QUANTUM CONTROL AND GENERATION OF QUANTUM ENTANGLEMENT

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ABSTRACT

QUANTUM CONTROL AND GENERATION OF QUANTUM ENTANGLEMENT

In this thesis, the generation of entanglement is studied in a controlled environment. The model system of interest includes a cavity field interacting with a pair of atoms. The cavity field is heavily damped and it is pumped in order to maintain a steady state field population. Thus, we can eliminate the cavity field adiabatically and obtain the master equation describing only the qubits evolution in time. At first, this system is analyzed in the steady state, without making any measurement on the photons leaking through the cavity walls. In this way, the ideal physical parameter set for maximum entanglement in this model is investigated. In the second step, we assume a direct measurement on the leaking cavity photons, and observe the evolution of entanglement in a quantum trajectories approach. We simulate quantum trajectories approach by applying Monte Carlo method. The amount of entanglement is obtained as a function of time and number of photon detections.

ÖZET

KUANTUM KONTROL VE KUANTUM DOLANIKLIĞIN OLUŞTURULMASI

Bu tezde, dolanıklığın oluşturulması kontrollü bir çevrede çalışılmıştır. İlgilenilen model system kavite alanı ile etkileşen bir çift atomdan olşmaktadır. Kavite alanı, kararlı durum alan populasyonunun elde edilmesi için şiddetli bir şekilde sönümlenmekte ve pompalanmaktadır. Bu nedenle, kavite alanını adyabatik olarak eleyebiliriz ve sadece kübitlerin zamanda gelişimini ifade eden yoğunluk operatörünü elde edebiliriz. Öncelikle ilgilenilen system, kavite duvarlarından sızan fotonlar üzerinde hiçbir ölçüm yapılmaksızın, kararlı durumda analiz edilmiştir. Bu şekilde, bu sistemdeki maksimum dolanıklık için gerekli ideal fiziksel parametreler belirlenmiştir. İkinci aşamada, kaviteden sızan fotonlar üzerinde doğrudan ölçümler yapıldığını varsayıyoruz ve dolanıklığın gelişimini kuantum yörünge yaklaşımı üzerinden gözlemliyoruz. Kuantum yörünge yaklaşımını Monte Carlo yöntemi ile simüle etmekteyiz. Dolanıklık miktarı, zamanın ve gözlenen foton sayısının bir fonksiyonu olarak elde edilmiştir.

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LIST OF SYMBOLS

$|\phi^{\pm}\rangle$ . . . . . Bell states

$\hat{H}_{total}(\hat{H})$ . . . . . The total Hamiltonian about a system-reservoir interaction)

$\hat{H}_{system}$ . . . . . The Hamiltonian only about the system

$\hat{H}_{reservoir}$ . . . . . The Hamiltonian about the reservoir(environment or bath)

$\hat{H}_{interaction}(\hat{H}_{int})$ . . . . The Hamiltonian about the interaction between the system and the reservoir(environment or bath)

$\hat{\rho}_{pure}$ . . . . . a density operator(density matrix) of a pure state

$\hat{\rho}_{mixed}$ . . . . . a density operator(density matrix) of a mixed state

$\hat{\rho}^{T_A}$ . . . . . Partial transpose of $\hat{\rho}$ with respect to A

$E_F(\hat{\rho})$ (or $E(\hat{\rho})$) . . . . . Entanglement of formation

$F(\hat{\rho})$ . . . . . Fidelity

$N(\hat{\rho})$ . . . . . Negativity

$C(\hat{\rho})$ . . . . . Concurrence

$\hat{\rho}$ . . . . . The density operator of a system

$\hat{L}_j$ . . . . . The operator of field about the interaction between the system and the reservoir

$\mathcal{L}$ . . . . . the Lindblad operator

$\hat{c}$ . . . . . Quantum jump operator for a system

CHAPTER 1

INTRODUCTION

In classical information theory, the basic unit is bit which takes just two possible value as 0 and 1. In quantum mechanics, the corresponding basic unit is called quantum bit or qubit [16],[14]. It describes the state of a simple quantum system as $|\psi\rangle = a\,|1\rangle + b\,|0\rangle$, for $a^2 + b^2 = 1$. It can be shown on a Bloch sphere,

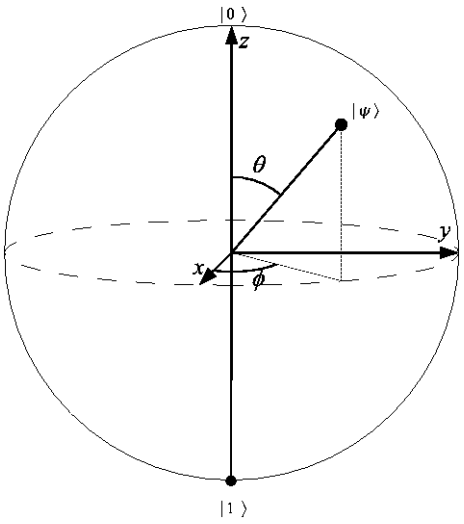

Figure 1.1. Bloch sphere

When we measure $|\psi\rangle$, we can find the probabilities of finding $|\psi\rangle$ in state $|1\rangle$ or $|0\rangle$ separately. But, if there are two or more systems, there can be states which cannot observe separately. For example, for a state as follows,

$$\left|\phi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle \pm |1\rangle \otimes |0\rangle) \tag{1.1}$$

These are one kind of the Bell states (also known as EPR states or EPR pairs). When we apply measurement The details about entangled states are studied in the first chapter.

In 1935, A. Einstein, B. Podolski and N.Rosen proposed an experiment with entangled particles [9]. They argued this experiment would demonstrate that quantum mechanics was an incomplete theory, because the 'hidden variables', the values you need to specify the state of a physical system, didn't be considered in quantum mechanics [4]. In 1964 J.S. Bell showed [2] with a similar experiment that quantum mechanics make predictions that are incompatible with the existence of hidden variables[4]. The measurement correlations in the Bell states are stronger than could ever exist between classical systems [14]. It is about the nature of the quantum measurement.

Quantum measurements disturbs the state of a quantum system. If we want to observe a system, the disturbing is a disadvantage. But to generate quantum entanglement, we can turn this into an advantage. We can effect the initial system to obtain a system with a ratio of entanglement we want to obtain by affecting on the system with quantum measurement. We can observe the change in the amount of entanglement according to the time which measurement applied on the system and how many times the measurement applied.

In this work, the application quantum control and the generation of an entanglement is studied on an open quantum system.

An open quantum system consists of a reservoir, it is also called environment or bath, and a system inside of it. The reservoir contains many degrees of freedom, so it is a difficulty to work on the system interacting with it. The difficulty comes from separating the degree of freedoms of reservoir and system[8]. The total Hamiltonian of open quantum systems can be define as,

$$\hat{H}_{total} = \hat{H}_{system} + \hat{H}_{reservoir} + \hat{H}_{interaction} \tag{1.2}$$

When we used the density operator, the degrees of reservoir can be trace out the density operator of reservoir and we only work on the system and interaction part.

$$\hat{\rho}_{total}(t) = \hat{\rho}_{system}(t) \otimes \hat{\rho}_{reservoir}(t) \tag{1.3}$$

In this way, the time-evolutions of the system can be examined by writing master equation. In this thesis, we study the dynamics of a system in cavity environment and to observe this dynamics, we study the derivation of master equation to be able to observe the time evolution of our system in the second part of the second chapter.

In the second chapter, we studied basic facts and structure of the system in which we interest.

In the third chapter, we described the details of the model system we use. We examined how the entanglement of the system evolve in time and according to the detection number. To examine these relations, we simulate the system by using Monte Carlo method on quantum trajectories approach as explained in the the third chapter.

There are different measures can be used to define the amount of entanglement like the concurrence, entanglement of formation, entanglement of distillation, relative entropy of entanglement and negativity. They have been proposed for the purpose of the amount of entanglement. We use concurrence method as the measure of entanglement.

Finally, by taking concurrence as the measure of entanglement, the relation of entanglement and measurement is studied. Also there is a part about what we have been studying and what we will study following the subject of this thesis.

CHAPTER 2

BASIC INFORMATIONS

In this chapter, we will explain the basic concepts we have used in our work.

2.1. Quantum Entanglement

Quantum entanglement is the quantum mechanical property that Schrödinger singed out many decades ago as the ”characteristic trait of quantum mechanics”[19] and that has been studied extensively in connection with Bell’s inequality[2],[26].

2.1.1. Quantum States

To define entanglement in a detailed way, we have to define the variation of quantum states. For example, entanglement can be defined for both pure states and mixed states. Also, we interest in a quantum system with two objects and this kind of systems are called bipartite systems (The state written about the interaction of two objects is called bipartite state.).

2.1.1.1. Pure and Mixed States

A pure quantum state is a state which can be described by a single ket vector, or as a sum of basis states. The expectation value $\langle a \rangle$ of a measurement $\hat{A}$ on a pure quantum state is given by

$$\langle a \rangle = \langle \psi | \hat{A} | \psi \rangle = \sum_i a_i \langle \psi | \alpha_i \rangle \langle \alpha_i | \psi \rangle = \sum_i a_i \left| \langle \alpha_i | \psi \rangle \right|^2 = \sum_i a_i p(\alpha_i) \tag{2.1}$$

For this example, we can write the density operator of the pure state $|\psi\rangle$;

$$\hat{\rho}_{pure} = |\psi\rangle \langle \psi| \tag{2.2}$$

A mixed quantum state is a statistical distribution of pure states. For example; the ensemble average of $\hat{A}$ for a mixed state can be written as,

$$\langle \overline{A} \rangle = \sum_s p_s \langle \psi_s | \hat{A} | \psi_s \rangle = \sum_s \sum_i p_s a_i \left| \langle \alpha_i | \psi_s \rangle \right|^2. \tag{2.3}$$

where $|\psi_s\rangle$ is a pure state. The density operator for this mixed state is;

$$\hat{\rho}_{mixed} = \sum_s p_s |\psi_s\rangle \langle \psi_s| . \tag{2.4}$$

2.1.1.2. Bipartite Systems

In most general definition, bipartite systems are the systems with two objects. We work on a system with two qubits, so the basic structure of the bipartite systems are very important. The decomposition of a bipartite state can be shown as following[7];

- Pure state decomposition: For $H = H_A \otimes H_B$,

$$|\psi\rangle = \sum_{n,m=0} c_{n,m} |n\rangle_A \times |m\rangle_B \tag{2.5}$$

- Mixed state decomposition:

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle \psi_i| \tag{2.6}$$

2.1.1.3. Separable and Entangled States

We will study the subject of entanglement by using the basic properties of bipartite systems. The states of bipartite systems can be study under two title[7];

- Pure and bipartite states:
  For $\psi \in H_A \otimes H_B$, if there exist two vectors $\varphi_1 \in H_A$ and $\varphi_2 \in H_B$ such that $|\psi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle$, ψ is a separable state. Otherwise it is an entangled state. As

an example of the pure and bipartite entangled state, Bell states can be given in Eq. (2.7) and Eq. (2.8),

$$\left|\Phi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle \pm |1,1\rangle) \tag{2.7}$$

$$\left|\Psi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle \pm |1,0\rangle) \tag{2.8}$$

- Mixed and bipartite states:
  For $\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B$, $\hat{\rho}$ can be written as a mixture of states. A state is separable if

$$\sum_i p_i \left|a_i, b_i\right\rangle \left\langle a_i, b_i\right|$$

  where $p_i > 0$. Otherwise it is an entangled state. As an example of the mixed and bipartite entangled state is,

$$\hat{\sigma} = \frac{1}{2}(\left|\Phi^{+}\right\rangle \left\langle\Phi^{+}\right| + \left|\Phi^{-}\right\rangle \left\langle\Phi^{-}\right|)$$

2.1.2. Entropy of Entanglement

To analyze the entanglement in bipartite states, we will use the Schmidt decomposition(SD). For $|u_k\rangle$ is orthonormal basis in H_A and $|v_k\rangle$ is orthonormal basis in H_B, we can write a state as,

$$|\psi\rangle = \sum_{k, d_k \leqslant d_A, d_B} d_k \left|u_k, v_k\right\rangle . \tag{2.9}$$

The reason is that the matrix C can be written as $C = UDV$, where U and V are isometries and D is a diagonal positive element[7]. This way of writing C is called singular decomposition with positive elements. This way of writing C is called 'singular decomposition', and it is valid for any matrix [10]. The U and V can be found by diagonalizing $CC^{\dagger}$ and $C^{\dagger}C$ respectively. The matrix D can be found by taking the square root of the resulting diagonal matrix, which coincides in both cases. The diagonal elements of D, d_k, are called *Schmidt coefficients*[7].

