ON δ-PERFECT AND δ-SEMIPERFECT RINGS

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ABSTRACT

ON δ-PERFECT AND δ-SEMIPERFECT RINGS

In this thesis, we give a survey of generalizations of right-perfect, semiperfect and semiregular rings by considering the class of all singular R-modules in place of the class of all R-modules. For a ring R and a right R-module M, a submodule N of M is said to be δ-small in M if, whenever $N + X = M$ with M/X singular, we have $X = M$. If there exists an epimorphism $p : P \to M$ such that P is projective and $\mathrm{Ker}(p)$ is δ-small in P, then we say that P is a projective δ-cover of M. A ring R is called δ-perfect (respectively, δ-semiperfect) if every R-module (respectively, simple R-module) has a projective δ-cover. In this thesis, various properties and characterizations of δ-perfect and δ-semiperfect rings are stated.

ÖZET

δ-MÜKEMMEL VE δ-YARIMÜKEMMEL HALKALAR ÜZERİNE

Bu tezde, tüm R-modül sınıfı yerine tüm tekil R-modül sınıfını alarak, sağ-mükemmel halka, yarımükemmel halka ve yarıdüzenli halkaların genellemesi üzerine bir inceleme yaptık. R bir halka ve M bir sağ R-modül için, eğer $N + X = M$ ve M/X tekil olduğunda $X = M$ oluyorsa, N'e M modülünün δ-küçük altmodülü denir. Eğer P projektif ve $\mathrm{Ker}(p)$ P'de δ-küçük olacak şekilde $p : P \to M$ bir epimorfizma var ise, P modülüne M'nin projektif δ-örtüsü denir. Eğer her R-modülün (sırasıyla, basit R-modülün) projektif δ-örtüsü varsa R halkasına δ-mükemmel (sırasıyla, δ-yarımükemmel) halka denir. Bu tezde, δ-mükemmel ve δ-yarımükemmel halkaların çeşitli özellikleri ve karakterizasyonları verilmiştir.

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LIST OF SYMBOLS

R	an associative ring with unit unless otherwise stated
$\mathcal{M}od$-R	the category of right R-modules
$\mathbb{Z}$	the ring of integers
J	Jacobson radical of the ring R
Z	right singular ideal
$\subseteq$	submodule
$\subset$	proper submodule
$<$	proper ideal
$\leq$	ideal
$\ll$	small (or superfluous) submodule
$\ll_\delta$	δ-small submodule
$\leq_\oplus$	direct summand
$\mathrm{Ker}(f)$	the kernel of the map f
$\mathrm{Im}(f)$	the image of the map f
$\mathrm{End}(M)$	the endomorphism ring of a module M
$\leq$e	essential ideal
$\leq$max	maximal ideal

CHAPTER 1

INTRODUCTION

Covers and envelopes of modules play an important role in Module and Ring Theory. In 1953, Eckmann and Schopf proved the existence of injective envelopes of modules over any associative ring. The existence of projective covers was studied by Bass in 1960. After that, different kinds of covers and envelopes have been described. For example, Enochs introduced the torsion free coverings of modules, and Warfield investigated the pure injective envelopes of modules. Then in 1981, Enochs gave a categorical definition of covers and envelopes for a class of modules.

Throughout this thesis, R denotes an associative ring with identity and modules are unitary right R-modules unless otherwise indicated.

A ring R is called perfect if every R-module has a projective cover. If every finitely generated R-module has this property, then R is called semiperfect. In this thesis, we study the generalizations of perfect and semiperfect rings by considering the class of all singular R-modules in place of the class of all R-modules.

In chapter 2 we give some results related with our work and used in following chapters. For the results in this chapter we refer to (Bland, Paul E., 2010), (Anderson, F.W., Fuller, K. R. 1992), (Wisbauer, R., 1991), (Nicholson, W. K., 1976) and (Goodearl, K. R., 1976).

In chapter 3 we give a survey of generalizations of right perfect, semiperfect and semiregular rings from (Zhou, Y., 2000). It is of interest to know how far the old theories extend to the new situation. The concept of small submodules which leads to the definition of projective covers, is certainly the key in introducing perfect, semiperfect and semiregular rings. As a generalization of small submodules, (Zhou, Y., 2000) introduces δ-small submodules and obtains various characterizations and properties for a ring R, for which every R-module (respectively simple R-module, cyclically presented R-module) has a projective δ-cover. From these properties, it is clear that if a ring R is semiperfect, then R is δ-semiperfect. For the converse, we need some extra condition.

In chapter 4 we study when δ-semiperfect rings are semiperfect. (Büyükaşık, E., Lomp, C., 2009) introduce that an arbitrary associative unital ring R is semiperfect if and only if it is semilocal and δ-semiperfect. They characterize finitely generated δ-supplemented modules M as those which are sums of simple and δ-local modules or equivalently which satisfy the property that every maximal submodule of M has a δ-supplement.

CHAPTER 2

PRELIMINARIES

In this chapter we give some fundamental properties of rings and modules that will be used later.

2.1. Radical of a Module

Definition 2.1 *For a right R-module M, a submodule S of M is said to be small or superfluous in M if for any submodule L of M, $S + L = M$ implies $L = M$. This is denoted by $S \ll M$. A right ideal of R is small if it is small when viewed as a submodule of R_R.*

Definition 2.2 *The Jacobson radical of R, denoted by $J(R)$, is the intersection of the maximal right ideals of R.*

If $J(R) = 0$, then R is said to be a *Jacobson semisimple* ring. It is also referred to as J*-semisimple* or *semiprimitive* ring. The concept of the Jacobson radical of R carries over to modules. If M is an R-module, then the radical of M, denoted by $\mathrm{Rad}(M)$, is the intersection of the maximal submodules of M. If M has no maximal submodules then $\mathrm{Rad}(M) = M$. For example, the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ has no maximal submodules, so $\mathrm{Rad}(\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$.

Proposition 2.1 *(Bland, Paul E., 2010)[Proposition 6.1.2] If M is a nonzero finitely generated R-module, then M has at least one maximal submodule.*

Since M has a maximal submodule, $\mathrm{Rad}(M) \neq M$. Thus, we have the following corollary.

Corollary 2.1 *(Bland, Paul E., 2010)[Corollary 6.1.3] If M is a nonzero finitely generated R-module, then* $\mathrm{Rad}(M) \neq M$.

Example 2.1 *Since R is generated by 1_R, $J(R) \neq R$.*

Proposition 2.2 *(Bland, Paul E., 2010)[Proposition 6.1.4] If $\{M_\alpha\}_\Delta$ is a family of R-modules, then*

$$\mathrm{Rad}(\bigoplus_{\Delta} M_\alpha) = \bigoplus_{\Delta} \mathrm{Rad}(M_\alpha).$$

Proposition 2.3 *(Bland, Paul E., 2010)[Proposition 6.1.5] If F is a free R-module, then $\mathrm{Rad}(F) = FJ(R)$.*

Proposition 2.4 *(Bland, Paul E., 2010)[Proposition 6.1.8] The following hold for any ring R.*

(i) *$J(R)$ is the intersection of the maximal right ideals of R.*

(ii) *$J(R)$ is an ideal of R that coincides with the intersection of the right annihilator ideals of the simple right R-modules.*

(iii) *$J(R)$ is the set of all $a \in R$ such that $1 - ar$ has a right inverse for all $r \in R$.*

Lemma 2.1 *(Bland, Paul E., 2010)[Lemma 6.1.9] If I is a right ideal of R such that $I \subseteq J(R)$, then $MI \subseteq \mathrm{Rad}(M)$ for every R-module M.*

Lemma 2.2 *(Nakayama's Lemma) If I is a right ideal of R such that $I \subseteq J(R)$, then the following two equivalent conditions hold for every finitely generated R-module M.*

(i) *If N is a submodule of M such that $N + MI = M$, then $N = M$.*

(ii) *If $MI = M$, then $M = 0$.*

Now we will give some properties of the radical of projective modules.

Proposition 2.5 *(Bland, Paul E., 2010) [Proposition 7.2.4, Proposition 7.2.8, Corollary 7.2.9] The following statements hold for a projective R-module M:*

(i) $\mathrm{Rad}(M) = MJ(R)$,

(ii) *M contains a maximal submodule,*

(iii) $\mathrm{Rad}(M) \subsetneq M$.

2.2. Local Rings

A ring R is a local ring in case the set of non-invertible elements of R is closed under addition.

Proposition 2.6 *(Anderson, F.W., Fuller, K. R. 1992)[Theorem 15.15] For a ring R, the following statements are equivalent:*

(i) *R is a local ring,*

(ii) *R has a unique maximal right ideal,*

(iii) *$J(R)$ is a maximal right ideal,*

(iv) *The set of elements of R without right inverses is closed under addition,*

(v) *$J(R) = \{x \in R \mid xR \neq R\}$,*

(vi) *$R/J(R)$ is a division ring,*

(vii) *$J(R) = \{x \in R \mid x$ is not invertible$\}$,*

(viii) *If $x \in R$, then either x or $1 - x$ is invertible.*

2.3. Covers of Modules

Let $\mathcal{X}$ be a class of right R-modules. We assume that $\mathcal{X}$ is closed under isomorphisms, i.e., if $M \in \mathcal{X}$ and $N \cong M$, then $N \in \mathcal{X}$. We also assume that $\mathcal{X}$ is closed under taking finite direct sums, and direct summands, i.e., if $M_1, \ldots, M_t \in \mathcal{X}$, then $M_1 \oplus \cdots \oplus M_t \in \mathcal{X}$; if $M = N \oplus L \in \mathcal{X}$, then $N, L \in \mathcal{X}$.

Definition 2.3 *On the class $\mathcal{X}$, for an R-module M, $X \in \mathcal{X}$ is called an $\mathcal{X}$-cover of M if there is a homomorphism $\varphi : X \to M$ such that the following hold.*

(i) *For any homomorphism $\varphi^{'} : X^{'} \to M$ with $X^{'} \in \mathcal{X}$, there exists a homomorphism $f : X^{'} \to X$ with $\varphi^{'} = \varphi f$, or equivalently*

$$\mathrm{Hom}_R(X^{'}, X) \longrightarrow \mathrm{Hom}_R(X^{'}, M) \longrightarrow 0$$

is exact for any $X^{'} \in \mathcal{X}$.

(ii) *If f is an endomorphism of X with $\varphi = \varphi f$, then f must be an automorphism.*

If (1) holds (and perhaps not (2)), $\varphi : X \to M$ is called an $\mathcal{X}$-precover. Note that an $\mathcal{X}$-cover (precover) is not necessarily surjective.

Theorem 2.1 *(Xu, J. 1996)[Theorem 1.2.6] Let M be an R-module. If $\varphi_i : X_i \to M$, i=1,2, are two different $\mathcal{X}$-covers, then $X_1 \cong X_2$.*

Theorem 2.2 *(Xu, J. 1996)[Theorem 1.2.9] Suppose $\mathcal{X}$ is closed under an arbitrary direct product, and for each i, $\varphi_i : X_i \to M_i$ is an $\mathcal{X}$-precover. Then the natural product $\prod \varphi_i : \prod X_i \to \prod M_i$ is an $\mathcal{X}$-precover.*

Theorem 2.3 *(Xu, J. 1996)[Theorem 1.2.10] If $\varphi_i : X_i \to M_i$ is an $\mathcal{X}$-cover for $i = 1, \ldots, n$, then $\bigoplus_{i=1}^{n} \varphi_i : \bigoplus_{i=1}^{n} X_i \to \bigoplus_{i=1}^{n} M_i$ is an $\mathcal{X}$-cover.*

Let Ω be the class of all projective R-modules. A homomorphism $f : F \to M$ is an Ω-cover of M if and only if F is projective and f is a superfluous epimorphism.

Definition 2.4 *A projective cover of an R-module M is a projective R-module $P(M)$ together with an epimorphism $\varphi : P(M) \to M$ such that $\operatorname{Ker}(\varphi)$ is small in $P(M)$.*

By Theorem 2.1, we know that an $\mathcal{X}$-cover of an R-module is unique up to isomorphism, so is a projective cover.

Example 2.2 *(Bland, Paul E., 2010)*

(i) ***Projective Modules.*** *Every projective module has a projective cover, namely, itself.*

(ii) ***Local Rings and Projective Covers.*** *Let R be a commutative ring that has a unique maximal ideal m. Then R together with the natural mapping $R \to R/m$ is a projective cover of R/m. If M is a finitely generated R-module, then M has a projective cover.*

Proposition 2.7 *(Bland, Paul E., 2010)[Proposition 7.2.3] Let $\{M_k\}_{k=1}^{n}$ be a finite family of R-modules.*

(i) If S_k is a small submodule of M_k, for $k = 1, 2, \ldots, n$, then $\bigoplus_{k=1}^{n} S_k$ is a small submodule of $\bigoplus_{k=1}^{n} M_k$.

(ii) If each M_k has a projective cover $\varphi_k : P_k \to M_k$, then $\bigoplus_{k=1}^{n} M_k$ has a projective cover $\bigoplus_{k=1}^{n} \varphi_k : \bigoplus_{k=1}^{n} P_k \to \bigoplus_{k=1}^{n} M_k$ and if $\varphi : P \to \bigoplus_{k=1}^{n} M_k$ is a projective cover of $\bigoplus_{k=1}^{n} M_k$, then there is a family $\{\overline{P_k}\}_{k=1}^{n}$ of submodules of P such that $\overline{P_k} \cong P_k$ for each k.

Now, we will see that a projective cover of a module may fail to exist.

Example 2.3 *Suppose that R is a Jacobson semisimple ring. If $\varphi : P \to M$ is a projective cover of M, then $\operatorname{Ker}(\varphi)$ is a small submodule of P and*

$$\operatorname{Ker}(\varphi) \subseteq \operatorname{Rad}(P) = PJ(R) = 0.$$

So φ is an isomorphism. Thus over a Jacobson semisimple ring R, an R-module M has a projective cover if and only if it is projective. For example, the ring $\mathbb{Z}$ is Jacobson semisimple, so the only $\mathbb{Z}$-modules with projective covers are the free $\mathbb{Z}$-modules. Thus, $\mathbb{Z}_n$ does not have a projective cover, since $\mathbb{Z}_n$ is not a free $\mathbb{Z}$-module for any integer $n \geq 2$.

2.4. Semiperfect Rings, Perfect Rings and Supplemented Modules

Since there are modules that do not have a projective cover, this brings up the question are there rings over which every module has a projective cover? Such rings exist, and we will characterize these rings. First, we define semiperfect rings.

Definition 2.5 *An R-module M is called finitely generated if there are finitely many elements $x_1, \ldots, x_n \in M$ such that $M = x_1R + x_2R + \cdots + x_nR$.*

Definition 2.6 *A ring R is said to be a semiperfect ring if every finitely generated R-module has a projective cover.*

Proposition 2.8 *(Wisbauer, R., 1991) The following are equivalent for a ring R:*

(i) R is semiperfect,

(ii) The ring $R/J(R)$ is semisimple and idempotents of $R/J(R)$ can be lifted modulo $J(R)$,

(iii) The right R-module R_R is a sum of local modules,

(iv) The ring R has a complete set $\{e_1, e_2, \ldots, e_n\}$ of orthogonal idempotents such that e_iRe_i is a local ring for $i = 1, 2, \ldots, n$,

(v) Every simple right R-module has a projective cover,

(vi) Every finitely generated right R-module has a projective cover.

Lemma 2.3 *(Bland, Paul E., 2010) Let I be a right ideal in a ring R. Then the following statements are equivalent:*

(i) $MI \neq M$ for every nonzero R-module M,

(ii) $MI \ll M$ for every nonzero R-module M,

(iii) $FI \ll F$ for the countably generated free R-module $F = R^{(\mathbb{N})}$,

(iv) I is right T-nilpotent.

We will now characterize the right perfect rings.

Definition 2.7 *A ring R is called right perfect if every R-module has a projective cover. Left perfect rings are defined similarly. A ring that is left and right perfect is called a perfect ring.*

Bass has given the following characterizations of perfect rings.

Proposition 2.9 *(Bland, Paul E., 2010)[Proposition 7.2.28] The following are equivalent for a ring R:*

(i) R *is a right perfect ring,*

(ii) $R/J(R)$ *is semisimple and every nonzero* R*-module contains a maximal submodule,*

(iii) $R/J(R)$ *is semisimple and* $J(R)$ *is right* T*-nilpotent.*

Proposition 2.10 *(Bland, Paul E., 2010)[Proposition 7.2.29] The following are equivalent for a ring R:*

(i) R *is a right perfect ring,*

(ii) R *satisfies the descending chain condition on principal left ideals,*

(iii) Every flat R*-module is projective,*

(iv) R *contains no infinite set of orthogonal idempotents and every nonzero right* R*-module contains a simple submodule.*

Definition 2.8 *Let U be a submodule of the R-module M. A submodule $V \subset M$ is called a supplement or addition complement of U in M if V is a minimal element in the set of submodules $L \subset M$ with $U + L = M$.*

Lemma 2.4 *(Wisbauer, R., 1991) V is a supplement of U if and only if $U + V = M$ and $U \cap V \ll V$.*

Proof If V is a supplement of U and $X \subset V$ with $(U \cap V) + X = V$, then we have

$$M = U + V = U + (U \cap V) + X = U + X,$$

hence $X = V$ by the minimality of V. Thus $U \cap V \ll V$.

