STRONGLY T-NONCOSINGULAR MODULES

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by
Ozan GÜNYÜZ

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İZMİR

We approve the thesis of **Ozan GÜNYÜZ**

Assist. Prof. Dr. Engin BÜYÜKAŞIK
Supervisor

Assoc. Prof. Dr. Dilek PUSAT YILMAZ
Committee Member

Assist. Prof. Dr. Engin MERMUT
Committee Member

29 June 2010

Prof. Dr. Oğuz YILMAZ
Head of the Department of
Mathematics

Assoc. Prof. Dr. Talat YALÇIN
Dean of the Graduate School of
Engineering and Sciences

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ABSTRACT

STRONGLY T-NONCOSINGULAR MODULES

This thesis is mainly concerned with the T-noncosingularity issue of a module. Derya Keskin Tütüncü and Rachid Tribak introduced the T-noncosingular modules and gave some properties of these modules. A module M is said to be T-noncosingular relative to N if, for every nonzero homomorphism f from M to N, the image of f is not small in N. Inspired by this study, we define a new kind of module, as a particular case of T-noncosingular modules, and call it strongly T-noncosingular modules. We define M to be strongly T-noncosingular relative to N if, for every nonzero homomorphism f from M to N, the image of f is not contained in the radical of N. Obviously, if a module is strongly T-noncosingular, then it is also T-noncosingular, but the converse is, in general, not true. In an attempt to identify the situation when a T-noncosingular module is strongly T-noncosingular, we give necessary and sufficient conditions in terms of the specific ring structures as well as well-known module types.

ÖZET

GÜÇLÜ T-EŞ TEKİL OLMAYAN MODÜLLER

Bu tez esas olarak modüllerde T-eş tekil olmama problemi ile ilgilidir. Derya Keskin Tütüncü ve Rachid Tribak T-eş tekil olmayan modülleri tanımladılar ve bazı özelliklerini verdiler.Bir M modülü ve M den N ye her sıfırdan farklı f homomorfizması için, f nin görüntüsü N de küçük alt modül değilse, bu M modülüne N modülüne göre T-eş tekil olmayan modül denir. Biz de bu çalışmadan esinlenerek bir anlamda T-eş tekil olmayan modüllerin özel bir durumu olarak güçlü T-eş tekil olmayan modülleri tanımlıyoruz. Bir M modülünün başka bir N modülüne göre güçlü T-eş tekil olmayan modül olmasını şöyle tanımlıyoruz: M den N ye sıfırdan farklı her f homomorfizması için, f nin görüntüsünün N modülünün radikalinin içinde kapsanmamasıdır. Açıkça eğer bir modül güçlü T-eş tekil olmayan modül ise aynı zamanda T-eş tekil olmayan modüldür, ancak bu durumun tersi genel olarak doğru değildir. Bir T-eş tekil olmayan modülün ne zaman güçlü T-eş tekil olmayan modül olacağı durumunu saptamak için bilinen modül örneklerinin yanısıra özel halka yapılarını da göz önünde tutarak gerekli ve yeterli koşullar veriyoruz.

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SYMBOLS AND ABBREVIATIONS

R	an associative ring with unit unless otherwise stated
$R_{\mathfrak{p}}$	the localization of a commutative ring R at a prime ideal $\mathfrak{p}$ of R
$\mathbb{Z}$, $\mathbb{Z}^{+}$	the ring of integers, the set of all positive integers
$\mathbb{Q}$	the field of rational numbers
$\mathbb{Z}_{p^\infty}$	the Prüfer (divisible) group for the prime p (the p-primary part of the torsion group $\mathbb{Q}/\mathbb{Z}$)
R-$\mathcal{M}od$	the category of *left* R-modules
$\mathrm{Hom}_R(M, N)$	all R-module homomorphisms from M to N
$\mathrm{Ker}\, f$	the kernel of the map f
$\mathrm{Im}\, f$	the image of the map f
$T(M)$	the torsion submodule of the R-module M: $T(M) = \{m \in M \mid rm = 0$ for some $0 \neq r \in R\}$ when R is a commutative domain
$\mathrm{Soc}(M)$	the socle of the R-module M
$\mathrm{Rad}(M)$	the radical of the R-module M
$\subseteq$	submodule
$\ll$	small (=superfluous) submodule
$\trianglelefteq$	essential(=large) submodule
$End(M)$	the endomorphism ring of a module M
$\sigma[M]$	the full subcategory of the R-$\mathcal{M}od$ subgenerated by M

CHAPTER 1

INTRODUCTION

Singular and nonsingular modules and rings are of importance for developing certain generalizations. Singular modules are the generalization of the torsion modules. There are many approaches to this concept related to diverse module and ring types. On the one hand cosingularity and noncosingularity became indispensable tools in module and ring theory. The rigorous study of small modules and essential modules established with the aid of singularity and cosingularity.

In this thesis, we deal with the strongly T-noncosingular modules. By a strongly T-noncosingular module M relative to a module N, we mean a module M such that for every nonzero homomorphism $f : M \longrightarrow N$, $\operatorname{Im} f \nsubseteq \operatorname{Rad}(N)$. Main results will be given in the last chapter.

Throughout this thesis all rings are associative and have an identity element. All modules are unitary left modules.

We chased the following order when we constituted our work:

In Chapter 2, we begin with the necessary notions and useful theorems, lemmas and propositions making up our thesis background.

In Chapter 3, the definitions and important properties of singular and cosingular modules are given. This chapter explains us well what singular and cosingular modules and rings are. We provide basic theorems stated about these modules.

Chapter 4 forms the main goal of our thesis. T-noncosingular and strongly T-noncosingular modules are studied. We characterize the torsion strongly T-noncosingular $\mathbb{Z}$-modules. This chapter mentions also that any direct sum of strongly T-noncosingular modules need not be a strongly T-noncosingular module with an example, we also give a necessary and sufficient condition when the direct sum of strongly T-noncosingular modules is strongly T-noncosingular. Finally, as our main theorem, we prove that, over a commutative noetherian ring R, the condition that every T-noncosingular R-module is strongly T-noncosingular R-module is equivalent to the condition that R is an artinian ring.

CHAPTER 2

PRELIMINARIES

This chapter of our thesis provides fundamental facts for us in module and ring theory to introduce our definitions and main theorems. For the proofs, we refer to the books being in references.

2.1. Noetherian, Artinian, Regular Rings and Some Basic Module-Theoretic Concepts

The ring R is said to satisfy the $descending\ chain\ condition\,(dcc)$ on left (right) ideals if every descending chain of left (right) ideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ becomes $stationary$ after a finite number of steps, i.e. for some $k \in \mathbb{N}$, we obtain

$$I_k = I_{k+1} = I_{k+2} = \ldots \tag{2.1}$$

The ring R is said to satisfy the $ascending\ chain\ condition\,(acc)$ on left (right) ideals if every ascending chain of left (right) ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ becomes $stationary$ after a finite number of steps, i.e. for some $k \in \mathbb{N}$, we have (2.1) again.

Definition 2.1 *A ring R is called left (right) $noetherian$ if R satisfies the ascending chain condition on left (right) ideals.*

Definition 2.2 *A ring R is called left (right) $artinian$ if R satisfies the descending chain condition on left (right) ideals.*

With the aid of modules, noetherian and artinian rings has equivalent definitions.

Definition 2.3 *A module M is a $noetherian$ module if every non-empty family of submodules of M has a maximal element.*

Definition 2.4 *A module M is called an $artinian$ module if every non-empty family of submodules of M has a minimal element.*

As we have mentioned before, with respect to the above definitions; a ring R is $left\ (right)\ noetherian$ if it is noetherian as a left (right) R-module and a ring R is $left\ (right)\ artinian$ if it is artinian as a left (right) R-module.

Theorem 2.1 ((Anderson & Fuller, 1992), 10.10.Proposition) *Let M be an R-module and $A \subseteq M$. Then the following are equivalent:*

1. *M is artinian.*

2. *A and M/A are artinian.*

Theorem 2.2 ((Anderson & Fuller, 1992), 10.9.Proposition) *Let M be an R-module and $A \subseteq M$. Then the following statements are equivalent:*

1. *M is noetherian.*

2. *A and M/A are noetherian.*

3. *Every submodule of M is finitely generated.*

An element a of the ring R is called $regular$ if there is an element $b \in R$ with $aba = a$.

We call a ring R $regular$ if every element $a \in R$ is regular.

Theorem 2.3 ((Lam, 1991), (4.23)Theorem) *For any ring R, the following are equivalent:*

1. *For any $a \in R$, there exists $x \in R$ such that $a = axa$.*

2. *Every principal left ideal of R is a direct summand of $_RR$.*

3. *Every principal left ideal of R is generated by an idempotent.*

4. *Every finitely generated left ideal of R is a direct summand of $_RR$.*

5. *Every finitely generated left ideal of R is generated by an idempotent.*

Lemma 2.1 ((Anderson & Fuller, 1992), Corollary 2.12) *Let M be a left R-module, and let I be an ideal of R contained in the annihilator of M. Then M can be considered to be an R/I-module naturally by defining $(r+I)m = rm$, where $r \in R$, $m \in M$, and a subgroup of M is an R-module iff it is an R/I-module. That is, the lattices of R-submodules and R/I-submodules coincide.*

Let R be an integral domain and M be an R-module. The submodule $T(M) = \{m \in M \mid rm = 0 \ for \ some \ 0 \neq r \in R\}$ is called the $torsion \ submodule$ of M. If $T(M) = M$, then M is said to be a $torsion \ module$, and if $T(M) = 0$, then M is said to be a $torsionfree$ module.

