^{i}{}_{j} \qquad (6.102)
\end{aligned}
$$

$$
K_F{}^{0i}_{\ j0} = F^{a0i}F^{a}{}_{j0} - \frac{1}{4}F^{a\rho\sigma}F^{a}{}_{\rho\sigma}\underbrace{\delta^{0}{}_{j}\delta^{i}{}_{0}}_{\delta^{i}{}_{j}} \qquad (6.103)
$$

By using Eq.(6.19)and (6.101), it becomes;

$$
K_F{}^{0i}_{\ j0} = \frac{a^{-2}}{2}\left(5\dot{\phi}^2 - 3a^{-2}g^2\phi^4\right)\delta^{i}{}_{j} \qquad (6.104)
$$

$$
\begin{aligned}
K_F{}^{0i}_{\ jk} &= F^{a0i}F^{a}{}_{jk} - \frac{1}{4}F^{a\rho\sigma}F^{a}{}_{\rho\sigma}\underbrace{\delta^{0}{}_{j}\delta^{i}{}_{k}}_{0} \\
&= F^{a0i}F^{a}{}_{jk} \\
&= a^{-2}g\dot{\phi}\phi^2\epsilon^{i}{}_{jk} \qquad (6.105)
\end{aligned}
$$

$$
\begin{aligned}
K_F{}^{ij}_{\ 0k} &= F^{aij}F^{a}{}_{0k} - \frac{1}{4}F^{a\rho\sigma}F^{a}{}_{\rho\sigma}\underbrace{\delta^{i}{}_{0}\delta^{j}{}_{k}}_{0} \\
&= F^{aij}F^{a}{}_{0k} \\
&= -a^{-4}g\dot{\phi}\phi^2\epsilon_{k}{}^{ij} \qquad (6.106)
\end{aligned}
$$

$$
\begin{aligned}
K_F{}^{ij}_{\ kl} &= F^{aij}F^{a}{}_{kl} - \frac{1}{4}F^{a\rho\sigma}F^{a}{}_{\rho\sigma}\delta^{i}{}_{k}\delta^{j}{}_{l} \\
&= \left[\left(-a^{-4}g\phi^2\epsilon^{aij}\right)\left(-g\phi^2\epsilon^{a}{}_{kl}\right)\right] - \frac{6a^{-2}}{4}\left(a^{-2}g^2\phi^4 - \dot{\phi}^2\right)\delta^{i}{}_{k}\delta^{j}{}_{l} \\
&= \frac{a^{-2}}{2}\left(3\dot{\phi}^2 - a^{-2}g^2\phi^4\right)\delta^{i}{}_{k}\delta^{j}{}_{l} - a^{-4}g^2\phi^4\delta^{i}{}_{l}\delta^{j}{}_{k} \qquad (6.107)
\end{aligned}
$$

since

$$\epsilon^{aij}\epsilon^{a}_{kl} = \delta^{i}{}_{k}\delta^{j}{}_{l} - \delta^{i}{}_{l}\delta^{j}{}_{k}$$

For the other components of ${K_F}^{\nu\beta}{}_{\nu'\beta'}$, we can use antisymmetry property, which supplies us equations given below, of Non Abelian Field Strength Tensor.

$$\begin{aligned}
{K_F}^{0i}{}_{0j} &= {K_F}^{i0}{}_{j0} \\
{K_F}^{0i}{}_{j0} &= {K_F}^{i0}{}_{0j} \\
{K_F}^{0i}{}_{jk} &= -{K_F}^{i0}{}_{jk} \\
{K_F}^{ij}{}_{0k} &= -{K_F}^{ij}{}_{k0}
\end{aligned} \tag{6.108}$$

Let us expand ${K_F}^{\nu\beta}{}_{\nu'\beta'}F^{\nu'\beta'}F_{\nu\beta}$ one by one.

$$\begin{aligned}
{K_F}^{\nu\beta}{}_{\nu'\beta'}F^{\nu'\beta'}F_{\nu\beta} &= {K_F}^{0\beta}{}_{\nu'\beta'}F^{\nu'\beta'}F_{0\beta} + {K_F}^{i\beta}{}_{\nu'\beta'}F^{\nu'\beta'}F_{i\beta} \\
&= \left({K_F}^{00}{}_{\nu'\beta'}F^{\nu'\beta'}\underbrace{F_{00}}_{0} + {K_F}^{0i}{}_{\nu'\beta'}F^{\nu'\beta'}F_{0i}\right) \\
&+ \left({K_F}^{i0}{}_{\nu'\beta'}F^{\nu'\beta'}F_{i0} + {K_F}^{ij}{}_{\nu'\beta'}F^{\nu'\beta'}F_{ij}\right) \\
&= \left({K_F}^{0i}{}_{0\beta'}F^{0\beta'}F_{0i} + {K_F}^{0i}{}_{j\beta'}F^{j\beta'}F_{0i}\right) \\
&+ \left({K_F}^{i0}{}_{0\beta'}F^{0\beta'}F_{i0} + {K_F}^{i0}{}_{j\beta'}F^{j\beta'}F_{i0}\right) \\
&+ \left({K_F}^{ij}{}_{0\beta'}F^{0\beta'}F_{ij} + {K_F}^{ij}{}_{k\beta'}F^{k\beta'}F_{ij}\right) \\
&= \left({K_F}^{0i}{}_{00}\underbrace{F^{00}}_{0}F_{0i} + {K_F}^{0i}{}_{0j}F^{0j}F_{0i}\right) \\
&+ \left({K_F}^{0i}{}_{j0}F^{j0}F_{0i} + {K_F}^{0i}{}_{jk}F^{jk}F_{0i}\right) \\
&+ \left({K_F}^{i0}{}_{00}\underbrace{F^{00}}_{0}F_{i0} + {K_F}^{i0}{}_{0j}F^{0j}F_{i0}\right) \\
&+ \left({K_F}^{i0}{}_{j0}F^{j0}F_{i0} + {K_F}^{i0}{}_{jk}F^{jk}F_{i0}\right) \\
&+ \left({K_F}^{ij}{}_{00}\underbrace{F^{00}}_{0}F_{ij} + {K_F}^{ij}{}_{0k}F^{0k}F_{ij}\right) \\
&+ \left({K_F}^{ij}{}_{k0}F^{k0}F_{ij} + {K_F}^{ij}{}_{kl}F^{kl}F_{ij}\right) \\
&= {K_F}^{0i}{}_{0j}F^{0j}F_{0i} + {K_F}^{0i}{}_{j0}F^{j0}F_{0i} + {K_F}^{0i}{}_{jk}F^{jk}F_{0i} + {K_F}^{i0}{}_{0j}F^{0j}F_{i0} \\
&+ {K_F}^{i0}{}_{j0}F^{j0}F_{i0} + {K_F}^{i0}{}_{jk}F^{jk}F_{i0} + {K_F}^{ij}{}_{0k}F^{0k}F_{ij} + {K_F}^{ij}{}_{k0}F^{k0}F_{ij} \\
&+ {K_F}^{ij}{}_{kl}F^{kl}F_{ij}
\end{aligned} \tag{6.109}$$