On the other hand, let $U + V = M$ and $U \cap V \ll V$. For $Y \subset V$ with $U + Y = M$, we have

$$V = M \cap V = (U \cap V) + Y,$$

that is, $V = Y$. Hence, V is minimal in the desired sense. □

Theorem 2.4 *(Wisbauer, R., 1991)[41.1 Properties of supplements] Let U, V be submodules of the R-module M. Assume V to be a supplement of U. Then:*

(i) If $K \ll M$, then V is a supplement of $U + K$.

(ii) For $K \ll M$ we have $K \cap V \ll V$ and so $\mathrm{Rad}(V) = V \cap \mathrm{Rad}(M)$.

(iii) For $L \subset U$, $(V + L)/L$ is a supplement of U/L in M/L.

Definition 2.9 *An R-module M is called supplemented if every submodule of M has a supplement in M.*

If every finitely generated submodule of M has a supplement in M, then we call M finitely supplemented or f-supplemented.

Theorem 2.5 *(Wisbauer, R., 1991)[41.2 Properties of supplemented modules] Let M be an R-module.*

(i) Let M_1, U be submodules of M with M_1 supplemented. If there is a supplement for $M_1 + U$ in M, then U also has a supplement in M.

(ii) If $M = M_1 + M_2$, with M_1, M_2 supplemented modules, then M is also supplemented.

(iii) If M is supplemented, then

(a) Every finitely M-generated module is supplemented.

(b) $M/Rad(M)$ is semisimple.

Theorem 2.6 *(Wisbauer, R., 1991)[41.6 Supplemented modules, characterizations]*

(i) For a finitely generated module M, the following are equivalent:

(a) M is supplemented,

(b) Every maximal submodule of M has a supplement in M,

(c) M is a (finite) sum of local submodules.

(ii) If M is supplemented and $Rad(M) \ll M$, then M is an irredundant sum of local modules.

Definition 2.10 *M is called an amply supplemented module if for any two submodules A and B of M with $A + B = M$, B contains a supplement of A.*

If every finitely generated submodule of M has ample supplements in M, then we call M amply finitely supplemented.

Theorem 2.7 *(Wisbauer, R., 1991) [42.6 Semiperfect Rings, characterizations] For a ring R the following statements are equivalent:*

(i) R_R *is semiperfect,*

(ii) R_R *is supplemented,*

(iii) every finitely generated R*-module is semiperfect in* $\mathcal{M}od$*-*R*,*

(iv) every finitely generated R*-module has a projective cover in* $\mathcal{M}od$*-*R*,*

(v) every finitely generated R*-module is (amply) supplemented,*

(vi) $R/J(R)$ *is right semisimple and idempotents in* $R/J(R)$ *can be lifted to* R*,*

(vii) every simple R*-module has a projective cover in* $\mathcal{M}od$*-*R*,*

(viii) every maximal right ideal has a supplement in R*,*

(ix) R_R *is a (direct) sum of local (projective covers of simple) modules,*

(x) $R = e_1R \oplus \cdots \oplus e_kR$ *for local orthogonal idempotents* e_i*,*

(xi) ${}_RR$ *is semiperfect.*

If R *satisfies one of these conditions, then* R *is called a semiperfect ring. The assertions (b) - (j) hold similarly for left* R*-modules.*

2.5. Semiregular Modules and Rings

In this section a class of semiregular modules is introduced which contains all regular and all semiperfect modules. In addition, several theorems about regular and semiperfect modules are extended. Also, these results are applied to the study of rings R (semiregular rings) such that R_R is semiregular.

If M is an R-module, the dual of M will be denoted by $M^* = \mathrm{Hom}_R(M, R)$. A dual basis for M is a pair of subsets $\{x_i \mid i \in I\} \subseteq M$ and $\{\varphi_i \mid i \in I\} \subseteq M^*$ (indexed by the same set I) such that, for each $x \in M$, $\varphi_i(x) = 0$ for all but finitely many $i \in I$ and $x = \sum_i x_i\varphi_i(x)$. It is well known that M is (finitely generated) projective if and only if it has a (finite) dual basis. An element x in a module M is called regular if $x\alpha(x) = x$ for some $\alpha \in M^*$. A module M is called regular if each of its elements is regular.

Definition 2.11 *A submodule* N *of a module* M *is said to lie over a summand of* M *if there exists a direct decomposition* $M = P \oplus Q$ *with* $P \subseteq N$ *and* $Q \cap N$ *is small in* M*.*

Lemma 2.5 *(Nicholson, W. K., 1976)[Lemma 1.2] If M is projective, a submodule N lies over a summand of M if and only if M/N has a projective cover.*

Proposition 2.11 *(Nicholson, W. K., 1976)[Proposition 1.3] If M is any module, the following conditions are equivalent for $x \in M$:*

(i) xR lies over a projective summand of M,

(ii) There exists $\alpha \in M^*$ such that $(\alpha(x))^2 = \alpha(x)$ and $x - x\alpha(x) \in \mathrm{Rad}(M)$,

(iii) There exists a regular element $y \in xR$ such that $x - y \in \mathrm{Rad}(M)$ and $xR = yR \oplus (x-y)R$,

(iv) There exists a regular element $y \in M$ such that $x - y \in \mathrm{Rad}(M)$,

(v) There exists $\gamma : M \to xR$ such that $\gamma^2 = \gamma$, $\gamma(M)$ is projective and

$$x - \gamma(x) \in \mathrm{Rad}(M).$$

Definition 2.12 *An element x in a module M is said to be semiregular (in M) if the conditions in Proposition 2.11 are satisfied. A module M is called a semiregular module if each of its elements is semiregular.*

The regular modules are precisely the semiregular modules with zero radical.

Theorem 2.8 *(Nicholson, W. K., 1976)[Theorem 1.6] The following conditions are equivalent for a module M:*

(i) M is semiregular,

(ii) If $N \subseteq M$ is a finitely generated submodule there exists $\gamma : M \to N$ such that $\gamma^2 = \gamma$, $\gamma(M)$ is projective and $(1-\gamma)(N) \subseteq \mathrm{Rad}(M)$,

(iii) Every finitely generated submodule of M lies over a projective summand of M.

Corollary 2.2 *(Nicholson, W. K., 1976)[Corollary 1.7] A projective module M is semiregular if and only if M/N has a projective cover for every finitely generated (cyclic) submodule N.*

Corollary 2.3 *(Nicholson, W. K., 1976)[Corollary 1.8] A module M is regular if and only if every finitely generated (cyclic) submodule is a projective summand.*

Theorem 2.9 *(Nicholson, W. K., 1976)[Theorem 1.10] If $M = \bigoplus_{i \in I} M_i$ is a direct sum of modules then M is semiregular if and only if each M_i is semiregular.*

Corollary 2.4 *(Nicholson, W. K., 1976)[Corollary 1.11] A direct sum $M = \bigoplus_{i\in I} M_i$ is regular if and only if each M_i is regular.*

An element a of a ring R is said to be regular (in the sense of von Neumann) if $aba = a$ for some $b \in R$. If each element of a ring R is regular, R is said to be a regular ring. It is clear that an element a in a ring R is regular if and only if it is regular in R_R.

Lemma 2.6 *(Nicholson, W. K., 1976)[Lemma 2.1] Let a be an element of a ring R. Then a is semiregular in R_R if and only if there exists $e^2 = e \in Ra$ such that $a(1-e) \in J(R)$. An analogues result holds for ${}_RR$.*

Proposition 2.12 *(Nicholson, W. K., 1976)[Proposition 2.2] The following are equivalent for an element a of a ring R:*

(i) There exists $e^2 = e \in aR$ such that $(1-e)a \in J(R)$,

(ii) There exists $e^2 = e \in Ra$ such that $a(1-e) \in J(R)$,

(iii) There exists a regular element $b \in R$ with $a - b \in J(R)$,

(iv) There exists a regular element $b \in R$ with $bab = b$ and $a - aba \in J(R)$.

Definition 2.13 *An element a of a ring R is called semiregular (in R) if it satisfies the conditions in Proposition 2.12. A ring is a semiregular ring if each of its elements is semiregular.*

Theorem 2.10 *(Nicholson, W. K., 1976)[Theorem 2.9] The following statements (and their left-right analogues) are equivalent for a ring R:*

(i) R is semiregular,

(ii) $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$,

(iii) Every finitely generated (cyclic) right ideal lies over a direct summand,

(iv) Every finitely related (finitely related and cyclic) right R-module has a projective cover,

(v) Every finitely generated (cyclic) right ideal has a complement in R.

2.6. The Singular Submodule

Definition 2.14 *We shall use $\Gamma(R)$ to stand for the set of all essential right ideals of the ring R.*

Also if I is a right ideal of R, and $r \in R$, we use $r^{-1}I$ to denote the right ideal $\{x \in R \mid rx \in I\}$.

Note that if r is invertible in R, then this definition coincides with the product of r^{-1} and I.

Proposition 2.13 *(Goodearl, K. R., 1976) For a ring R, the following conditions hold.*

(i) $R \in \Gamma(R)$.

(ii) *If $I \leq J \leq R_R$ and $I \in \Gamma(R)$, then $J \in \Gamma(R)$.*

(iii) *If $I \in \Gamma(R)$ and $r \in R$, then $r^{-1}I \in \Gamma(R)$.*

Definition 2.15 *Given any right R-module A, we set*

$$Z(A) = \{x \in A \mid xI = 0 \text{ for some } I \leq_e R\}$$

Equivalently, $Z(A)$ is the set of those $x \in A$ for which the right ideal $\{r \in R \mid xr = 0\}$ belongs to $\Gamma(R)$. It can be easily checked that $Z(A)$ is a submodule of A, and it is called the singular submodule of A.

In a similar fashion, we define the singular submodule of any left R-module B:

$$Z(B) = \{x \in B \mid Jx = 0 \text{ for some } J \leq_e R\}.$$

Actually, $Z(-)$ defines a functor from $\mathcal{M}od\text{-}R \to \mathcal{M}od\text{-}R$. Given any map $f : A \to B$ in $\mathcal{M}od\text{-}R$, it follows directly from our definitions that $f(Z(A)) \leq Z(B)$ and hence we define $Z(f) : Z(A) \to Z(B)$ to be the restriction of f to $Z(A)$. In particular, for any module A, we have $f\big(Z(A)\big) \leq Z(A)$ for all $f \in \operatorname{End}_R(A)$, so that $Z(A)$ is a fully invariant submodule of A.

Considering R as a module, we see that $Z(R_R)$ is thus a 2-sided ideal of R. The ideal $Z(R_R)$ is known as the right singular ideal of R, and is denoted by $Z_r(R)$. Likewise we have the left singular ideal $Z_l(R)$ which is the singular submodule of ${}_RR$.

Definition 2.16 *A module A is called a singular module provided $Z(A) = A$, and is called a nonsingular module if $Z(A) = 0$.*

Remark 2.1 *The ring R is a nonsingular right module if and only if $Z_r(R) = 0$ and in this event R is called a right nonsingular ring, and R is called a left nonsingular ring if $Z_l(R) = 0$. $Z(R) \neq R$ (i.e., R is not singular) unless $R = 0$.*

$$Z_r(R) = \{r \in R \mid rI = 0, I \leq_e R\} \lneq R.$$

Example 2.4 *Let $R = \mathbb{Z}$. (The details are similar for any commutative integral domain.) We know that all nonzero ideals of $\mathbb{Z}$ are essential in $\mathbb{Z}$; hence $\Gamma(\mathbb{Z})$ is just the set of all nonzero ideals of $\mathbb{Z}$. Given a $\mathbb{Z}$-module A and an element $x \in A$, we thus have $x \in Z(A)$ if and only if $x(n\mathbb{Z}) = 0$ for some positive integer n, i.e., if and only if x has finite order. Therefore, $Z(A)$ is just the torsion subgroup of A. It follows that A is singular if and only if it is a torsion group, and that A is nonsingular if and only if it is a torsion-free group. In particular, $\mathbb{Z}_{\mathbb{Z}}$ is nonsingular; hence $\mathbb{Z}$ is a nonsingular ring.*

Proposition 2.14 *(Goodearl, K. R., 1976)*

(i) A module C is nonsingular if and only if $\mathrm{Hom}_R(A, C) = 0$ for all singular modules A.

(ii) A module C is singular if and only if there exists a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

such that f is an essential monomorphism.

Proof

(i) If A is singular, C is nonsingular and $f : A \to C$ is an R-homomorphism, then

$$f(A) = f(Z(A)) \leq Z(C) = 0.$$

So, $f = 0$. Therefore, $\mathrm{Hom}_R(A, C) = 0$ whenever A is singular, C is nonsingular.

Conversely, if $\mathrm{Hom}_R(A, C) = 0$ for all singular modules A, then in particular $\mathrm{Hom}_R(Z(C), C) = 0$. Now, the inclusion map $Z(C) \to C$ is zero and hence $Z(C) = 0$.

(ii) First assume that we have such an exact sequence. Given any $b \in B$, we have a map $\varphi : R \to B$ given by $r \mapsto br$. Since $f(A) \leq_{\text{e}} B$, we have $\varphi^{-1}(f(A)) \leq_{\text{e}} R_R$, that is, the right ideal

$$I = \{r \in R \mid br \in f(A)\} \leq_{\text{e}} R.$$

Now $bI \leq f(A) = \mathrm{Ker}(g)$, hence $g(bI) = g(b)I = 0$. So $g(b) \in Z(C)$. Since g is epic, we have $Z(C) = C$.

Conversely assume that C is singular, and choose a short exact sequence

$$0 \longrightarrow A \xrightarrow{\mathrm{f}} B \xrightarrow{\mathrm{g}} C \longrightarrow 0$$

such that B is free. If $\{b_\alpha\}$ is a basis for B, then for each α, we have $g(b_\alpha)I_\alpha = 0$ for some $I_\alpha \leq_{\text{e}} R_R$. Hence, $b_\alpha I_\alpha \leq A$. Since $I_\alpha \leq_e R_R$ for all α, we get $b_\alpha I_\alpha \leq_{\text{e}} b_\alpha R_R$ for all α.

Therefore, $\bigoplus b_\alpha I_\alpha \leq_{\text{e}} \bigoplus b_\alpha R = B$. In as much as $\bigoplus b_\alpha I_\alpha \leq A$, we obtain $A \leq_{\text{e}} B$ and thus, the inclusion map $A \to B$ is an essential monomorphism. $\square$

Example 2.5 *Proposition 3.1 shows that, B/A is singular whenever $A \leq_e B$. Thus, for example, $E(A)/A$ is always singular.*

The converse of this can easily fail; for example, let $B = \mathbb{Z}/2\mathbb{Z}$ and $A = 0$. Then B/A is a singular $\mathbb{Z}$-module, and yet $A \nleq_e B$.

There are, however, two special cases in which this converse does work: One is given by the next Proposition, and the other is the case $B = R_R$. Namely, if I is a right ideal of R such that R/I is singular, then we must have $\overline{1}J = 0$ for some $J \in \Gamma(R)$. Then $J \leq I$ and so $I \in \Gamma(R)$, that is $I \leq_e R_R$. Since we already know that R/I is singular whenever $I \leq_e R_R$, we conclude that

$$\Gamma(R) = \{I \leq R_R \mid R/I \text{ is singular}\}.$$

Proposition 2.15 *(Goodearl, K. R., 1976) Let B be nonsingular and let $A \leq B$. Then B/A is singular if and only if $A \leq_e B$.*

Proof If B/A is singular and x is a nonzero element of B, then $\overline{x}I = 0$ for some $I \leq_{\text{e}} R$. That is, $xI \leq A$. In as much as B is nonsingular, we have $xI \neq 0$. Thus, $xR \cap A \neq 0$. Therefore, $A \leq_{\text{e}} B$. $\square$

Proposition 2.16 *(Goodearl, K. R., 1976)*

(i) The class of all nonsingular right R-modules is closed under submodules, direct products, essential extensions and module extensions.

(ii) The class of all singular right R-modules is closed under submodules, factor modules and direct sums.

Proposition 2.17 *(Goodearl, K. R., 1976) Assume that $Z_r(R) = 0$.*

(i) $Z(A/Z(A)) = 0$ for all right R-modules A.

(ii) A right R-module A is singular if and only if $\operatorname{Hom}_R(A, C) = 0$ for all nonsingular right R-modules C.

(iii) The class of all singular right R-modules is closed under module extensions and essential extensions.

(iv) $\Gamma(R)$ is closed under finite products.