Since $\mathbb{Z}$-modules are exactly abelian groups, torsion $\mathbb{Z}$-modules are just torsion abelian groups. The following theorem is well-known from the theory of abelian groups:

Theorem 2.4 ((Fuchs, 1970), Theorem 8.4) *Let T be a torsion $\mathbb{Z}$-module. Then*

$$T = \bigoplus_{prime \ p} T_p$$

where $T_p = \{x \in T \mid p^n x = 0 \ for \ some \ n \in \mathbb{Z}^+\}$. The $\mathbb{Z}$-submodules T_p are called $p-components$ or $torsion \ components \ of \ T$.

2.2. Small and Essential Submodules

A submodule K of an R-module M is called *superfluous* or *small* in M, written $K \ll M$, if, for every submodule $L \subseteq M$, the equality $K + L = M$ implies $L = M$.

A submodule L of a module M is called *essential or large* in M, written $L \trianglelefteq M$, if for every submodule $U \subseteq M$, the equality $L \cap U = 0$ implies $U = 0$.

Proposition 2.1 ((Warfield & Goodearl, 1989), Proposition 3.21)

- *Let A, B and C be modules with $A \subseteq B \subseteq C$. Then $A \trianglelefteq C$ if and only if $A \trianglelefteq B$ and $B \trianglelefteq C$.*

- *Let A, B, C and D be submodules of a module C. If $A \trianglelefteq C$ and $B \trianglelefteq D$, then $A \cap B \trianglelefteq C \cap D$.*

- *Let A be a submodule of a module C and let $f : B \to C$ be a homomorphism. If $A \trianglelefteq C$, then $f^{-1}(A) \trianglelefteq B$.*

- *Let $\{A_i \mid i \in I\}$ and $\{B_i \mid i \in I\}$ be a collection of submodules of a module C. If the A_i are independent, that is the sum of the A_i is a direct sum, and for each A_i, $A_i \trianglelefteq B_i$, then the B_i are independent and $\bigoplus_{i\in I} A_i \trianglelefteq \bigoplus_{i\in I} B_i$.*

Lemma 2.2 ((Kasch, 1982), 5.1.3.Lemma)

- *If $A \subseteq B \subseteq M \subseteq N$ and $B \ll M$, then $A \ll N$*

- *If $A_i \ll M$, where $i = 1, 2, ..., n$, then $\sum_{i=1}^{n} A_i \ll M$*

- *If $A \ll M$ and $\varphi \in Hom(M, N)$, then $\varphi(A) \ll N$*

Lemma 2.3 ((Kasch, 1982), 5.1.4.Lemma) *If M is a left R-module, then for any $m \in M$, Rm is not small in M if and only if there is a maximal submodule $K \subseteq M$ with $m \notin K$.*

Lemma 2.4 ((Kasch, 1982), 5.1.6.Lemma) *Let $N \subseteq M$. Then we have $N \trianglelefteq M$ if and only if for every $0 \neq m \in M$, there is an element $r \in R$ such that $rm \neq 0$ and $rm \in N$.*

2.3. Injectivitiy, Divisibility, Essential Extensions and Injective Hull

A module I is $injective$ in case whenever there is given the solid part of a diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[dl, dashed, "h"] \\ & I & \end{tikzcd}$$

with exact row, there is a homomorphism h such that the whole diagram commutes; i.e. $hf = g$.

Theorem 2.5 ((Alizade & Pancar, 1999), Theorem 8.11.) *For a left R-module I, the following are equivalent:*

1. *I is injective.*

2. *The functor $Hom(.\,,\ I)$ is exact.*

3. *Every short exact sequence of the form*

$$0 \to I \to A \to B \to 0$$

is splitting.

Any $\mathbb{Z}$-module D is $divisible$ provided that $nD = D$ for all non-zero $n \in \mathbb{Z}$.

Lemma 2.5 ((Warfield & Goodearl, 1989), Proposition 4.2)

- *D is an injective $\mathbb{Z}$-module if and only if it is divisible.*
- *Every $\mathbb{Z}$-module is a submodule of a divisible module.*

Theorem 2.6 ((Warfield & Goodearl, 1989), Theorem 4.4) *Every module is a submodule of an injective module.*

Corollary 2.1 ((Warfield & Goodearl, 1989), Corollary 4.5) *A module M is injective if and only if it is a direct summand of every module that contains it.*

A $proper\ essential\ extension$ of a module N is any module M such that $N \trianglelefteq M$ while N is a proper submodule of M.

Proposition 2.2 ((Warfield & Goodearl, 1989), Proposition 4.6) *A module M is injective if and only if M has no proper essential extensions.*

Let C be a module and A a submodule of C. We say that A is $essentially\ closed$ in C provided A has no proper essential extensions within C, that is, the only submodule B of C for which $A \trianglelefteq B$ is A.

Proposition 2.3 ((Warfield & Goodearl, 1989), Proposition 4.7) *Let A be a submodule of an injective module E. Then A is injective if and only if A is essentially closed in E.*

An $injective\ envelope$ for a module A is any injective module which is an essential extension of A.

Theorem 2.7 ((Anderson & Fuller, 1992), 18.10.Theorem) *Every module has an injective envelope. It is unique to within isomorphism.*

Proposition 2.4 ((Anderson & Fuller, 1992), 18.12.Proposition) *Let M be an R-module and $E(M)$ be its injective hull. Then in the category of R-$\mathcal{M}od$:*

1. *M is injective if and only if $M = E(M)$.*

2. *If $M \trianglelefteq N$, then $E(M) = E(N)$.*

3. *If $M \subseteq Q$, with Q injective, then $Q = E(M) \oplus E'$ for some E'.*

4. *If $\bigoplus_{\alpha\in A} E(M_\alpha)$ is injective, then*

$$E(\bigoplus_{\alpha\in A} M_\alpha) = \bigoplus_{\alpha\in A} E(M_\alpha)$$

A module M is $simple$ if it has no non-trivial submodules.

Lemma 2.6 ((Büyükaşık, 2005), Lemma 1.6.4.) *For every non-zero module U, there exists a non-zero homomorphism $f : U \longrightarrow E$, where E is the injective hull of a simple module.*

Theorem 2.8 ((Matlis, 1960), Proposition 3) *Let M be a module over a commutative noetherian ring R. Then the following are equivalent:*

1. *M has the descending chain condition.*

2. *M is a submodule of $E_1 \oplus E_2 \oplus ... \oplus E_n$, where $E_i = E(R/M_i)$ with M_i a maximal ideal of R.*

2.4. Semisimple Modules

Let $(T_\alpha)_{\alpha\in A}$ be an indexed set of simple submodules of a module M. If M is the direct sum of this set, then

$$M = \bigoplus_{\alpha\in A} T_\alpha$$

is a $semisimple\ decomposition$ of M. A module M is said to be $semisimple$ in case it has a semisimple decomposition.

Proposition 2.5 ((Anderson & Fuller, 1992), 9.1.Proposition) *A left R-module T is simple if and only if $T \cong R/K$ for some maximal left ideal K of R.*

Theorem 2.9 ((Anderson & Fuller, 1992), 9.6.Theorem) *For a left R-module, the following statements are equivalent:*

1. *M is semisimple.*
2. *M is the sum of some set of simple submodules.*
3. *M is the sum of its simple submodules.*
4. *Every submodule of M is a direct summand.*

Proposition 2.6 ((Kasch, 1982), 8.2.2 Corollary) *For a ring R, the following are equivalent:*

1. *R is semisimple.*
2. *Every left R-module is semisimple.*

The following lemma is clear, we include it for completeness.