By using Eq. (6.102), (6.104), (6.105), (6.106), (6.107), (6.108) and (6.20)

$$K_{F\ \ \nu'\beta'}^{\ \nu\beta}F^{\nu'\beta'}F_{\nu\beta} = -4a^{-2}\dot{\phi}^2F^{0i}F_{0i} - 4a^{-4}g\dot{\phi}\phi^2F_{ij}F^{0k}\epsilon_k^{\ ij} + \frac{a^{-2}}{2}\left(3\dot{\phi}^2 + a^{-2}g^2\phi^4\right)F^{ij}F_{ij} \tag{6.110}$$

and also we have

$$K_{F\ \ \nu'\beta'}^{\ \nu\beta}F^{b\nu'\beta'}F^{b}_{\ \nu\beta} = 12a^{-4}\dot{\phi}^4 - 15a^{-6}g^2\dot{\phi}^2\phi^4 + 3a^{-8}g^4\phi^8 \tag{6.111}$$

and finally inserting Eq. (A.11), (6.19), (6.97), (6.99), (6.110), (6.111) in (6.89)

$$\begin{aligned} S = & \int d^4x M_{pl}^2\kappa^2 c_{DD}\sqrt{-g}\Big\{1 + \underbrace{\frac{3\lambda_R}{2\kappa^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)}_{1} + \underbrace{\frac{3\lambda_{\tilde{F}}}{2\kappa^2}\left(a^{-4}g^2\phi^4 - a^{-2}\dot{\phi}^2\right)}_{2} \\ + & \underbrace{\frac{\lambda_F}{4\kappa^2}F^{\nu\beta}F_{\nu\beta}}_{3} - \frac{1}{8\kappa^4}\Big[2\lambda_R\lambda_F\Big(8\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right)F^{0i}F_{i0} - \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)F^{ij}F_{ij}\Big) \\ + & 6\lambda_R\lambda_{\tilde{F}}\Big(4a^{-2}\dot{\phi}^2\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right) - a^{-4}g^2\phi^4\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\Big) \\ + & \frac{3\lambda_F^2}{4}F^{\nu\beta}F_{\nu\beta}F^{\nu'\beta'}F_{\nu'\beta'} \\ + & \lambda_F\lambda_{\tilde{F}}\Big(-8\left(a^{-2}\dot{\phi}^2F^{0i}F_{0i} + a^{-4}g\dot{\phi}\phi^2F^{ij}F_{0k}\epsilon_k^{\ ij}\right) \\ + & \left(a^{-4}g^2\phi^4 + 3a^{-2}\dot{\phi}^2\right)F^{ij}F_{ij}\Big) \\ + & 3\lambda_{\tilde{F}}^2\Big(4a^{-4}\dot{\phi}^4 - 5a^{-6}g^2\dot{\phi}^2\phi^4 + a^{-8}g^4\phi^8\Big)\Big]\Big\} \end{aligned} \tag{6.112}$$

This is action of Born-Infeld-Riemann Gravity in FRW Background (for simplicity, we avoid $\mathcal{O}(A^3)$). Here we have three different coefficients which are λ_R, λ_F and $\lambda_{\tilde{F}}$. To determine these coefficients, let us use well-known actions- Einstein-Hilbert, Electromagnetism and Yang-Mills.Because the first way to understand we have a consistent theory or not is to seek Einstein-Hilbert action in the limit of small curvature (Escobar, 2012). Thus curvature terms in the action, which is signed as 1, should be the same as Einstein-Hilbert term (Carroll, 2004).

$$S_{EH} = \frac{1}{2} \int d^4x \sqrt{-g} M_{pl}^2 R \tag{6.113}$$

The term labeled as 2 should be the same as Yang-Mills action (Quigg, 1983)

$$S_{YM} = \frac{-1}{4} \int d^4x \sqrt{-g} F^{a\alpha\beta} F^{a}{}_{\alpha\beta} \tag{6.114}$$

and 3 should be the same as Electromagnetism Action (Quigg, 1983)

$$S_{EM} = \frac{-1}{4} \int d^4x \sqrt{-g} F^{\alpha\beta} F_{\alpha\beta} \tag{6.115}$$

If we compare these terms one by one, it does not hard to find coefficients. Then

$$\lambda_R = \frac{2}{c_{DD}} \quad , \quad \lambda_F = \frac{-1}{M_{pl}^2 c_{DD}} \quad , \quad \lambda_{\tilde{F}} = \frac{-1}{M_{pl}^2 c_{DD}}$$

Inserting these in Eq.(6.112),

$$\begin{aligned} S = \int d^4x \sqrt{-g} \quad \Big\{ \quad & M_{pl}^2 \kappa^2 c_{DD} + 3M_{pl}^2 a^{-2} \left(\ddot{a}a + \dot{a}^2 \right) + \frac{3a^{-2}}{2} \left(\dot{\phi}^2 - a^{-2} g^2 \phi^4 \right) \\ - \quad & \frac{1}{4} F^{\nu\beta} F_{\nu\beta} - \frac{a^{-2}}{4\kappa^2 c_{DD}} \Big[8 \left(\ddot{a}a + 2\dot{a}^2 \right) F^{0i} F_{0i} - \left(\ddot{a}a - \dot{a}^2 \right) F^{ij} F_{ij} \\ + \quad & 4a^{-2} \left(\ddot{a}a + 2\dot{a}^2 \right) \dot{\phi}^2 - a^{-4} \left(\ddot{a}a - \dot{a}^2 \right) g^2 \phi^4 \Big] \\ - \quad & \frac{3}{32 M_{pl}^2 \kappa^2 c_{DD}} F^{\nu\beta} F_{\nu\beta} F^{\nu'\beta'} F_{\nu'\beta'} \\ + \quad & \frac{a^{-2}}{8 M_{pl}^2 \kappa^2 c_{DD}} \Big[8 \left(\dot{\phi}^2 F^{0i} F_{0i} + a^{-2} g^2 \phi^4 F_{ij} F^{0k} \epsilon_k{}^{ij} \right) \\ - \quad & \left(3\dot{\phi}^2 + a^{-2} g^2 \phi^4 \right) F^{ij} F_{ij} \Big] \\ - \quad & \frac{3a^{-4}}{8 M_{pl}^2 \kappa^2 c_{DD}} \Big[4\dot{\phi}^4 - 5a^{-2} g^2 \dot{\phi}^2 \phi^4 + a^{-4} g^4 \phi^8 \Big] \Big\} \end{aligned} \tag{6.116}$$

This is the exact Born-Infeld-Riemann action in FRW background.