Proposition 2.18 *(Goodearl, K. R., 1976) If A is any simple right R-module, then A is either singular or projective, but not both.*

Proof In as much as $A \cong R/M$ for some maximal right ideal M of R, we see that A is singular if and only if $M \leq_{\mathrm{e}} R$. Thus, if A is not singular, we must have $K \cap M = 0$ for some nonzero right ideal K of R. Since M is a maximal right ideal, $K \oplus M = R$, whence A is projective.

Now, if A is projective, we have $K \oplus M = R$ for some right ideal K, whence M is not essential in R and so A is not singular. □

Corollary 2.5 *(Goodearl, K. R., 1976) Every nonsingular semisimple right R-module is projective.*

Proof Any semisimple right R-module has the form $\bigoplus S_\alpha$, where each S_α is simple. If $\bigoplus S_\alpha$ is nonsingular, then every S_α is nonsingular and thus projective, by Proposition 2.18. Therefore, $\bigoplus S_\alpha$ is projective. □

Corollary 2.6 *(Goodearl, K. R., 1976) If A is any nonsingular right R-module, then* $\mathrm{Soc}(A) = A\,\mathrm{Soc}(R)$.

2.7. The Reject

Let $\wp$ be a class of modules. The reject of $\wp$ in M is defined by

$$\mathrm{Rej}_M(\wp) = \bigcap\{\mathrm{Ker}\, h \mid h : M \to U \text{ for some } U \text{ in } \wp\}.$$

Example 2.6 *If M is an abelian group, then $\mathrm{Rej}_M(\mathbb{Q})$ is the intersection of all $K \leq M$ with M/K torsion free. So $\mathrm{Rej}_M(\mathbb{Q})$ is just the torsion subgroup $\mathrm{T}(M)$ of M, the unique smallest subgroup with $M/\mathrm{T}(M)$ torsion free. And of course*

$$T(M/\mathrm{T}(M)) = 0.$$

Clearly, $\mathrm{Rej}_M(\mathbb{Q})$ *is a left R-submodule of M.*

Proposition 2.19 *(Anderson, F.W., Fuller, K. R. 1992)[Proposition 8.16] Let $\wp$ be a class of modules, let M and N be modules and let $f : M \to N$ be a homomorphism. Then*

$$f(\mathrm{Rej}_M(\wp)) \leq \mathrm{Rej}_N(\wp).$$

Corollary 2.7 *(Anderson, F.W., Fuller, K. R. 1992)[Corollary 8.17] If $f : M \to N$ is epic and* $\mathrm{Ker}(f) \subseteq \mathrm{Rej}_M(\wp)$, *then*

$$f(\mathrm{Rej}_M(\wp)) = \mathrm{Rej}_N(\wp).$$

Definition 2.17 *If ${}_RM$ is a module, then its (left) annihilator is*

$$\ell_R(M) = \{r \in R \mid rx = 0 \quad (x \in M)\},$$

and that M is faithful in case $\ell_R(M) = 0$.

Proposition 2.20 *(Anderson, F.W., Fuller, K. R. 1992)[Proposition 8.22] For each left R-module M,*

$$\mathrm{Rej}_R(M) = \ell_R(M).$$

In particular, M is faithful if and only if M cogenerates R.

Motivated by this fact, we define, for a class $\wp$ of left R-modules, its annihilator:

$$\ell_R(\wp) = \mathrm{Rej}_R(\wp).$$

Thus, $\ell_R(\wp)$ is simply the intersection of all left ideals I of R such that R/I embeds in some element of $\wp$.

Corollary 2.8 *(Anderson, F.W., Fuller, K. R. 1992)[Corollary 8.23] For each class $\wp$ of left R-modules, the reject*

$$\mathrm{Rej}_R(\wp) = \ell_R(\wp) \text{ is a two-sided ideal.}$$

Let $\wp$ be the class of simple R-modules. Then $\mathrm{Rad}(M)$ is just the reject of $\wp$ in M.

CHAPTER 3

δ-SEMIPERFECT AND δ-PERFECT RINGS

As a generalization of small submodules, (Zhou, Y., 2000) defined δ-small submodules.

3.1. δ-small Submodules

Definition 3.1 *(Zhou, Y., 2000) A submodule N of M is said to be δ-small in M if $N + K \neq M$ for any proper submodule K of M with M/K singular. We use $N \ll_\delta M$ to indicate that N is a δ-small submodule of M.*

Examples 3.1

(i) Every small submodule of M is δ-small in M.

(ii) Every nonsingular semisimple submodule of M is δ-small in M:
Let N be a nonsingular semisimple submodule of M. Let $N + X = M$ with M/X singular. Then

$$M/X = (N + X)/X \cong N/(N \cap X)$$

is singular. Since N is nonsingular $N \cap X \leq_e N$. But N is semisimple, so $N \cap X = N$, i.e., $N \subseteq X$. This gives that $X = M$. Thus, N is δ-small in M.

(iii) The δ-small submodules of a singular module are small submodules:
Since factor module of a singular module is singular, we obtain that for a singular module, δ-small submodules and small submodules coincide.

The second singular submodule, in other words, the Goldie torsion submodule $Z_2(M)$ of M is defined by $Z\big(M/Z(M)\big) = Z_2(M)/Z(M)$. The module M is called Z_2-torsion (or Goldie torsion) if $M = Z_2(M)$.

Lemma 3.1 *(Zhou, Y., 2000) Let N be a submodule of M. The following are equivalent:*

(i) $N \ll_\delta M$,

(ii) If $X + N = M$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$,

(iii) If $X + N = M$ with M/X Goldie torsion, then $X = M$.

Proof

(i)$\Rightarrow$ (ii) Let $X + N = M$. By Zorn's Lemma, there exists a submodule Y of N with respect to the property $X \cap Y = 0$. First we need to show that $(N \cap X) + Y \leq_{\mathsf{e}} N$. Let $0 \neq a \in N$. Assume a is not an element of Y. By the maximality of Y, we have $X \cap (Y + aR) \neq 0$. Take $0 \neq x = y + ar \in X$, where $y \in Y, r \in R$. Then $ar = x - y$, so $ar \in (N \cap X) + Y$. Since $X \cap Y = 0$, we have $ar \neq 0$. Therefore $(N \cap X) + Y \leq_{\mathsf{e}} N$. Thus,

$$M/(X + Y) = (X + N)/(X + Y) = (X + Y + N)/(X + Y) \cong N/(Y + (N \cap X))$$

is singular. Since $(X \oplus Y) + N = M$ and $N \ll_\delta M$, we have $M = X \oplus Y$.

To see that Y is semisimple, let $A \leq Y$. Then $X + A + N = M$. Arguing as above with $X + A$ replacing X, we have $X \oplus A = X + A$ is a direct summand of M. That is, $M = (X \oplus A) \oplus K$ for some submodule K of M. Then

$$Y = Y \cap M = Y \cap [(X \oplus A) \oplus K] = Y \cap [(A \oplus X) \oplus K] = A \oplus [Y \cap (X \oplus K)]$$

So A is a direct summand of Y. Therefore, Y is semisimple. Now, we will show that Y is projective. Write $Y = Z(Y) \oplus Y_n$ where Y_n is nonsingular. Then

$$M/(X \oplus Y_n) = (X \oplus Y)/(X \oplus Y_n) = (X \oplus Y_n \oplus Z(Y))/(X \oplus Y_n) \cong Z(Y)$$

which is singular. Since $M = (X + Y_n) + N$ and $N \ll_\delta M$, we have $X \oplus Y_n = M$. This shows that $Z(Y) = 0$. Since Y is semisimple and nonsingular, Y is projective by Corollary 2.5.

(ii)$\Rightarrow$ (iii) Let $M = X + N$ with M/X Goldie torsion. By (ii), $M = X \oplus Y$ where Y is projective and semisimple. It follows that $M/X \cong Y$ is Goldie torsion. Since Y is semisimple, we can write $Y = \bigoplus S_\alpha$, where S_α is simple for all α. Now S_α is simple and projective. Thus by Proposition 2.18, $Z(S_\alpha) \neq S_\alpha$, i.e., $Z(S_\alpha)$ is a proper submodule of S_α. Since S_α is simple $Z(S_\alpha) = 0$. Therefore $Z(Y) = \bigoplus Z(S_\alpha) = 0$. Since Y is Goldie torsion, $Y = Z(Y/Z(Y))$. From these we have $Z(Y) = 0$ and $Z(Y) = Y$, so $Y = 0$. Therefore, $M = X$.

(iii)$\Rightarrow$ (i) By (iii), we have $X + N = M$ with M/X Goldie torsion, then $X = M$. Since every singular module is Goldie torsion, (i) is true. □

Lemma 3.2 *(Zhou, Y., 2000) Let M be a module.*

(i) For submodules N, K, L of M with $K \subseteq N$, we have

(a) $N \ll_\delta M$ if and only if $K \ll_\delta M$ and $N/K \ll_\delta M/K$,

(b) $N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.

(ii) If $K \ll_\delta M$ and $f : M \to N$ is a homomorphism, then $f(K) \ll_\delta N$. In particular, if $K \ll_\delta M \subseteq N$ then $K \ll_\delta N$.

(iii) Let $K_1 \subseteq M_1 \subseteq M$ and $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$, then $K_1 \oplus K_2 \ll_\delta M$ if and only if $K_1 \ll_\delta M_1$ and $K_2 \ll_\delta M_2$.

Proof Let N, K, L are submodules of M with $K \subseteq N$.

(i)(a) Suppose that $N \ll_\delta M$. Then $K \subseteq N \ll_\delta M$. Let $K + X = M$ with M/X singular. Since $K+X = M$, we have $N+X = M$. Since $N+X = M$, M/X singular and $N \ll_\delta M$, we have $X = M$. Therefore, $K \ll_\delta M$. Let

$$N/K + X/K = M/K$$

with $(M/K)/(X/K) \cong M/X$ singular. Then $N/K+X/K = M/K$ implies that $N+X = M$. Since M/X is singular and $N \ll_\delta M$, we have $X = M$, i.e., $X/K = M/K$. Therefore $N/K \ll_\delta M/K$.

Conversely, suppose that $K \ll_\delta M$ and $N/K \ll_\delta M/K$. Let $N+X = M$ with M/X singular. Then $(N+X)/K = M/K$, so $N/K+(X+K)/K = M/K$. Since M/X singular, we have $M/(X + K)$ is singular by Proposition 2.16. Therefore, $(X + K)/K = M/K$, that is, $X + K = M$. Since $K \ll_\delta M$, $X + K = M$, and M/X singular we have $X = M$. Therefore $N \ll_\delta M$.

(i)(b) Suppose that $N + L \ll_\delta M$. Let $N + X = M$ with M/X singular. Then we have $N + L + X = M$ with M/X singular. By assumption, $X = M$. Therefore, $N \ll_\delta M$. Similarly, $L \ll_\delta M$.

Conversely, suppose that $N \ll_\delta M$ and $L \ll_\delta M$. Let $(N + L) + X = M$ with M/X singular. Then $N + (L + X) = M$ with M/X singular. Since M/X is singular, we have $M/(L + X)$ is singular, by Proposition 2.16. Now, $N \ll_\delta M$ gives $L + X = M$. Since $L \ll_\delta M$, we have $X = M$. Therefore $N + L \ll_\delta M$.

(ii) Suppose that $K \ll_\delta M$ and $f : M \to N$ is a homomorphism. Suppose that $f(K) + X = N$ with N/X singular. Then for all $m \in M$, we can write $f(m) = f(k) + x$, for some $k \in K, x \in X$. Then $m - k \in f^{-1}(X)$. Thus $m \in K + f^{-1}(X)$. Hence $M = K + f^{-1}(X)$. Since $K \ll_\delta M$, we have $M = K' \oplus f^{-1}(X)$, where K' is a projective and semisimple submodule of K, by Lemma 3.1. Thus, $M/f^{-1}(X) \cong K'$ is nonsingular. So, $\mathrm{Hom}(N/X, M/f^{-1}(X)) = 0$, by Proposition 3.1.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & N & \longrightarrow & N/X & \longrightarrow & 0 \\
\downarrow & & \uparrow & & \uparrow & & {\scriptstyle 0}\downarrow & & \downarrow \\
0 & \longrightarrow & f^{-1}(X) & \longrightarrow & M & \xrightarrow{0} & M/f^{-1}(X) & \longrightarrow & 0
\end{array}$$

Therefore, $M = f^{-1}(X)$. Thus, $f(K) \subseteq f(M) \subseteq X$. Hence, $X = N$.

For the last part, let f be the inclusion map from M to N. Then $K \ll_\delta M$ implies $K = f(K) \ll_\delta N$.

(iii) Let $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Suppose that $K_1 \oplus K_2 \ll_\delta M$. We know that δ-small submodules are preserved under homomorphisms. Now, consider the canonical projection $\pi_1 : M \to M_1$. Since $K_1 \oplus K_2 \ll_\delta M$, $\pi_1(K_1 \oplus K_2) \ll_\delta M_1$, i.e., $K_1 \ll_\delta M_1$. Also we have the canonical projection $\pi_2 : M \to M_2$. Since $K_1 \oplus K_2 \ll_\delta M$, $\pi_2(K_1 \oplus K_2) \ll_\delta M_2$, i.e., $K_2 \ll_\delta M_2$.

Conversely, suppose that $K_1 \ll_\delta M_1$ and $K_2 \ll_\delta M_2$. Then $K_1 \ll_\delta M$ and $K_2 \ll_\delta M$ by (ii). Therefore, $K_1 \oplus K_2 = K_1 + K_2 \ll_\delta M$, by (i)(b). □

Definition 3.2 *(Zhou, Y., 2000) Let $\wp$ be the class of all singular simple modules. For a module M, let*

$$\delta(M) = \mathrm{Rej}_M(\wp) = \bigcap\{N \subseteq M \mid M/N \in \wp\}$$

be the reject in M of $\wp$.

Lemma 3.3 *(Zhou, Y., 2000) Let M and N be modules.*

(i) $\delta(M) = \sum\{L \subseteq M \mid L$ *is a δ-small submodule of* $M\}$.

(ii) If $f : M \to N$ is an R-homomorphism, then $f(\delta(M)) \subseteq \delta(N)$. Therefore $\delta(M)$ is a fully invariant submodule of M and $M\delta(R) \subseteq \delta(M)$.

(iii) If $M = \bigoplus_{i\in I} M_i$, then $\delta(M) = \bigoplus_{i\in I} \delta(M_i)$.

(iv) If every proper submodule of M is contained in a maximal submodule of M, then $\delta(M)$ is the unique largest δ-small submodule of M.

Proof

(i) We know that

$$\delta(M) = \mathrm{Rej}_M(\wp) = \bigcap\{\mathrm{Ker}\, f \subseteq M \mid f : M \to U, U \text{ is a singular simple module}\}.$$

Let $A \ll_\delta M$. Then $f(A) \ll_\delta U$, i.e., $f(A) \neq U$. Since U is simple and $f(A) \subset U$, $f(A) = 0$, i.e., $A \subseteq \mathrm{Ker}\, f$. Therefore, $A \subseteq \delta(M)$.

Conversely, let

$$\begin{aligned} U_1 &= \sum\{L \subseteq M \mid L \text{ is a } \delta\text{-small submodule of } M\} \\ U_2 &= \bigcap\{N \subseteq M \mid M/N \in \wp\} \end{aligned}$$

Let $a \in U_2$. Suppose aR is not a δ-small submodule. Then there exists a maximal ideal K of M such that a is not an element of K, $aR+K = M$ with M/K singular. Since $K \leq_{\text{max}} M$, M/K is simple. But then we have M/K is singular and simple and a is not an element of K, this contradicts with $a \in U_2$. Therefore, aR is a δ-small submodule.

(ii) Suppose that $f : M \to N$ is an R-homomorphism. Then

$$f(\delta(M)) = \textstyle\sum_{L \ll_\delta M} f(L).$$

Since $L \ll_\delta M$, $f(L) \ll_\delta N$. Thus, $f(\delta(M)) \subseteq \delta(N)$. Let m be a fixed element of M. Then $f_m : R_R \to M_R$ given by $f(r) = mr$, $r \in R$ is a homomorphism. Then $f_m(\delta(R_R)) = m\delta(R_R)$. Since $f_m(\delta(R_R)) \subseteq \delta(M_R)$, we have $m\delta(R_R) \subseteq \delta(M_R)$. Hence

$$\sum_{m \in M} m\delta(R_R) = M\delta(R_R) \subseteq \delta(M_R).$$

(iii) Let $M = \bigoplus_{i \in I} M_i$. Then $\delta(M_i) \subseteq M$ by (ii). Since $\delta(M_i) \subseteq (M_i)$, we have

$$\sum \delta(M_i) = \bigoplus \delta(M_i) \subseteq \delta(M).$$

Thus, $\bigoplus \delta(M_i) \subseteq \delta(M)$.