Lemma 2.7 *Let M be an R-module. If M/N is semisimple, then N is the intersection of some maximal submodules of M.*

Proof Let $\pi : M \to M/N$ be the canonical epimorphism. Since M/N is semisimple, $M/N = \bigoplus_{i\in I} S_i$, where S_i is simple for each $i \in I$. Let $M_i = \bigoplus_{i\neq j\in I} S_j$. Then, $\bigcap_{i\in I} M_i = 0_{M/N}$ and by using the second isomoprhism theorem M_i is maximal in M/N. We must show that the inverse images of the M_i for each $i \in I$ is maximal in M. Since M_i is maximal in M/N, it is of the form $M_i = X_i/N$, where $X_i \subseteq M$. By the third isomorphism theorem, $(M/N)/(X_i/N) \cong M/X_i$ is simple by the maximality of M_i. Therefore, X_i is maximal in M. Hence, $(\bigcap_{i\in I} X_i)/N = \bigcap_{i\in I} M_i = 0_{M/N}$. That is $\bigcap_{i\in I} X_i = N$. □

2.5. Radical, Socle and Projectivity

Let M be a left R-module. The *radical* of M is defined by

$$\begin{aligned} \mathrm{Rad}\,(M) &= \bigcap\{K \subseteq M \mid K \ is\, maximal \ in \ M\} \\ &= \sum\{L \subseteq M \mid L \ is \ small \ in \ M\} \end{aligned}$$

and the *socle* of M is defined by

$$\begin{aligned} \mathrm{Soc}\,(M) &= \sum\{K \subseteq M \mid K \ is \ minimal \ in \ M\} \\ &= \bigcap\{L \subseteq M \mid L \ is \ essential \ in \ M\} \end{aligned}$$

Lemma 2.8 ((Kasch, 1982), 9.1.3.Corollary)

- *For $m \in M$, we have : $Rm \ll M$ if and only if $m \in \mathrm{Rad}\,(M)$.*
- $\mathrm{Soc}\,(M)$ *is the largest semisimple submodule of M.*

Lemma 2.9 ((Kasch, 1982), 9.1.5.Corollaries)

1. *Let M be an R-module and $N \subseteq M$, then* $\mathrm{Rad}\,(N) \subseteq \mathrm{Rad}\,(M)$ *and* $\mathrm{Soc}\,(N) \subseteq \mathrm{Soc}\,(M)$.
2. *Let $M = \bigoplus_{i \in I} M_i$, then* $\mathrm{Rad}\,(M) = \bigoplus_{i \in I} \mathrm{Rad}\,(M_i)$.
3. *Let $M = \bigoplus_{i \in I} M_i$, then* $\mathrm{Soc}\,(M) = \bigoplus_{i \in I} \mathrm{Soc}\,(M_i)$.

Proposition 2.7 ((Anderson & Fuller, 1992), 9.14.Proposition) *Let M and N be two left R-modules and let $f : M \to N$ be a R-module homomorphism. Then* $f(\mathrm{Rad}\,(M)) \subseteq \mathrm{Rad}\,(N)$.

Lemma 2.10 *If* $\mathrm{Rad}\,(M) = M$, *then* $\mathrm{Rad}\,(M/U) = M/U$ *for any submodule U of M.*

Proof Let $f : M \to M/U$ be the natural epimorphism with $U \subseteq M$. Clearly, $\mathrm{Rad}\,(M/U) \subseteq M/U$. On the other hand, $f(\mathrm{Rad}\,(M)) = f(M) = M/U \subseteq \mathrm{Rad}\,(M/U)$ by Proposition 2.7. So we have the desired equality $\mathrm{Rad}\,(M/U) = M/U$.

□

Lemma 2.11 ((Kasch, 1982), 9.3.1.Lemma) *The following statements are equivalent for $A \subseteq_R R$:*

1. $A \ll_R R$.
2. $A \subseteq \mathrm{Rad}\,({}_R R)$.
3. *For every $a \in A$, $1 - a$ has a right inverse in R.*
4. *For every $a \in A$, $1 - a$ has an inverse in R.*

Theorem 2.10 ((Kasch, 1982), 9.3.2.Theorem) $\mathrm{Rad}\,(R_R) = \mathrm{Rad}\,({}_R R)$

Theorem 2.11 ((Kasch, 1982), 9.1.4.Theorem) $\mathrm{Rad}\,(M/\,\mathrm{Rad}\,M) = 0$ *and for every submodule C of a module M, if* $\mathrm{Rad}\,(M/C) = 0$*, then* $\mathrm{Rad}\,(M) \subseteq C$.

We denote the (Jacobson)radical of a ring R by $J(R)$.
The following lemma is well-known:

Lemma 2.12 *Let R be a commutative ring and Ω be the set of all maximal ideals of R. Then for an R-module M,* $\mathrm{Rad}\,(M) = \bigcap_{\mathfrak{p} \in \Omega} \mathfrak{p}M$.

Proof For a maximal ideal $\mathfrak{p}$, we can consider $M/\mathfrak{p}M$ as a module over $R/\mathfrak{p}$ by Lemma 2.1 because $\mathfrak{p} \subseteq \mathrm{Ann}^l\, M$. $M/\mathfrak{p}M$ is semisimple by Proposition 2.6, therefore $\mathrm{Rad}\,(M) \subseteq \mathfrak{p}M$ by Lemma 2.7. Then we obtain $\mathrm{Rad}(M) \subseteq \bigcap_{\mathfrak{p} \in \Omega} \mathfrak{p}M$.
Conversely, let $x \in M$ be such that $x \notin \mathrm{Rad}\,(M)$. By Lemma 2.8, Rx is not small in M and by Lemma 2.3, there is a maximal submodule K in M such that $x \notin K$. M/K is a simple module, so $\mathfrak{q}M \subseteq K$ for some $\mathfrak{q} \in \Omega$. Then we obtain $x \notin \mathfrak{q}M$, hence $x \notin \bigcap_{\mathfrak{p} \in \Omega} \mathfrak{p}M$, contradicting our assertion. □

Theorem 2.12 ((Kasch, 1982), 9.2.1 Lemma) *Let $M =_R M$, then we have:*

1. *If M is semisimple, then* $\mathrm{Rad}\,(M) = 0$.
2. $J(R)\,M \subseteq \mathrm{Rad}\,(M)$.
3. *$J(R)$ is a two-sided ideal of R.*
4. *If M is finitely generated, then* $\mathrm{Rad}\,(M) \ll M$*, in particular, $J(R) \ll R$.*
5. *Let M be a finitely generated and $A \subseteq J(R)$, then $AM \ll M$.*

A left R-module P is $projective$ if the given solid part of a diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\ B \arrow[r, "f"] & C \arrow[r] & 0 \end{tikzcd}$$

with exact row, there is a homomorphism h such that the whole diagram commutes, i.e. $fh = g$.

Theorem 2.13 ((Anderson & Fuller, 1992), 16.11.Corollary) *A direct sum $\bigoplus_{i \in I} P_i$ of modules P_i is projective iff each P_i is projective.*

Theorem 2.14 ((Wisbauer, 1991), 22.3) *Let P be a non-zero projective module. Then:*

1. *There are maximal submodules in P, i.e. $\operatorname{Rad}(P) \neq P$.*

2. *If $P = P_1 \oplus P_2$ with $P_2 \subseteq \operatorname{Rad}(P)$, then $P_2 = 0$.*

CHAPTER 3

SINGULAR AND COSINGULAR MODULES

In this chapter, we explain singular and cosingular modules in regard to our aims. Singularity problem began with the right singular ideal of a ring and is introduced by Johnson, R.E. in his paper (Johnson, 1951). Later on he introduced the singular submodule of a module in another paper (Johnson, 1957). Although the proofs are able to be found in the book “An Introduction to Noncommutative Noetherian Rings” (Warfield & Goodearl, 1989), we again give them here for the completeness of our study.

3.1. Singular and Nonsingular Modules

Let M be a left R-module. Consider the following set:

$$Z(M) = \{x \in M \mid Ix = 0 \; for\ some\ I \trianglelefteq_R R\} = \{x \in M \mid \operatorname{Ann}^l x \trianglelefteq_R R\}$$

Lemma 3.1 ((Warfield & Goodearl, 1989), Lemma 3.25.) *$Z(M)$ is a submodule of M.*

Proof Since R is an essential left ideal of itself, we get $0 \in Z(M)$. Given any x , $y \in Z(M)$, there are essential left ideals I , J in R such that $Ix = Jy = 0$. Since $I \cap J$ is an essential right ideal of R by Proposition 2.1 and $x \mp y \in Z(M)$. Now for any $t \in R$ and $x \in Z(M)$, we will show that $tx \in Z(M)$. Consider the left ideal $K = \{r \in R \mid rt \in I\}$ is essential by Lemma 2.4, and we have $Ktx \leq Ix = 0$, whence $tx \in Z(M)$. Thus $Z(M)$ is a submodule of M. □

Definition 3.1 *The submodule $Z(M)$ defined in Lemma 3.1 is called the $(maximal) singular$ submodule of M. If $Z(M) = M$ then M is said to be a $singular$ module, whereas if $Z(M) = 0$ then, M is said to be a $nonsingular$ module.*

For example, suppose R is a commutative domain. Then the essential ideals of R are exactly the nonzero ideals, and so the singular submodule of any R-module is just its torsion submodule. In this case the nonsingular R-modules are exactly the torsionfree R-modules.

There is an alternative definition for $Z(M)$ using trace but even if we regard this definition, the definition of singular and nonsingular modules is the same again with that of modules that we gave above:

$$Z(M) = Tr(\mathbf{A}, M) = \sum\{\operatorname{Im} f \mid f \in \operatorname{Hom}(A, M), A \in \mathbf{A}\} \tag{3.1}$$

where $\mathbf{A}$ is the class of all singular modules.

Proposition 3.1 ((Warfield & Goodearl, 1989), Proposition 3.26) *A module M is singular if and only if $M \cong K/L$ for some module K and some essential submodule L of K.*

Proof We may assume that M is a left module over a ring R. First suppose that $M \cong K/L$ for some left R-modules $L \trianglelefteq K$. Given any $k \in K$, the left ideal $I = \{r \in R \mid rk \in L\}$ is essential in R by Lemma 2.4, and $I(k + L) = 0_{K/L}$. Thus K/L, and hence M, is singular.