6.3.1. Inflation from BIR?

To examine inflationary setup, let us set $\lambda_F = 0$. Then the action becomes,

$$
\begin{aligned}
S = \int d^4x\sqrt{-g} \quad \Big\{ \quad & M_{pl}^2\kappa^2 c_{DD} + 3M_{pl}^2 a^{-2}\left(\ddot{a}a + \dot{a}^2\right) + \frac{3a^{-2}}{2}\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right) \\
- \quad & \frac{a^{-2}}{4\kappa^2 c_{DD}}\left[4a^{-2}\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi}^2 - a^{-4}\left(\ddot{a}a - \dot{a}^2\right)g^2\phi^4\right] \\
- \quad & \frac{3a^{-4}}{8M_{pl}^2\kappa^2 c_{DD}}\left[4\dot{\phi}^4 - 5a^{-2}g^2\dot{\phi}^2\phi^4 + a^{-4}g^4\phi^8\right]\Big\}
\end{aligned} \tag{6.117}
$$

For $\sqrt{-g}$ and $\int d^4x$, we can use Eq. (6.47)and (6.49) respectively. Then, we obtain a theory which only depends on two scalar field, $a(t)$ and $\phi(t)$. Moreover, let us label $\kappa^2 = \epsilon M^2$ where $\epsilon = \pm 1$

$$
\begin{aligned}
S = \int dt \quad \Big\{ \quad & \xi^3 a^6 \epsilon M^2 M_{pl}^2 c_{DD} + 3\xi^3 a^4 M_{pl}^2\left(\ddot{a}a + \dot{a}^2\right) + \frac{3\xi^3 a^4}{2}\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right) \\
- \quad & \frac{\xi^3 a^2}{4\kappa^2 c_{DD}}\left[4\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi}^2 - a^{-2}\left(\ddot{a}a - \dot{a}^2\right)g^2\phi^4\right] \\
- \quad & \frac{3\xi^3 a^2}{8\epsilon M^2 M_{pl}^2 c_{DD}}\left[4\dot{\phi}^4 - 5a^{-2}g^2\dot{\phi}^2\phi^4 + a^{-4}g^4\phi^8\right]\Big\}
\end{aligned} \tag{6.118}
$$

To understand the dynamics of the system, we should examine equations of the motions. We have two dynamical variable which are scalar field $\phi(t)$ and scalar factor $a(t)$. Let us start with $\phi(t)$. Effective Lagrange of the system is given as

$$
\begin{aligned}
L_{eff} \quad = \quad & \Big\{\xi^3 a^6 \epsilon M^2 M_{pl}^2 c_{DD} + 3\xi^3 a^4 M_{pl}^2\left(\ddot{a}a + \dot{a}^2\right) + \frac{3\xi^3 a^4}{2}\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right) \\
- \quad & \frac{\xi^3 a^2}{4\kappa^2 c_{DD}}\left[4\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi}^2 - a^{-2}\left(\ddot{a}a - \dot{a}^2\right)g^2\phi^4\right] \\
- \quad & \frac{3\xi^3 a^2}{8\epsilon M^2 M_{pl}^2 c_{DD}}\left[4\dot{\phi}^4 - 5a^{-2}g^2\dot{\phi}^2\phi^4 + a^{-4}g^4\phi^8\right]\Big\}
\end{aligned} \tag{6.119}
$$

By using Euler-Lagrange equation (see Appendix B)

$$
\frac{d}{dt}\left(\frac{\partial L_{eff}}{\partial \dot{\phi}}\right) - \frac{\partial L_{eff}}{\partial \phi} = 0 \tag{6.120}
$$

Subsituting Eq. (6.120) in (6.119)

$$
\begin{aligned}
0 \;=\; & \frac{d}{dt}\left[3\xi^3 a^4\dot{\phi} - \frac{2\xi^2 a^3}{\epsilon M^2 c_{DD}}\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi} - \frac{6\xi^3 a^2}{\epsilon M^2 M_{pl}^2 c_{DD}}\dot{\phi}^3 + \frac{15\xi^3}{4\epsilon M^2 M_{pl}^2 c_{DD}}g^2\dot{\phi}\phi^4\right] \\
- & \Big[-6\xi^3 a^2 g^2 \phi^3 + \frac{\xi^3}{\epsilon M^2 c_{DD}}\left(\ddot{a}a - \dot{a}^2\right)g^2\phi^3 + \frac{15\xi^3}{2\epsilon M^2 M_{pl}^2 c_{DD}}g^2\dot{\phi}^2\phi^3 \\
- & \frac{3\xi^3 a^{-2}}{\epsilon M^2 M_{pl}^2 c_{DD}}g^4\phi^7\Big] \qquad (6.121)
\end{aligned}
$$

By taking derivative with respect to time,

$$
\begin{aligned}
0 \;=\; & 3\xi^3\left(4a^3\dot{a}\dot{\phi} + a^4\ddot{\phi}\right) - \frac{2\xi^3}{\epsilon M^2 c_{DD}}\Big[\left(a^3\dddot{a} + 3a^2\dot{a}\ddot{a} + 4\dot{a}^3 a + 4a^2\ddot{a}\dot{a}\right)\dot{\phi} \\
+ & a^2\left(\ddot{a}a + 2\dot{a}^2\right)\ddot{\phi}\Big] - \frac{6\xi^3}{\epsilon M^2 M_{pl}^2 c_{DD}}\left(2\dot{a}a\dot{\phi}^3 + 3a^2\dot{\phi}^2\ddot{\phi}\right) \\
+ & \frac{15g^2\xi^3}{4\epsilon M^2 M_{pl}^2 c_{DD}}\left(\ddot{\phi}\phi^4 + 4\dot{\phi}^2\phi^3\right) + 6\xi^3 a^2 g^2\phi^3 - \frac{\xi^3 g^2\phi^3}{\epsilon\, M^2 c_{DD}}\left(\ddot{a}a - \dot{a}^2\right) \\
- & \frac{15\xi^3 g^2\dot{\phi}^2\phi^3}{2\epsilon M^2 M_{pl}^2 c_{DD}} + \frac{3\xi^3 a^{-2} g^4\phi^7}{\epsilon M^2 M_{pl}^2 c_{DD}} \qquad (6.122)
\end{aligned}
$$