- In case one of the Schmidt coefficients about the state in which we interest is one and the rest are zero, we have a product state. Otherwise, the state is entangled.
- The SD is also very useful to determine the reduced density operators for subspaces A and B alone.

The reduced density operators of the subspaces A and B can be written by using Schmidt decomposition as follows,

$$\hat{\rho}_A = Tr_B(|\psi\rangle \langle\psi|) = \sum_k d_k^2 |u_k\rangle \langle u_k| , \tag{2.10}$$

$$\hat{\rho}_B = Tr_A(|\psi\rangle \langle\psi|) = \sum_k d_k^2 |v_k\rangle \langle v_k| . \tag{2.11}$$

Entanglement is related to the mixedness of the reduced density operators. So, we can introduce a measure of entanglement by using any measure of mixedness of a state.

As a measure of mixedness the von Neumann entropy can be used,

$$S(\hat{\rho}) = -Tr[\hat{\rho}\, log_2(\hat{\rho})] \tag{2.12}$$

Thus, we can define the entropy of entanglement of a state for

$$|\psi\rangle = \sum_{k, d_k \leqslant d_A, d_B} d_k |u_k, v_k\rangle$$

as follows,

$$E(\psi) = -Tr[\hat{\rho}_A\, log_2(\hat{\rho}_A)] = -Tr[\hat{\rho}_B\, log_2(\hat{\rho}_B)] = -\sum_k d_k^2 log_2(d_k^2). \tag{2.13}$$

Another way detecting entanglement is using partial transposition. Given ρ, we can always write it in terms of an orthonormal basis $\{|]n, m\rangle\}$, as

$$\hat{\rho} = \sum_{i,j;k,l} |i, k\rangle \langle j, l| . \tag{2.14}$$

We define its partial transpose with respect to A in the basis $\{|n\rangle\}$,

$$\hat{\rho}^{T_A} = \sum_{i,j;k,l} |j, k\rangle \langle i, l| . \tag{2.15}$$

For example, for a density matrix as follows[21],

$$\hat{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} & 0 & 0 \\ \rho_{21} & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & \rho_{34} \\ 0 & 0 & \rho_{43} & \rho_{44} \end{bmatrix} \tag{2.16}$$

The partial transpose matrix can be written as

$$\hat{\rho}^{T_A} = \begin{bmatrix} \rho_{11} & \rho_{43} & 0 & 0 \\ \rho_{34} & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & \rho_{21} \\ 0 & 0 & \rho_{12} & \rho_{44} \end{bmatrix} \tag{2.17}$$

whose eigenvalues are,

$$\begin{aligned} \{v_i\} =& \{\frac{1}{2}\big(\rho_{11} + \rho_{22} \pm \sqrt{(\rho_{11} + \rho_{22})^2 + 4(|\rho_{34}|^2 - \rho_{11}\rho_{22})}\big), \\ & \frac{1}{2}\big(\rho_{11} + \rho_{22} \pm \sqrt{(\rho_{33} + \rho_{44})^2 + 4(|\rho_{12}|^2 - \rho_{33}\rho_{44})}\big)\} \end{aligned} \tag{2.18}$$

There are two roots which can be negative. However,they cannot be negative simultaneously. Because, there must be the inequality,

$$|\rho_{34}| - \sqrt{\rho_{11}\rho_{22}} > 0, \tag{2.19}$$

implies that

$$|\rho_{12}| - \sqrt{\rho_{33}\rho_{44}} < 0, \tag{2.20}$$

and vice versa. So, eigenvalues take negative values for entangled states.

2.1.3. Measures of Entanglement

Most common measures of entanglement are as follows;

- Entanglement of formation: The most basic of these measures is the *entanglement of formation*, which is intended to quantify the resources needed to create a given entangled state[3]. The entanglement of formation for a state $\hat{\rho}$ is defined as the average entanglement of the pure states of the decomposition, minimized all over decompositions of ρ,

$$E(\hat{\rho}) = min \sum_i p_i E(\psi_i). \tag{2.21}$$

- Fidelity: The fidelity with a maximally entangled state,

$$F(\hat{\rho}) = max \left| \langle \Phi^+ | \hat{U}_A \otimes \hat{V}_B | \psi \rangle \right|^2 \tag{2.22}$$

where the maximization is with respect to the unitary operators U and V. It measures in a sense how close we are to a maximally entangled state: U and V just correspond to a basis change. [7]

- Negativity: Another measure of entanglement is negativity, which is based on the Peres-Horodecki [15, 11] criterion and it is defined by the formula

$$N(\hat{\rho}) = max(||hatrho^{T_A}|| - 1) \tag{2.23}$$

where ρ^{T_A} is partial transpose operator. Here, the partial transpose means transportation with respect to the system in subspace A[21].

- Concurrence: Concurrence is associated with the entanglement of formation, but it is a good measure of entanglement for the bipartite systems. The concurrence introduced by Wooters[26] is defined as[21],

$$C(\hat{\rho}) = max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \tag{2.24}$$

where λ_i are the eigenvalues of matrix $R = \widetilde{\rho}\rho$. Here $\widetilde{\rho}$ is,

$$\widetilde{\rho} = \hat{\sigma}_y \otimes \hat{\sigma}_y \hat{\rho}^* \hat{\sigma}_y \otimes \hat{\sigma}_y \tag{2.25}$$

2.1.3.1. Concurrence

Concurrence is the form of entanglement of formation, which can only apply on bipartite systems. It makes use of what called the "*spin flip*" transformation. Spin flip transformation is a function applicable to states of an arbitrary number of qubits. For a pure state of a single qubit, the spin flip is defined by

$$\left|\tilde{\psi}\right\rangle = \hat{\sigma}_y \left|\psi^*\right\rangle \tag{2.26}$$

where $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$ when it is expressed in a fixed basis such as $|\uparrow\rangle, |\downarrow\rangle$, and $\hat{\sigma}_y$ expressed in the same basis is the Pauli matrix[26]. For spin-1/2 particle, this is the standard time reversal operation and reverses the direction of spins [17, 26].

To perform a spin flip on n qubits, the transformation given in Eq. (2.26) must be applied. For example, for a general state of ρ of two qubits, the spin flip state is

$$\widetilde{\rho} = \hat{\sigma}_y \otimes \hat{\sigma}_y \hat{\rho}^* \hat{\sigma}_y \otimes \hat{\sigma}_y \tag{2.27}$$

As the measure of entanglement, we use concurrence method. So, the calculation of concurrence is basically[26] as follows,

- First, we calculate $\widetilde{\rho} = \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y$,
- Then we calculate $R = \widetilde{\rho}\rho$
- As the eigenvalues of R are λ_i for i=1,2,3,4 while λ_i's line in decreasing order. The concurrence becomes,

$$\mathcal{C}(\hat{\rho}) = max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{2.28}$$

2.2. Open Quantum Systems

For a closed quantum system, the time derivation of the density operator can be define in Heisenberg picture as[17],

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] \tag{2.29}$$

where $\hat{\rho}$ is the density operator about the system and $\hat{H}$ is the Hamiltonian of the system. But for an open quantum system, terms about the interaction have to be added into this relation. So, the time derivation of the density operator can be written as[25],

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \sum_j \left[2\hat{L}_j \hat{\rho} \hat{L}_j^\dagger - \{\hat{L}_j^\dagger \hat{L}_j, \rho\}\right]. \tag{2.30}$$

where $\hat{\rho}$ is the density operator about the system and $\hat{H}$ is the Hamiltonian of the system. The relation in Eq. (2.30) is called as the 'master equation'.

2.2.1. Master Equation

The main objective is to describe the time evolution of an open system with a differential equation which properly describes non-unitary behaviour. This description is provided by master equation , which can be written most generally in the Lindblad[12] form as,

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \sum_j \left[2\hat{L}_j \rho \hat{L}_j^\dagger - \{\hat{L}_j^\dagger \hat{L}_j, \hat{\rho}\}\right] \equiv \mathcal{L}\hat{\rho}, \tag{2.31}$$

where $\{\hat{x}, \hat{y}\} = \hat{x}\hat{y} + \hat{y}\hat{x}$ denotes an anticommutator, $\hat{H}$ is the system's Hamiltonian which is a Hermitian operator representing the coherent part of the dynamics, and $\hat{L}_j$ are the operators representing the coupling of the system to the environment.

For $\mathcal{D}[\hat{L}_j]\rho = \left[2\hat{L}_j \hat{\rho} \hat{L}_j^\dagger - \{\hat{L}_j^\dagger \hat{L}_j, \hat{\rho}\}\right]$, it can be written as

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \sum_j \mathcal{D}[\hat{L}_j]\hat{\rho} \tag{2.32}$$

2.2.2. Derivation of Master Equation on an Example

In order to derive a master equation for a system, one begins with a system-environment model Hamiltonian , and then makes the Born and Markov approximations in order to determine $\hat{L}_j$. We will show this process on an example which consists a two level atom interacting with thermal radiation as the environment.