Conversely, assume that M is singular, and write $M \cong F/K$ for some free left R-module F and some submodule $K \subseteq F$. Choose a basis $\{x_j \mid j \in J\}$ for F. For each $j \in J$, there is an essential left ideal I_j in R such that $I_j x_j \subseteq K$, because F/K is singular. By Proposition 2.1, $\bigoplus I_j x_j \trianglelefteq \bigoplus R x_j = F$, and thus $K \trianglelefteq F$. □

Proposition 3.2 ((Warfield & Goodearl, 1989), Proposition 3.27) *Let N be a submodule of a nonsingular module M. Then M/N is singular if and only if $N \trianglelefteq M$.*

Proof We may assume that M is a left module over a ring R. If $N \trianglelefteq M$, then M/N is singular by Proposition 3.1. Conversely, assume that M/N is singular. Given a nonzero submodule $L \subseteq M$, choose a nonzero element $x \in L$, since M/N is singular, there is some $I \trianglelefteq R_R$ such that $Ix \leq N$. As M is nonsingular, $Ix \neq 0$, whence $N \cap L \neq 0$.

□

Proposition 3.3 *If $f : M \to N$ is a homomorphism of left R-modules, then $f(Z(M)) \subseteq Z(N)$.*

Proof Let $x \in Z(M)$. Then there exists an essential left ideal of R such that $Ix = 0$. That is for every $a \in I$, $ax = 0$, so $f(ax) = af(x) = 0$ for every $a \in I$, that is $I \subseteq \operatorname{Ann}^l(f(x))$. Since $I \trianglelefteq R$, by Proposition 2.1, $\operatorname{Ann}^l(f(x)) \trianglelefteq R$, so we get $f(x) \in Z(N)$, which was to be shown. $\square$

Corollary 3.1 *If N is a submodule of a module M, then $Z(N) \subseteq Z(M)$.*

Proof This is clear from Proposition 3.3. $\square$

Lemma 3.2 *If N is a submodule of a module M, then $Z(M) \cap N = Z(N)$.*

Proof By Corollary 3.1, $Z(N) \subseteq Z(M)$ and $Z(N) = Z(N) \cap N \subseteq Z(M) \cap N$. Conversely, let $x \in Z(M) \cap N$. Then there is an essential left ideal I such that $Ix = 0$, on the other hand, $x \in N$, so $x \in Z(N)$. $\square$

Proposition 3.4 ((Warfield & Goodearl, 1989), Proposition 3.28)

1. *All submodules and sums(direct or not) of singular modules are singular.*

2. *All submodules, direct products and essential extensions of nonsingular modules are nonsingular.*

3. *Let N be a submodule of a module M. If N and M/N are both nonsingular, then M is nonsingular.*

Proof

1. It is easily seen by definition that all submodules and factor modules of singular modules are singular. If $\{M_i \mid i \in I\}$ is a family of submodules of a module M, and each M_i is singular, then M_i is contained in $Z(M)$, whence $\sum_{i\in I} M_i \subseteq Z(M)$ and so $\sum_{i\in I} M_i$ is singular.

2. Obviously all submodules of nonsingular modules are nonsingular. Given a family $\{M_i \mid i \in I\}$ of modules, by Proposition 3.3, each of the projections $\prod_{i\in I} M_i \to M_j$ maps $Z(\prod_{i\in I} M_i)$ into $Z(M_j)$. Thus if each M_j is nonsingular, then $Z(\prod_{i\in I} M_i)$ is contained in the kernel of all the projections $\prod_{i\in I} M_i \to M_j$, whence $Z(\prod_{i\in I} M_i) = 0$.

If N is a submodule of a module M, then, by Lemma 3.2, $N \cap Z(M) = Z(N)$. Therefore if M is nonsingular, then $N \cap Z(M) = 0$, whence if $N \trianglelefteq M$ we deduce that $Z(M) = 0$.

3. Consider the canonical epimorphism $M \to M/N$. This epimorphism carries $Z(M)$ into $Z(M/N)$ by Proposition 3.3. Since $Z(M/N) = 0$, we have $Z(M) \subseteq N$, and since N is also nonsingular, it follows that $Z(M) = 0$

□

Generally speaking, the essential extensions of singular modules does not have to be singular. The subsequent example shows this case:

Example 3.1 *If $R = \mathbb{Z}/4\mathbb{Z}$ and $M = 2R$, then M is a singular R-module and R is an essential extension of M, but R is not a singular R-module (even though it is singular as a $\mathbb{Z}$-module). Also, R/M is a singular module.*

Definition 3.2 *The right singular ideal of a ring R is the ideal $Z_r(R) = Z(R_R)$, and the left singular ideal of R is the ideal $Z_l(R) = Z({}_RR)$.*

Definition 3.3 *A right(left) nonsingular ring is any ring whose right(left) singular ideal is zero. Of course, a nonsingular ring is a ring which is both right and left nonsingular.*

For instance, every domain is a nonsingular ring. Also, every semisimple ring R is nonsingular. (Since R has no proper essential one-sided ideals, all R-modules are nonsingular).

Proposition 3.5 ((Warfield & Goodearl, 1989), Proposition 3.29) *Let R be a left nonsingular ring. Then,*

1. *For every left R-module M, the factor module $M/Z(M)$ is nonsingular.*
2. *If N is a submodule of a left R-module M such that N and M/N are both singular, then M is singular.*
3. *All essential extensions of singular left R-modules are singular.*

Proof

1. Let $N/Z(M) = Z(M/Z(M))$. We first claim that $Z(M) \trianglelefteq N$. If K is a submodule of N such that $Z(M) \cap K = 0$, then K is nonsingular. On the other hand, K embeds

in the singular module $N/Z(M)$, whence K is singular. Consequently $K = 0$, which proves that $Z(M) \trianglelefteq N$.

Now consider any $x \in N$, and set $I = \operatorname{Ann}^l x$. If J is any left ideal of R for which $I \cap J = 0$, then $J \cong Jx$. Since $Z(Jx) = Jx \cap Z(M) \trianglelefteq Jx \cap N = Jx$, we see that $Z(J) \trianglelefteq J$. However, $Z(J) = 0$ because ${}_RR$ is nonsingular, whence $J = 0$. Thus $I \trianglelefteq_R R$, and so $x \in Z(M)$.

Therefore $N/Z(M) = 0$, and hence $M/Z(M)$ is nonsingular.

2. Since N is singular, $N \subseteq Z(M)$, whence M/N maps onto $M/Z(M)$, and so $M/Z(M)$ is singular. On the other hand, $M/Z(M)$ is nonsingular by (1), and hence $M/Z(M) = 0$. Thus M is singular.

3. Let N be an essential submodule of a left R-module M, and suppose that N is singular. By Proposition 3.1, M/N is singular, and so (2) shows that M is singular. □

We finish singular modules here, for further information we refer to the book "An Introduction to Noncommutative Noetherian Rings" as we stated in the beginning of this chapter and the papers of Johnson, R.E. [(Johnson, 1951) and (Johnson, 1957)] in references.

3.2. Cosingular and Noncosingular Modules

Dual to the notion of singular submodule of a module M, $\overline{Z}(M)$ is defined by Talebi and Vanaja in the paper (Talebi & Vanaja, 2002) as follows:

$$\overline{Z}(M) = Rej(M, \mathbf{S}) = \bigcap \{\operatorname{Ker} f \mid f \in \operatorname{Hom}(M, K), K \in \mathbf{S}\} \tag{3.2}$$

where S denotes the class of all small modules.

Definition 3.4 *A module M is called $cosingular$ if $\overline{Z}(M) = 0$ and is called $noncosingular$ if $\overline{Z}(M) = M$.*

Actually, the definition 3.2 considers the category R-$\mathcal{M}od$, while Talebi and Vanaja work in the full subcategory $\sigma[M]$ of R-$\mathcal{M}od$ subgenerated by M. But we will pertain to cosingular modules in R-$\mathcal{M}od$.

Now we are going to resume basic properties of cosingular modules that can be proven readily.

Proposition 3.6 ((Talebi & Vanaja, 2002), Proposition 2.1) *Let M and N be R-modules, and $\{M_i \mid i \in I\}$ be a collection of modules. Then we have the following:*

1. *If $M \subseteq N$, then $\overline{Z}(M) \subseteq \overline{Z}(N)$ and $\overline{Z}(N/M) \supseteq (\overline{Z}(N) + M)/M$.*
2. *If $f : M \to N$ is a homomorphism, then $f(\overline{Z}(M)) \subseteq \overline{Z}(N)$.*
3. *$\overline{Z}(M/\overline{Z}(M)) = 0$.*
4. *$\overline{Z}(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} \overline{Z}(M_i)$.*
5. *$\overline{Z}(\prod_{i\in I} M_i) \subseteq \prod_{i\in I} \overline{Z}(M_i)$.*

Corollary 3.2 ((Talebi & Vanaja, 2002), Corollary 2.2) *For any ring R, the class of all cosingular R-modules is closed under submodules, direct products and direct sums.*

CHAPTER 4

STRONGLY T-NONCOSINGULAR MODULES

In this final chapter, initially we will give the definition of T-noncosingular module introduced by Tütüncü, D.K. and Tribak, R. within the paper "On T-noncosingular Modules" and then we mention some significant properties of this kind of a module. This module is origin of the "Strongly T-noncosingular module" introduced by us in this chapter .