Dividing both sides with ξ^3 and arranging all terms,

$$
\begin{aligned}
0 = & \left[3a^4 - \frac{2a^2}{\epsilon M^2 c_{DD}}\left(\ddot{a}a + 2\dot{a}^2\right) - \frac{18a^2\dot{\phi}^2}{\epsilon M^2 M_{pl}^2 c_{DD}} + \frac{15g^2\phi^4}{4\epsilon M^2 M_{pl}^2 c_{DD}}\right]\ddot{\phi} \\
& - \frac{12\dot{a}a}{\epsilon M^2 M_{pl}^2 c_{DD}}\dot{\phi}^3 + \frac{15g^2\phi^3}{2\epsilon M^2 M_{pl}^2 c_{DD}}\dot{\phi}^2 \\
& + \left[12a^3\dot{a} - \frac{2a^3\dddot{a}}{\epsilon M^2 c_{DD}} - \frac{6a^2\dot{a}\ddot{a}}{\epsilon M^2 c_{DD}} - \frac{8\dot{a}^3 a}{\epsilon M^2 c_{DD}}\right]\dot{\phi} \\
& + \left[6g^2 a^2 - \frac{g^2\left(\ddot{a}a - \dot{a}^2\right)}{\epsilon M^2 c_{DD}}\right]\phi^3 - \frac{3a^{-2}g^4}{\epsilon M^2 M_{pl}^2 c_{DD}}\phi^7 \qquad (6.123)
\end{aligned}
$$

This is the equation of $\phi(t)$. In this step, we should write the equation of motion in terms of Hubble Parameter and its derivatives which are given in Eq.(6.5) and (6.9) respectively.

$$0 = \left[3a^4 - \frac{2a^4}{\epsilon M^2 c_{DD}}\left(\dot{H} + 3H^2\right) - \frac{18a^2\dot{\phi}^2}{\epsilon M^2 M_{pl}^2 c_{DD}} + \frac{15g^2\phi^4}{4\epsilon M^2 M_{pl}^2 c_{DD}}\right]\ddot{\phi} - \frac{12a^2 H}{\epsilon M^2 M_{pl}^2 c_{DD}}\dot{\phi}^3 + \frac{15g^2\phi^3}{2\epsilon M^2 M_{pl}^2 c_{DD}}\dot{\phi}^2 + \left[12a^4 H - \frac{2a^4\left(\ddot{H} + 6\dot{H}H + 8H^3\right)}{\epsilon M^2 c_{DD}}\right]\dot{\phi} + \left[6g^2a^2 - \frac{g^2a^2\dot{H}}{\epsilon M^2 c_{DD}}\right]\phi^3 - \frac{3a^{-2}g^4}{\epsilon M^2 M_{pl}^2 c_{DD}}\phi^7 \quad (6.124)$$

Now, we should be able to analysis whether the theory enables us to *slow-roll inflation* or not. We explain inflationary model at the beginning of this chapter. As you see in (Section 6.1), in the case of slow roll inflation, scalar field $\phi(t)$ should be vary slowly with time. Thus we can avoid the term which contains derivatives of $\phi(t)$ in Eq.(6.124). Then it becomes,

$$0 = \left[6a^2g^2 - \frac{a^2g^2}{\epsilon M^2 c_{DD}}\dot{H}\right]\phi^3 + \frac{3a^{-2}}{\epsilon M^2 M_{pl}^2 c_{DD}}g^4\phi^7$$

Here, we can also avoid $\frac{a^2g^2}{\epsilon M^2 c_{DD}}\dot{H}$ since it is so small due to M^2. Then

$$0 = 6a^2g^2\phi^3 + \frac{3a^{-2}}{\epsilon M^2 M_{pl}^2 c_{DD}}g^4\phi^7 \quad (6.125)$$

$$\left(\frac{\phi}{a}\right)^4 = \frac{-2\epsilon M^2 M_{pl}^2 c_{DD}}{g^2} \quad (6.126)$$

One can easily see that ϵ should be -1.

After finding $\left(\frac{\phi}{a}\right)^4$ in terms of M,M_{pl} and g, let us find density and pressure of the theory in case of perfect fluid (see §6.1). Density and pressure of the system is given in Eq. (6.65) (Maleknejad, 2011). Also by taking into account Eq. (6.117)

$$\begin{aligned}\mathcal{L}_{eff} = & \Big\{ \epsilon M^2 M_{pl}^2 c_{DD} + 3M_{pl}^2 a^{-2}\left(\ddot{a}a + \dot{a}^2\right) + \frac{3a^{-2}}{2}\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right) \\ - & \frac{a^{-2}}{4\epsilon M^2 c_{DD}}\Big[4a^{-2}\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi}^2 - a^{-4}\left(\ddot{a}a - \dot{a}^2\right)g^2\phi^4\Big] \\ - & \frac{3a^{-4}}{8M_{pl}^2\epsilon M^2 c_{DD}}\Big[4\dot{\phi}^4 - 5a^{-2}g^2\dot{\phi}^2\phi^4 + a^{-4}g^4\phi^8\Big]\Big\}\end{aligned} \tag{6.127}$$

Thus,

$$\frac{\partial \mathcal{L}_{eff}}{\partial \dot{\phi}} = 3a^{-2}\dot{\phi} - \frac{2a^{-4}}{\epsilon M^2 c_{DD}}\left(\ddot{a}a + \dot{a}^2\right)\dot{\phi} - \frac{6a^{-4}}{\epsilon M^2 M_{pl}^2}\dot{\phi}^3 + \frac{15a^{-6}g^2\phi^4}{4\epsilon M^2 M_{pl}^2}\dot{\phi} \tag{6.128}$$