2.2.2.1. Two Level Atom Interacting with an Electric Field

The Hamiltonian for the system of interest;

$$\hat{H}_A = \hbar w_{eg} \left|e\right\rangle \left\langle e\right| = \hbar w_{eg} \hat{S}_+ \hat{S}_- \tag{2.33}$$

The Hamiltonian for the electric field which effect the system of interest is;

$$\hat{H}_F = \sum_{\mathbf{k},\lambda} \hbar w_k (\hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + \frac{1}{2}) \tag{2.34}$$

And the definition of the interaction between the system and the electric field is

$$H_{int} = -\mathbf{d}\cdot\mathbf{E} \approx -\left\langle e\right| \mathbf{d}\cdot\mathbf{E}^{(+)} \left|g\right\rangle S_+ - \left\langle e\right| \mathbf{d}\cdot\mathbf{E}^{(-)} \left|g\right\rangle S_- \tag{2.35}$$

where $S_+ = \left|g\right\rangle \left\langle e\right|$ and $S_- = \left|e\right\rangle \left\langle g\right|$

For the usual expansion of electromagnetic field,

$$\mathbf{E}^{(+)}(\mathbf{x}) = -i \sum_{\mathbf{k},\lambda} \sqrt{\frac{2\pi\hbar w_{\mathbf{k}}}{V}} \vec{\epsilon}_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.36}$$

when we put the electromagnetic field definition in the interaction Hamiltonian equation, we find;

$$H_{int} = \hbar(F S_+ + F^\dagger S_-) \tag{2.37}$$

where

$$F = -i \sum_{\mathbf{k},\lambda} \sqrt{\frac{2\pi\hbar w_{\mathbf{k}}}{V}} \frac{\left\langle e\right| \vec{\epsilon}_{\mathbf{k},\lambda}\cdot\mathbf{d} \left|g\right\rangle}{\hbar} a_{\mathbf{k},\lambda} = \sum_{\mathbf{k},\lambda} \frac{V_{\mathbf{k},\lambda}}{\hbar} a_{\mathbf{k},\lambda} \tag{2.38}$$

$V_{\mathbf{k},\lambda}$ is the dipole interaction energy of the atom with a single photon.

In order to remove the trivial free evolution of the atom and the field from our problem we will transform into the "interaction picture" via the unitary operator,

$$U^{(I)}(t_1, t_2) = exp\{-\frac{i}{\hbar}(H_A + H_F)(t_2 - t_1)\} \tag{2.39}$$

For $t_1 = 0$ and $t_2 = t$, the transformation operator is

$$U^{(I)}(t, 0) = U^{(I)}(t) = exp\{-\frac{i}{\hbar}(H_A + H_F)(t)\} \tag{2.40}$$

According to this transformation, the density operator in interaction picture becomes

$$\rho^{(I)}_{total}(t) = (U^{(I)})^{\dagger}(t, 0)\rho_{total}U^{(I)}(t, 0) \tag{2.41}$$

In the same way,

$$H^{(I)}_{int}(t) = (U^{(I)})^{\dagger}H_{int}U^{(I)} = \hbar(F^{(I)}(t)S^{(I)}_{+}(t) + F^{(I)^{\dagger}}(t)S^{(I)}_{-}(t)) \tag{2.42}$$

while

$$F^{(I)}(t) = \sum_{\mathbf{k},\lambda} \frac{V_{\mathbf{k},\lambda}}{\hbar} a^{(I)}(t)_{\mathbf{k},\lambda} \tag{2.43}$$

$$a^{(I)}_{\mathbf{k},\lambda}(t) = a_{\mathbf{k},\lambda}e^{-iw_k t} \tag{2.44}$$

$$S^{(I)}_{+}(t) = S_{+}e^{-iw_{eg}t} \tag{2.45}$$

Using these definitions and the Liouville equation in the Schrödinger picture, we arrive at the Liouville equation in the interaction picture.

$$\dot{\rho}^{(I)}_{total}(t) = -\frac{i}{\hbar}[H^{(I)}_{int}(t), \rho^{(I)}_{total}(t)] \tag{2.46}$$

If we integrate this equation from t to $(t + \delta t)$,

$$\rho^{(I)}_{total}(t + \Delta t) = \rho(t)^{(I)}_{total}(t) - \frac{i}{\hbar}\int_{t}^{t+\Delta t} dt'[H^{(I)}_{int}(t'), \rho^{(I)}_{total}(t)] \tag{2.47}$$

This can be expand to a series solution. If this series is terminated at some point, a perturbation solution will be obtained. Since the coupling between the system and reservoir is assumed to be weak, we will expand it only up to second order

$$\begin{aligned}\rho^{(I)}_{total}(t+\Delta t) =& \rho^{(I)}_{total}(t) \\ &- \frac{i}{\hbar}\int_t^{t+\Delta t} dt'[H^{(I)}_{int}(t'), \rho^{(I)}_{total}(t')] \\ &- \frac{1}{\hbar^2}\int_t^{t+\Delta t} dt' \int_t^{t'} dt''[H^{(I)}_{int}(t'), [H^{(I)}_{int}(t''), \rho^{(I)}_{total}(t)]\end{aligned} \tag{2.48}$$

We will drop superscript (I), because all of the calculations will be in interaction picture.

We want to work on the dynamics of the reduced density matrix of atom alone, so we can trace over the modes of the electromagnetic field

$$\begin{aligned}\rho_A(t+\Delta t) =& Tr_R(\rho(t)_{total}(t+\Delta t)) \\ =& \rho_A(t) \\ &- \frac{i}{\hbar}\int_t^{t+\Delta t} dt'[H_{int}(t'), \rho_A(t)\otimes\rho_R(t)] \\ &- \frac{1}{\hbar^2}\int_t^{t+\Delta t} dt' \int_t^{t'} dt''[H_{int}(t'), [H_{int}(t''), \rho_A(t)\otimes\rho_R(t)]\end{aligned} \tag{2.49}$$

Here we have used the fact that at the initial time the atom and field are uncorrelated so that the total density matrix is a product of the two subsystems. Substituting the relation for $H_{int}(t)$ in (2.42), we arrive at the following expression by using the cyclic property of the trace $[Tr(ABC) = Tr(CAB) = Tr(BCA)]$,

$$Tr_R(F(t')\rho_R(t)F^\dagger(t')) = Tr_R(F^\dagger(t')F(t')\rho_R(t)) = \langle F(t)^\dagger F(t)\rangle \tag{2.50}$$

$$\begin{aligned}\rho_A(t+\Delta t)-\rho_A(t) =& -i\int_t^{t+\Delta t} dt'([S_+(t'),\rho_A(t)]\langle F(t')\rangle_R + [S_-(t'),\rho_A(t)]\langle F^\dagger(t')\rangle_R) \\ & -\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \\ & \times \{(S_-(t')S_+(t'')\rho_A(t) - S_+(t'')\rho_A(t)S_-(t'))\langle F^\dagger(t')F(t'')\rangle_R \\ & + (S_+(t')S_-(t'')\rho_A(t) - S_-(t'')\rho_A(t)S_+(t'))\langle F(t')F^\dagger(t'')\rangle_R \\ & + H.c.\} \\ & + \int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \\ & \times \{(S_+(t')S_+(t'')\rho_A(t) - S_+(t'')\rho_A(t)S_+(t'))\langle F(t')F(t'')\rangle_R \\ & (S_-(t')S_-(t'')\rho_A(t) - S_-(t'')\rho_A(t)S_-(t'))\langle F(t')F(t'')\rangle_R \\ & + H.c.\} \end{aligned} \tag{2.51}$$

If we use the definition of $F(t)$, we can see some terms will vanish

$$\langle F(t')\rangle_R \equiv \sum_{\mathbf{k},\lambda} \frac{V_{\mathbf{k},\lambda}}{\hbar} e^{-iw_k t'} \langle a_{\mathbf{k},\lambda}\rangle_R = 0 \tag{2.52}$$

Similarly

$$\langle F(t')F(t'')\rangle_R \equiv \sum_{\mathbf{k},\lambda} \frac{V^2_{\mathbf{k},\lambda}}{\hbar^2} e^{-iw_k (t'+t'')} \langle a^2_{\mathbf{k},\lambda}\rangle_R = 0 \tag{2.53}$$

Also

$$\langle F^\dagger(t')F^\dagger(t'')\rangle_R \equiv \sum_{\mathbf{k},\lambda} \frac{V^2_{\mathbf{k},\lambda}}{\hbar^2} e^{iw_k (t'+t'')} \langle (a^\dagger_{\mathbf{k},\lambda})^2\rangle_R = 0 \tag{2.54}$$

2.2.2.2. Correlation Function and Markov Approximation

The terms $\langle F^\dagger(t')F(t'')\rangle_R$ and $\langle F(t')F^\dagger(t'')\rangle_R$ are known as reservoir correlation function. For example,

$$K(t',t'') \equiv \langle F^\dagger(t')F(t'')\rangle_R \tag{2.55}$$

Physically this measures how an excitation of the reservoir at time t' is correlated with the excitation at t''.

Here some properties of K[8];

- Stationery correlations

  For a reservoir describing thermal equilibrium the correlations should not depend on any origin of time. This is if we ask how an excitation at t' is correlated with an excitation at $t'' = t' + \tau$, the result only depend on the time difference τ,

$$K(t', t'') = K(t' - t'') = \langle F^\dagger(t' - t'')F(0)\rangle_R \tag{2.56}$$

- Time reversal

  From the properties of stationary and the Hermitian conjugate we have

$$K(t'' - t') = \langle F^\dagger(t'')F(t')\rangle_R = K^*(t' - t'') \tag{2.57}$$

  Thus, the real part of K is symmetric in its argument: $Re(K(-\tau)) = Re(K(\tau))$

  For sufficiently large values of $|\tau|$, the correlation function should approach zero. The characteristic temporal width τ_c is known as correlation time. Physically, it sets the scale of time over which the reservoir relaxes back to thermal equilibrium given some excitation. It can be said that the correlation time is as a time scale over which the reservoir has "memory". After this time the reservoir is back to equilibrium and effectively has no memory that there was any excitation.

  One can also define a spectral density of the correlation function by as the Fourier transform

$$S(w) \equiv \int d\tau K(\tau)e^{iw\tau} \tag{2.58}$$

  The bandwidth of the reservoir, Δw_R, given by the characteristic width of $S(w)$, is related to the correlation time by the usual Fourier duality

$$\tau_c \sim \frac{1}{\Delta w_R} \tag{2.59}$$

  As $\Delta w_R \to \infty$, (i.e. the reservoir is "white noise"), the correlation time $\tau_c \to 0$. The real part of the correlation function

$$Re(K(\tau)) \sim \delta(\tau) \tag{2.60}$$

Thus a "white noise" reservoir has no memory; it instantaneously relaxes to its equilibrium value. In general, the correlation time will be very small, but not exactly zero.

The other important time scale in the problem is the time for the system to relax to the equilibrium as it dissipates energy to the reservoir. If we call this rate of relaxation Γ, then it will generally be the case that $\Gamma \ll \Delta w_R$, or equivalently $\tau_c \ll 1/\Gamma$. Furthermore, this inequality is generally be the case that the reservoir correlation time is effectively zero when compared with the system decay time . We want to study the evolution of the reduced density matrix for the system alone, which changes on a time scale $1/\Gamma$. Thus we want to look on time scales $\Delta t \ll 1/\Gamma$. On the other hand, this time scale may still be very large compared to the reservoir correlation. Therefore, we have

$$\tau_c \ll \Delta t \ll 1/\Gamma \tag{2.61}$$

Stated alternatively, we want to study the dynamics of the reduced density matrix for the system "coarse-grained" over the reservoir correlation time. In this way we approximate the reservoir as it has no memory. So it is "delta-correlated". Such an approximation is known **Markov approximation**. This approximation will allow us to take the time-reversible equations of motion into irreversible equation which give rise to dissipation.