Conversely, let $m \in \delta(M)$. Then $m = \sum_{i \in I' \subseteq I} m_i$, I' is finite, and let $\pi_i : M \to M_i$ be the i^{th} projection. Then $\pi_i(m) = m_i \in \delta(M_i)$. Thus, $m \in \bigoplus \delta(M_i)$. So $\delta(M) \subseteq \bigoplus \delta(M_i)$. Therefore, we obtain that

$$\delta(M) = \bigoplus \delta(M_i).$$

(iv) Suppose that every proper submodule of M is contained in a maximal submodule of M. Let $L \lneqq M$ and M/L singular. Then there exists $K \leq_{\text{max}} M$ such that $L \subseteq K$. Since M/L is singular, we have $M/(L+K) = M/K$ is singular. Thus, $\delta(M) \subseteq K$. Then $L + \delta(M) \subseteq K \neq M$. Therefore, $L + \delta(M) \neq M$, i.e., $\delta(M)$ is δ-small in M. Since $\delta(M)$ is the sum of all δ-small submodules of M and $\delta(M)$ is δ-small in M, $\delta(M)$ is the largest δ-small submodule of M. □

Next we give some descriptions of $\delta(R_R)$ and some properties of R related to $\delta(R_R)$. From now on, let $\delta(R) = \delta(R_R), \mathrm{Soc}(R) = \mathrm{Soc}(R_R)$. For a module M, with $I \subseteq R$ and $X \subseteq M$, let

$$\begin{aligned} r_R(X) &= \{a \in R \mid Xa = 0\} \\ l_M(I) &= \{x \in M \mid xI = 0\}. \end{aligned}$$

Theorem 3.1 *(Zhou, Y., 2000) Given a ring R, each of the following sets is equal to $\delta(R)$:*

(i) $R_1 =$ *the intersection of all essential maximal right ideals of* R*,*

(ii) $R_2 =$ *the unique largest* δ*- small right ideal of* R*,*

(iii) $R_3 = \{x \in R \mid xR + K = R \Rightarrow K \leq_{\oplus} R\}$*,*

(iv) $R_4 = \bigcap\{$ *ideals* P *of* $R \mid R/P$ *has a faithful singular simple module*$\}$*,*

(v) $R_5 = \{x \in R \mid \forall y \in R, \exists$ *a semisimple right ideal* Y *of* R *such that* $(1 - xy)R \oplus Y = R\}$.

Proof

(i) For a right ideal I of R, R/I is a singular simple module if and only if I is an essential maximal right ideal of R. Thus $\delta(R) = R_1$.

(ii) By Lemma 3.3, $\delta(R) = R_2$.

(iii) Because of Lemma 3.1, it is easy to check that for $x \in R$, $xR \ll_\delta R$ if and only if $x \in R_3$. Thus $\delta(R) = R_3$.

(iv) An ideal P of R is such that R/P has a faithful singular simple module if and only if $P = r_R(M) = Rej_R(M)$ for a singular simple module M. Let F be a complete set of representatives of the singular simple modules. Then

$$\delta(R) = \mathrm{Rej}_R(\wp) = \mathrm{Rej}_R(\textstyle\prod_F M) = \displaystyle\bigcap_F \mathrm{Rej}_R(M)$$

Thus $\delta(R) = R_4$.

(v) Let $x \in \delta(R)$. For $y \in R$, we have $xy \in \delta(R)$. So $(xy)R \ll_\delta R$. Since $R = (1 - xy)R + (xy)R$ and $(xy)R \ll_\delta R$, we have by Lemma 3.1, $(1 - xy)R \oplus Y = R$ for a semisimple right ideal Y of R. Thus, $x \in R_5$.

Conversely, suppose $x \in R - \delta(R)$. Then x is not an element of N for some essential maximal right ideal N of R. So, $xR + N = R$. Write $1 = xy + n$ where $y \in R$, $n \in N$. If $x \in R_5$, then

$$R = (1 - xy)R \oplus Y = nR \oplus Y$$

for some semisimple right ideal Y of R. Since $n \in N$, we have $nR \subseteq N$, and since Y is a semisimple right ideal, we have $Y \subseteq \mathrm{Soc}(R) \subseteq N$. Therefore, $R = nR + Y \subseteq N$, a contradiction. So, x is not an element of R_5. This shows that $\delta(R) = R_5$. $\square$

Corollary 3.1 *(Zhou, Y., 2000) For a ring* R*,* $\delta(R)/\mathrm{Soc}(R) = J\big(R/\mathrm{Soc}(R)\big)$*. In particular,* $R = \delta(R)$ *if and only if* R *is a semisimple ring.*

Proof Let $x + \mathrm{Soc}(R) \in \delta(R)/\mathrm{Soc}(R)$, $x \in \delta(R)$. By Theorem 3.1 (v), for all $y \in R$ there exists a semisimple right ideal Y of R such that $R = (1 - xy)R \oplus Y$. Then we can write $1 + \mathrm{Soc}(R) = [(1 - xy)r + y] + \mathrm{Soc}(R)$. Since $y \in Y$ and Y is semisimple, $y \in \mathrm{Soc}(R)$, so

$$1+\operatorname{Soc}(R)=(1-xy)r+\operatorname{Soc}(R)=[(1-xy)+\operatorname{Soc}(R)][r+\operatorname{Soc}(R)].$$

Thus, $1-xy+\operatorname{Soc}(R)$ is right invertible in $R/\operatorname{Soc}(R)$ for all $y\in R$. Hence, $x\in J(R/\operatorname{Soc}(R))$. Therefore, $J\big(R/\operatorname{Soc}(R)\big)\supseteq\delta(R)/\operatorname{Soc}(R)$.

Conversely, since $J\big(R/\operatorname{Soc}(R)\big)$ is the intersection of all maximal submodules of R which contains $\operatorname{Soc}(R)$, it is contained in the intersection of all maximal essential ideals of R which contains $\operatorname{Soc}(R)$. Therefore, $J\big(R/\operatorname{Soc}(R)\big)\subseteq\delta(R)/\operatorname{Soc}(R)$. □

Theorem 3.2 *(Zhou, Y., 2000) The following are equivalent for a ring R:*

(i) $R/\delta(R)$ *is a semisimple ring,*

(ii) Every direct product of singular semisimple modules is semisimple,

(iii) $\operatorname{Soc}(M)\cap Z(M)=l_M(\delta(R))$ *for any R-module M.*

In this case, $M\delta(R)=\delta(M)$ for any module M.

Proof

(i)⇒ (iii) Suppose that $R/\delta(R)$ is a semisimple ring. We know that

$$\big(\operatorname{Soc}(M)\cap Z(M)\big)\delta(R)\subseteq\delta\big(\operatorname{Soc}(M)\cap Z(M)\big).$$

Let $L\subseteq\operatorname{Soc}(M)\cap Z(M)$. That is, L is semisimple and singular. So $\delta(L)=0$. Therefore, $\delta\big(\operatorname{Soc}(M)\cap Z(M)\big)=0$, i.e.,

$$\operatorname{Soc}(M)\cap Z(M)\subseteq l_M(\delta(R)).$$

Since $l_M(\delta(R))$ is an $R/\delta(R)$-module, $l_M(\delta(R))$ is semisimple. Thus,

$$l_M(\delta(R))\subseteq\operatorname{Soc}(M).$$

By (i) and Definition 3.2, $\delta(R)$ is a finite intersection of essential maximal right ideals. So $\delta(R)\leq_{\text{e}} R$ and thus

$$l_M(\delta(R))\subseteq Z(M).$$

Therefore,

$$\operatorname{Soc}(M)\cap Z(M)=l_M(\delta(R)).$$

(iii)⇒ (ii) Let M be a product of singular semisimple modules. Since $\delta(R)$ annihilates every singular semisimple module, we have $M=l_M(\delta(R))$. By (iii), $M=\operatorname{Soc}(M)$.

(ii)⇒ (i) $R/\delta(R)$ is embeddable in a product of singular simple modules, and so $R\delta(R)$ is semisimple by (ii).

For the last statement, note that $M/M\delta(R)$ is a semisimple $R/\delta(R)$-module and hence a semisimple R-module. Write $M/M\delta(R)=S\oplus N$, where S is singular and N is non-singular. Since $N\delta(R)=0$, we have $N=0$, by (iii). Thus,

$$\delta(M/M\delta(R)) = \delta(S) = 0.$$

But by Lemma 3.3,

$$[\delta(M) + M\delta(R)]/M\delta(R) \subseteq \delta(M/M\delta(R)).$$

It follows that $\delta(M) \subseteq M\delta(R)$, and so $\delta(M) = M\delta(R)$, by Lemma 3.3. $\square$

Lemma 3.4 *(Zhou, Y., 2000) If P is a projective module, then $\delta(P) = P\delta(R)$ and $\delta(P)$ is the intersection of all essential maximal submodules of P.*

Proof Since P is a projective module, P is a direct summand of a free module. Assume that $P \oplus P' = R^{(\Delta)}$. Then by Lemma 3.3,

$$\delta(P) \oplus \delta(P') = \delta\big(R^{(\Delta)}\big) = \big(\delta(R)\big)^{(\Delta)} = R^{(\Delta)}\delta(R) = P\delta(R) \oplus P'\delta(R).$$

Since $P\delta(R) \subseteq \delta(P)$ and $P'\delta(R) \subseteq \delta(P')$, we must have $P\delta(R) = \delta(P)$. We know that $\delta(R)$ is the intersection of all essential maximal right ideals of R, so $\delta(P) = P\delta(R)$ is the intersection of all essential maximal submodules of P. $\square$

3.2. Projective δ-covers

In this section, the notion of projective δ-covers is defined. Unlike projective covers, the projective δ-covers of a module are not unique up to isomorphism, but they differ by only a projective semisimple direct summand.

Definition 3.3 *(Zhou, Y., 2000) A pair $(P; p)$ is called a projective δ-cover of the module M if P is projective and p is an epimorphism of P onto M with $\mathrm{Ker}(p) \ll_\delta P$.*

Every projective cover of M is a projective δ-cover of M. As we will see later, some modules may not have projective δ-covers and some modules have projective δ-covers but no projective covers.

Lemma 3.5 *(Zhou, Y., 2000) Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be such that all $\rho_i : P_i \to M_i$ are projective δ-covers. Let $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$. Then $\rho = \bigoplus \rho_i : P \to M$ is a projective δ-cover.*

Proof By Lemma 3.3, $\mathrm{Ker}(\rho) = \bigoplus \mathrm{Ker}(\rho_i)$ is δ-small in M. Since P_i, $i = 1, 2, \ldots, n$ are projective R-modules, we have $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ is a projective R-module. Therefore, $\rho = \bigoplus \rho_i : P \to M$ is a projective δ-cover of M. $\square$

Lemma 3.6 *(Zhou, Y., 2000) Let $p : P \to M$ be a projective δ-cover. If Q is projective and $q : Q \to M$ is an epimorphism, then there exist decompositions $P = A \oplus B$ and $Q = X \oplus Y$ such that*

(i) $A \cong X$,

(ii) $p|_A : A \to M$ *is a projective δ-cover,*

(iii) $q|_X : X \to M$ *is a projective δ-cover,*

(iv) B *is a projective semisimple module with* $B \subseteq \mathrm{Ker}(p)$ *and* $Y \subseteq \mathrm{Ker}(q)$.

Proof

$$\begin{tikzcd} & Q \arrow[dl, dashed, "h"'] \arrow[d, "q"] & \\ P \arrow[r, "p"] & M \arrow[r] & 0 \end{tikzcd}$$

Since Q is projective, there exists $h : Q \to P$ such that $q = ph$. Thus, we have

$$P = h(Q) + \mathrm{Ker}(p).$$

By Lemma 3.3, since $\mathrm{Ker}(p) \ll_\delta P$, we have $P = h(Q) \oplus B$, where B is a projective semisimple submodule with $B \subseteq \mathrm{Ker}(p)$.

$$p|_A(A) = p(A) = p(A + B) = p(P) = M,$$

that is, $p|_A$ is an epimorphism. Since $h(Q) = A$ is a direct summand of P, A is projective. Since $\mathrm{Ker}(p|_A) \subseteq \mathrm{Ker}(p) \ll_\delta P$, $\mathrm{Ker}(p|_A) \ll_\delta P$. So, $p|_A : A \to M$ is a projective δ-cover.

Since A is projective, $h : Q \to A$ splits. So, there exists $g : A \to Q$ such that $hg = 1_A$. Thus

$$Q = X \oplus Y = \mathrm{Im}\, g \oplus \mathrm{Ker}(h).$$

This gives $A \cong g(A) = X$. Since $\mathrm{Ker}(p|_A) \ll_\delta A$, we have that

$$\mathrm{Ker}(q|_X) = \mathrm{Ker}(p|_A) \ll_\delta g(A) = X,$$

by Lemma 3.3. Note that

$$q(X) = (ph)(X) = (ph)(X + Y) = (ph)(Q) = q(Q) = M.$$

Thus, $q|_X : X \to M$ is a projective δ-cover. $\square$

Lemma 3.7 *(Zhou, Y., 2000) Let P be a projective module and N be a submodule of P. Then the following are equivalent:*

(i) *P/N has a projective δ-cover,*

(ii) *$P = P_1 \oplus P_2$ for some P_1 and P_2 with $P_1 \subseteq N$ and $P_2 \cap N \ll_\delta P$.*

Proof (i)⇒(ii) Consider a projective δ-cover $q : Q \to P/N$. Let $p : P \to P/N$ be the canonical epimorphism.

$$\begin{tikzcd} & Q \arrow[dl, dashed, "h"'] \arrow[d, "q"] & \\ P \arrow[r, "p"] & P/N \arrow[r] & 0 \end{tikzcd}$$

Then we have $q = ph$. So,

$$p = \mathrm{Ker}(p) + \mathrm{Im}\, h = N + \mathrm{Im}\, h.$$

By Lemma 3.3 , there exists a decomposition $P = X \oplus Y$ such that $p|_X : X \to P/N$ is a projective δ-cover and $Y \subseteq \mathrm{Ker}(p) = N$. Thus $X \cap N = \mathrm{Ker}(p|_X) \ll_\delta X$.Since X is a direct summand of P, $X \cap N \ll_\delta P$ by Lemma 3.3. Now let $P_1 = Y$ and $P_2 = X$.
(ii)⇒(i) Suppose (ii) holds. Let $p : P_2 \to P/N$ be the canonical epimorphism. Then $\mathrm{Ker}\, p = N \cap P_2 \ll_\delta P$. Hence, $\mathrm{Ker}\, p$ is δ-small in P_2, by Lemma 3.3. So (P_2, p) is a projective δ-cover of P/N. □

3.3. Rings Over Which Certain Modules Have Projective δ-covers

In this section various characterizations and properties are obtained for a ring R, for which every R-module (respectively simple R-module, cyclically presented R-module) has a projective δ-cover.

Definition 3.4 *(Zhou, Y., 2000) A ring R is called δ-perfect (respectively δ-semiperfect, δ-semiregular) if every R-module (respectively simple R-module, cyclically presented R-module) has a projective δ-cover.*

Examples 3.2

(i) *Every right perfect ring is δ-perfect.*

(ii) *Semiperfect rings and δ-perfect rings are δ-semiperfect.*

We will show later that a ring R is called δ-semiperfect if and only if every finitely generated R-module has a projective δ-cover. First we characterize the δ-semiregular rings.

Lemma 3.8 *(Zhou, Y., 2000) Let $x \in M$. The following are equivalent:*

(i) There exists a decomposition $M = A \oplus B$ such that A is projective, $A \subseteq xR$ and $xR \cap B \ll_\delta M$,

(ii) There exists $\alpha \in M^*$ such that $(\alpha(x))^2 = \alpha(x)$ and $x - x\alpha(x) \in \delta(M)$.

Lemma 3.9 *(Zhou, Y., 2000) The following are equivalent for a module M:*

(i) For any finitely generated submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2 \ll_\delta M$,

(ii) For any cyclic submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2 \ll_\delta M$,

(iii) Every finitely generated (or cyclic) submodule A of M can be written as $A = N \oplus S$, where N is a direct summand of M and $S \ll_\delta M$.

Proof (i)⇒(iii) Suppose that for any finitely generated submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $M_2 \cap A \ll_\delta M$. Then

$$A = A \cap (M_1 \oplus M_2) = M_1 \oplus (A \cap M_2).$$

(iii)⇒(ii) Let A be a cyclic submodule of M. Then by assumption, $A = N \oplus S$, where N is a direct summand of M and S is δ-small in M. Write $M = N \oplus N'$ and let $\pi : N \oplus N' \to N'$ be the projection. Then $A = N \oplus (A \cap N')$ and

$$A \cap N' = \pi(A \cap N') = \pi(N + (A \cap N')) = \pi(A) = \pi(N + S) = \pi(S).$$

Since $S \ll_\delta M$, $\pi(S) \ll_\delta N'$. Therefore $A \cap N' \ll_\delta N'$.
(i)⇒(ii) Clear.
(ii)⇒(i) The proof is obtained by induction on the number of generating elements of the submodules of M. The assertion in (ii) provides the basis.