By inserting Eq. (6.127) and (6.128) in (6.65),

$$\begin{aligned}\rho = & -\epsilon M^2 M_{pl}^2 c_{DD} - 3M_{pl}^2 a^{-2}\left(\ddot{a}a + \dot{a}^2\right) + \frac{3a^{-2}}{2}\left(\dot{\phi}^2 + a^{-2}g^2\phi^4\right) \\ - & \frac{a^{-4}}{4\epsilon M^2 c_{DD}}\left[4\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi}^2 + a^{-2}\left(\ddot{a}a - \dot{a}^2\right)g^2\phi^4\right] \\ - & \frac{3a^{-4}}{8\epsilon M^2 M_{pl}^2 c_{DD}}\left[12\dot{\phi}^4 + 5a^{-2}g^2\dot{\phi}^2\phi^4 - a^{-4}g^4\phi^8\right]\end{aligned} \tag{6.129}$$

This is density which the theory enable us. To analyze inflation, we can replace $\ddot{a}$ and $\dot{a}^2$ with Hubble Parameter given in Eq. (6.5) and (6.9). Then ρ becomes

$$\begin{aligned}\rho = & -\epsilon M^2 M_{pl}^2 c_{DD} - 3M_{pl}^2\left(\dot{H} + 2H^2\right) + \frac{3a^{-2}}{2}\left(\dot{\phi}^2 + a^{-2}g^2\phi^4\right) \\ - & \frac{a^{-2}}{4\epsilon M^2 c_{DD}}\left[4\left(\dot{H} + 3H^2\right)\dot{\phi}^2 + a^{-2}g^2\phi^4\dot{H}\right] \\ - & \frac{3a^{-4}}{8\epsilon M^2 M_{pl}^2 c_{DD}}\left[12\dot{\phi}^4 + 5a^{-2}g^2\dot{\phi}^2\phi^4 - a^{-4}g^4\phi^8\right]\end{aligned} \tag{6.130}$$

To find P,

$$\begin{aligned} a^3\mathcal{L}_{eff} &= a^3\epsilon M^2 M_{pl}^2 c_{DD} + 3M_{pl}^2 a\left(\ddot{a}a + \dot{a}^2\right) + \frac{3a}{2}\left(\dot{\phi}^2 - a^{-2}g^2\phi^4\right) \\ &- \frac{a}{4\epsilon M^2 c_{DD}}\left[4a^{-2}\left(\ddot{a}a + 2\dot{a}^2\right)\dot{\phi}^2 - a^{-4}\left(\ddot{a}a - \dot{a}^2\right)g^2\phi^4\right] \\ &- \frac{3a^{-1}}{8M_{pl}^2\epsilon M^2 c_{DD}}\left[4\dot{\phi}^4 - 5a^{-2}g^2\dot{\phi}^2\phi^4 + a^{-4}g^4\phi^8\right]\Big\} \end{aligned} \tag{6.131}$$

By inserting Eq. (6.131) in (6.65)

$$\begin{aligned} P &= \epsilon M^2 M_{pl}^2 c_{DD} + M_{pl}^2 a^{-2}\left(2\ddot{a}a + \dot{a}^2\right) + \frac{a^{-2}}{2}\left(\dot{\phi}^2 + a^{-2}g^2\phi^4\right) \\ &+ \frac{a^{-4}}{12\epsilon M^2 c_{DD}}\left[8\dot{a}^2\dot{\phi}^2 - a^{-2}\left(2\ddot{a}a - \dot{a}^2\right)g^2\phi^4\right] \\ &+ \frac{a^{-4}}{8\epsilon M^2 M_{pl}^2 c_{DD}}\left[4\dot{\phi}^4 - 15a^{-2}g^2\dot{\phi}^2\phi^4 + 5a^{-4}g^4\phi^8\right] \end{aligned} \tag{6.132}$$

Also, we replace $\ddot{a}$ and $\dot{a}^2$ with Hubble Parameter given in Eq. (6.5) and (6.9) for P too. Then it becomes

$$\begin{aligned} P &= \epsilon M^2 M_{pl}^2 c_{DD} + M_{pl}^2\left(2\dot{H} + 3H^2\right) + \frac{a^{-2}}{2}\left(\dot{\phi}^2 + a^{-2}g^2\phi^4\right) \\ &+ \frac{a^{-2}}{12\epsilon M^2 c_{DD}}\left[8H^2\dot{\phi}^2 - a^{-2}\left(2\dot{H} + H^2\right)g^2\phi^4\right] \\ &+ \frac{a^{-4}}{8\epsilon M^2 M_{pl}^2 c_{DD}}\left[4\dot{\phi}^4 - 15a^{-2}g^2\dot{\phi}^2\phi^4 + 5a^{-4}g^4\phi^8\right] \end{aligned} \tag{6.133}$$

Finally Eq. (6.130) and (6.133) is what our theory gives us as density and pressure of our universe.

We find that density and pressure in terms of Hubble parameter in Eq. (6.13). If we insert it in Eq. (6.130) and (6.133) we will have two equation which both of them only depend on Hubble parameter. Hence Eq. (6.130) becomes,

$$
\begin{aligned}
3M_{pl}^2 &= -\epsilon M^2 M_{pl}^2 c_{DD} - 3M_{pl}^2\left(\dot{H}+2H^2\right) + \frac{3a^{-2}}{2}\left(\dot{\phi}^2 + a^{-2}g^2\phi^4\right) \\
&- \frac{a^{-2}}{4\epsilon M^2 c_{DD}}\left[4\left(\dot{H}+3H^2\right)\dot{\phi}^2 + a^{-2}g^2\phi^4\dot{H}\right] \\
&- \frac{3a^{-4}}{8\epsilon M^2 M_{pl}^2 c_{DD}}\left[12\dot{\phi}^4 + 5a^{-2}g^2\dot{\phi}^2\phi^4 - a^{-4}g^4\phi^8\right]
\end{aligned} \tag{6.134}
$$

By arranging,

$$
\begin{aligned}
&\left[\quad 3M_{pl}^2 + \frac{a^{-2}}{4\epsilon M^2 c_{DD}}\left(4\dot{\phi}^2 + a^{-2}g^2\phi^4\right)\right]\dot{H} + \left[9M_{pl}^2 + \frac{3a^{-2}}{\epsilon M^2 c_{DD}}\dot{\phi}^2\right]H^2 \\
&= -\epsilon M^2 M_{pl}^2 c_{DD} + \frac{3a^{-2}}{2}\left[\dot{\phi}^2 + a^{-2}g^2\phi^4\right] \\
&- \frac{3a^{-4}}{8\epsilon M^2 M_{pl}^2 c_{DD}}\left[12\dot{\phi}^4 + 5a^{-2}g^2\dot{\phi}^2\phi^4 + a^{-4}g^4\phi^8\right]
\end{aligned} \tag{6.135}
$$