2.2.2.3. Master Equation in the Markov Approximation

With our characterization of the correlation function for the reservoir forcing function, we make the following change of variables so that, only time differences appear

For $\tau \equiv t'' - t'$ and $\xi \equiv t'' - t$, the Jacobian for the change of area element for out two dimensional integral is

$$\frac{\partial(\xi,\tau)}{\partial(t',t'')} = det \begin{bmatrix} \frac{\partial \xi}{\partial t'} & \frac{\partial \tau}{\partial t''} \\ \frac{\partial \tau}{\partial t'} & \frac{\partial \xi}{\partial t''} \end{bmatrix} = det \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = 1 \tag{2.62}$$

To get the limits of integration, we must change the two dimension integration region from the (t', t'') plane to the (τ, ξ) plane. Thus,

$$\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \rightarrow \int_0^{\Delta t} d\xi \int_0^{\Delta t} d\tau \tag{2.63}$$

The equation for the reduced density operator for the atom in these new variables becomes

$$\begin{aligned}\rho_A(t+\Delta t) - \rho_A(t) = &- \int_0^{\Delta t} d\xi \int_0^{\Delta t} d\tau \\ &\times \{(S_- S_+ \rho_A(t) - S_+ \rho_A(t) S_-)\langle F^\dagger(0) F(\tau)\rangle_R \\ &+ (S_+ S_- \rho_A(t) - S_- \rho_A(t) S_+)\langle F(0) F^\dagger(\tau)\rangle_R \\ &+ H.c.\}\end{aligned} \tag{2.64}$$

Now we can apply the Markov approximation. According to the discussion above, the integrand will be non-negligible only for times τ smaller than correlation time. As we are course graining the dynamics on time scales $\Delta t \gg t_c$, the upper limit of integration over τ is effectively infinity, ($\Delta t \rightarrow \infty, \tau$ integral). On the other hand, the coarse grained time step is chosen to be much more smaller than the rate at ρ_A changes, as it is effectively a differential. By using the Markov approximation we have

$$\begin{aligned}\frac{d\rho_A}{dt} =&(-S_- S_+ \rho_A(t) + S_+ \rho_A(t) S_-) \int_0^{\infty} d\tau\, e^{iw_{eg}\tau} \langle F^\dagger(0) F(\tau)\rangle_R \\ &+ (-S_+ S_- \rho_A(t) + S_- \rho_A(t) S_+) \int_0^{\infty} d\tau\, e^{-iw_{eg}\tau} \langle F(0) F^\dagger(\tau)\rangle_R \\ &+ H.c.\end{aligned} \tag{2.65}$$

2.2.2.4. Fluctuation-Dissipation Theorem

It is left to evaluate the Fourier transforms of the correlation functions. [1] Substituting for the reservoir forcing $F(t)$, in Eq. (2.10), gives

$$\int_0^\infty d\tau e^{iw_{eg}\tau}\langle F^\dagger(0)F(\tau)\rangle_R = \sum_{\mathbf{k},\lambda,\mathbf{k}',\lambda'} \frac{V^*_{\mathbf{k},\lambda}V_{\mathbf{k}',\lambda'}}{\hbar^2}\langle a^\dagger_{\mathbf{k},\lambda}a_{\mathbf{k}',\lambda'}\rangle_R \int_0^\infty d\tau\, e^{-i(w_{\mathbf{k}}-w_{eg})\tau} \tag{2.66}$$

For $\langle a^\dagger_{\mathbf{k},\lambda}a_{\mathbf{k}',\lambda'}\rangle_R = \overline{n}(w_{\mathbf{k}})\ \delta_{\mathbf{k},\mathbf{k}'}\ \delta_{\lambda,\lambda'}$,

$$\int_0^\infty d\tau\, e^{iw_{eg}\tau}\langle F^\dagger(0)F(\tau)\rangle_R = \sum_{\mathbf{k},\lambda} \frac{|V_{\mathbf{k},\lambda}|^2}{\hbar^2}\overline{n}(w_{\mathbf{k}}) \int_0^\infty d\tau\, e^{-i(w_{\mathbf{k}}-w_{eg})\tau} \tag{2.67}$$

The next step is obtaining equations of motions to let the reservoir consist of an uncountable finite numbers (i.e. continuous) number of degrees of freedom.

Under this approximation the sum over reservoir modes goes to an integral over the density modes $D(w_{\mathbf{k}})$,

$$\int_0^\infty d\tau\, e^{iw_{eg}\tau}\langle F^\dagger(0)F(\tau)\rangle_R = \sum_\lambda \int dw_{\mathbf{k}} \frac{|V_{\mathbf{k},\lambda}|^2}{\hbar^2} D(w_{\mathbf{k}})\overline{n}(w_{\mathbf{k}}) \int_0^\infty d\tau\, e^{-i(w_{\mathbf{k}}-w_{eg})\tau} \tag{2.68}$$

It can integrate over τ. In order to do this, we add convergence factor $e^{\varepsilon\tau}$, and then take the limit as $\varepsilon \to 0$,

$$\begin{aligned} lim_{\varepsilon\to 0}\int_0^\infty d\tau\, e^{-i(w_{\mathbf{k}}-w_{eg}-i\varepsilon)\tau} &= lim_{\varepsilon\to 0}\frac{1}{i(w_{\mathbf{k}}-w_{eg})+\varepsilon} \\ &= lim_{\varepsilon\to 0}\frac{\varepsilon}{(w_{\mathbf{k}}-w_{eg})^2+\varepsilon^2} \\ &- i\, lim_{\varepsilon\to 0}\frac{w_{\mathbf{k}}-w_{eg}}{(w_{\mathbf{k}}-w_{eg})^2+\varepsilon^2} \end{aligned} \tag{2.69}$$

For the real part,

$$lim_{\varepsilon\to 0}\frac{\varepsilon}{(w_{\mathbf{k}}-w_{eg})^2+\varepsilon^2} = \pi\delta(w_{\mathbf{k}}-w_{eg}) \tag{2.70}$$

And for the imaginary part,

$$lim_{\varepsilon\to 0}\frac{w_{\mathbf{k}}-w_{eg}}{(w_{\mathbf{k}}-w_{eg})^2+\varepsilon^2} = \mathbb{P}\Big(\frac{1}{w_{\mathbf{k}}-w_{eg}}\Big) \tag{2.71}$$

The principle part, the part define with $\mathbb{P}$, allows one to exclude the singularity at x=0, thus,

$$\int_0^\infty d\tau\, e^{-i(w_{\mathbf{k}}-w_{eg})\tau} = \pi\delta(w_{\mathbf{k}}-w_{eg}) - i\mathbb{P}\Big(\frac{1}{w_{\mathbf{k}}-w_{eg}}\Big). \tag{2.72}$$

Eq. (2.68) becomes,

$$\begin{aligned}\int_0^\infty d\tau\, e^{iw_{eg}\tau}\langle F^\dagger(0)F(\tau)\rangle =& \frac{1}{2}\sum_\lambda \frac{2\pi}{\hbar^2}|V_\lambda(w_{eg})|^2 D(w_{eg})\overline{n}(w_{eg}) \\ &- \frac{i}{\hbar}\mathbb{P}\int dw_{\mathbf{k}}\frac{|V_\lambda(w_{\mathbf{k}})|^2}{\hbar w_{\mathbf{k}}-\hbar w_{eg}} D(w_{\mathbf{k}})\overline{n}(w_{\mathbf{k}})\end{aligned} \tag{2.73}$$

. For

$$\sum_\lambda \frac{2\pi}{\hbar^2}|V_\lambda(w_{eg})|^2 D(w_{eg}) = \Gamma \tag{2.74}$$

and

$$\mathbb{P}\int dw_{\mathbf{k}}\frac{|V_\lambda(w_{\mathbf{k}})|^2}{\hbar w_{\mathbf{k}}-\hbar w_{eg}} D(w_{\mathbf{k}})\overline{n}(w_{\mathbf{k}}) = \delta E^g_{light} \tag{2.75}$$

It can be written in a simpler form as

$$\int_0^\infty d\tau\, e^{iw_{eg}\tau}\langle F^\dagger(0)F(\tau)\rangle = \frac{1}{2}\overline{n}\,\Gamma - \frac{i}{\hbar}\delta E^g_{light} \tag{2.76}$$

In the same way, for $\langle F(0)F^\dagger(\tau)\rangle$, by using the relation of annihilation-creation operators of thermal reservoir, $\langle a_{\mathbf{k},\lambda}a^\dagger_{\mathbf{k}',\lambda'}\rangle_R = (\overline{n}(w_{\mathbf{k}})+1)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}$,

$$\int_0^\infty d\tau\, e^{-iw_{eg}\tau}\langle F(0)F^\dagger(\tau)\rangle = \frac{1}{2}\overline{n}\,\Gamma + \frac{i}{\hbar}\delta E^e_{light} \tag{2.77}$$

Here, $\delta E^e_{light} = \delta E^g_{light} + \delta E_{vacuum}$. Also, it can be taken as $\delta E^g_{light} \to \delta E_{light}$.

As a result,

$$\begin{aligned}\frac{d\rho_A}{dt} =& -\frac{i}{\hbar}[H'_A,\rho_A] - \frac{\Gamma}{2}\,\overline{n}\{S-S_+,\rho_A\} + S_+ + \Gamma\,\overline{n}\rho_A S_- \\ &- \frac{\Gamma}{2}\,(\overline{n}+1)\{-S_+S_-,\rho_A\} + -\Gamma\,(\overline{n}+1)S_-\rho_A S_+,\end{aligned} \tag{2.78}$$

while,

$$H'_A = (\hbar w_{eg} - \delta E_{light})S_+S_- + (\delta E_{light} + \delta E_{vacuum})S_-S_+ \tag{2.79}$$

.

In this way, we obtained master equation for a two state atom in an electric field.

2.3. Quantum Jumps and Evolution of a Quantum System

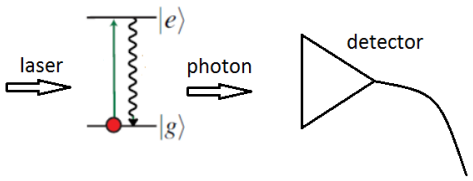

Figure 2.1. Quantum Jump. Here, measurement means quantum jump. This measurement includes information about the system.

The quantum jump is basically the effectively instantaneous transition of an atom from one state to another[23]. It was shown by Carmichael that quantum jumps are an implicit part of standard photodetection theory[6]. During transitions from a state to another, photons are emmited and absorbed. These photons contain the knowledge about the system. When we make measurement on an emmited photon, we can obtain information about the state of the system. So we have to interested in a quantum measurement on the system to obtain the definition of quantum jump.

The evolution of an isolated quantum system in the absence of measurement is Markovian[25],

$$|\psi(t+dt)\rangle = \hat{U}(dt)\,|\psi(t)\rangle = e^{-i\hat{H}\,dt}\,\psi(t) \tag{2.80}$$

where $\hat{H}$ is the Hamiltonian. The evolution of the system for an infinitesimal dt can be describe with a finite differential,

$$lim_{dt\to 0}\frac{|\psi(t+dt)\rangle - |\psi(t)\rangle}{dt} = \left|\dot{\psi}(t)\right\rangle = -i\hat{H}(t)\,|\psi(t)\rangle = finite \tag{2.81}$$

So, the evolution of $\hat{\rho}$ becomes

$$lim_{dt\to 0}\frac{\hat{\rho}(t+dt) - \hat{\rho}(t)}{dt} = \dot{\hat{\rho}}(t) = finite \tag{2.82}$$

In this limit, we can measure the system. The state matrix can be written in the average of the possible results

$$\hat{\rho}(t+dt) = \sum_i \hat{M}_i^\dagger(dt)\hat{\rho}(t)\hat{M}_i(dt) \tag{2.83}$$

If we can define a $\hat{M}_i$ time evolution operator as,

$$\hat{M}_0(dt) = \hat{1} - (\hat{R}/2 + i\hat{H})dt \tag{2.84}$$

The system in which we interest has interaction with the environment. So, there is an additional term with $\hat{R}$ operator in the time evolution operator.