Assume the assertion to be proved for submodules with $n-1$ generating elements and consider

$$U = u_1R + \cdots + u_nR.$$

We choose an idempotent $e \in \mathrm{End}(M)$ with $e(M) \subseteq u_nR$. We can write

$$M = e(M) \oplus (1-e)M.$$

First we need to show that $u_nR \cap (1-e)(M) = (1-e)(u_nR)$. If $x \in u_nR \cap (1-e)(M)$, then we have

$$x = u_nr = (1-e)m = m - em.$$

Since $e(M) \subseteq u_nR$, $em \in u_nR$, thus $m \in u_nR$ and so $x \in (1-e)u_nR$. Conversely, if $x \in (1-e)u_nR$, then $x \in u_nR$. Also we have $(1-e)u_nR \subseteq (1-e)(M)$. Therefore, $x \in (1-e)M \cap u_nR$. Hence,

$$u_nR \cap (1-e)(M) = (1-e)(u_nR) \ll_\delta M.$$

Now we form $K = \sum_{i<n}(1-e)u_iR$. From $e(U) \subseteq u_nR \subset U$, we obtain the relation

$$U = (1-e)U + e(U) = K + u_nR.$$

By induction hypothesis, we find an idempotent $f \in \mathrm{End}(M)$ with

$$f(M) \subseteq K \text{ and } K \cap (1-f)(M) = (1-f)(K) \ll_\delta M.$$

Then we have $M = f(M) \oplus (1-f)(M)$. From $f(M) \subset K \subset (1-e)(M)$, we can write

$$f(m) \in K = \textstyle\sum_{i<n}(1-e)u_iR,$$

so

$$f(m) = (1-e)u_1r_1 + \cdots + (1-e)u_{n-1}r_{n-1}$$

and $(1-e)f(m) = f(m)$. Thus, $f(m) \in (1-e)f(M)$. Similarly, we can show that $(1-e)f(m) \in f(M)$. Hence, $(1-e)f = f$, that is, $ef = 0$. Let $g = e + f - fe$. Then

$$g^2 = (e+f-fe)(e+f-fe) = e+f-fe = g,$$

that is, $g = e+f-fe$ is an idempotent. So we can write $M = g(M) \oplus (1-g)(M)$. Then,

$$g(M) \subseteq f(M) + e(M) \subseteq K + u_nR = U$$

We have $1-g = 1-e-f+ef = (1-f) + e(f-1) = (1-f)(1-e)$. So,

$$\begin{aligned}(1-f)(1-e)(U) &= (1-f)(1-e)(K) + (1-f)(1-e)(u_nR)\\ &\subseteq (1-f)(K) + (1-f)(1-e)(u_nR)\end{aligned}$$

Since $(1-f)(K) \ll_\delta M$ and $(1-e)(u_nR) \ll_\delta M$, we have

$$(1-f)(K) + (1-f)(1-e)(u_nR) \ll_\delta M$$

and hence $(1-f)(1-e)(U) \ll_\delta M$. Therefore,

$$(1-g)(M) \cap U = (1-f)(1-e)(U) \subset (1-f)(K) + (1-f)(1-e)u_nR \ll_\delta M.$$

□

Lemma 3.10 *(Zhou, Y., 2000) The following are equivalent for a module M:*

(i) For any submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $A \cap M_2 \ll_\delta M$,

(ii) Every submodule A of M can be written as $A = N \oplus S$ with N is a direct summand of M and S is δ-small in M.

Proof (i)⇒(ii) Suppose that there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq A$ and $M_2 \cap A \ll_\delta M$. Then

$$A = A \cap (M_1 \oplus M_2) = M_1 \oplus (A \cap M_2).$$

Say $N = M_1$ and $S = A \cap M_2$.
(ii)⇒(i) Let $A \leq M$, $A = N \oplus S$ with N is a direct summand of M and S is δ-small in M. Write $M = N \oplus N'$ and let $\pi : N \oplus N' \to N'$ be the projection. Then $A = N \oplus (A \cap N')$ and

$$A \cap N' = \pi(A \cap N') = \pi(N + (A \cap N')) = \pi(A) = \pi(N + S) = \pi(S).$$

Since $S \ll_\delta M$, we have $\pi(S) \ll_\delta N'$, that is,

$$\pi(S) = A \cap N' \ll_\delta N' \subseteq M.$$

Therefore, $A \cap N' \ll_\delta M$. □

Theorem 3.3 *(Zhou, Y., 2000) The following are equivalent for a ring R:*

(i) R is a δ-semiregular ring,

(ii) Every finitely presented R-module has a projective δ-cover,

(iii) Every finitely generated (or cyclic) right ideal I of R can be written as $I = eR \oplus S$, where $e = e^2 \in R$ and $S \subseteq \delta(R)$,

(iv) $R/\delta(R)$ is a Von-Neumann regular ring and idempotents lift modulo $\delta(R)$,

(v) For any $a \in R$, there exists $b \in R$ such that $(ba)^2 = ba$ and $a - aba \in \delta(R)$,

(vi) For any $a \in R$, there exists $b \in R$ such that $(ab)^2 = ab$ and $a - aba \in \delta(R)$.

Proof Let $\overline{R} = R/\delta(R)$ be the factor ring and for any $x \in R$ let $\overline{x} = x + \delta(R)$.
(i)⇒(iv) Let $a \in R$. Since R is δ-semiregular, every cyclically presented R-module has a projective δ-cover. So, R/aR has a projective δ-cover. Thus, by Lemma 3.7, aR lies over a projective direct summand of R. Hence, there exists a decomposition $R = I \oplus J$ such that $I \subseteq aR$ and $aR \cap J \ll_\delta R$. Then $aR = I \oplus (aR \cap J)$. By Theorem 3.1, $aR \cap J \subseteq \delta(R)$. Write $I = eR$ for an idempotent $e \in R$. Then

$$\big(aR+\delta(R)\big)/\delta(R)=\big(eR+(aR\cap J)+\delta(R)\big)/\delta(R)=\big(eR+\delta(R)\big)/\delta(R).$$

So, $\overline{a}\overline{R}=\overline{e}\overline{R}$ is a direct summand of $\overline{R}$. Since every finitely generated (cyclic) submodule is a projective direct summand, $\overline{R}$ is a regular ring.

To see the second part, let $\overline{a}$ be an idempotent in $\overline{R}$. Then $a^2+\delta(R)=a+\delta(R)$. As above $\overline{a}\overline{R}=\overline{e}\overline{R}$. We can write

$$\overline{R}=\overline{a}\overline{R}\oplus(\overline{1}-\overline{a})\overline{R} \text{ and } \overline{R}=\overline{e}\overline{R}\oplus(\overline{1}-\overline{e})\overline{R}.$$

Since $\overline{a}\overline{R}=\overline{e}\overline{R}$ we have $(\overline{1}-\overline{e})\overline{a}=\overline{0}$, that is, $\overline{ea}=\overline{a}$ and $(\overline{1}-\overline{a})\overline{e}=\overline{0}$, that is, $\overline{ae}=\overline{e}$. Let $f=e+ea(1-e)$. Then

$$f^2=(e+ea(1-e))(e+ea(1-e))=e+ea(1-e)=f.$$

So, f is an idempotent in R.

$$\overline{f}=\overline{e}+\overline{ea}(\overline{1}-\overline{e})=\overline{e}+\overline{a}(\overline{1}-\overline{e})=\overline{e}+\overline{a}-\overline{ae}=\overline{a}.$$

Therefore, $\overline{f}=\overline{a}$, that is, idempotents lift module $\delta(R)$.

(iv)⇒(i) Let $a\in R$. Since $\overline{R}$ is regular, $\overline{a}\overline{R}$ is a direct summand of $\overline{R}$. By (iv), there exists an idempotent $e\in R$ such that $\overline{a}\overline{R}=\overline{e}\overline{R}$. Thus,

$$\overline{R}=\overline{a}\overline{R}\oplus(\overline{1}-\overline{e})\overline{R}.$$

It follows that $R=aR+(1-e)R+\delta(R)$ and $aR\cap(1-e)R\subseteq\delta(R)$. Note that $\delta(R)\ll_\delta R$. By Lemma 3.1, $R=[aR+(1-e)R]\oplus X$ where X is a projective semisimple right ideal of R. Thus,

$$\begin{aligned}R/aR &= \big(aR+(1-e)R\big)\oplus X)/aR\cong\big(aR+(1-e)R\big)/aR\oplus X\\ &\cong (1-e)R/(aR\cap(1-e)R)\oplus X.\end{aligned}$$

Since $aR\cap(1-e)R\subseteq\delta(R)$, $aR\cap(1-e)R\ll_\delta(1-e)R$, by Lemma 3.2. So,

$$\phi:(1-e)R\to(1-e)R/\big(aR\cap(1-e)R\big)$$

is a projective δ-cover of $(1-e)R/\big(aR\cap(1-e)R\big)$. Therefore,

$$\varphi:(1-e)R\oplus X\to(1-e)R/\big(aR\cap(1-e)R\big)\oplus X\cong R/aR$$

is a projective δ-cover of R/aR.

(i)⇒(ii) It suffices to show that for any finitely generated free module F, and any finitely generated submodule X of F, F/X has a projective δ-cover. Because of Lemma 3.9, we can assume $X=xR$ is a cyclic submodule. By (i), we can write $F=F_1\oplus F_2$ and assume any factor module F_i $(i = 1, 2)$ modulo a cyclic submodule has a projective δ-cover. Write $x = x_1 + x_2$, where $x_1 \in F_1$ and $x_2 \in F_2$. By Lemmas 3.7 and 3.8, there exists $\alpha \in F_1^*$ such that

$$(\alpha(x_1))^2 = \alpha(x_1) \text{ and } x_1 - x_1\alpha(x_1) \in \delta(F_1).$$

Extend α to F by defining $\alpha(F_2) = 0$. Let

$$y = x - x\alpha(x) \text{ and } y_i = x_i - x_i\alpha(x_1) \text{ for } i = 1, 2.$$

Since $y_2 \in F_2$, and F_2/y_2R has a projective δ-cover, by Lemmas 3.7 and 3.8, there exists $\beta \in F_2^*$ such that $\beta(y_2)$ is an idempotent of R and $y_2 - y_2\beta(y_2) \in \delta(F_2)$. Extend β to F by letting $\beta(F_1) = 0$. Let

$$e = \alpha(x) = \alpha(x_1) \text{ and } f = \beta(y) = \beta(y_2).$$

Since $fe = 0$, $e + f - ef$ is an idempotent. Define

$$\gamma = \alpha + (1 - e)(\beta - \beta(x)\alpha) \in F^*.$$

Then

$$\gamma(x) = e + (1 - e)f = e + f - ef$$

is an idempotent and

$$\begin{aligned} x - x\gamma(x) &= x - xe - xf + xef = (x - xe) - (x - xe)f = y - y\beta(y) \\ &= y_1(1 - \beta(y_2)) + (y_2 - y_2\beta(y_2)) \in \delta(F_1) + \delta(F_2) \subseteq \delta(F). \end{aligned}$$

By Lemmas 3.7 and 3.8, F/xR has a projective δ-cover.

(ii)$\Rightarrow$(iii) Suppose that every finitely presented module has a projective δ-cover. Then R/aR has a projective δ-cover. By Lemma 3.7, $R = A \oplus B$ for some A and B with $A \subseteq aR$ and $B \cap aR \ll_\delta R$. Then $A = eR$ for an idempotent $e \in R$. Thus,

$$aR = A \oplus (aR \cap B) = eR \oplus S \text{ where } S \subseteq \delta(R).$$

(iii)$\Rightarrow$(i) Suppose that every cyclic right ideal $I = aR$ of R can be written as $aR = eR \oplus S$, where $e^2 = e \in R$ and $S \subseteq \delta(R)$. Then by Lemma 3.9, $R = A \oplus B$ such that $A \subseteq aR$ and $B \cap aR \subseteq \delta(R)$. Therefore, by Lemma 3.7, R/aR has a projective δ-cover. Thus R is δ-semiregular.

(iii)$\Rightarrow$(vi) For any $a \in R$, there exists $e^2 = e \in R$ such that $aR = eR \oplus S$, where $S \subseteq \delta(R)$. Since $eR \leq aR$, write $e = ab$ and $a = er + s$ where $b, r \in R$ and $s \in S$. Then we have, $e^2 = (ab)^2 = e = ab$ and $ea = er + es$.

$$a - aba = a - ea = er + s - er - es = s - es = s \in S \subseteq \delta(R),$$

that is, $a - aba \in \delta(R)$.
(vi)⇒(iii) For $I = aR$, we have $b \in R$ such that $(ab)^2 = ab$ and $a - aba \in \delta(R)$. Let $e = ab$. For any $r \in R$, $ar = ear + (a - ea)r$. Therefore $I = eR \oplus S$, where $S = (a - ea)R \subseteq \delta(R)$.
(i)⇒(v) Suppose that R is a δ-semiregular ring. Let $a \in R$. Then R/aR has a projective δ-cover. By Lemma 3.7, $R = A \oplus B$ such that $A \subseteq aR$ and $B \cap aR \subseteq \delta(R)$. So by Lemma 3.8, there exists $b \in R$ such that $(ba)^2 = ba$ and $a - aba \in \delta(R)$.
(v)⇒(i) Suppose that for any $a \in R$, there exists $b \in R$ such that $(ba)^2 = ba$ and $a - aba \in \delta(R)$. Then by Lemma 3.8, there exists a decomposition $R = A \oplus B$ such that $A \subseteq aR$ and $B \cap aR \ll_\delta R$. Then by Lemma 3.7, R/aR has a projective δ-cover, that is, R is a δ-semiregular ring. □

Next we characterize the δ-semiperfect rings.

Theorem 3.4 *(Zhou, Y., 2000) The following statements are equivalent for a ring R:*

(i) R is a δ-semiperfect ring,

(ii) Every finitely generated R-module has a projective δ-cover,

(iii) Every right ideal I of R can be written as $I = eR \oplus S$, where $e = e^2 \in R$ and $S \subseteq \delta(R)$,

(iv) $R/\delta(R)$ is a semisimple ring and idempotents lift modulo $\delta(R)$,

(v) There exists a complete orthogonal set of idempotents $e_1, e_2, \ldots, e_n$ such that, for each i, either e_iR is a simple R-module or e_iR has a unique essential maximal submodule,

(vi) For any countably generated right ideal I, R/I has a projective δ-cover.

Proof (i)⇒(ii) Suppose that R is a δ-semiperfect ring, that is, every simple R-module has a projective δ-cover. So we can form a set Γ of R-modules such that every module in Γ is a projective δ-cover of some simple module and every simple module has a projective δ-cover in Γ. Thus, Γ generates every R-module. Let M be a finitely generated R-module. We may assume M is not semisimple. Then M has a proper essential submodule N. Since M is finitely generated, there exists a maximal submodule $L \leq M$ such that $N \subseteq L$. By Lemma 3.3 (ii),

$$M\delta(R) \subseteq \delta(M) \subseteq L \subset M.$$

Thus, $M/M\delta(R) \neq \overline{0}$. There exists $P_i \in \Gamma$ $(i = 1, 2, \ldots, n)$ such that

$$P = P_1 \oplus P_2 \oplus \cdots \oplus P_n \xrightarrow{\mu} M \longrightarrow 0$$

Since $\mu(P\delta(R)) = \mu(P)\delta(R) = M\delta(R)$, μ induces an epimorphism

$$P_1/P_1\delta(R) \oplus P_2/P_2\delta(R) \oplus \cdots \oplus P_n/P_n\delta(R) \cong P/P\delta(R) \to M/M\delta(R) \to 0.$$

Since P_i is a projective δ-cover of a simple module, P_i contains a δ-small maximal submodule X_i. Thus,

$$X_i \subsetneq \delta(P_i) = P_i\delta(R) \subseteq P_i.$$

This shows that $P_i/P_i\delta(R)$ is simple or 0. Hence, $M/M\delta(R)$ is a finite direct sum of simple modules. By Lemma 3.5, $M/M\delta(R)$ has a projective δ-cover. Note that $\nu\mu : P \to M/M\delta(R)$ is onto, where $\nu : M \to M/M\delta(R)$ is the natural homomorphism of M onto $M/M\delta(R)$. By Lemma 3.6, P has a decomposition $P = X \oplus Y$ such that $(\nu\mu)|_X : X \to M/M\delta(R)$ is a projective δ-cover.

$$\begin{tikzcd}
& X \arrow[d, "\mu"] \arrow[dl, dashed, "\mu"'] & \\
M \arrow[r, "\nu"] & M/M\delta(R) \arrow[r] & 0
\end{tikzcd}$$

$$M = \mu(X) + M/M\delta(R).$$

By (ii) and (iv) in Lemma 3.3, we have $M\delta(R) \ll_\delta M$ and so by Lemma 3.1, we have $M = \mu(X) \oplus Z$ for a projective submodule $Z \leq M\delta(R)$. Note that

$$\mathrm{Ker}(\mu|_X) = \mathrm{Ker}((\nu\mu)|_X) \ll_\delta X.$$

So $\mu|_X : X \to \mu(X)$ is a projective δ-cover of $\mu(X)$. Thus

$$\mu|_X \oplus f : X \oplus Z \to \mu(X) \oplus Z = M$$

is a projective δ-cover of M.