4.1. T-noncosingular Modules

In this section we will give the definition and basic properties of T-noncosingular modules defined by Tütüncü, D.K. and Tribak, R. (Tütüncü & Tribak 2009)).

Definition 4.1 *Let M be an R-module. M is called T-noncosingular relative to N if, for every nonzero homomorphism $f : M \to N$, $\operatorname{Im} f$ is not small in N. If M is T-noncosingular relative to M, we say that M is T-noncosingular .The ring R is said to be right(left) T-noncosingular if the right(left) R-module R is T-noncosingular.*

In previous section, we gave the following set defined by Talebi and Vanaja :

$$\overline{Z}(M) = Rej(M, \mathbf{S}) = \bigcap \{\operatorname{Ker} f \mid f \in \operatorname{Hom}(M, K), K \in \mathbf{S}\}$$

As in (Tütüncü & Tribak 2009)), consider the set $\nabla(M) = \{f \in End(M) \mid \operatorname{Im} f \ll M\}$. Observe that $\nabla(M)$ is an ideal of the endomorphism ring of M. With the help of this notation, the $T - noncosingular submodule$ of M is defined by:

$$\overline{Z}_T(M) = \bigcap_{\varphi \in \nabla(M)} \operatorname{Ker} \varphi \tag{4.1}$$

Proposition 4.1 ((Tütüncü & Tribak 2009)), Proposition 2.2) *Let M be an R-module. Then we have:*

1. *M is T-noncosingular if and only if $\overline{Z}_T(M) = M$,*

2. *$\overline{Z}_T(M)$ is a fully invariant submodule of M; moreover, $\overline{Z}(M) \subseteq \overline{Z}_T(M)$,*

3. *If $M = \bigoplus_{i\in I} M_i$, then $\overline{Z}_T(M) \subseteq \bigoplus_{i\in I} \overline{Z}_T(M_i)$.*

Proposition 4.2 ((Tütüncü & Tribak 2009)), Proposition 2.3) *Let M be a T-noncosingular module and let N be a direct summand of M. Then N is T-noncosingular.*

In general, a direct sum of T-noncosingular modules is not a T-noncosingular module, as the following example shows. A dedekind domain R is $proper$ if it is not a field. $R(P^\infty)$ will denote the P-primary component of the torsion R-module K/R, where K is the quotient field of R.

Example 4.1 ((Tütüncü & Tribak 2009)), Example 2.12) *Let R be a proper Dedekind domain. Let P be any nonzero prime ideal of R. Consider the module $M = R(P^\infty) \oplus R/P$ and the endomorphism $f : M \to M$ defined by $f(x+\overline{y}) = cy$ with $x \in R(P^\infty)$, $y \in R$ and c is a nonzero element of $R(P^\infty)$ such that $cP = 0$. It is clear that $\operatorname{Im} f = cR$ which is nonzero and small in M. So M is not a T-noncosingular module. In particular, for any prime integer p, the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}/p\mathbb{Z}$ is not a T-noncosingular $\mathbb{Z}$-module.*

Proposition 4.3 ((Tütüncü & Tribak 2009)), Proposition 2.11) *Let $\{M_i\}_{i\in I}$ be a family of modules. Then $M = \bigoplus_{i\in I} M_i$ is a T-noncosingular module if and only if M_i is a T-noncosingular module relative to M_j for all $i,\ j \in I$.*

4.2. Strongly T-noncosingular Modules

Definition 4.2 *Let M and N be two R-modules. We call M strongly T-noncosingular relative to N if, for every nonzero homomorphism $f : M \to N$, $\operatorname{Im} f \nsubseteq \operatorname{Rad}(N)$. If M is strongly T-noncosingular relative to M, we call M strongly T-noncosingular.*

Motivated by T-noncosingular modules we define the following set :

$$\nabla_{rad}(M) = \{f \in End(M) \mid \operatorname{Im} f \subseteq \operatorname{Rad}(M)\}$$

and $\nabla_{rad}(M)$ is an ideal of $End(M)$ as $\nabla(M)$ is.

Proposition 4.4 *If M is a strongly T-noncosingular module and N is a direct summand of M, i.e. $M = N \oplus K$ for some $K \subseteq M$, then N is a strongly T-noncosingular module as well.*

Proof Let $M = N \oplus K$ and a homomorphism $f : N \to N$ for which $\operatorname{Im} f \subseteq \operatorname{Rad}(N)$. Let us look at the following homomorphism

$$f \oplus 0_K : N \oplus K \to N \oplus K$$

by $f \oplus 0_K(n + k) = f(n)$, where $n \in N$, $k \in K$. This is an endomorphism of M. From this we have $(f \oplus 0_K)(N \oplus K) = f(N) \subseteq \operatorname{Rad}(M)$, but by assumption that M is a strongly T-noncosingular, we obtain $f \oplus 0_K = 0$ and thus $f = 0$. This is what we wish to prove. □

Proposition 4.5 *If M is strongly T-noncosingular module, then M is also T-noncosingular module.*

Proof By definitions $\nabla_{rad}(M) = \{f \in End(M) : \operatorname{Im} f \subseteq \operatorname{Rad}(M)\}$, $\nabla(M) = \{f \in End(M) : \operatorname{Im} f \ll M\}$ we immediately get $\nabla(M) \subseteq \nabla_{rad}(M)$, but since M is a strongly T-noncosingular, then $\nabla_{rad}(M) = 0$ and hence $\nabla(M) = 0$, consequently M is a strongly T-noncosingular module implies that M is a T-noncosingular module.

□

Proposition 4.6 *For a module M with $\operatorname{Rad}(M) \ll M$, the following are equivalent:*

1. *M is strongly T-noncosingular.*
2. *M is T-noncosingular.*

Proof $(1 \Rightarrow 2)$: This obviously follows from Proposition 4.5.

$(2 \Rightarrow 1)$: Let M be a T-noncosingular module and let $f \in End(M)$ with $\operatorname{Im} f \subseteq \operatorname{Rad}(M)$, since $\operatorname{Rad}(M) \ll M$, it follows that $\operatorname{Im} f \subseteq \operatorname{Rad}(M) \ll M$, this implies that $f = 0$ since M is a T-noncosingular module. Thus M is strongly T-noncosingular module.

□

We call an R-module M π-$projective$ (or co-$continuous$) if, for every two submodules U, V of M with $U + V = M$, there exists $f \in End(M)$ with

$$\operatorname{Im} f \subseteq U \ \ and \ \ \operatorname{Im}(1 - f) \subseteq V$$

By this definition we obtain the following proposition:

Proposition 4.7 *Let M be a π-projective module, and suppose that M is a strongly T-noncosingular module. Then* $\operatorname{Rad}(M) \ll M$.

Proof We wish to show that $K + \operatorname{Rad}(M) = M$ implies that $K = M$. By $\pi - projectivity$ of M, there exists an $\alpha \in End(M)$ such that $\alpha(M) \subseteq K$ and $(1-\alpha)(M) \subseteq \operatorname{Rad}(M)$, since $\nabla_{rad}(M) = 0$, obtaining $1 - \alpha = 0$ and hence $\alpha(M) = M \subseteq K$, but this forces $K = M$.

□

There is a natural question as to whether any direct sum of strongly T-noncosingular modules is again a strongly T-noncosingular module or not. The answer is no generally and the following example shows that any direct sum of strongly T-noncosingular modules does not have to be strongly T-noncosingular module.

Example 4.2 *Let*

$$M = \sum_{prime\ p} \frac{1}{p}\mathbb{Z}$$

Then we have

$$M/\mathbb{Z} = (\sum_{prime\ p} \frac{1}{p}\mathbb{Z})/\mathbb{Z} = \sum_{prime\ p} (\frac{1}{p} + \mathbb{Z})/\mathbb{Z} = \sum_{prime\ p} \mathbb{Z}/p\mathbb{Z}$$

That is $M/\mathbb{Z}$ is semisimple. Thus $\mathrm{Rad}\,(M/\mathbb{Z}) = 0$ *by Theorem 2.12,* $\mathrm{Rad}\,(M) \subseteq \mathbb{Z}$ *by Theorem 2.11. On the other hand, for any prime q, $qM \supseteq \mathbb{Z}$. Now by Lemma 2.12,* $\mathrm{Rad}\,(M) \supseteq \mathbb{Z}$ *and so we have* $\mathrm{Rad}\,(M) = \mathbb{Z}$*. Let $f : M \longrightarrow M$ be an endomorphism of M with* $\mathrm{Im}\, f \subseteq \mathrm{Rad}\,(M) = \mathbb{Z}$*.*
Now $f(1) = f(\frac{p}{p}) = pf(\frac{1}{p})$, so that $p \mid f(1)$, where $f(1) \in \mathbb{Z}$, for every prime p. Thus

$$f(1) \in \bigcap_{prime\, p} p\mathbb{Z} = 0$$

and thus f(1)=0. From this $f(1) = pf(\frac{1}{p}) = 0$, so that $f(\frac{1}{p}) = 0$ for every prime p since $\mathbb{Z}$ is an integral domain and $p \neq 0$. Now pick any element $m \in M$, which is of the form

$$m = \frac{a_1}{p_1} + \frac{a_2}{p_2} + \ldots + \frac{a_n}{p_n}$$

then

$$f(m) = a_1 f(\frac{1}{p_1}) + \ldots + a_n f(\frac{1}{p_n}) = 0$$

therefore we have f=0. So M is a strongly T-noncosingular module.
After this identification, we will be able to form the diagram below to obtain an endomorphism of $M \oplus \mathbb{Z}$,

$$M \oplus \mathbb{Z} \xrightarrow{\pi_{\mathbb{Z}}} \mathbb{Z} \overset{\imath_1}{\hookrightarrow} M \overset{\imath_2}{\hookrightarrow} M \oplus \mathbb{Z},$$

where $\pi_{\mathbb{Z}} : M \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$ by $(x, y) \mapsto y$, $\imath_1 : \mathbb{Z} \longrightarrow M$ by $y \mapsto y$,

$\imath_2 : M \longrightarrow M \oplus \mathbb{Z}$ *by* $y \mapsto (y, 0)$.