For,

$$\hat{M}_0^\dagger(dt)\hat{M}_0(dt) = \hat{1} - \hat{R}dt \neq \hat{1} \tag{2.85}$$

Even $\hat{M}_0$ have to be a Hermitian operator, it does not satisfy the completeness condition in this structure. So, there is two option:(i)$\hat{R}$ operator must be a zero operator,or (ii) if $\hat{R}$ is not a zero operator, it can be define in terms of another operator

If we consider the relation $\sum_i \hat{M}_i^\dagger(dt)\hat{M}_i(dt) = \hat{1}$, we can define a operator as $\hat{M}_1(dt)$,

$$\hat{M}_1(dt) = \sqrt{dt}\hat{c} \tag{2.86}$$

$\hat{R}$ becomes

$$\hat{R} = dt\, \hat{c}^\dagger \hat{c} \tag{2.87}$$

Here, $\hat{M}_1(dt)$ is about detections of photon. It defines a quantum jump.

2.4. Quantum Trajectory Approach

For $\hat{c} = \sqrt{\gamma}\hat{a}$, we can write that[25],

$$\begin{aligned} \hat{\mathcal{J}}\hat{\rho} \equiv & \hat{c}\hat{\rho}\hat{c}^{\dagger} \\ \langle \hat{c}^{\dagger}\hat{c} \rangle = & Tr[\hat{\mathcal{J}}\hat{\rho}(t)] \end{aligned} \tag{2.88}$$

where $\hat{c}$ is the quantum jump operator and $\hat{\mathcal{J}}$ is the quantum jump superoperator. If we denote the number of photodetections up to time t by $N_t(t)$, there is two meaning of $dN(t)$. dN is either zero or one, because it is in an infinitesimal time. The other means of dN is that it is identical with the probability of detecting a photon. These definitions of dN are about the properties of the stochastic process.

If $dN(t) = 1$, the state vector changes under the effect of jump operator. Under the normalization,

$$|\psi_1(t+dt)\rangle = \frac{\hat{M}_1(dt)\,|\psi(t)\rangle}{\sqrt{\langle \hat{M}_1^{\dagger}(dt)\hat{M}_1(dt)(t)\rangle}} = \frac{\hat{c}\,|\psi(t)\rangle}{\sqrt{\langle \hat{c}^{\dagger}\hat{c}\rangle(t)}} \tag{2.89}$$

If $dN(t) = 0$, the state vector changes with the time-evolution operator. It can be normalized as,

$$\begin{aligned} |\psi_0(t+dt)\rangle &= \frac{\hat{M}_0(dt)\,|\psi(t)\rangle}{\sqrt{\langle \hat{M}_0^{\dagger}(dt)\hat{M}_0(dt)\rangle(t)}}\,|\psi(t)\rangle \\ &= \frac{(1-\frac{\hat{c}^{\dagger}\hat{c}}{2} - i\hat{H})dt\,|\psi(t)\rangle}{\sqrt{\langle(1-\frac{\hat{c}^{\dagger}\hat{c}}{2}+i\hat{H})((1-\frac{\hat{c}^{\dagger}\hat{c}}{2}-i\hat{H}))dt\rangle(t)}}\,|\psi(t)\rangle \\ &\approx 1 - dt[i\hat{H} + \frac{1}{2}\hat{c}^{\dagger}\hat{c} - \langle\frac{1}{2}\hat{c}^{\dagger}\hat{c}\rangle](t)\,|\psi(t)\rangle \end{aligned} \tag{2.90}$$

So, the time derivation of the state becomes,

$$d\,|\psi\rangle = \left[dN(t)\Big(\frac{\hat{c}}{\sqrt{\langle \hat{c}^{\dagger}\hat{c}(t)\rangle}}\Big) + [1-dN(t)]dt\Big(\frac{\langle \hat{c}^{\dagger}\hat{c}(t)\rangle}{2} - \frac{\hat{c}^{\dagger}\hat{c}}{2} - i\hat{H}\Big)\right]|\psi(t)\rangle \tag{2.91}$$

This equation is a non-linear stochastic Schrödinger equation and a solution of this equation is a *quantum trajectory* for the system.

If dt is so small as there can be only one photodetection happen in each time step, dN has only two possible value as 1 or 0. So,there are possibilities for $d\,|\psi\rangle$

- If $dN(t) = 0$,

$$d\left|\psi\right\rangle = \left[dt\Big(\frac{\langle\hat{c}^{\dagger}\hat{c}(t)\rangle}{2} - \frac{\hat{c}^{\dagger}\hat{c}}{2} - i\hat{H}\Big)\right]\left|\psi(t)\right\rangle \tag{2.92}$$

  The term about quantum jump will be vanish and just the term about time-evolution operator will affect the state of the system.

- If $dN(t) = 1$,

$$d\left|\psi\right\rangle = \Big(\frac{\hat{c}}{\sqrt{\langle\hat{c}^{\dagger}\hat{c}(t)\rangle}}\Big)\left|\psi(t)\right\rangle \tag{2.93}$$

  The term about time-evolution will be vanish and the system collapse into a state by the effect of the measurement operator (quantum jump operator).

If this process continues for an exact time as t, where $t \ggg dt$, we obtain the Dyson expantion for the superoperators[5],

$$\hat{\rho(t)} = \sum_{m=0}^{\infty}\int_{0}^{t} dt_m \int_{0}^{t_m} dt_{m-1}... \int_{0}^{t_2} dt_1 \hat{\mathcal{S}}(t-t_m)\hat{\mathcal{J}}\hat{\mathcal{S}}(t_m - t_{m-1})...\hat{\mathcal{J}}\hat{\mathcal{S}}(t_1)\hat{\rho}(0) \tag{2.94}$$

where $\hat{\mathcal{S}}(t) = e^{(\hat{\mathcal{L}}-\hat{\mathcal{J}})t}$ can be defined according to the notation of Srinivas and Davies[20]. This process called 'quantum trajectories approach'.

CHAPTER 3

CAVITY QED ON A MODAL SYSTEM

We are taking two atoms with two states, it means two qubits, as our model system. These two qubits are interacting with each other by the cavity field. The steps of application are as follows,

(a) Defining master equation of the system in Lindblad form

(b) Eliminating cavity field adiabatically

(c) Using the master equation to define the time evolution of the system and the jump operators to be able to use Quantum Trajectory Approach

(d) Defining the initial state (as the steady state of the system)

(e) Applying Monte Carlo Method on the Quantum Trajectories Approach

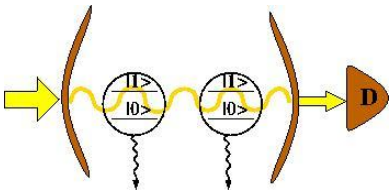

Figure 3.1. A cavity system with two qubits which are pumped with a cavity field.

3.1. Quadrature Measurement-Atomic Mode and Cavity Mode

We can take one of the quadrature modes as atomic mode which just responsible from transferring between exited and ground states for two qubits because of it cannot escape from the cavity[22]. The other quadrature mode will be responsible from the photon modes (it is called ”cavity mode”) distributed through cavity.

According to the following relations between the qubits

$$\hat{J}^{\pm} = \hat{\sigma}_1^{\pm} + \hat{\sigma}_2^{\pm}$$

$$\hat{J}^{\pm} = \hat{J}_x \pm i\hat{J}_y$$

for

$$\hat{\sigma}^{\pm} = \hat{\sigma}_x + i\hat{\sigma}_y$$

By using the dipole approximation, the definition of the interaction Hamiltonian becomes similar to the quadrature modes' and it can be written as

$$\hat{H}_{int} = \hbar g \hat{J}_x \hat{Y}$$

where

$$\hat{Y} = [(\hat{b}e^{-i\omega_k t} + \hat{b}^{\dagger}e^{i\omega_k t})/2]$$

When we put Y in the equation,

$$\hat{H}_{int} = \frac{g}{2}(\hat{J}^{-}e^{-i\omega_a t} + \hat{J}^{+}e^{i\omega_a t})(\hat{b}e^{-i\omega_k t} + \hat{b}^{+}e^{i\omega_k t})$$

If we apply the the rotating-wave approximation in the Jaynes-Cummings model,

$$|\omega_k - \omega_a| << \omega_k + \omega_a$$

$$\hat{H}_{int} = \frac{g}{2}[\hat{J}^{-}\hat{b}^{\dagger} + \hat{J}^{+}\hat{b}] \tag{3.1}$$

where b describes the cavity mode and g is the coupling constant between the qubits and cavity.

The possible states for the two qubit system are,

$$|1\rangle = |e\rangle_1 |e\rangle_2, \qquad |2\rangle = \frac{1}{\sqrt{2}}(|g\rangle_1 |e\rangle_2 + |e\rangle_1 |g\rangle_2), \qquad |3\rangle = |g\rangle_1 |g\rangle_2$$

as the triplet states (j=1) and

$$|4\rangle = \frac{1}{\sqrt{2}}(|g\rangle_1 |e\rangle_2 - |e\rangle_1 |g\rangle_2)$$

as the singlet state(j=0). This latter subspace will not change under any of transformations we perform.

3.2. Adiabatic Elimination of the Cavity Mode

In our model, we assume the system is pumped with a laser whose mode is set to be same with the cavity mode. We can add a term to Hamiltonian to show the effect of pumping on the system;

$$\hat{H}' = \frac{g}{2}[\hat{J}^-\hat{b}^\dagger + \hat{J}^+\hat{b}] - \frac{i\alpha}{2}((\hat{b}) - \hat{b}^\dagger) \tag{3.2}$$

For two-qubit system, if cavity mode is heavily damped, it can be eliminated adiabatically from the master equation. [22] For a system like this, a density operator w is given as follows:

$$\dot{\hat{w}} = -\frac{1}{2}\alpha[\hat{b} - \hat{b}^\dagger, \hat{w}] - i\frac{g}{2}[\hat{J}^-\hat{b}^+ + \hat{J}^+\hat{b}, \hat{w}] + \gamma_1\mathcal{D}[\hat{\sigma}_1]\hat{w} + \gamma_2\mathcal{D}[\hat{\sigma}_2]\hat{w} + \gamma_p\mathcal{D}[b]\hat{w} \tag{3.3}$$

where $\mathcal{D}[\hat{X}]\hat{w} = \hat{X}^\dagger\hat{w}\ \hat{X} - \frac{1}{2}\hat{w}\hat{X}^\dagger\hat{X} - \frac{1}{2}\hat{X}^\dagger\hat{X}\hat{w}$.

A displacement operator for the cavity field can be written as;

$$\hat{D}_b = e^{\mu(\hat{b}^\dagger - \hat{b})}. \tag{3.4}$$

where $\mu = \alpha/\gamma_p$.