(ii)⇒(iii) Let I be a right ideal of R. Then $R/I =< 1 + I >$ is cyclic. So it has a projective δ-cover. By Lemma 3.7, $R = eR \oplus (1-e)R$ such that

$$eR \subseteq I \text{ and } (1-e)R \cap I \ll_\delta R.$$

Then

$$I = I \cap [eR \oplus (1-e)R] = eR \oplus [I \cap (1-e)R].$$

Say $S = I \cap (1-e)R$.

(iii)⇒(i) Let S be a simple R-module. By (iii),

$$S \cong eR \text{ where } R = eR \oplus (1-e)R.$$

Thus, S is projective. So, every simple R-module has a projective δ-cover, i.e., R is a δ-semiperfect ring.

(iii)⇒(iv) Suppose that every finitely generated module has a projective δ-cover and every right ideal I of R can be written as $I = eR \oplus S$, where $e = e^2 \in R$ and $S \subseteq \delta(R)$. By (iii), and Theorem 3.3 (iii), every idempotent of $R/\delta(R)$ can be lifted to an idempotent of R. Let $I + \delta(R)/\delta(R)$ be a submodule of $R/\delta(R)$. Then by assumption, $I = eR \oplus S$, where $e = e^2 \in R$ and $S \subseteq \delta(R)$. Then

$$R/\delta(R) = (e + \delta(R)) \oplus \big((1-e) + \delta(R)\big)/\delta(R) = I \oplus \big(1 - e + \delta(R)\big)/\delta(R).$$

Therefore $\overline{R}$ is semisimple.

(iv)⇒(i) Suppose that $R/\delta(R)$ is a semisimple ring and idempotents lift modulo $\delta(R)$. Let X be a singular simple R-module. Then $X\delta(R) = 0$, so X is a simple $R/\delta(R)$-module. Since $R/\delta(R)$ is semisimple, $X \cong I/\delta(R)$ as $R/\delta(R)$-module, where $I/\delta(R)$ is a direct summand of $R/\delta(R)$. Then there exists $e = e^2 \in R$ such that

$$X \cong I/\delta(R) = (eR + \delta(R))/\delta(R).$$

Thus we have

$$X \cong eR + \delta(R)/\delta(R) = eR/eR \cap \delta(R) = eR/e\delta(R)$$

as R-modules. By Lemma 3.3 (iv), $\delta(eR) = eR\delta(R) = e\delta(R) \ll_\delta eR$. So eR is a projective δ-cover of X.

To prove (iv)⇒(v), we need the following proposition:

Proposition 3.1 *(Bland, Paul E., 2010) Let $I_1, I_2, \ldots, I_n$ be left ideals of the ring R. Then the following are equivalent about the left R-module R:*

(i) $R = I_1 \oplus I_2 \oplus \cdots \oplus I_n$,

(ii) Each element $r \in R$, has a unique expression $r = r_1 + \cdots + r_n, r_i \in I_i (i = 1, 2, \ldots, n)$,

(iii) There exists a (necessarily unique) complete set $e_1, \ldots, e_n$ of pairwise orthogonal idempotents in R with $I_i = Re_i (i = 1, 2 \ldots, n)$.

Note in particular that if $e_1, \ldots, e_n$ are idempotents in R that satisfy (iii), then for each $r \in R$, $r = re_1 + re_2 + \cdots + re_n$.

Now we can give the proof.

(iv)⇒(v) Let $\overline{R} = R/\delta(R)$ be the factor ring and $\overline{a} = a + \delta(R)$ for any $a \in R$. Since $\overline{R}$ is semisimple, $\overline{R}$ is a direct sum of k minimal right ideals for some k. Let $I_1/\delta(R)$ be a minimal right ideal of $\overline{R}$ and hence a direct summand of $\overline{R}$. By assumption, there exists an idempotent f_1 of R such that $I_1 = f_1R + \delta(R)$. Thus, we have

$$I_1/\delta(R) \cong f_1R/f_1R \cap \delta(R) = f_1R/f_1\delta(R) = f_1R/\delta(f_1R).$$

Since $\mathrm{Soc}(R) \subseteq \delta(R)$ and $\overline{R}$ is semisimple, $\delta(R) \leq_{\mathrm{e}} R$. Thus, $f_1R/\delta(f_1R)$ is singular. It follows that $\delta(f_1R)$ is an essential maximal submodule of f_1R. By Lemma 3.4, $\delta(f_1R)$ is the unique essential maximal submodule of f_1R. If $(\overline{1} - \overline{f_1})\overline{R} \neq \overline{0}$, then it has a direct summand $I_2/\delta(R)$ which is a minimal right ideal of $\overline{R}$. It follows that

$$I_2 = [I_2 \cap (1 - f_1)R] + \delta(R).$$

Let $I = I_2 \cap (1 - f_1)R$. By Lemma 3.7, there exists a decomposition $(1 - f_1)R = X \oplus Y$ such that $X \subseteq I$ and $I \cap Y \ll_\delta (1 - f_1)R$. Thus, $I \cap Y \subseteq \delta(R)$. Write $X = fR$ with $f^2 = f$. Since $fR \subseteq (1 - f_1)R$, we have $f_1fR \subseteq 0$. Thus, $f_1f = 0$. Let $f_2 = f(1 - f_1)$. Then

$$\begin{aligned} f_2^2 = f(1-f_1)f(1-f_1) &= (f - ff_1)(f - ff_1) = f - ff_1 - ff_1f - ff_1ff_1 \\ &= f - ff_1 = f(1 - f_1) = f_2. \end{aligned}$$

Thus, $f_2^2 = f_2$ and $f_2f_1 = f_1f_2 = 0$. Also we have

$$f_2f = f(1 - f_1)f = (f - ff_1)f = f - ff_1f = f.$$

If $f_2 \in \delta(R)$, then $f = f_2f \in \delta(R)$. Thus, $I = X \oplus (I \cap Y) \subseteq \delta(R)$ and so $I_2 = \delta(R)$ which gives a contradiction. Hence, f_2 is not an element of $\delta(R)$. Since $I_2/\delta(R)$ is simple,

$$I_2 = \delta(R) = \big(f_2R + \delta(R)\big)/\delta(R) \cong f_2R/\big(f_2R \cap \delta(R)\big) = f_2R/f_2\delta(R) = f_2R/\delta(f_2R)$$

As above $\delta(f_2R)$ is the unique essential maximal submodule of f_2R. By a simple induction, we can choose idempotents $f_1, f_2, \ldots, f_k$ in R such that

$$f_{i+1}(f_1 + \cdots + f_i) = (f_1 + \cdots + f_i)f_{i+1} = 0$$

for $i = 1, \ldots, k - 1$, each f_iR has a unique essential maximal submodule, each $[f_iR + \delta(R)]/\delta(R)$ is a minimal right ideal of $\overline{R}$, and $\overline{R} = \bigoplus_{i=1}^k [f_iR + \delta(R)/\delta(R)]$. It follows that $f_if_j = 0$ if $i \neq j$ and $1 \leq i, j \leq k$. Thus, $\sum_{i=1}^k f_iR = \bigoplus_{i=1}^k f_iR$ and $R = \sum_{i=1}^k f_iR + \delta(R)$. By Lemma 3.1,

$$R = (\textstyle\bigoplus_{i=1}^k f_iR) \oplus Y_{k+1} \oplus \cdots \oplus Y_n,$$

where each Y_j is a simple R-module. Now by Proposition 3.1, there exists a complete orthogonal set $\{e_i \mid i = 1, \ldots, n\}$ of idempotents such that $e_iR = f_iR$ for $i = 1, \ldots, k$ and $e_jR = Y_j$ for $j = k + 1, \ldots, n$.
(ii)$\Rightarrow$(vi) Suppose that every finitely generated R-module has a projective δ-cover. Let I be a countably generated right ideal of R. Since $R/I =< 1 + I >$ is finitely generated, by

assumption R/I has a projective δ-cover.

(v)⇒(i) Let $R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$. Let M be a simple R-module. Then either $e_iR \to M$ is a projective δ-cover of M or $e_iR \to e_iR/\delta(e_iR) \to M$ is a projective δ-cover of M. So, every simple R-module has a projective δ-cover. Therefore, R is δ-semiperfect.

(vi)⇒(iv) By Theorem 3.3, R is δ-semiregular. So, $R/\delta(R)$ is regular and idempotents of $R/\delta(R)$ lift to R. We need to show that $R/\delta(R)$ is semisimple. It is enough to show that $\overline{R}$ is right Noetherian. If not, there exists a family $\{u_i \mid i = 1, 2, \ldots, n\}$ of nonzero idempotents of $\overline{R}$ such that

$$u_1\overline{R} \subset u_2\overline{R} \subset \cdots$$

is a strictly ascending chain. For each i, write $u_{i+1}\overline{R} = u_i\overline{R} \oplus A_i$ for a right ideal A_i of $\overline{R}$ and $u_{i+1} = v_i + l_i$ with $v_i \in u_i\overline{R}$ and $l_i \in A_i$. Then $\overline{0} \neq l_i = l_i^2$ and $u_i\overline{R} = v_i\overline{R}$. For $k > i$, we have $l_i \in u_k\overline{R} = v_k\overline{R} \subseteq u_{k+1}\overline{R}$ and so

$$l_i = u_{k+1}l_i = (v_k + l_k)l_i = v_kl_i + l_kl_i.$$

shows $l_kl_i = \overline{0}$. For each i, there exists an idempotent e_i of R such that $\overline{e_i} = l_i$. Thus $e_ie_j \in \delta(R)$ for $i > j$. Let $L = e_1R + e_2R + \cdots$. By Lemma 3.7, $R = eR \oplus (1-e)R$, where $e = e^2$ such that $eR \subseteq L$ and $(1-e)R \cap L \ll_\delta R$. So $L = eR \oplus [(1-e)R \cap L]$ with $(1-e)R \cap L \subseteq \delta(R)$. Write $e = e_1r_1 + \cdots + e_nr_n$ for some n, where $r_i \in R$. For $i > n$, write $e_i = es_i + t_i$, where $s_i \in R$ and $t_i \in \delta(R)$. Then

$$\begin{aligned} e_i &= e_ie_i = e_i(es_i + t_i) = e_i[(e_1r_1 + \cdots + e_nr_n)s_i + t_i] \\ &= e_ie_1r_1s_i + \cdots + e_ie_nr_ns_i + e_it_i \in \delta(R), \end{aligned}$$

shows $l_i = \overline{e_i} = \overline{0}$ for all $i > n$, a contradiction. This contradiction shows that $\overline{R}$ is right Noetherian. Therefore $\overline{R}$ is semisimple. □

Theorem 3.5 *(Zhou, Y., 2000) The following statements are equivalent for a ring R:*

(i) $R/\operatorname{Soc}(R)$ is a right perfect ring,

(ii) $R/\delta(R)$ is semisimple and $\delta(M) \neq M$ for every non-semisimple module M,

(iii) $R/\delta(R)$ is semisimple and $\delta(M) \ll_\delta M$ for every non-semisimple module M,

(iv) $R/\delta(R)$ is semisimple and $\delta(M) \ll_\delta M$ for every module M.

Proof (i)⇒(ii) Since $R/\operatorname{Soc}(R)$ is a right perfect ring, $(R/\operatorname{Soc}(R))/J(R/\operatorname{Soc}(R))$ is semisimple. We know that $J(R/\operatorname{Soc}(R)) = \delta(R)/\operatorname{Soc}(R)$. Therefore, $R/\delta(R)$ is semisimple. Suppose $\delta(M) = M$ for a non-semisimple module M. Note that, since $R/\delta(R)$ is semisimple, $\delta(M) = M\delta(R)$. Thus, $M = \delta(M) = M\delta(R)$ is non-semisimple. So there exists a non-semisimple submodule. Then Ma_1R is not semisimple for some $a_1 \in \delta(R)$. But $Ma_1R = M\delta(R)a_1R$, so there exists $a_2 \in \delta(R)$ such that Ma_2a_1R is not semisimple. A simple induction shows that there exists a sequence $a_1, a_2, \ldots \in \delta(R)$ such that $Ma_na_{n-1}\cdots a_2a_1R$ is not semisimple for all n. Thus $Ma_na_{n-1}\cdots a_2a_1R \nsubseteq \operatorname{Soc}(R)$ for all n, i.e., $a_na_{n-1}\cdots a_2a_1R \nsubseteq \operatorname{Soc}(R)$ for all n. Therefore, $J(R/\operatorname{Soc}(R))$ is not right T-nilpotent and this gives us a contradiction.

(ii)⇒(iii) Let M be a non-semisimple module and $M = \delta(M) + K$ with M/K singular. Suppose M/K is not semisimple. By (ii),

$$(\delta(M) + K)/K = (M\delta(R) + K)/K = (M/K)\delta(R) = \delta(M/K) \neq M/K,$$

which implies that $M \neq \delta(M) + K$ and this gives a contradiction. So M/K is singular semisimple. Thus, $(M/K)\delta(R) = \overline{0}$, which shows that $\delta(M) \subseteq K$ and so $M = K$. Therefore, $\delta(M) \ll_\delta M$.

(iii)⇒(iv) It suffices to show that $\delta(M) \ll_\delta M$ for any semisimple module M. Write $M = S \oplus N$ with S singular and N nonsingular. Then $\delta(M) = \delta(S \oplus N) = \delta(S) \oplus \delta(N)$. Nonsingular submodule of a semisimple module is δ-small in M, so $N \ll_\delta M$, i.e., $\delta(N) = N$. We know that semisimple modules has no nonzero small submodule and if X is a singular module and K is a δ-small submodule of X then K is a small submodule of X. Thus, since S is singular and semisimple submodule of M, $\delta(S) = 0$. Hence, $\delta(M) = 0 + N = N \ll_\delta M$. Therefore, $\delta(M) \ll_\delta M$.

To prove (iv)⇒(i) we need the following Lemmas:

Lemma 3.11 *(Anderson, F.W., Fuller, K. R. 1992)[Lemma 28.1] Let $a_1, a_2, \ldots$ be a sequence in R. Let F be the free left R-module with basis $x_1, x_2, \ldots$, let $y_n = x_n - a_nx_{n+1}, (n \in \mathbb{N})$ and finally, let G be the submodule of F spanned by $y_1, y_2, \ldots$. Then*

(i) G is free with basis $y_1, y_2, \ldots$

(ii) $G = F$ if and only if for each $k \in \mathbb{N}$, there is $n \geq k$ such that $a_k \cdots a_n = 0$.

Lemma 3.12 *(Anderson, F.W., Fuller, K. R. 1992)[Lemma 28.2] With the hypothesis of Lemma 3.11, if G is a direct summand of F, then the chain*

$$a_1R \geq a_1a_2R \geq \cdots$$

of principal right ideals terminates.

Now we can give the proof.

(iv)⇒(i) Let $F \cong R^{(\aleph_0)}$ have a free basis $x_1, x_2, \ldots$. Let $a_1, a_2, \ldots$ be a sequence in $\delta(R)$ and $G = \sum_{i=1}^{\infty}(x_i - x_{i+1}a_i)$. Then $F = G + \delta(F)$. By assumption, $\delta(F) \ll_\delta F$. Thus,

$F = G \oplus Y$ for a semisimple submodule Y. Thus, by Lemma 3.12, there exists a number n such that $Ra_{n+1}a_n \cdots a_1 = Ra_n \cdots a_1$. Then $a_n \cdots a_1 = ra_{n+1} \cdots a_1$ for some $r \in R$ and so $(1 - ra_{n+1})a_n \cdots a_1 = 0$. Since $ra_{n+1} \in \delta(R)$, we have

$$ra_{n+1} + \operatorname{Soc}(R) \in \delta(R)\big/\operatorname{Soc}(R) = J(R\big/\operatorname{Soc}(R)).$$

Therefore, $(1 - ra_{n+1}) + \operatorname{Soc}(R)$ is right invertible. Thus, $a_n \cdots a_1 \in \operatorname{Soc}(R)$. Hence, $\delta(R)\big/\operatorname{Soc}(R) = J(R\big/\operatorname{Soc}(R))$ is right T-nilpotent. Therefore, $R\big/\operatorname{Soc}(R)$ is a right perfect ring. □

Theorem 3.6 *(Zhou, Y., 2000) The following are equivalent for a ring R:*

(i) R is a δ-perfect ring,

(ii) Every semisimple R-module has a projective δ-cover,

(iii) R is a δ-semiperfect ring and $\delta(M) \ll_\delta M$ for any module M,

(iv) $R\big/\operatorname{Soc}(R)$ is right perfect ring and idempotents lift modulo $\delta(R)$.