By applying slow roll condition given in Eq.(6.17)

$$
\begin{aligned}
&\left[\quad 3M_{pl}^2 + \frac{a^{-2}}{4\epsilon M^2 c_{DD}}\left(4\eta^2H^2\phi^2 + a^{-2}g^2\phi^4\right)\right]\dot{H} + \left[9M_{pl}^2 + \frac{3a^{-2}}{\epsilon M^2 c_{DD}}\eta^2H^2\phi^2\right]H^2 \\
&= -\epsilon M^2 M_{pl}^2 c_{DD} + \frac{3a^{-2}}{2}\left[\eta^2H^2\phi^2 + a^{-2}g^2\phi^4\right] \\
&- \frac{3a^{-4}}{8\epsilon M^2 M_{pl}^2 c_{DD}}\left[12\eta^4H^4\phi^4 + 5a^{-2}g^2\eta^2H^2\phi^6 + a^{-4}g^4\phi^8\right]
\end{aligned} \tag{6.136}
$$

Since $\eta << 1$, we can neglect all terms which contains η. Then

$$
\begin{aligned}
\left[3M_{pl}^2 + \frac{1}{4\epsilon M^2 c_{DD}}g^2\left(\frac{\phi}{a}\right)^4\right]\dot{H} + 9M_{pl}^2H^2 &= -\epsilon M^2 M_{pl}^2 c_{DD} + \frac{3}{2}g^2\left(\frac{\phi}{a}\right)^4 \\
&- \frac{3}{8\epsilon M^2 M_{pl}^2 c_{DD}}g^4\left(\frac{\phi}{a}\right)^8
\end{aligned} \tag{6.137}
$$

and inserting Eq.(6.126),

$$
\frac{5}{2}\dot{H} + 9H^2 = \frac{-11}{3}\epsilon M^2 c_{DD} \tag{6.138}
$$

This is the equation given ρ under slow roll- conditions. We have another equation which is P. We can follow the same procedure which is done for ρ. let us start with inserting Eq. (6.13) in (6.133)

$$\begin{aligned} -M_{pl}^2\left(2\dot{H}+3H^2\right) &= \epsilon M^2 M_{pl}^2 c_{DD} + M_{pl}^2\left(2\dot{H}+3H^2\right) + \frac{a^{-2}}{2}\left(\dot{\phi}^2 + a^{-2}g^2\phi^4\right) \\ &+ \frac{a^{-2}}{12\epsilon M^2 c_{DD}}\left[8H^2\dot{\phi}^2 - a^{-2}\left(2\dot{H}+H^2\right)g^2\phi^4\right] \\ &+ \frac{a^{-4}}{8\epsilon M^2 M_{pl}^2 c_{DD}}\left[4\dot{\phi}^4 - 15a^{-2}g^2\dot{\phi}^2\phi^4 + 5a^{-4}g^4\phi^8\right] \end{aligned} \tag{6.139}$$

and again arranging,

$$\begin{aligned} &\left[-4M_{pl}^2 + \frac{a^{-4}}{6\epsilon M^2 c_{DD}}g^2\phi^4\right]\dot{H} + \left[-6M_{pl}^2 - \frac{a^{-2}}{12\epsilon M^2 c_{DD}}\left(8\dot{\phi}^2 - a^{-2}g^2\phi^4\right)\right]H^2 \\ &= \epsilon M^2 M_{pl}^2 c_{DD} + \frac{a^{-2}}{2}\left[\dot{\phi}^2 + a^{-2}g^2\phi^4\right] \\ &+ \frac{a^{-4}}{8\epsilon M^2 M_{pl}^2 c_{DD}}\left[4\dot{\phi}^4 - 15a^{-2}g^2\dot{\phi}^2\phi^4 + 5a^{-4}g^4\phi^8\right] \end{aligned} \tag{6.140}$$

applying slow roll condition (Eq.(6.17)),

$$\begin{aligned} &\left[-4M_{pl}^2 + \frac{a^{-4}}{6\epsilon M^2 c_{DD}}g^2\phi^4\right]\dot{H} + \left[-6M_{pl}^2 - \frac{a^{-2}}{12\epsilon M^2 c_{DD}}\left(8\eta^2 H^2\phi^2 - a^{-2}g^2\phi^4\right)\right]H^2 \\ &= \epsilon M^2 M_{pl}^2 c_{DD} + \frac{a^{-2}}{2}\left[\eta^2 H^2\phi^2 + a^{-2}g^2\phi^4\right] \\ &+ \frac{a^{-4}}{8\epsilon M^2 M_{pl}^2 c_{DD}}\left[4\eta^4 H^4\phi^4 - 15a^{-2}g^2\eta^2 H^2\phi^6 + 5a^{-4}g^4\phi^8\right] \end{aligned} \tag{6.141}$$

and omitting terms with η

$$\begin{aligned} &\left[-4M_{pl}^2 + \frac{1}{6\epsilon M^2 c_{DD}}g^2\left(\frac{\phi}{a}\right)^4\right]\dot{H} + \left[-6M_{pl}^2 - \frac{1}{12\epsilon M^2 c_{DD}}g^2\left(\frac{\phi}{a}\right)^4\right]H^2 \\ &= \epsilon M^2 M_{pl}^2 c_{DD} + \frac{1}{2}g^2\left(\frac{\phi}{a}\right)^4 + \frac{5}{8\epsilon M^2 M_{pl}^2 c_{DD}}g^4\left(\frac{\phi}{a}\right)^8 \end{aligned} \tag{6.142}$$

inserting Eq. (6.126) in (6.142),

$$\frac{-13}{3}\dot{H} - \frac{37}{6}H^2 = \frac{5}{2}\epsilon M^2 c_{DD} \tag{6.143}$$

In this step, to find $\dot{H}$ and H^2 in terms of M, the only thing we should do to solve Eq. (6.138) and (6.143). Hence the solution is given below:

$$H^2 = \frac{-347}{849}\epsilon M^2 c_{DD} \qquad \& \qquad \dot{H} = \frac{4}{849}\epsilon M^2 c_{DD} \tag{6.144}$$

At the beginning of this section, where we find $\left(\frac{\phi}{a}\right)^4$, we state that $\epsilon = -1$. By using this argument

$$H^2 = \frac{347}{849}M^2 c_{DD} \qquad \& \qquad \dot{H} = \frac{-4}{849}M^2 c_{DD} \tag{6.145}$$

In the end, our last step is to examine whether the theory gives us inflation or not. The way to understand this is to find slow roll parameter ε. If ε is so small from 1, the theory gives us inflation. Thus let us insert Eq. (6.145) in (6.16).

$$\begin{aligned} \varepsilon &= \frac{-\dot{H}}{H^2} \\ &= \frac{\left(\frac{-4}{849}M^2 c_{DD}\right)}{\left(\frac{347}{849}M^2 c_{DD}\right)} \\ &= \frac{4}{347} \end{aligned} \tag{6.146}$$

Then finally,

$$\varepsilon \simeq 0,01 \tag{6.147}$$

The result is Born-Infeld-Riemann Gravity with Non-Abelian Fields explains Inflation.