Letting $\varphi = \imath_2 \imath_1 \pi_{\mathbb{Z}}$*, we obtain an endomorphism of* $M \oplus \mathbb{Z}$ *i.e.*

$\varphi : M \oplus \mathbb{Z} \longrightarrow M \oplus \mathbb{Z}$

$\varphi(x, y) \longrightarrow (y, 0)$. *But on the other hand, by Lemma 2.9,* $\mathrm{Rad}\,(M \oplus \mathbb{Z}) = \mathrm{Rad}\,(M) \oplus \mathrm{Rad}\,(\mathbb{Z}) = \mathbb{Z} \oplus 0$ *and so* $\mathrm{Im}\,\varphi \subseteq \mathrm{Rad}\,(M \oplus \mathbb{Z})$. *Hence we have found an endomorphism of* $M \oplus \mathbb{Z}$ *being different than zero with* $\mathrm{Im}\,\varphi \subseteq \ Rad(M \oplus \mathbb{Z})$*,* $M \oplus \mathbb{Z}$ *is not a strongly T-noncosingular module though M and* $\mathbb{Z}$ *are so.*

Lemma 4.1 *Let* $\{M_i\}_{i \in I}$ *be a family of modules.Then* $M = \oplus_{i \in I} M_i$ *is a strongly T-noncosingular module if and only if* M_i *is a strongly T-noncosingular module relative to* M_j *for all* $i, j \in I$

Proof (⇒) Let $(i, j) \in I \times I$ and $f \in Hom(M_i, M_j)$ with $\mathrm{Im}\, f \subseteq \mathrm{Rad}\,(M_j)$. Consider the homomorphism $\phi : M_i \oplus M_j \longrightarrow M_i \oplus M_j$ defined by $\phi(x_i + x_j) = f(x_i)$, where $x_i \in M_i$, $x_j \in M_j$, so $\mathrm{Im}\,\phi = f(M_i) \subseteq \mathrm{Rad}\,(M_i \oplus M_j)$, now by Proposition 4.4, $M_i \oplus M_j$ is a strongly T-noncosingular module since it is a direct summand of M, but our hypothesis says that $\phi = 0$, so $f = 0$, concluding this direction.

(⇐) Let $f \in End(M)$ with $\mathrm{Im}\, f \subseteq \mathrm{Rad}\,(M)$. We have homomorphisms $\pi_i : M \to M_i$ (the i^{th} projections) and $\varphi_i : M_i \to M$ (the inclusion mappings).

Consider the following diagram

$$M \xrightarrow{\pi_i} M_i \xrightarrow{\varphi_i} M \xrightarrow{f} M \xrightarrow{\pi_j} M_j$$

We observe that $\mathrm{Im}\,\pi_j f \varphi_i \subseteq \mathrm{Im}\, f\varphi_i \subseteq \mathrm{Im}\, f \subseteq \mathrm{Rad}\,(M_j)$ and from this $\pi_j f \varphi_i = 0$ because of $\pi_j f \varphi_i \in Hom(M_i, M_j)$ and our assumption. Let $x \in M$, then consider $f(\varphi_i(\pi_i(x)))$ by our diagram:

$$M \xrightarrow{\pi_i} M_i \xrightarrow{\varphi_i} M \xrightarrow{f} M \xrightarrow{\pi_j} M_j$$

$f(\varphi_i(\pi_i(x))) = f(\varphi_i(x_i)) = f(x_i)$, where $x_i \in M_i$, from this composition we obtain $\pi_j(f(\varphi_i(\pi_i(x)))) = 0$ for all $i, j \in I$ since $\pi_j f \varphi_i = 0$, now passing to the sums, (which are, of course, finite)

$$\begin{aligned}
\sum_{j\in J}\pi_j f(x) &= \sum_{j\in J}\pi_j(f(\sum_{i\in I}x_i))\\
&= \sum_{j\in J}\sum_{i\in I}\pi_j(f(x_i))\\
&= \sum_{i\in I}\sum_{j\in J}\pi_j(f(x_i))\\
&= \sum_{i\in I}\sum_{j\in J}\pi_j(f(\varphi_i(\pi_i(x))) = 0
\end{aligned}$$

for all $j \in J$. Therefore we get $f(x) = 0$ for any $x \in M$ and thus $f = 0$. Hence M is a strongly T-noncosingular module. □

Proposition 4.8 ((Tütüncü & Tribak 2009)), Proposition 2.14) *Let M be a T-noncosingular module. If $N \subseteq X$, $X/N \ll M/N$ and N is direct summand of M, then N is unique.*

Lemma 4.2 *Let M be a strongly T-noncosingular module,if $N \subseteq X$ is a direct summand of M with $X/N \subseteq \mathrm{Rad}\,(M/N)$, then N is unique. That is, if $K \subseteq X$ is a direct summand in M such that $X/K \subseteq \mathrm{Rad}\,(M/K)$, then $K = N$.*

Proof Suppose that $X/N_1 \subseteq \mathrm{Rad}\,(M/N_1)$, $X/N_2 \subseteq \mathrm{Rad}\,(M/N_2)$, $M = N_1 \oplus P_1 = N_2 \oplus P_2$ and $N_1 \neq N_2$, i.e. $N_1 \nsubseteq N_2$ or $N_2 \nsubseteq N_1$. Without loss of generality, we can assume that $N_1 \nsubseteq N_2$. We try to make up a non-zero endomorphism φ of M by projections $\pi_{N_1} : M \longrightarrow N_1$ and $\pi_{P_2} : M \longrightarrow P_2$ as follows:

$$M \xrightarrow{\pi_{N_1}} N_1 \xrightarrow{\imath_1} M \xrightarrow{\pi_{P_2}} P_2 \xrightarrow{\imath_2} M$$

If we let $\varphi' = \pi_{P_2}\imath_1\pi_{N_1}$, then $\mathrm{Im}\,\varphi = \mathrm{Im}\,\varphi'$. Now let us see what happens to M under φ'.(φ' is nonzero, or else $\pi_{N_1} = N_1 \subseteq \mathrm{Ker}\,\pi_{P_2} = N_2$, contradicting our assertion). $\varphi'(M) = \varphi'(N_1 \oplus P_1) = \pi_{P_2}\pi_{N_1}(N_1 \oplus P_1) = \pi_{P_2}(N_1) = (N_1 + N_2) \cap P_2$. To see this, let $n_1' = n_2' + p_2'$. Then $p_2' = n_1' - n_2' \in (N_1 + N_2) \cap P_2$, and by projection $\pi_{P_2}(n_2' + p_2') = p_2' = n_1' - n_2' \in (N_1 + N_2) \cap P_2$.

We also have $\mathrm{Im}\,\varphi = \mathrm{Im}\,\varphi' = (N_1 + N_2) \cap P_2 \subseteq X \cap P_2$. Since $M = N_2 \oplus P_2$, we have $X \cap M = X = N_2 \oplus (X \cap P_2)$. Since also $X/N_2 \subseteq \mathrm{Rad}\,(M/N_2)$, it follows that $X \cap P_2 \subseteq \mathrm{Rad}\,(P_2)$ with the help of isomorphisms. From these, $\mathrm{Im}\,\varphi \subseteq \mathrm{Rad}\,(P_2) \subseteq \mathrm{Rad}\,(M)$ implies that $\varphi = 0$ since M is a strongly T-noncosingular module, hence we have obtained a contradiction with assumption above pertaining to φ that φ is non-zero. Consequently $N_1 = N_2$.

□

A submodule N of a module M is said to be a $fully\ invariant$ if, for every endomorphism $f \in End(M)$, $f(N) \subseteq N$.