By applying the displacement operator, the effect of the cavity field can be partially derive out from the master equation. After similarity transformation the density becomes;

$$\hat{v} = \hat{D}_b^\dagger\hat{w}\hat{D}_b \tag{3.5}$$

$$\dot{\hat{v}} = -i\frac{g\alpha}{2\gamma_p}[(\hat{J}^+ + \hat{J}^-), \hat{v}] - i\frac{g}{2}[\hat{J}^+\hat{b} + \hat{J}^-\hat{b}^\dagger] + \gamma_1\mathcal{D}[\sigma_1]\hat{v} + \gamma_2\mathcal{D}[\sigma_2]\hat{v} + \gamma\mathcal{D}[J^-]\hat{v} \tag{3.6}$$

$$\dot{\hat{v}} = \mathcal{L}\hat{v} - i\frac{g}{2}[\hat{J}^+\hat{b} + \hat{J}^-\hat{b}^\dagger] + \gamma\mathcal{D}[\hat{J}^-]\hat{v} \tag{3.7}$$

where $\mathcal{L}\hat{v}$ includes only the terms about atomic modes.

Since the amplitude of $\hat{b}$ is small, $\hat{v}$ needs to be carried out to small photon numbers. Therefore;

$$\hat{v} = \rho_0 |0\rangle\langle 0| + \rho_1 |1\rangle\langle 0| + H.c.) + \rho_2 |1\rangle\langle 1| + (\rho_{2'} |2\rangle\langle 0| + H.c.) + \mathcal{O}(\lambda^3) \tag{3.8}$$

where λ is a very small number.

We can obtain the values of ρ_k by using the steady state parameters.

Steady state is the situation of the system when the system is stable (stationery) in time. In this situation, there is any transition between states.

From the master equation , we can write down the equation of motion for the components of the three by three density matrix of the state of the system, taking into account that $Tr(\hat{\rho}) = 1$ and that $\hat{\rho}$ is Hermitian[18].

For Eq.(3.7) and Eq.(3.8), if the terms greater than second order are neglected, the equations of motions for the components of the density matrix can be written as,

$$\dot{\rho}_0 = \mathcal{L}\rho_0 - i\frac{g}{2}[J^+\rho_1 + \rho_1^\dagger J^-] + \gamma_p/\rho_2, \tag{3.9}$$

$$\dot{\rho}_1 = \mathcal{L}\rho_1 - i\frac{g}{2}[J^+\rho_0 + \sqrt{2}(J^+)^2 - (J^-)^2\rho_2'] - \frac{\gamma_p}{2}\rho_1, \tag{3.10}$$

$$\dot{\rho}_2 = \mathcal{L}\rho_2 - i\frac{g}{2}[J^+\rho_1^\dagger + \rho_1(J^+)^2] - \gamma_p\rho_2, \tag{3.11}$$

$$\dot{\rho}_2' = \mathcal{L}\rho_2' - i\frac{g}{2}[\sqrt{2}J^-\rho_1] - \gamma_p\rho_2'. \tag{3.12}$$

The density terms ρ_1 and ρ_2' are about transitions. As before, steady state is a stable state, so the time derivatives of transition terms must be zero,

$$\dot{\rho}_1 = 0 \quad \Rightarrow \quad \rho_1 = -i\frac{g}{\gamma_p}[J^-\rho_0 - J^-\rho_2], \tag{3.13}$$

$$\dot{\rho}_2' = 0 \quad \Rightarrow \quad \rho_1 = -i\frac{g}{\sqrt{2}\gamma_p}J^-\rho_1. \tag{3.14}$$

By using these relations, the Eq. (3.9) and Eq. (3.11) can be written as

$$\dot{\rho}_0 = \mathcal{L}\rho_0 - \frac{g^2}{2}[J^+J^-\rho_0 + \rho_0 J^+J^- - J^+J^-\rho_2 - \rho_2 J^+J^-] + \gamma_p/\rho_2, \tag{3.15}$$

$$\dot{\rho}_2 = \mathcal{L}\rho_2 + \frac{g^2}{\gamma_p}[J^-\rho_0 J^+ - J^-\rho_2 J^+] + \gamma_p/\rho_2. \tag{3.16}$$

By considering $Tr(\rho) = 1 \cong \rho_0 + \rho_2$, if we add Eq.(3.15) and Eq.(3.16),

$$\dot{\hat{\rho}} = \hat{\mathcal{L}}\hat{\rho} + \frac{g^2}{\gamma_p}\mathcal{D}[\hat{J}^-]\hat{\rho} \tag{3.17}$$

The coefficient g^2/γ_p describes the strength of collective damping of the two-qubit and the cavity mode, b, and will be called γ.

We can obtain the following master equation

$$\dot{\hat{\rho}} = -i\frac{g\alpha}{2\gamma_p}[(\hat{J}^+ + \hat{J}^-), \hat{\rho}] + \gamma_1\mathcal{D}[\hat{\sigma}_1]\hat{\rho} + \gamma_2\mathcal{D}[\hat{\sigma}_2]\hat{\rho} + \gamma\mathcal{D}[\hat{J}^-]\hat{\rho} \tag{3.18}$$

This is the super-fluorescence master equation. For $\gamma \gg \gamma_1, \gamma_2$, the equation becomes

$$\dot{\hat{\rho}} = -i\frac{g\alpha}{2\gamma_p}[(\hat{J}^+ + \hat{J}^-), \hat{\rho}] + \gamma\mathcal{D}[\hat{J}^-]\hat{\rho}. \tag{3.19}$$

If we take $a = \frac{g\alpha}{2\gamma_p}$ and $g' = \gamma$,

$$\dot{\hat{\rho}} = -ia[(\hat{J}^+ + \hat{J}^-), \hat{\rho}] + g'\mathcal{D}[\hat{J}^-]\hat{\rho}. \tag{3.20}$$

This is the master equation we used to simulate our modal system.

3.3. Defining Quantum Trajectory of the System

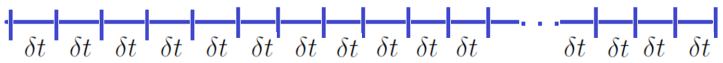

Figure 3.2. δt time steps from t=0 to t.

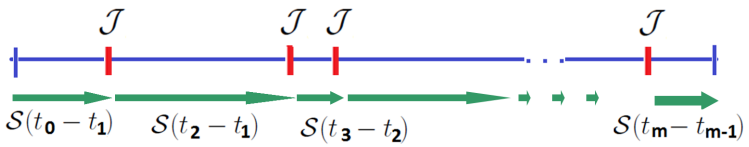

Figure 3.3. A quantum trajectory from t=0 to t. $\mathcal{S}(t_i - t_j)$ is the superoperator about time-evolution of the system and $\mathcal{J}$ is the superoperator about the quantum jumps

For the general definition of the master equation, the relation about the quantum trajectories approach is given as,

$$\hat{\rho}(t) = \sum_{m=0}^{\infty}\int_0^t dt_m \int_0^{t_m} dt_{m-1}... \int_0^{t_2} dt_1 \;\; \hat{\mathcal{S}}(t - t_m)\hat{\mathcal{J}}\hat{\mathcal{S}}(t_m - t_{m-1})...\hat{\mathcal{J}}\hat{\mathcal{S}}(t_1)\hat{\rho}(0). \tag{3.21}$$

Here, the final form of the master equation about our system is given as Eq. (3.20),

$$\dot{\hat{\rho}} = -ia[(\hat{J}^+ + \hat{J}^-), \hat{\rho}] + g'\mathcal{D}[\hat{J}^-]\hat{\rho}. \tag{3.22}$$

According the master equation, the operators about the system can be written to define the quantum trajectory about our system.

- The quantum jump superoperator about the system is,

$$\hat{\mathcal{J}}\hat{\rho} = g'\hat{J}^+\hat{\rho}\hat{J}^- \tag{3.23}$$

  So the quantum jump operator about the system can be written as,

$$\hat{c}\,|\psi\rangle = \sqrt{g'}\hat{J}^-\,|\psi\rangle \tag{3.24}$$

- In the same way, the superoperator defining time-evolution without quantum jump is;

$$\hat{\mathcal{S}}(t)\hat{\rho} = \hat{N}(t)\hat{\rho}\hat{N}^\dagger(t). \tag{3.25}$$

  So, time-evolution operator about the system can be written as;

$$\hat{N}(t)\,|\psi\rangle = exp\left[\left(\frac{1}{i\hbar}a(\hat{J}^+ + \hat{J}^-) - \frac{1}{2}g'\hat{J}^+\hat{J}^-\right)t\right]|\psi\rangle \tag{3.26}$$

3.4. The Steady State as the Initial State of the System

We defined the initial state as the steady state of our system and applied the quantum trajectories approach after that point.

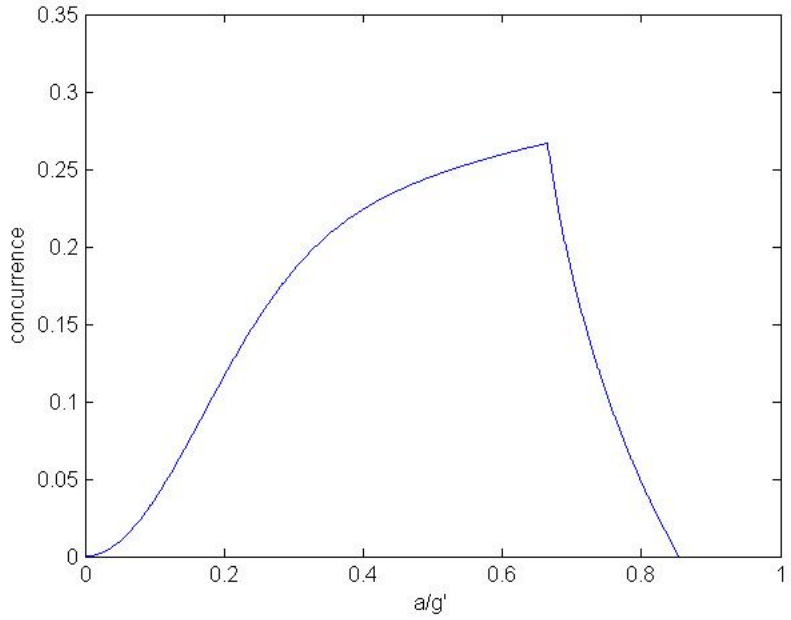

Figure 3.4. concurrence versus a/g' for steady state of the system of interest.

Initially we make calculations for the steady state of the system. For

$$\dot{\hat{\rho}} = -ia[(\hat{J}^{+} + \hat{J}^{-}), \hat{\rho}] + g'\mathcal{D}[\hat{J}^{-}]\hat{\rho}.$$

A relation between concurrence and the ratio of the pumping rate to the damping rate as in the Figure 3.4. As it can be seen from the graph, while the rate of pumping on the system increasing, the entanglement of the system goes to zero at a ratio $a/g' \simeq 1$. It can be explain with the Hermitian and anti-Hermitian's terms of the density of the system's states. While the ratio of the pumping increase, the probability of anti-Hermitian terms decrease. The probability of the states of the system stock into the states about the Hermitian terms. So, amount of the entanglement of the system goes to zero.