CHAPTER 7

CONCLUSION

In this thesis work we have studied Born-Infeld type extensions of the GR. We analyzed BIE known in literature with our own BIR extension. In the former the invariant volume involves determinant of a rank-two tensor, in the latter a rank-four tensor. Their implications can thus differ form each other.

In the thesis we have first given a general framework in which Abelian and Non-Abelian fields mix with each other and with the background geometry. Here we have also indicated how GR appears as a limiting case.

Then, we turned to the analysis of the inflationary cosmology. For successful inflation we need universe to slowly roll down on flat potential of the inflaton. The inflaton field here is provided by the SU(2) non-Abelian gauge field. We find that this gauge-inflation is successfully realized in BIR but not in BIE. Inflation thus provides an explicit example of the difference between two extensions.

The Abelian-Non Abelian mixing gives rise to photon emission during inflation. This process is exceedingly slow, and its effects can be hard to detect. In this thesis study we have not been able to analyze this point in the depth it requires.

The major result of the thesis is that, when we consider both BIE and BIR in homogeneous, isotropic and spatially flat backgrounds, the BIR explains inflationary phase while BIE does not. Therefore not only electromagnetism and SU(2) Yang-Mills theory but also inflaton itself is embedded in geometry.

REFERENCES

Banados M., 2008: Eddington-Born-Infeld action for Dark Energy and Dark Matter, [arXiv: hep-ph/ 0801.4103v5]

Beisert N., 2013: Quantum Field Theory II, ETH Zurich.

Born M. and Infeld L., 1934: Foundations of the New Field Theory. Proc. Roy. Soc. A144, 425-451.

Carroll S. M., 2004: Spacetime and geometry: An introduction to general relativity. San Francisco, USA: Addison-Wesley, 513 pp.

Deser S. *et al.*, 1998: Born-Infeld-Einstein Actions? Class. Quant. Grav. 15, L35 (1998), [arXiv:hep-th/9803049].

Eddington A. S., 1924: The Mathematical Theory of Relativity(CUP 1924).

Enberg R., 2013: Advanced Particle Physics: Lecture notes on Gauge Theory.

Escobar D., 2012: Born-Infeld Type Modification of the Gravity. [arXiv:physics.gen-ph/1209.6559v4]

Griffiths D., 1999: Introduction to Electrodynamics. ISBN: 0-13-805326-X.

Griffiths D., 1987: Introduction to Elementary Particles. ISBN 0-471-60386-4.

Guth A., 1980: Inflationary Universe: A Possible Solution to the Flatness and Horizon Problem. Phys. Review D, 23,2 (1981).

Hubble E., 1929: A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae. 10.1073/pnas.15.3.168

Karahan C. N., 2010: Conformal Transformations in Metric-Affine Gravity. IZTECH.

Koerber P., 2004: Abelian and non-Abelian D-brane Effective Actions. [arXiv:hep-th/0405227v2].

Lewis C., 2009: Explicit Gauge Covariant Euler-Lagrange Equation. Am.J.Phys.77:839-843,2009, [arXiv:physics.class-ph/0907.2301v2]

Makeenko Y. M., 1996: Introduction to Gauge Theory, Lectures at ITEP Winter School.

Maleknejad, A. *et al.* Gauge Flation: Inflation from Non-Abelian Gauge Fields. [arXiv:1102.1513 [hep-ph]].

Nieto J. A., 2004: Born-Infeld Gravity in any dimension, [arXiv:hep-th/0402071v2].

Riotto A., 2002: Inflation and The Theory of the Cosmological Perturbations. Lectures given at the Summer School on Astroparticle Physics and Cosmology.

Quigg C., 1983: Gauge Theories of the strong, weak and electromagnetic Interactions. ISBN: 0-201-32832-1

Soysal S., 2012: Born-Infeld-Riemann Gravity. IZTECH.

Stelle K. S., 1978: Gen. Rel. Grav. 9 (1978) 353.

Vaudrevange P., 2010: Slow roll Inflation

Weinberg S., 1972: Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. ISBN 0-471-92567-5

Wohlfarth M. N. R., 2004: Gravity a la Born-Infeld. Class. Quantum Grav. 21, 1927-1939

APPENDIX A

METRIC THEORY OF GRAVITATION

The most fundamental formulation of General Relativity (GR) was suggested by Einstein in 1916 and it was called as metrical formulation of Gravity. Naturally, modifications of the GR has been done since that time and mainly these are called as Metric-Affine gravity and Affine gravity. However, in this thesis we have followed the Einstein's route. Therefore, all of calculations given in this thesis was made by using Metrical theory of Gravity.

According to Einstein's formulation of GR, spacetime is a differentiable manifold, and metric tensor, which is symbolized as $g_{\alpha\beta}$, is responsible for curving and twirling of the spacetime (Carroll, 2004). This is just a collection of clocks and rulers needed for measuring distances and angles.

If one wants to examine any smooth manifold in a general framework, it obtains two independent dynamical objects: metric tensor and connection- connection is a guiding force for geodesic motion. However, in the light of Metrical theory of gravity, metric tensor is the most important concept since all of the notions including connection are related to itself. Connection can be written in terms of metric tensor just as given below:

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\rho} \left(\partial_{\alpha} g_{\beta\rho} + \partial_{\beta} g_{\rho\alpha} - \partial_{\rho} g_{\alpha\beta} \right) \tag{A.1}$$

which is especially known as the Levi-Civita connection. Since metric tensor is symmetric, it is also symmetric in lower indices (Carroll, 2004) (Weinberg, 1972). Hence, as you see in Eq.(A.1), GR is described by a single variable and it is the metric tensor.