Proposition 4.9 ((Tütüncü & Tribak 2009)), Proposition 2.16) *Let M be a T-noncosingular module and X fully invariant in M. Let $N \subseteq X$ such that $X/N \ll M/N$ and N a direct summand of M. Then N is unique fully invariant submodule in M*

Motivated by Proposition 4.9, we have the following lemma for strongly T-noncosingular modules:

Lemma 4.3 *Let M be a strongly T-noncosingular module and X be a fully invariant in M. Let $N \subseteq X$ such that $X/N \subseteq \mathrm{Rad}\,(M/N)$ and N be a direct summand of M. Then N is unique fully invariant submodule in M.*

Proof By Lemma 4.2, such a submodule N is unique, so that we have to show that N is fully invariant submodule of M. Suppose for contradiction that N is not a fully invariant submodule of M. Then there exists an endomorphism $f \in End(M)$ and an element $x \in N$ with $f(x) \notin N$. Since N is a direct summand in M, there are projections π_P and π_N subject to the decomposition $M = N \oplus P$, $\pi_P : M \longrightarrow P$, $\pi_N : M \longrightarrow N$, look at the following diagram

$$M \xrightarrow{\pi_N} N \xrightarrow{f} M \xrightarrow{\pi_P} P$$

If we put $\alpha = \pi_P f \pi_N$, then α is non-zero. Indeed if we assume contrary that $\alpha = 0$, then, for $m = x+p$, where $x \in N$, $p \in P$, we have $\pi_P f \pi_N(x+p) = \pi_P f(x) = f(x) = 0 \in N$, a contradiction with $f(x) \notin N$. ($f(x) \in P$ since $f(x) \notin N$ and $M = N \oplus P$).

Now let us identify $\mathrm{Im}\,\alpha$, i.e. $\mathrm{Im}\,\pi_P f \pi_N$, for this $\pi_P f \pi_N(M) = \pi_P f(N) \subseteq \pi_P f(X) \subseteq \pi_P(X)$ since X is a fully invariant submodule of M. Because $M = N \oplus P$, $X \cap M = X = X \cap (N \oplus P) = N \oplus (X \cap P)$ by modular law. From this $\pi_P(X) = X \cap P$ and $X/N \cong X \cap P \subseteq \mathrm{Rad}\,(M/N) \cong \mathrm{Rad}\,(P) \subseteq \mathrm{Rad}\,(M)$, combining these together to obtain $\operatorname{Im}\alpha \subseteq \operatorname{Rad}(M)$, but on the other hand M is a strongly T-noncosingular module, this forces $\alpha = 0$, a contradiction. Consequently, N is fully invariant in M. □

Theorem 4.1 *For any torsion $\mathbb{Z}$-module T, the following are equivalent:*

1. *T is a strongly T-noncosingular module.*
2. *T is a semisimple module.*

Proof $(2 \Rightarrow 1)$: This is obvious because of $\operatorname{Rad}(T) = 0$ by Theorem 2.12.
$(1 \Rightarrow 2)$: Since T is torsion $\mathbb{Z}$-module, by Theorem 2.4, there is a decomposition

$$T = \bigoplus_{prime\, p} T_p$$

where T_p's are torsion components of T, that is to say,

$$T_p = \{x \in T \mid p^n x = 0 \; for \;\; some \;\; n \in \mathbb{Z}^+\}$$

Let $x \in T_p$ for a prime number q, consider qT_p, if $q \neq p$, then, $(q,p) = 1$ and we get $(q, p^n) = 1$, so there exists $u, v \in \mathbb{Z}$ such that $qu + p^n v = 1$, multiplying by both side by x, we have $xqu + p^n xv = x$, implying $x = xqu \in qT_p$, thus $T_p = qT_p$. If $q = p$, then $qT_p = pT_p$. Now we have

$$\operatorname{Rad}(T_p) = \bigcap_{prime\, q} qT_p = pT_p$$

Therefore, there is an endomorphism of T_p:

$$f : T_p \to T_p$$

$$x \to px \; ,$$

where $px \in pT_p$, so that $\operatorname{Im} f = pT_p = \operatorname{Rad}(T_p)$, but according to our assumption and Proposition 4.4, T_p is also strongly T-noncosingular module, thus we have

$$\operatorname{Im} f = pT_P = 0$$

We see easily that $p\mathbb{Z}T_p = 0$ since $pT_p = 0$; therefore T_p is also $\mathbb{Z}/p\mathbb{Z}$-module by Lemma 2.1. Since $\mathbb{Z}/p\mathbb{Z}$ is a field, and thus simple ring T_p is a semisimple module by Proposition 2.6 . We are through. (In fact, T_p is a $\mathbb{Z}/p\mathbb{Z}$-vector space) □

Proposition 4.10 *Let M be a module over a commutative ring R with unique maximal submodule, then M is strongly T-noncosingular module if and only if M is simple module*

Proof ($\Leftarrow$): This is clear since $\operatorname{Rad}(M) = 0$ by Theorem 2.11.
($\Rightarrow$): Since M has a unique maximal submodule, this unique maximal submodule is $\operatorname{Rad}(M)$ itself, since also $M/\operatorname{Rad}(M)$ is simple and so is cyclic, we get $M/\operatorname{Rad}(M) \cong R/P$ for some maximal ideal P of R by Proposition 2.5. Because of $P(R/P) = (PR + P)/P = 0$, $P(R/P) = 0 = P(M/\operatorname{Rad}(M)) = (PM + \operatorname{Rad}(M))/\operatorname{Rad}(M)$. This implies

$$PM \subseteq \operatorname{Rad}(M) \tag{4.2}$$

Since P is contained in the annihilator of M/PM, M/PM is also an R/P-module by Lemma 2.1. Because R/P is simple, M/PM is a semisimple R/P-module by Proposition 2.6. Since M/PM is semisimple, it follows that PM is the intersection of some maximal submodules of M by Lemma 2.7. We thus obtain

$$\operatorname{Rad}(M) \subseteq PM \tag{4.3}$$

From (4.2) and (4.3), $PM = \operatorname{Rad}(M)$.

Let now $f : M \longrightarrow M$ be an endomorphism of M defined by $f(m) = rm$, where $r \in P$. Then $\operatorname{Im} f = rM \subseteq PM$, that is $\operatorname{Im} f \subseteq \operatorname{Rad}(M)$. On the other hand, by assumption that M is strongly T-noncosingular module, we must have $f = 0$, and

$rM = 0$ for any $r \in P$, this means that $PM = 0 = \operatorname{Rad}(M)$, now returning to the beginning of the proof, we have $M/\operatorname{Rad}(M) \cong M$ is simple as desired.

$\square$

Proposition 4.11 *Let M be a left R-module such that $End(M)$ is Von Neumann regular and let $P(M) = \{N \subseteq M \mid \operatorname{Rad}(N) = N\}$. If $P(M) = 0$, then M is a strongly T-noncosingular module.*

Proof Let $f \in End(M)$ with $\operatorname{Im} f \subseteq \operatorname{Rad}(M)$. Since $End(M)$ is regular, there exists an endomorphism $g \in End(M)$ such that $f = fgf$, so fg is an idempotent and the decomposition $M = \operatorname{Im} fg \oplus \operatorname{Ker} fg$ exists by Theorem 2.3. From $\operatorname{Im} fg \subseteq \operatorname{Im} f \subseteq \operatorname{Rad}(M)$ and applying modular law to $\operatorname{Rad}(M) = \operatorname{Rad}(\operatorname{Im} fg \oplus \operatorname{Ker} fg) = \operatorname{Rad}(\operatorname{Im} fg) \oplus \operatorname{Rad}(\operatorname{Ker} fg)$, we obtain, $\operatorname{Im} fg \cap \operatorname{Rad}(M) = \operatorname{Im} fg = \operatorname{Rad}(\operatorname{Im} fg) \oplus (\operatorname{Im} fg \cap \operatorname{Rad}(\operatorname{Ker} fg))$ and so, $\operatorname{Im} fg = \operatorname{Rad}(\operatorname{Im} fg)$. Therefore $\operatorname{Im} fg \in P(M) = 0$, so $\operatorname{Im} fg = 0$ from this, $fg = 0$, and thus $f = 0$, but this says that M is a strongly T-noncosingular module. $\square$

Corollary 4.1 *Every non-zero projective module whose endomorphism ring is Von Neumann regular is a strongly T-noncosingular module.*

Proof Let $f \in End(P)$ with $\operatorname{Im} f \subseteq \operatorname{Rad}(P)$. By the process exploited in Proposition 4.11, we have a direct decomposition $P = \operatorname{Im} fg \oplus \operatorname{Ker} fg$ with respect to the idempotent fg, and $\operatorname{Rad}(\operatorname{Im} fg) = \operatorname{Im} fg$. On the other hand, both direct summand are projective since P is projective by Theorem 2.13. Now by Theorem 2.14, $\operatorname{Im} fg = 0$ since $\operatorname{Im} fg \subseteq \operatorname{Im} f \subseteq \operatorname{Rad}(P)$ and so $f = fgf = 0$; therefore P is a strongly T-noncosingular module.

$\square$

Proposition 4.12 *Any regular module is a strongly T-noncosingular module.*

Proof Let M be a regular module, then for any $x \in M$, we have $M = Rx \oplus N$ for some $N \subseteq M$. We claim that $\operatorname{Rad}(M) = 0$. In order to prove this, suppose for contradiction that $\operatorname{Rad}(M) \neq 0$, then there is a non-zero element $x \in \operatorname{Rad}(M)$ and by Lemma 2.8, $Rx \ll M$ but M is regular so this leads to $Rx = 0$ for every $r \in R$, so $x = 0$, a contradiction. Thus $\operatorname{Rad}(M) = 0$. Therefore from this M is a strongly T-noncosingular module clearly. $\square$

4.3. Commutative Noetherian Max Rings are Artinian

At the end of this section we shall state and prove our main theorem. Although the proof of the following proposition is clear, we include it for completeness.