Proof (ii)⇒(iii) Suppose that every semisimple R-module has a projective δ-cover. Then every simple R-module has a projective δ-cover, i.e., R is a δ-semiperfect ring and so $R\big/\delta(R)$ is semisimple. By Theorem 3.5, it suffices to show that $\delta(M) \neq M$ for any non-semisimple module M. Suppose for the contrary that $\delta(M) = M$ for some non-semisimple module M. Since every module is an epimorphic image of a free module, there exists an epimorphism $f : P \to M$ with P projective. Since M is non-semisimple, we have P is non-semisimple. Then we obtain that

$$f(\delta(P)) = f(P\delta(R)) = f(P)\delta(R) = M\delta(R) = M.$$

We have the following diagram:

$$\begin{tikzcd} & \delta(P) \arrow[dl, dashed, "i"'] \arrow[d, "f"] & \\ P \arrow[r, "f"] & M \arrow[r] & 0 \end{tikzcd}$$

It follows that $P = \delta(P) + \operatorname{Ker}(f)$. We now show that $\delta(P) \ll_\delta P$. Since $P\big/\delta(P) = P\big/P\delta(R)$ is an $R\big/\delta(R)$-module and hence a semisimple R-module, it has a projective δ-cover. By Lemma 3.7, there exists a decomposition $P = A \oplus B$ such that $A \subseteq \delta(P)$ and $\delta(P) \cap B \ll_\delta P$. So $\delta(P) = A \oplus (\delta(P) \cap B)$. But by Lemma 3.3 (iii),

$$\delta(P) = \delta(A) \oplus \delta(B).$$

This implies that $A = \delta(A)$ and $\delta(P) \cap B = \delta(B) \ll_\delta B$. Since A is projective, and $\delta(A) = A$, A must be semisimple. Thus, $A \ll_\delta A$. By Lemma 3.2, $\delta(P) = A \oplus \delta(B) \ll_\delta A \oplus B = P$. From $P = \delta(P) + \operatorname{Ker}(f)$, by Lemma 3.1, we have that $P = Q \oplus \operatorname{Ker}(f)$ for some semisimple Q. Then

$$M \cong P/\operatorname{Ker}(f) \cong Q$$

is semisimple, a contradiction.

(i)⇒(ii) Obvious.

(iii)⇒(iv) Suppose that R is a δ-semiperfect ring and $\delta(M) \ll_\delta M$ for any R-module M. Then $R/\delta(R)$ is semisimple. Since R is a δ-semiperfect ring, $R/\delta(R)$ is semisimple and idempotents lift modulo $\delta(R)$, by Theorem 3.4. Since R is a δ-semiperfect ring and $\delta(M) \ll_\delta M$ for any R-module M, we have $R/\operatorname{Soc}(R)$ is right perfect by Theorem 3.5.

(iv)⇒(iii) Suppose that $R/\operatorname{Soc}(R)$ is right perfect and idempotents lift modulo $\delta(R)$. Then we have

$$\frac{R/\operatorname{Soc}(R)}{J(R/\operatorname{Soc}(R))} = \frac{R/\operatorname{Soc}(R)}{\delta(R)/\operatorname{Soc}(R)}$$

is semisimple. So $R/\delta(R)$ is semisimple. Therefore, by Theorem 3.4, R is a δ-semiperfect ring and by Theorem 3.5, $\delta(M) \ll_\delta M$ for any R-module M. □

3.4. Examples

In this section, some examples are given to illustrate the concepts introduced earlier.

Example 3.1 *(Zhou, Y., 2000) A δ-semiperfect and semiregular ring is not necessarily semiperfect.*

Let $Q = \prod_{i=1}^{\infty} F_i$, where each $F_i = \mathbb{Z}_2$. Let R be the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. Then R is δ-semiregular but not semiperfect.

The simple R-modules are $F_0 = R/(\bigoplus_{i=1}^{\infty} F_i), F_1, F_2, \ldots$. To check that R is δ-semiperfect, we only need to verify that each singular simple module has a projective δ-cover. F_0 is the only singular simple module. Since R is not semisimple, $\delta(R) \neq R$. Note that $\bigoplus_{i=1} F_i = \operatorname{Soc}(R) \subseteq \delta(R)$. Thus, $\operatorname{Soc}(R) = \delta(R)$ is δ-small in R. So F_0 has a projective δ-cover. Thus, R is δ-semiperfect.

Example 3.2 *(Zhou, Y., 2000) A semiregular ring is not necessarily δ-semiperfect.*

Let R be as in Example 3.1. Let

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in R, b \in \operatorname{Soc}(R) = \bigoplus_i F_i \right\}$$

Then T is a ring under the matrix addition and multiplication. We have that

$$\delta(T) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathrm{Soc}(R) \right\}$$

$T/\delta(T) \cong R$ is regular but not semisimple. So T is not δ-semiperfect. Clearly $J(T) = \delta(T)$ and idempotents of $T/\delta(T)$ lift to idempotents of T. Since $T/J(T) = T/\delta(T)$ is regular and idempotents of $T/\delta(T)$ lift to idempotents of T, we have T is a semiregular ring.

Example 3.3 *(Zhou, Y., 2000) A δ-perfect ring is not necessarily semiregular.*

Let F be a field,

$$I = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$$

and, $R = \{(x_1, x_2, .., x_n, x, x, \ldots) \mid n \in \mathbb{N}, x_i \in M_2(F), x \in I\}$. With componentwise operations, R is a ring. R is not a semiregular ring. We see that

$$\mathrm{Soc}(R) = \{(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \mid n \in \mathbb{N}, x_i \in M_2(F)\},$$
$$\delta(R) = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) \mid n \in \mathbb{N}, x_i \in M_2(F), x \in J\}$$

where $J = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Thus,

$$R/\mathrm{Soc}(R) \cong \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} = I$$

is a right perfect ring. It is easy to check that idempotents of $R/\delta(R)$ lift to idempotents of R. So R is δ-perfect.

Example 3.4 *(Zhou, Y., 2000) A local ring is not necessarily δ-perfect.*

Let R be the ring of polynomials over a field K in countably many commuting indeterminates $x_1, x_2, \ldots$ modulo the ideal generated by $\{x_1^2, x_2^2 - x_1, x_3^2 - x_2, \ldots\}$.

$$J(R) = (x_1, x_2, \ldots)/(x_1^2, x_2^2 - x_1, x_3^2 - x_2, \ldots)$$

is the unique maximal ideal of R and R has no minimal ideal. Thus, R is a local ring and $\mathrm{Soc}(R) = 0$. It is easy to see that $J(R)$ is not T-nilpotent. So $R/\mathrm{Soc}(R)$ is not perfect, and hence R is not δ-perfect.

CHAPTER 4

WHEN δ-SEMIPERFECT RINGS ARE SEMIPERFECT

Zhou defined δ-semiperfect rings as a proper generalization of semiperfect rings. The purpose of this chapter is to discuss relative notions of supplemented modules and to show that the semiperfect rings are precisely the semilocal rings which are δ-supplemented.

4.1. Introduction

H. Bass characterized those rings R whose right R-modules have projective covers and termed them right perfect rings. He characterized them as those semilocal rings which have a right T-nilpotent Jacobson radical $J(R)$. Bass's semiperfect rings are those whose finitely generated R-modules have projective covers. Kasch and Mares transferred the notions of perfect and semiperfect rings to modules and characterized semiperfect modules by a lattice theoretical condition as follows.

Definition 4.1 *A module M is called supplemented if for any submodule N of M, there exists a submodule X of M minimal with respect to $M = N + X$.*

Definition 4.2 *For $N, X \leq M$, X is a supplement of N in M if $N+X = M$ and $N \cap X \ll X$.*

The right perfect rings are then shown to be exactly those rings whose right R-modules are supplemented while the semiperfect rings are those whose finitely generated right R-modules are supplemented. Equivalently, it is enough for a ring R to be semiperfect if the right (or left) R-module R is supplemented. Recall that a submodule $N \leq M$ is said to be small, denoted by $N \ll M$, if $N + X \neq M$ for all proper submodules X of M, and that $N \leq M$ is said to be essential in M, denoted by $N \leq_e M$, if $N \cap X \neq 0$ for each nonzero submodule X of M. Recall that a module M is said to be singular if $M \cong N/X$ for some module N and a submodule $X \leq N$.

Definition 4.3 *A module M is called δ-supplemented if every submodule N of M has a δ-supplement X in M, i.e., $M = N + X$ and $N \cap X \ll_\delta X$.*

It is known that a ring R is δ-semiperfect if and only if it is a δ-supplemented module.

4.2. δ-supplements

In this section we have seen that some of the technicalities on supplement submodules have their relative equivalent. Let S be a nonsingular simple module. Then it is easy to see that $\delta(S) = S$. Also note that if K is a maximal submodule which is essential in M, then M/K is singular simple so $\delta(M) \leq K$.

Definition 4.4 *A submodule N of M is said to be coclosed if $N/K \ll M/K$ implies $K = N$ for each $K \leq N$.*

Example 4.1 *Every supplement submodule of a module is coclosed:*
Let N be a supplement submodule of a module $X \leq M$. Then $X + N = M$. So

$$M/K = (X + N)/K = (X + K)/K + N/K.$$

If $N/K \ll M/K$, then $(X + K)/K = M/K$, i.e., $M = X + K$. Since N is minimal with respect to $X + N = M$, we have $K = N$. Therefore, N is coclosed.

Definition 4.5 *Let M be an R-module and $N \leq M$. We call N a δ-coclosed submodule of M if, whenever N/X is singular and $N/X \ll_\delta M/X$ for some $X \leq N$, we have $X = N$.*

Supplements are coclosed and so are their δ-equivalents:

Lemma 4.1 *(Büyükaşık, E., Lomp, C., 2009)[Lemma 2.3] Let M be any module and $N \leq M$ be a δ-supplement in M. Then N is δ-coclosed.*

Proof Let N be a δ-supplement of a module $K \leq M$. Then $N+K = M$ and $N \cap K \ll_\delta N$. Suppose N/X is singular and $N/X \ll_\delta M/X$ for some $X \leq N$. Then we have

$$N/X + (K + X)/X = M/X,$$

and

$$M/(K + X) \cong N/(N \cap (K + X))$$

is singular as a factor module of the singular module N/X. Therefore, we have $(K + X)/X = M/X$ as $N \cap X \ll_\delta M/X$. Then we get $K + X = M$, and so by modular law $N = (N \cap K) + X$. Since $N \cap K \ll_\delta N$ and N/X is singular we have $X = N$. So N is a δ-coclosed submodule of M. $\square$

In the following proposition some properties of δ-coclosed submodules are given.

Proposition 4.1 *(Büyükaşık, E., Lomp, C., 2009)[Proposition 2.4] Let N be a δ-coclosed submodule of M. Then the following hold.*

(i) If $K \leq N \leq M$ and $K \ll_\delta M$, then $K \ll_\delta N$. Hence, $\delta(N) = N \cap \delta(M)$.

(ii) If X is a proper submodule of N such that $N/X \ll_\delta M/X$, then $N = X \oplus X'$ for some $X' \leq N$.

(iii) If N is singular, then N is coclosed.

Proof (i) Let $K \ll_\delta M$ and suppose that $K + X = N$ for some $X \leq N$ with N/X singular. Then

$$N/X = (K + X)/X \ll_\delta M/X.$$

Since N is a δ-coclosed submodule of M, we have $X = N$. Therefore, $K \ll_\delta N$. Now we have,

$$\delta(N) \leq N, \delta(N) = \sum\{K \leq N \mid K \ll_\delta N\} \subseteq \sum\{K \leq M \mid K \ll_\delta M\} = \delta(M)$$

So $\delta(N) \subseteq N \cap \delta(M)$. Therefore, we need to prove that $N \cap \delta(M) \subseteq \delta(N)$. Let $x \in N \cap \delta(M)$. Then $Rx \ll_\delta M$ and so by the first part of the proof $Rx \ll_\delta N$, that is, $x \in \delta(N)$. Hence, $\delta(N) = N \cap \delta(M)$.
(ii) Let $X \leq N$ with $N/X \ll_\delta M/X$. Let $X' \leq N$ be a maximal submodule in N such that $X \cap X' = 0$. Then $X \oplus X' \leq_e N$ and so $N/(X \oplus X')$ is singular. On the other hand $N/(X \oplus X') \ll_\delta M$. Since N is δ-coclosed, we have $N = X \oplus X'$.
(iii) Let N be a δ-coclosed submodule of M. Suppose N is singular. Since singular modules are closed under factor modules, N/X is singular. If $N/X \ll M/X$, then $N/X \ll_\delta M/X$. Since N/X is singular, $N/X \ll_\delta M/X$ and N is δ-coclosed, we have $X = N$. Therefore, N is coclosed. □

Corollary 4.1 *(Büyükaşık, E., Lomp, C., 2009)[Corollary 2.5] Let N be a δ-supplement submodule of M. Then $\delta(N) = N \cap \delta(M)$.*

Proof Suppose N is a δ-supplement submodule of M. By Lemma 4.1, N is δ-coclosed. Thus, by Proposition 4.1, $\delta(N) = N \cap \delta(M)$. □

Corollary 4.2 *(Büyükaşık, E., Lomp, C., 2009)[Corollary 2.6] For a module M and a submodule $N \leq M$, consider the following statements.*

(i) N is a δ-supplement submodule of M,

(ii) N is δ-coclosed in M,

(iii) For all $X \leq N$, $X \ll_\delta M$ implies $X \ll_\delta N$.

If N has a weak δ-supplement in M, i.e., $N + K = M$ and $N \cap K \ll_\delta M$ for some submodule $K \leq M$, then $(iii) \Rightarrow (i)$ holds.

Proof (i)⇒(iii) By Lemma 4.1.
(ii)⇒(iii) By Proposition 4.1.
(iii)⇒(i) Suppose N has a weak δ-supplement in M. Then $N + L = M$ and $N \cap L \ll_\delta M$ for some submodule $L \leq M$. Then $N \cap L \subseteq N$, $N \cap L \ll_\delta M$ implies $N \cap L \ll_\delta N$. Therefore, N is a δ-supplement of L in M. □

4.3. On the Structure of δ-supplemented Modules

Definition 4.6 *A module M is said to be local if M has a largest proper submodule.*

Lemma 4.2 *M is local if and only if $\mathrm{Rad}(M)$ is a maximal submodule of M and $\mathrm{Rad}(M) \ll M$.*

Proof Suppose that M is local. Then M has a largest proper submodule, say N, so $\mathrm{Rad}\, M = N \leq M$. Since $\mathrm{Rad}\, M$ is the largest proper submodule of M, $\mathrm{Rad}\, M$ contains every proper submodule of M, i.e., $\mathrm{Rad}\, M + N \neq M$ for any $N \leq M$.

Conversely suppose that $\mathrm{Rad}\, M \leq_{max} M$ and $\mathrm{Rad}(M) \ll M$. If a is not an element of $\mathrm{Rad}\, M$, then $\mathrm{Rad}\, M + aR = M$ implies that $aR = M$. Therefore, $\mathrm{Rad}\, M$ is the largest proper submodule of M. □

Definition 4.7 *Let M be an R-module. M is said to be δ-local if $\delta(M) \ll_\delta M$ and $\delta(M)$ is a maximal submodule of M.*

Examples 4.1 *(i) Every simple module is local:*

Let S be a simple module. Then $\mathrm{Rad}\, S = 0 \leq_{max} S$ and $0 \ll S$. Thus S is local.

(ii) A simple module is δ-local if and only if it is singular:

Let S be a simple module and singular module. Then $\delta(S) = 0$, so $\delta(S) \ll_\delta S$ and $\delta(S) \leq_{max} S$.

Conversely let S be a simple module. Suppose that S is δ-local. Then $\delta(S) \leq_{max} S$ and $\delta(S) \ll_\delta S$. So $\delta(S) = 0$, i.e., S is singular.
(iii) Let S be a nonsingular simple module and S' be a singular simple module. Then by above argument, S is local. But S is not δ-local. Because $\delta(S) = S$. On the other hand, let $M = S \oplus S'$. Then

$$\operatorname{Rad} M = \operatorname{Rad} S \oplus \operatorname{Rad} S' = 0 \ll M.$$

But 0 is not a maximal submodule of M so M is not local. Since $\delta(S) = S$ and $\delta(S') = 0$, we have

$$\delta(M) = \delta(S) \oplus \delta(S') = S.$$

Since $M/S \cong S'$ simple, $\delta(M) = S \leq_{max} M$. Suppose $\delta(M) + K = M$ with M/K singular. Since $\delta(M) = S$ is nonsingular and

$$M/K \cong \delta(M)/(\delta(M) \cap K)$$

is singular, we have $\delta(M) \cap K \leq_e \delta(M)$. But $\delta(M)$ is simple, so $\delta(M) \cap K = \delta(M)$, i.e., $\delta(M) \subseteq K$, $K = M$. Hence, $\delta(M) \ll_\delta M$, i.e., M is δ-local.