The main success of General relativity is to define of curvature of spacetime. Spacetime curvature is basically related to the connection of smooth manifold.

$$R^{\lambda}{}_{\alpha\mu\beta} = \partial_{\mu} \Gamma^{\lambda}{}_{\beta\alpha} - \partial_{\beta} \Gamma^{\lambda}{}_{\mu\alpha} + \Gamma^{\lambda}{}_{\mu\tau} \Gamma^{\tau}{}_{\beta\alpha} - \Gamma^{\lambda}{}_{\beta\tau} \Gamma^{\tau}{}_{\mu\alpha} \tag{A.2}$$

Then, since connection depends on metric tensor, not only connection on smooth manifold, but also curvature is related to the metric tensor. By using Kronecker delta δ^{μ}_{ν}

and metric tensor $g_{\mu\nu}$ we obtain different forms of Curvature tensor. Let us write these respectively. The first form is also called as Ricci Tensor.

$$\begin{aligned} R_{\alpha\beta} &= R^{\lambda}{}_{\alpha\mu\beta}\delta^{\mu}_{\lambda} \\ &= R^{\lambda}{}_{\alpha\lambda\beta} \end{aligned} \tag{A.3}$$

The other is obtained by using Ricci Tensor and called as Ricci Scalar or Scalar Curvature:

$$R = g^{\alpha\beta}R_{\alpha\beta} \tag{A.4}$$

These are the fundamental equations of General Relativity.The field equations of GR are obtained by varying the Einstein-Hilbert action which is

$$S\left[g\right] = \int d^4x\sqrt{-g}\left\{\frac{1}{2}M^2_{Pl}R(g) + \mathcal{L}_{mat}\left(g,\psi\right)\right\} \tag{A.5}$$

Since this action only depends on metric tensor, variation is only taken with respect to it. Here, $M_{Pl} = (8\pi G_N)^{-1/2}$ is the Planck scale or the fundamental scale of gravity.

Variation of equation Eq.(A.5) gives the Einstein equations of gravitation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{M^2_{Pl}}T_{\mu\nu} \tag{A.6}$$

whose right-hand side

$$T_{\mu\nu} = \frac{\delta}{\delta g^{\mu\nu}}S_{mat}\left[g,\psi\right] \tag{A.7}$$

is the energy-momentum tensor of matter and radiation. Here,

$$S\left[g\right] = \int d^4x\sqrt{-g}\mathcal{L}_{mat}\left(g,\psi\right) \tag{A.8}$$

is the action of the matter and radiation fields ψ. The curvature scalar and matter Lagrangian $\mathcal{L}_{mat}$ both involve the same metric tensor $g_{\mu\nu}$.

It is important to that the field equations (A.6) arises from the gravitational action (A.5) by adding an extrinsic curvature term. The reason is that, curvature scalar $R(g)$ involves second derivatives of the metric tensor, and in applying the variational equations it is not sufficient to specify $\delta g_{\mu\nu}$ at the boundary. One must also specify its derivatives $\delta\partial_\alpha g_{\mu\nu}$ at the boundary. This additional piece does not admit construction of the Einstein-Hilbert action directly; one adds a term (extrinsic curvature) to cancel the excess term.

The metric formalism is the most common approach to gravitation because equivalence principle is automatic, geodesic equations are plain, and tensor algebra is simplified (metric tensor is covariantly constant). Equivalence principle means that gravitational force acting on a point mass can be altered by choosing an accelerated (non-inertial) coordinate system. In other words, there is always a frame where connection can be set to zero. This point corresponds to a locally-flat coordinate system. The curvature tensor involves both Levi-Civita connection and its derivatives, and making the connection vanish does not mean that curvature vanishes.

On the other hand, experiments and observation show that, our universe is homogeneous, isotropic and locally flat. These properties can be achieved by FRW metric tensor which is

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] \tag{A.9}$$

in spherical coordinates. Here,

- $k < 0$ corresponds to open universe
- $k = 0$ corresponds to (spatially) flat universe
- $k > 0$ corresponds to closed universe

In this thesis we work on locally flat spacetime and so we set $k = 0$ then also in Cartesian coordinates,

$$ds^2 = -dt^2 + a(t)^2 \delta^{ij} dx_i dx_j \tag{A.10}$$

By inserting this metric tensor into the Ricci tensor and Ricci Scalar, we obtain

$$\begin{aligned}
R_{00} &= \frac{-3\ddot{a}}{a} \\
R_{0i} &= 0 \\
R_{ij} &= \left(\ddot{a}a + 2\dot{a}^2\right)\delta_{ij} \\
R &= 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)
\end{aligned} \tag{A.11}$$

As a consequence, metrical theory of gravity is the most fundamental theory and contains only one dynamical variable which is metric tensor. Thus all notions are related to metric tensor. In FRW universe, metric tensor defined by scalar factor $a(t)$ which is related o the radius of the universe. Then curvature tensor is also related to the scalar factor and its first and second derivatives.

APPENDIX B

CLASSICAL FIELD THEORY

In this step we examine how the equations of motions are obtained. In the stage of understanding the dynamics of theories, the most important role is played by equations of motions. In classical field theory, the way of finding equations of motion is called variational method. So, we should firstly explain variational method. To achieve this aim, let us consider a Lagrangian density $\mathcal{L}(\phi, \dot{\phi})$ where the action is given as;

$$S = \int dtL = \int d^4x\mathcal{L}\left(\phi^i, \partial_\mu\phi^i\right) \tag{B.1}$$

By considering small variation in this field;

$$\begin{aligned} \phi^i &\to \phi^i + \delta\phi^i \\ \partial_\mu\phi^i &\to \partial_\mu\phi^i + \delta(\partial_\mu\phi^i) = \partial_\mu\phi^i + \partial_\mu(\delta\phi^i) \end{aligned} \tag{B.2}$$

Lagrangian Density varies;

$$\mathcal{L}(\phi^i, \partial_\mu\phi^i) \to \mathcal{L}(\phi^i + \delta\phi^i, \partial_\mu\phi^i + \partial_\mu(\delta\phi^i)) \tag{B.3}$$

thus action varies by virtue of this small variation $S \to S + \delta S$. Hence;

$$\delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^i}\right)\right] \tag{B.4}$$

We assume that ϕ^i is the same at the end points of the integral. Thus, by using this assumption, we conclude that δS should be zero. On the other hand $\delta\phi^i$ should be different from zero. Thus the only choice to obtain $\delta S = 0$ is

$$\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^i} \right) = 0 \tag{B.5}$$

This equation is called as Euler-Lagrange Equation and it leads to field equations (Carroll, 2004). Therefore, by using variational principle, we obtain the equation of motion for a Lagrangian which has one dynamical variable such as metrical theory of gravity.