3.5. Application of the Monte Carlo Method

One can simulate the ”trajectory” of the atom initially in the excited state by a deterministic decay by the non-Hermitian Hamiltonian, followed by a quantum jump. Averaging over many realization of this random event gives the solution. This method is known as the ”quantum Monte Carlo wave function simulation” and very useful for extremely large problems when there are too many matrix elements to solve for using more traditional techniques [8],[13].

We can use an operator as $\hat{c} = \sqrt{a}\hat{J}^{-}$. Then, the probability to detect a photon in the time interval $[t, t + \delta t]$, given the previous detection history can be written as follows,

$$\hat{\rho}_c dt = Tr[\hat{\mathcal{J}}\hat{\rho}_c(t)]dt \tag{3.27}$$

as expected. Using a numerical solution with a finite time step δt, one generates a random number r on the unit interval ,evolves the state forward according to

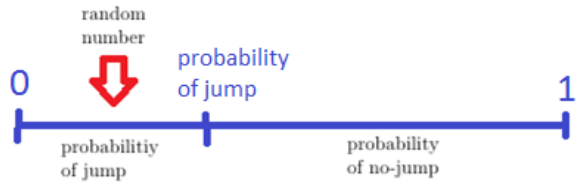

Figure 3.5. The case a photodetection is recorded.

$\tilde{\rho}_c(t + \delta t) = \hat{\mathcal{J}}\tilde{\rho}_c(t)\delta t$ if $r \leq p_c\delta t$

in which case a photodetection is recorded to have occurred in the interval $[t + \delta t]$, or

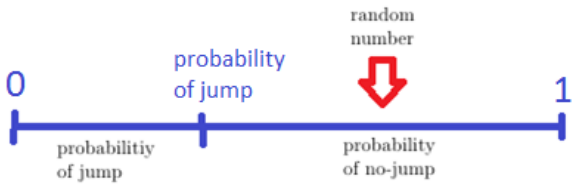

Figure 3.6. The case no photodetection is recorded.

$\tilde{\rho}_c(t + \delta t) = \hat{\mathcal{S}}(\delta t)\tilde{\rho}_c(t)$ if $r > p_c\delta t$

when no photon is detected. ($\tilde{\rho}_c$: the unnormalized density operator)

For

$$\hat{\mathcal{S}}(\delta t)\hat{\rho} = \hat{N}(\delta t)\hat{\rho}\hat{N}^{\dagger}(\ deltat) \tag{3.28}$$

$$\hat{N}(\delta t) = exp\left[\left(\frac{1}{i\hbar}a(\hat{J}^{+} + \hat{J}^{-}) - \frac{1}{2}g'\hat{J}^{\dagger}\hat{J}\right)\delta t\right] \tag{3.29}$$

By using this method and taking small steps of time, we can simulate the trajectory of the system as a random walk. So we obtain a statistical change and take the average behaviour of the system under direct detections.

The results of the simulation are as follows,

3.6. Results

In this section,we will study the details of the graphs of the data which obtained the simulation of our system.

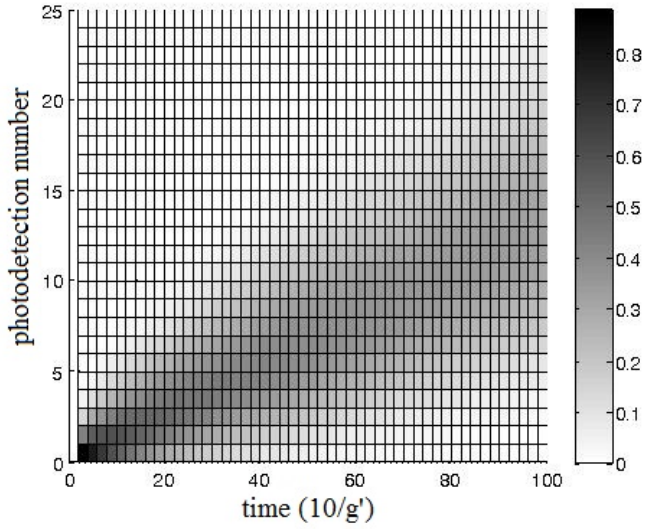

Figure 3.7. The change at the probability of jump defined by using random numbers basically and the number of jumps accounted and the measurement time. In our notation, we take a measurement time t which is much more longer than the time step δt. Here, the definite variables in the master equation are taken as $g' = 0.1\ a$ and $\delta t = 0.0025t$

The detected photons are increasing in time. In beginning, there are a few photons escaping from the cavity, but the probability of photodetection is quite high. When we continue to measure, we can see more photons with detection probabilities getting lower.

According to this graph, the photodetection number has a Gaussian-like distiribution. By using general equations, we can calculate the mean value and the variance of photodetection about the system.

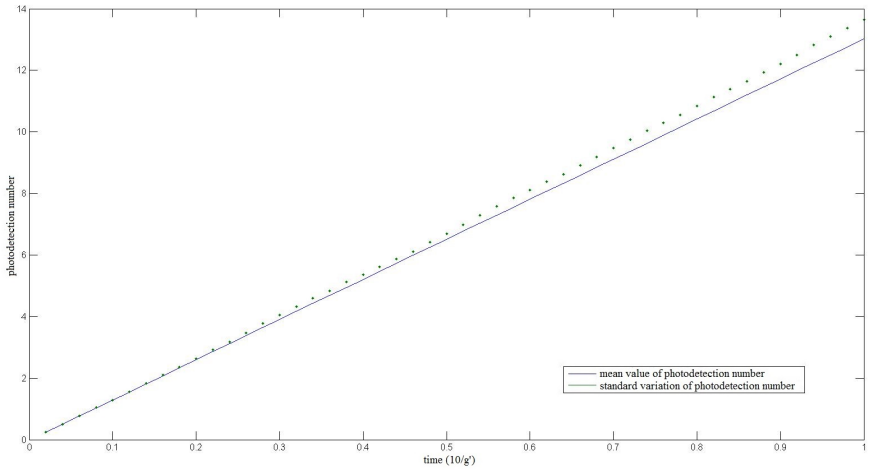

Figure 3.8. The mean value of photodetection number and the standard variation of photodetection number

To calculate the mean value of photodetection at each time step, we use the following equation,

$$\overline{n}(t) = \sum_i n_i p_i(t), \tag{3.30}$$

where i is the number of detection, n_i is the number of detected photons in that time step and p_i is the probability to detect these photons. In the same way, the variance of photodetection can be calculated as,

$$\sigma^2 = \left(\overline{n^2} - \overline{n}^2\right), \tag{3.31}$$

and the standard variation of the photodetection,

$$\sigma = \sqrt{\overline{n^2} - \overline{n}^2}. \tag{3.32}$$

The results of the calculations of mean value and standard variation are shown in the graph. The mean value and the variation of the photodetection is increase linearly with time. But, the standard variation of the detections change proportional to $\sqrt{t}$.

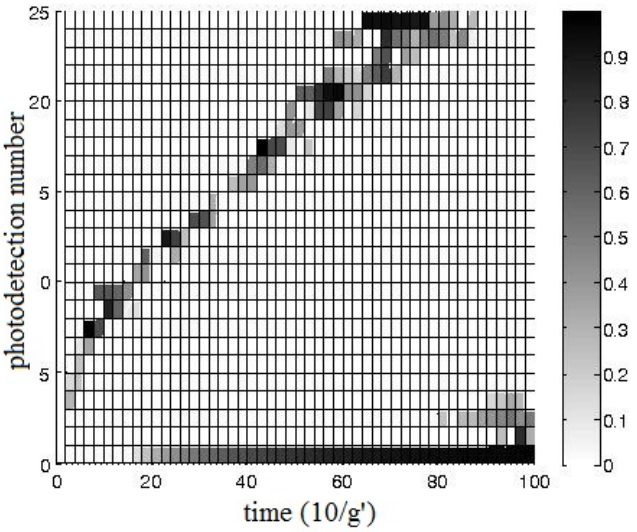

Figure 3.9. The change at the amount of the entanglement in terms of concurrence with the number of jumps accounted and the measurement time. In our notation, we take a measurement time t which is much more longer than the time step δt. Here, the definite variables in the master equation are taken as $g' = 0.1\ a$ and $\delta t = 0.0025t$

The counts about the time-evolution without making jump is increasing with time. When more photons counted, we can obtain more entangled states of the system. It is an expected result, because the system is getting more entangled in time under the effect of the pumping field.

For some values of the time of measurement and photodetections, the value of the concurrence of the system is even become 1. We couldn't find a regular distribution about concurrence. This must be examined in a more detailed way.

Using the data about the graph, we can obtain the system with the amount of entanglement that we want by setting the variables gives that amount of entanglement.

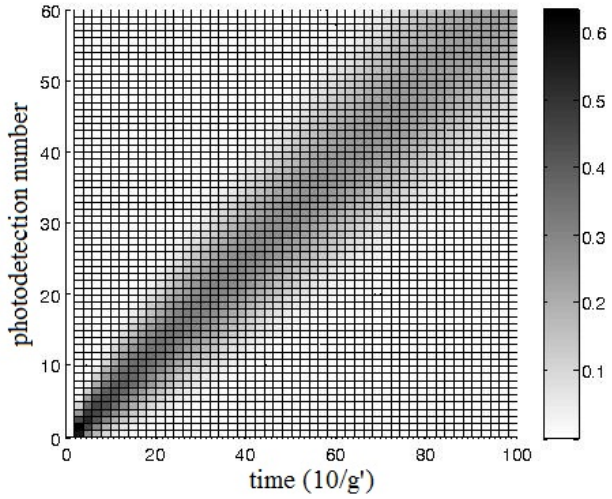

Figure 3.10. The change at the probability of jump defined by using random numbers basically and the number of jumps accounted and the measurement time. In our notation, we take a measurement time t which is much more longer than the time step δt. Here, the definite variables in the master equation are taken as $g' = 0.5\ a$ and $\delta t = 0.0025t$

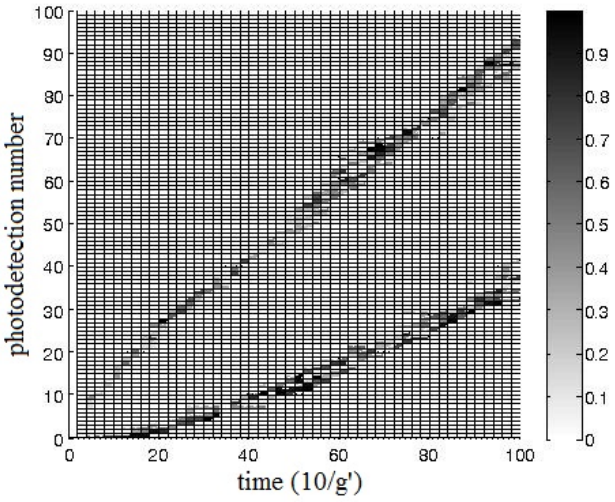

Figure 3.11. The change at the amount of the entanglement in terms of concurrence with the number of jumps accounted and the measurement time. In our notation, we take a measurement time t which is much more longer than the time step δt. Here, the definite variables in the master equation are taken as $g' = 0.5\ a$ and $\delta t = 0.0025t$.

CHAPTER 4

CONCLUSION

In this thesis, we studied the generation of quantum entanglement and the effect of quantum measurements on an open quantum system. We used a cavity with two qubits as our system. These qubits interacts with each other by means of the cavity field. The system is also pumped with a field which has the same strength with the cavity field. We simulate direct measurements on the system.