Lemma 4.3 *(Büyükaşık, E., Lomp, C., 2009)[Lemma 3.2] Let M be a module and H a local submodule of M. Then H is a supplement of each proper submodule $K \leq M$ with $H + K = M$.*

Proof Since K is a proper submodule of M and $H + K = M$, we have $K \cap H$ is a proper submodule of H. Since H is local, $\operatorname{Rad} H$ is the unique maximal submodule of H and $\operatorname{Rad} H \ll H$. Thus, we have $K \cap H \subseteq \operatorname{Rad} H \ll H$. So $K \cap H \ll H$. That is, H is a supplement of K in M. □

Lemma 4.4 *(Büyükaşık, E., Lomp, C., 2009)[Lemma 3.3] Any δ-local module is δ-supplemented.*

Proof Let M be a δ-local module and N be a proper submodule of M. Since $\delta(M) \leq_{\max} M$, we have either $N \leq \delta(M)$ or $\delta(M) + N = M$. If $N \leq \delta(M)$, then $N \ll_\delta M$. So M is a δ-supplement of N in M. Now suppose $\delta(M) + N = M$. Since $\delta(M) \ll_\delta M$, we have $N \oplus Y = M$ for some semisimple submodule $Y \leq \delta(M)$. Hence, Y is a δ-supplement of N in M. Therefore, M is δ-supplemented. □

Lemma 4.5 *(Büyükaşık, E., Lomp, C., 2009)[Lemma 3.4] Let M be an R-module and let K be a maximal submodule with $\operatorname{Soc}(M) \leq K$. If L is a δ-supplement of K in M, then L is δ-local.*

Proof $K + L = M$ and $K \cap L \ll_\delta L$, by assumption. We claim that $K \cap L \leq_e L$. If $(K \cap L) \cap T = 0$ for some nonzero submodule $T \leq L$, then $L = (K \cap L) \oplus T$. We get

$$M = K + L = K + (K \cap L) + T = K + T,$$

and so $T \nleq K$ giving a contradiction since $T \subseteq \operatorname{Soc} M \subseteq K$. Therefore, $K \cap L \leq_e L$ so $\delta(L) \subseteq K \cap L$. Hence, $\delta(L) = K \cap L \ll_\delta L$ and $\delta(L) = K \cap L \leq_{\max} L$, i.e., L is δ-local. □

Definition 4.8 *A submodule $N \leq M$ is called cofinite if M/N is finitely generated.*

Definition 4.9 *M is called cofinitely δ-supplemented if every cofinite submodule of M has a δ-supplement in M.*

In case M is finitely generated, clearly every submodule of M is cofinite; so M is δ-supplemented if and only if M is cofinitely δ-supplemented. If a finitely generated module M is a sum of δ-supplemented modules, then M is δ-supplemented.

Proposition 4.2 *(Büyükaşık, E., Lomp, C., 2009)[Proposition 3.5] For a finitely generated module M, the following are equivalent:*

(i) M is δ-supplemented,

(ii) Every maximal submodule of M has a δ-supplement,

(iii) $M = H_1 + H_2 + \cdots + H_n$ where H_i is either simple or δ-local.

Proof (i)⇒(ii) Clear.
(ii)⇒(iii) Let $\lambda(M) \leq M$ be the sum of all δ-supplement submodules of maximal submodules $N \leq M$ with $\operatorname{Soc}(M) \leq N$. Then by Lemma 4.5, $\lambda(M)$ is a sum of δ-local submodules of M. We claim that $M = \operatorname{Soc}(M) + \lambda(M)$. Suppose to the contrary that $M \neq \operatorname{Soc}(M) + \lambda(M)$. Since M is finitely generated, $\operatorname{Soc}(M) + \lambda(M) \leq K$ for some maximal submodule $K \leq M$. By assumption, K has a δ-supplement L in M. Since $\operatorname{Soc}(M) \subseteq K$, L is δ-local by Lemma 4.5. Hence $L \leq \lambda(M) \leq K$. Since $L + K = M$ and $L \subseteq K$, we have $K = M$. But $K \leq_{\max} M$, a contradiction. Therefore, $M = \operatorname{Soc}(M) + \lambda(M)$. Since M is finitely generated, M is a finite sum of simple submodules and δ-local submodules, as desired.
(iii)⇒(i) By Lemma 4.4 , δ-local modules are δ-supplemented, and clearly simple modules are also δ-supplemented. Therefore, M is δ-supplemented as a finite sum of δ-supplemented modules. □

4.4. When δ-supplemented Modules are Supplemented

We will turn to the problem of characterizing when a δ-semiperfect ring is semiperfect. Recall that a module M is called semilocal if $M/\operatorname{Rad} M$ is semisimple.

For any module M, let $X(M) = \operatorname{Soc} M/\big(\operatorname{Soc} M \cap \operatorname{Rad} M\big)$.

Lemma 4.6 *(Büyükaşık, E., Lomp, C., 2009)[Lemma 4.1] Let R be a ring and M a finitely generated, δ-supplemented right R-module. Then M is semilocal if and only if*

$$\operatorname{Soc}(M)/\big(\operatorname{Soc}(M)\cap\operatorname{Rad}(M)\big)$$

is finitely generated.

Proof If M is semilocal and finitely generated, then $M/\operatorname{Rad}(M)$ is semisimple Artinian. Moreover,

$$\operatorname{Soc}(M)/\big(\operatorname{Soc}(M)\cap\operatorname{Rad}(M)\big)\cong\big(\operatorname{Soc}(M)+\operatorname{Rad}(M)\big)/\operatorname{Rad}(M)\subseteq M/\operatorname{Rad}(M)$$

implies that $X(M)$ is semisimple Artinian; so $X(M)$ is finitely generated.

To show the converse we use induction on the length of $X(M)$. Suppose $X(M)=0$, i.e., $\operatorname{Soc}(M)\subseteq\operatorname{Rad}(M)$. Then $\operatorname{Rad}(M)=\delta(M)$ and hence $M/\delta(M)$ is semisimple. Assume that any finitely generated δ-supplemented module N with $X(N)$ of length $n\geq 0$ is semilocal and let M be a finitely generated δ-supplemented module with $X(M)$ having length $n+1$. Since $\operatorname{Soc}(M)\nsubseteq\operatorname{Rad}(M)$, there exists a simple direct summand $E\subseteq M$ with $M=E\oplus N$ for some $N\subseteq M$. Moreover $\operatorname{Rad}(M)=\operatorname{Rad}(N)$ and $\operatorname{Soc}(M)=E\oplus\operatorname{Soc}(N)$. Hence

$$X(M)=\operatorname{Soc}M/\big(\operatorname{Soc}M\cap\operatorname{Rad}M\big)\cong E\oplus\big(\operatorname{Soc}N/\big(\operatorname{Soc}N\cap\operatorname{Rad}N\big)\big)=E\oplus X(N).$$

Since direct summands of δ-supplemented modules are δ-supplemented, N is a finitely generated δ-supplemented module. $X(N)$ has length n, so by induction hypothesis N is semilocal and hence $M=E\oplus N$ is semilocal. $\square$

δ-semiperfect rings are exactly those rings R, that are δ-supplemented as a right (or left) R-module. Similarly a ring R is semiperfect if and only if R is supplemented as a right (or left) R-module. Recall that projective δ-supplemented modules M are δ-lifting, i.e., for every submodule N of M there exists a decomposition $M=D_1\oplus D_2$ such that $D_1\subseteq N$ and $N\cap D_2\ll_\delta D_2$.

Proposition 4.3 *(Büyükaşık, E., Lomp, C., 2009)[Proposition 4.2] A projective semilocal, δ-supplemented module with small radical is supplemented.*

Proof Since $S\cap\operatorname{Rad}(M)\subseteq\operatorname{Soc}(M)$, we have $S\cap\operatorname{Rad}(M)\leq_\oplus S$. Let

$$S=\operatorname{Soc}(M)=D\oplus\big(S\cap\operatorname{Rad}(M)\big).$$

Since M is semilocal, there exists $N\subseteq M$ such that $M=D+N$ and $N\cap D\subseteq\operatorname{Rad}(M)$. But since $D\cap\operatorname{Rad}(M)=0$, $M=D\oplus N$ with D semisimple and $\operatorname{Rad}(M)=\operatorname{Rad}(N)$.

Note that,

$$\mathrm{Soc}(N) = \mathrm{Soc}(M) \cap N = \big(D \oplus (S \cap \mathrm{Rad}(M)\big) \cap N = (D \cap N) \oplus (S \cap \mathrm{Rad}(M)) = S \cap \mathrm{Rad}(N) \subseteq \mathrm{Rad}(N).$$

Hence, if $K \subseteq N$ is a maximal submodule, then N/K must be singular, since otherwise N/K would be isomorphic to a simple direct summand of N, which is impossible, as $\mathrm{Soc}(N) \subseteq \mathrm{Rad}(N)$. Thus, $\mathrm{Rad}(N) = \delta(N)$. N is δ-lifting since it is projective and δ-supplemented. Hence, for any submodule $L \subseteq N$, there exists $A, B \subseteq N$ such that $N = A \oplus B$ and $A \subseteq L$ and $L \cap B \ll_\delta N$. In particular,

$$L \cap B \subseteq \delta(B) \subseteq \delta(N) \subseteq \mathrm{Rad}(N).$$

As M has a small radical, so has N and hence $L \cap B \ll N$. But since B is a direct summand of N, $L \cap B \ll B$. This shows that B is a supplement of L in N, i.e., N is a supplemented module. We showed that $M = D \oplus N$ is the direct sum of two supplemented modules. Hence, M is δ-supplemented. □

Corollary 4.3 *(Büyükaşık, E., Lomp, C., 2009)[Corollary 4.3] Let R be a ring with $J = J(R)$ and $S = \mathrm{Soc}(R)$. Then the following are equivalent:*

(i) R is semiperfect,

(ii) R is δ-semiperfect and semilocal,

(iii) R is δ-semiperfect and $S/S \cap J$ is finitely generated.

Proof (i)⇒(ii) Clear.
(ii)⇒(iii) Suppose that R is δ-semiperfect and semilocal. Then $S/(S \cap J)$ is finitely generated by Lemma 4.6.
(iii)⇒(ii) Suppose that R is δ-semiperfect and $S/S \cap J$ is finitely generated. Then by Lemma 4.6, R is semilocal.
(ii)⇒(i) Suppose that R is δ-semiperfect and semilocal. Since R is projective and $J \ll R$, R is supplemented, by Proposition 4.3. Therefore, R is semiperfect. □

Remark 4.1 *If $\mathrm{Soc}(R)$ is finitely generated, then $S/(S \cap J)$ is finitely generated. So we have, any ring R with finitely generated left socle (e.g. R is left Noetherian) is semiperfect if and only if it is δ-semiperfect. There are δ-semiperfect rings which are not semilocal and hence not semiperfect.*

We finish this section by showing that the last remark also holds for modules, i.e., finitely generated modules with finitely generated socle are supplemented if and only if they are δ-supplemented.

Lemma 4.7 *(Büyükaşık, E., Lomp, C., 2009)[Lemma 4.5] Let M be a module and K be a maximal submodule of M. Suppose $\mathrm{Soc}(M)$ is finitely generated and K has a δ-supplement H in M. Then K has a supplement in M contained in H.*

Proof By hypothesis, H is a δ-supplement of K in M, that is $K + H = M$ and $K \cap H \ll_\delta H$. In particular, $K \cap H \subseteq \delta(H)$. Since

$$M/K = (H + K)/K \cong H/(H \cap K)$$

is simple, $K \cap H$ is a maximal submodule of H. Therefore, we have either $\delta(H) = H$ or $\delta(H) = K \cap H$. First suppose that $\delta(H) = H$. Then $K \cap H$ is not essential in H. So there exists a submodule T of H such that $H = (K \cap H) \oplus T$. In this case, $M = K + H = K \oplus T$, so T is a supplement of K in M and T is contained in H.

Now, let $\delta(H) = K \cap H$. If $K \cap H \ll H$, then H is a supplement of K in M. Suppose $K \cap H = \delta(H)$ is not small in H, that is, $H = \delta(H) + L_1$ for some proper submodule $L_1 \lneq H$. Then $H = L_1 \oplus Y$ for some semisimple submodule $Y \leq \delta(H)$. Since L_1 is a direct summand H, we have

$$\delta(L_1) = L_1 \cap \delta(H) = L_1 \cap H \cap K = L_1 \cap K$$

and $\delta(L_1) \ll_\delta L_1$. We also have

$$K + H = K + L_1 + Y = K + L_1.$$

Therefore, L_1 is a δ-supplement of K.

Since L_1 is a proper submodule of H and Y is a nonzero semisimple module contained in H, we have $\mathrm{Soc}(L_1) \lneq \mathrm{Soc}(H)$. Now, if $\delta(L_1) \ll L_1$, then L_1 is a supplement of K in M and we are done. Suppose $\delta(L_1)$ is not small in L_1. Then $L_1 = \delta(L_1) + L_2$ for some $L_2 \lneq L_1$. Arguing as above, we get L_2 is a δ-supplement of K in M with $\mathrm{Soc}(L_1) \gneq \mathrm{Soc}(L_2)$. Continuing in this way, if none of the L_i's is a supplement of K we shall get a strictly descending chain of submodules

$$\mathrm{Soc}(L_1) > \mathrm{Soc}(L_2) > \cdots$$

of $\mathrm{Soc}(M)$. This will contradict the fact that $\mathrm{Soc}(M)$ is finitely generated. ($\mathrm{Soc}(M)$ is semisimple and finitely generated. Thus, $\mathrm{Soc}(M)$ is Artinian and Noetherian.) Therefore, K has a supplement in M. $\square$

Corollary 4.4 *(Büyükaşık, E., Lomp, C., 2009)[Corollary 4.6] Let M be a finitely generated module. Suppose $\mathrm{Soc}(M)$ is finitely generated. Then M is supplemented if and only if M is δ-supplemented.*

Proof Suppose that M is supplemented. Then M is δ-supplemented.

Conversely, suppose that M is δ-supplemented. So every submodule of M has a δ-supplement in M. In particular, every maximal submodule of M has a δ-supplement in M. Every maximal submodule of M has a supplement in M, by Lemma 4.7. Since M is finitely generated and every maximal submodule of M has a supplement in M, M is supplemented. $\square$

Corollary 4.5 *(Büyükaşık, E., Lomp, C., 2009)[Corollary 4.7] Let M be a module with finitely generated socle. Then M is cofinitely supplemented if and only if M is cofinitely δ-supplemented.*

Proof Necessity is clear.

To prove sufficiency, suppose M is cofinitely δ-supplemented. Let K be a maximal submodule of M. If $\operatorname{Soc}(M)$ is not contained in K, then we have $M = K + \operatorname{Soc}(M)$ by maximality of K in M. Then $K + S = M$ for some simple submodule of M. Since S is simple and $S \nleq K$, we have $K \oplus S = M$, and hence S is a supplement of K in M. Now, if $\operatorname{Soc}(M) \subseteq K$ and H is a δ-supplement of K in M, then K has a supplement in M, by Lemma 4.7. Hence, M is cofinitely supplemented. $\square$

CHAPTER 5

CONCLUSION

Right perfect, semiperfect and semiregular rings constitute the classes of rings that possess beautiful homological and nonhomological properties. Since Bass' pioneering work on right perfect and semiperfect rings, there has been a great deal of work on them by many other authors. In this thesis a generalization of right perfect, semiperfect and semiregular rings is studied.

Firstly, we have given the basic definitions and characterizations of right perfect, semiperfect and semiregular rings. We have studied supplemented modules and we have seen that, for a ring R, R_R is semiperfect if and only if R_R is supplemented.

Secondly, we have studied the paper (Zhou, Y., 2000). The generalizations of right perfect, semiperfect and semiregular rings are introduced by (Zhou, Y., 2000) by considering the class of all singular R-modules in place of the class of all R-modules. The concept of small submodules which leads to the definition of projective covers is certainly the key in introducing right perfect, semiperfect and semiregular rings. As a generalization of small submodules, (Zhou, Y., 2000) defined δ-small submodules. The definition of projective δ-cover is given and we have seen various characterizations and properties for δ-perfect, δ-semiperfect and δ-semiregular rings. From these definitions, it is clear that if a ring R is semiperfect, then it is δ-semiperfect.

Finally, we were interested in when δ-semiperfect rings are semiperfect. For this purpose, we have studied δ-supplemented modules, which are the generalizations of supplemented modules, to see the relation between δ-semiperfect rings and δ-supplemented modules. We have seen that δ-semiperfect rings are exactly those rings R that are δ-supplemented as a right (or left) R-module. We have studied the paper (Büyükaşık, E., Lomp, C., 2009) and we observed that an arbitrary associative unital ring R is semiperfect if and only if it is semilocal and δ-semiperfect.

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