Proposition 4.13 *If M is a strongly T-noncosingular module, then* $\operatorname{Rad}(M) \neq M$

Proof Let M be a strongly T-noncosingular module and suppose for the contrary $\operatorname{Rad}(M) = M$. Let $f \in End(M)$ be any non-zero endomorphism of M, then $\operatorname{Im} f \subseteq M = \operatorname{Rad}(M)$, now by assumption $f = 0$, a contradicton. Therefore $\operatorname{Rad}(M) \neq M$. □

Lemma 4.4 *Let M be an R-module such that* $\operatorname{Rad}(M) = M$ *and* $\operatorname{Rad}(U) \neq U$ *for every non-zero proper submodule U of M. Then M is T-noncosingular module, but not strongly T-noncosingular.*

Proof Let $f \in End(M)$ be a non-zero endomorphism of M with $\operatorname{Im} f \ll M$. Then $\operatorname{Im} f \neq M$ and so $\operatorname{Im} f$ is a proper submodule of M. By assumption $\operatorname{Rad}(\operatorname{Im} f) \neq \operatorname{Im} f$. On the other hand, $M/\operatorname{Ker} f \cong \operatorname{Im} f$, now by Lemma 2.10, $\operatorname{Rad}(\operatorname{Im} f) = \operatorname{Im} f$, a contradiction. Therefore $f = 0$. So M is a T-noncosingular module. Since $M = \operatorname{Rad}(M)$, M is not strongly T-noncosingular by Proposition 4.13. □

A right, left or two-sided ideal I of a ring R is called a nil ideal, resp. $nilpotent$ ideal if, for every $a \in I$, there exists an $n \in \mathbb{N}$ such that $a^n = 0$, resp. $I^n = 0$. A subset I of a ring R is a left $T-nilpotent$ in case for every sequence a_1 , a_2 , ..., in I, there is an n such that $a_1 a_2 ..., a_n = 0$. If I is an ideal, then I is called left $T-nilpotent$ ideal. Observe that if I is a left or right T-nilpotent then it is nil because a , a , a , a , ... , is a sequence in I whenever $a \in I$.

Not every left T-nilpotent ideal is a nilpotent ideal, but the subsequent lemma is useful for this transition,

Lemma 4.5 ((Kasch, 1982), Corollary 9.3.7) *If the left R-module R is noetherian, then every two-sided nil ideal is nilpotent.*

A ring R is said to be $semilocal$ if $R/\operatorname{Rad}(R)$ is a left artinian ring, or, equivalently, $R/\operatorname{Rad} R$ is a semisimple ring.

Lemma 4.6 ((Lam, 1991), Proposition 20.2) *For a ring R, consider the following two conditions:*

1. *R is semilocal.*

2. *R has finitely many maximal left ideals*

We have, in general, $(2) \Rightarrow (1)$, *but the converse holds if* $R/\operatorname{Rad}(R)$ *is commutative.*

Theorem 4.2 ((Anderson & Fuller, 1992), Theorem 15.20) *Let R be a ring with* $\operatorname{Rad}(R) = J(R)$. *Then R is left artinian if and only if R is noetherian, semilocal and $J(R)$ is nilpotent.*

Proposition 4.14 ((Anderson & Fuller, 1992),Remark 28.5) *If every left R-module has a maximal submodule, then $J(R)$ is left $T-nilpotent$.*

We call a module M max module if every nonzero submodule has a maximal submodule, and we say that ${}_RR$ is a max module if every nonzero left ideal contains a maximal submodule.

Lemma 4.7 ((Clark, 2006), 2.19(1)) *Let M be an R-module, then M is a max module if and only if* $\operatorname{Rad}(N) \neq N$ *(or* $\operatorname{Rad}(N) \ll N$*) for every non-zero* $N \subseteq M$

Lemma 4.8 ((Büyükaşık & Yılmaz, 2009), Lemma 6.1) *Let R be a ring and A be a finitely generated ideal of R. Let $X = \prod_{i\in I} X_i$ be the direct product of the R-modules X_i. Suppose that $X_i = AX_i$ for all $i \in I$. Then $X = AX$*

Lemma 4.9 *Let R be a commutative ring such that every maximal ideal is finitely generated. Suppose R has infinitely many distinct maximal ideals say $\{P_i\}_{i\in I}$ with I infinite index set. Set $S_i = R/P_i$. Then the module,*

$$M = (\prod_{i\in I} S_i)/(\bigoplus_{i\in I} S_i)$$

has no maximal submodules, i.e. $\operatorname{Rad}(M) = M$.

Proof Let P be a maximal ideal of R. Since $P \in \{P_i\}_{i\in I}$, $P = P_i$ for some $i \in I$, Then $PS_i = 0$ and $PS_j = S_j$ for all $i \neq j$ and i , $j \in I$. From this, we can write

$$P(\prod_{i\in I} S_i) = P[(\prod_{j\neq i} S_j) \oplus S_i] = P(\prod_{j\neq i} S_j) + PS_i = P(\prod_{i\neq j} S_j) = \prod_{i\neq j} S_j$$

by Lemma 4.8. Now

$$P(\prod_{i\in I} S_i) + (\bigoplus_{i\in I} S_i) = P(\prod_{j\neq i} S_j) + (\bigoplus_{i\in I} S_i) = (\prod_{j\neq i} S_j) + S_i = \prod_{i\in I} S_i$$

Therefore

$$PM = [P(\prod_{i\in I} S_i) + (\bigoplus_{i\in I} S_i)]/\bigoplus_{i\in I} S_i = (\prod_{i\in I} S_i)/(\bigoplus_{i\in I} S_i)$$

We have found $PM = M$ for each maximal ideal P of R. Hence, by Lemma 2.12, $\mathrm{Rad}\,(M) = M$. □

Lemma 4.10 *Let R be a commutative, noetherian and max ring, then R is semilocal.*

Proof By Lemma 4.6, it is enough to show that R has finitely many maximal ideals. Suppose that R has infinitely many distinct maximal ideals $\{P_i\}_{i\in I}$, then for $S_i = R/P_i$, the module

$$M = (\prod_{i\in I} S_i)/(\bigoplus_{i\in I} S_i)$$

has no maximal submodules by Lemma 4.9, but, on the other hand R is a max ring, hence $M = 0$, and so

$$\prod_{i\in I} S_i = \bigoplus_{i\in I} S_i$$

that is I is finite, therefore R has finitely many maximal ideals. □

The proof of the following theorem can be found in the paper of Ross, M. Hamsher (Theorem 1), but we shall give another proof by using Lemma 4.10.

Theorem 4.3 *Let R be a commutative, noetherian ring. Then the following are equivalent:*

1. *Every nonzero R-module has a maximal submodule.*

2. *R is artinian.*

Proof $(1 \Rightarrow 2)$ Suppose that every nonzero R-module has a maximal submodule, i.e. R is a max ring. By Lemma 4.10, R is semilocal, and by Proposition 4.14, $J(R)$ is left T-nilpotent and by Lemma 4.5, $J(R)$ is nilpotent. Finally by Theorem 4.2, R is artinian.

$(2 \Rightarrow 1)$ Trivial.

□

Theorem 4.4 *Let R be a commutative, noetherian ring. The following statements are equivalent:*

1. *Every T-noncosingular module is strongly T-noncosingular module.*

2. *R is artinian ring.*

Proof $(1 \Rightarrow 2)$: First of all, we show that R is a max ring (that is every non-zero R-module has a maximal submodule) and then by Theorem 4.3, the result will follow. Let (1) hold and suppose for contradiction that R is not a max ring, then there exists a non-zero module M with $\operatorname{Rad}(M) = M$. By Lemma 2.6, there exists a non-zero homomorphism $f : M \longrightarrow E(S)$, where $E(S)$ is an injective hull of a simple module S. Since $M/\operatorname{Ker} f \cong \operatorname{Im} f$ and $\operatorname{Rad}(M) = M$, by Lemma 2.10, $\operatorname{Rad}(M/\operatorname{Ker} f) = M/\operatorname{Ker} f$ and so $\operatorname{Rad}(\operatorname{Im} f) = \operatorname{Im} f$. Now by Theorem 2.8, $E(S)$ is artinian and $\operatorname{Im} f$ is artinian by Theorem 2.1 because of $\operatorname{Im} f \subseteq E(S)$.

Let $\Omega = \{N \subseteq f(M) \mid \operatorname{Rad}(N) = N \, and \, N \; is \; nonzero\}$. This set is non-empty since $f(M) \in \Omega$. Because $f(M)$ is artinian, Ω has a minimal element K, say. By Lemma 4.4, K is a T-noncosingular module, but not strongly T-noncosingular. Thus we have found a T-noncosingular module which is not strongly T-noncosingular. This contradicts (1). Accordingly R is a max ring, now Theorem 4.3 finishes the proof.

$(2 \Rightarrow 1)$: Suppose that R is an artinian ring. By Theorem 4.3, every R-module has a maximal submodule, i.e. R is a max ring. Then $\operatorname{Rad}(M) \ll M$ by Lemma 4.7. By Proposition 4.6, every T-noncosingular module is also a strongly T-noncosingular module. This concludes the proof. □

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