We made observations on the leaking photons which contains the information of the state. The repeated measurements on the system and the Hamiltonian of the system gives us the master equation of it. By applying adiabatic elimination on the master equation, we derived out the terms about the cavity field. In this way, we obtained a relation which indicates only the the atomic terms(the terms about the atomic states). This relation is in the same form with the basic construction of the master equation, so we could easily write the time-evolution operator and quantum jump operator about the system.

By using the master equation with only atomic terms, we simulated quantum trajectories approach with Monte Carlo method. We appointed random numbers for each infinitesimal time steps of the quantum trajectory about the final form of the master equation. As using the steady state as initial state, we took the evolution of the state for each time intervals and normalized it in each running of the simulation. We repeated these runnings and took the average of the result of the runnings. In this way, we obtained a general picture about the state of our system.

We used ”concurrence” method to measure the amount of entanglement. After calculating concurrence, we obtained the data to plot the relation of the amount of entanglement with time and detection numbers. Also we have plotted the relation between the quantum jump probability of our system changing with time and detection number.

According the graphs, we can see that the evolution of entanglement and the relations with the other parameters are as expected. The system becomes more entangled as long as we pump the cavity field and it has a direct relation between the jump number. Because, the jumps contains the information of interaction between two qubits.

Initially we have taken a steady state, but it changed under the effect of direct measurement. According to the relation between detected photons and entanglement, we can say that the amount of entanglement can be controlled by making measurements on the relevant system. By using the relations between time and measurement number, we see that we can obtain an entangled state with the entanglement ration as we want.

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APPENDIX A

THE PROPERTIES OF THE DISPLACEMENT OPERATOR

The displacement operator for the cavity field is,

$$\hat{D}_b = e^{\mu(\hat{b}^\dagger - \hat{b})} \tag{A.1}$$

where $\mu = \alpha/\gamma_p$.

Here, some properties of the displacement operator D_b:

- For the Hermitian of $\hat{D}_b$, these relations can be written,

$$\hat{D}_b^\dagger(\mu) = e^{\mu(\hat{b} - \hat{b}^\dagger)} = e^{-\mu(\hat{b}^\dagger - \hat{b})} = \hat{D}_b(-\mu), \tag{A.2}$$

$$\hat{D}_b^\dagger \hat{D}_b(\mu) = e^{\mu(\hat{b}^\dagger - \hat{b})} e^{-\mu(\hat{b}^\dagger - \hat{b})} = \hat{1}. \tag{A.3}$$

- The similarity transformation for $\hat{b}$ is,

$$\begin{aligned} \hat{D}_b^\dagger \hat{b} \hat{D}_b &= e^{\mu(\hat{b}^\dagger - \hat{b})} \hat{b} e^{-\mu(\hat{b}^\dagger - \hat{b})} \\ &= \hat{b} + \mu[\hat{b}^\dagger, \hat{b}] \\ &= \hat{b} + \mu. \end{aligned} \tag{A.4}$$

- The similarity transformation for $\hat{b}$ is,

$$\begin{aligned} \hat{D}_b^\dagger \hat{b}^\dagger \hat{D}_b &= e^{\mu(\hat{b}^\dagger - \hat{b})} \hat{b}^\dagger e^{-\mu(\hat{b}^\dagger - \hat{b})} \\ &= \hat{b}^\dagger + \mu. \end{aligned} \tag{A.5}$$

- The interaction with the vacuum state gives,

$$\begin{aligned} \hat{b}\hat{D}_b \left|0\right\rangle &= \hat{D}_b \hat{D}_b^\dagger \hat{b} \hat{D}_b \left|0\right\rangle \\ &= \hat{D}_b(\hat{b} + \mu) \left|0\right\rangle \\ &= \hat{D}_b \mu \left|0\right\rangle . \end{aligned} \tag{A.6}$$

So,

$$\hat{b} \left|0\right\rangle = \mu \left|0\right\rangle . \tag{A.7}$$

APPENDIX B

QUADRATURE MEASUREMENT

We take a cavity with two modes; one of them is inside of the cavity and cannot escape from the walls of the cavity, and the other mode is escaping from cavity but it is heavily damped. These two modes can define with an interaction Hamiltonian as follows

$$H_{int} = \hbar\chi X_1 Y_1 \tag{B.1}$$

where $X_1 = (ae^{-iw_a t} + a^\dagger e^{iw_a t})/2$ and $Y_1 = (be^{-iw_k t} + b^\dagger e^{iw_k t})/2$,

$$H_{int} = \hbar\frac{\chi}{4}(ae^{-iw_a t} + a^\dagger e^{iw_a t})(be^{-iw_k t} + b^\dagger e^{iw_k t}) \tag{B.2}$$

According to the rotating wave approximation of Jaynes-Cummings model[14];

$$|w_k - w_a| << w_k + w_a \tag{B.3}$$

The interaction Hamiltonian becomes

$$H_{int} = \hbar\frac{\chi}{4}[ab^\dagger + a^\dagger b] \tag{B.4}$$

The density operator for both modes obeys the following master equation[24]:

$$\dot{W} = \mathcal{L}_o - i\chi[X_1 Y_1, W] + \frac{\gamma}{2}(2bWb^\dagger - b^\dagger bW - Wb^\dagger W) \equiv \mathcal{L}W \tag{B.5}$$

We assume that mode b is heavily damped, so it has a few photons are slaved to mode a. This allow the dynamics of mode b to be eliminated adiabatically.

$$|\frac{\langle\mathcal{L}_o\rangle}{\gamma}| \sim |\frac{\chi\langle X_1\rangle}{\gamma}| = \epsilon \ll 1 \tag{B.6}$$

Now, we can expand W in powers of ϵ:

$$W = \rho_0 \otimes |0\rangle_b \langle 0| + (\rho_1 \otimes |1\rangle_b \langle 0| + H.c.) + \rho_2 \otimes |1\rangle_b \langle 1| + (\rho_{2'} \otimes |2\rangle_b \langle 0| + H.c.) + \mathcal{O}(\epsilon^3) \tag{B.7}$$

Substituting W into the master equation gives

$$\dot{\rho}_0 = \mathcal{L}_o\rho_0 - i\frac{\chi}{2}[X_1\rho_1 - \rho_1^\dagger X_1] + \gamma\rho_2 \tag{B.8}$$

$$\dot{\rho}_1 = \mathcal{L}_o\rho_1 - i\frac{\chi}{2}[X_1\rho_0 + \sqrt{2}X_1\rho_{2'} - \rho_2 X_1] - \frac{\gamma}{2}\rho_1 + \gamma\mathcal{O}(\epsilon^4) \tag{B.9}$$

$$\dot{\rho}_2 = \mathcal{L}_o\rho_2 - i\frac{\chi}{2}[X_1\rho_1^\dagger + -\rho_1 X_1] - \gamma\rho_2 + \gamma\mathcal{O}(\epsilon^4) \tag{B.10}$$

$$\dot{\rho}_{2'} = \mathcal{L}_o\rho_0 - i\frac{\chi}{2}[\sqrt{2}X_1\rho_1] + \gamma\rho_{2'}\gamma\mathcal{O}(\epsilon^4) \tag{B.11}$$

The off-diagonal elements ρ_1 and $\rho_{2'}$ can be adiabatically eliminated by slaving them to the on-diagonal elements ρ_0 and $\rho_{2'}$. Putting $\dot{\rho}_{2'} = 0$ gives

$$\rho_{2'} - i\frac{\chi}{\sqrt{2}\gamma}X_1\rho_1 + \mathcal{O}(\epsilon^3) = \mathcal{O}(\epsilon^2) \tag{B.12}$$

Substituting this into the equation of $\dot{\rho}_1$ and putting $\dot{\rho}_1 = 0$

$$\rho_1 - i\frac{\chi}{\gamma}[X_1\rho_0 - \rho_2 X_1] + \mathcal{O}(\epsilon^4) = \mathcal{O}(\epsilon) \tag{B.13}$$

Substituting this into the equation of $\dot{\rho}_0$ and $\dot{\rho}_2$ gives

$$\dot{\rho}_0 = \mathcal{L}_o\rho_0 - \frac{\chi^2}{2\gamma}[X_1^2\rho_0 - X_1\rho_2 X_1 + \rho_0 X_1^2 - X_1\rho_2 X_1] + \gamma\rho_2 \tag{B.14}$$

$$\dot{\rho}_2 = \mathcal{L}_o\rho_2 + \frac{\chi^2}{2\gamma}[X_1\rho_0 X_1 - X_1^2\rho_2 + X_1\rho_0 X_1 - \rho_2 X_1^2] - \gamma\rho_2 \tag{B.15}$$

The reduced density operator for mode a consist only diagonal terms $\rho = Tr_b(W) = \rho_0 + \rho_2 + \mathcal{O}(\epsilon^4)$. So, the master equation of the system becomes

$$\dot{\rho} = \mathcal{L}_o\rho - \frac{\chi^2}{2\gamma}[X_1, [X_1, \rho]] \equiv \mathcal{L}_a\rho \tag{B.16}$$

Under the adiabatic assumption, ρ_2 can also be slaved to ρ_0, giving

$$\rho_2 = \frac{\chi^2}{\gamma^2}X_1\rho_0 X_1 + \mathcal{O}(\epsilon^3) = \mathcal{O}(\epsilon^2) \tag{B.17}$$

Thus, to leading order the density operator for mode a is

$$\rho = \rho_0 + \frac{\chi^2}{\gamma^2}X_1\rho_0 X_1 \tag{B.18}$$

So,

$$\rho_0 = \rho - \frac{\chi^2}{\gamma^2} X_1 \rho X_1$$

Substituting this expression into ρ_1, $\rho_{2'}$ and ρ_2 yields the following expression for the density operator W for modes a and b

$$\begin{aligned} W =&(\rho - \frac{\chi^2}{\gamma^2} X_1 \rho X_1) \otimes |0\rangle_b \langle 0| + (-\frac{\chi}{\gamma} X_1 \rho \otimes |1\rangle_b \langle 0| + H.c.) \\ &+ \frac{\chi^2}{\gamma^2} X_1 \rho X_1 \otimes |1\rangle_b \langle 1| + (-\frac{\chi^2}{\sqrt{2}\gamma^2} X_1^2 \rho \otimes |2\rangle_b \langle 0| + H.c.) + \mathcal{O}(\epsilon^3) \end{aligned} \tag{B.19}$$

A photodetection in the output field of mode b is determined by the superoperator $\mathcal{J}$, defined by

$$\mathcal{J} W = \gamma b W b^\dagger$$

To determine the effect of this on mode a, we trace over mode b using W we have derived. In this way, it can be found that

$$Tr_b(\mathcal{J} W) = \frac{\chi^2}{\gamma} X_1 \rho X_1 = \mathcal{J}_a \rho \tag{B.20}$$

In the same way,

$$Tr_b[((\mathcal{L} - \mathcal{J})W)] = Tr_b[\mathcal{L}_o W - i\frac{\chi}{2}[X_1 Y_1, W] - \frac{\gamma}{2}(b^\dagger b W + W b^\dagger b)] \tag{B.21}$$

According to these, we can write a definition of quantum trajectories just by using mode a as $X_1 = (a + a^\dagger)/2$.