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q-Deformed and c-Deformed Harmonic Oscillators

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Hamilton functions of classical deformed oscillators (c-deformed oscillators) are derived from Hamiltonians of q-deformed oscillators of the Macfarlane and Dubna types. A new scale parameter, l_q, with the dimension of length, is introduced to relate a dimensionless parameter characterizing the deformation with the natural length of the harmonic oscillator. Contraction from q-deformed oscillators to c-deformed oscillators is accomplished by keeping l_q finite while taking the limit $\hbar \to 0$. The c-deformed Hamilton functions for both types of oscillators are found to be invariant under discrete translations: the step of the translation for the Dubna oscillator is half of that for the Macfarlane oscillator. The c-deformed oscillator of the Macfarlane type has propagating solutions in addition to localized ones. Reinvestigation of the q-deformed oscillator carried out in the light of these findings for the c-deformed systems proves that the q-deformed systems are invariant under the same translation symmetries as the c-deformed systems and have propagating waves of the Bloch type.

§1. Introduction

Macfarlane first formulated a coordinate representation of quantum deformed oscillators (q-deformed oscillators) in terms of difference operator.[1)] Around the same time, the Dubna group derived a coordinate representation of q-deformed oscillators as an exactly solvable model[2)] for the difference Schrödinger equation in the relativistic configurational space.[3)] Detailed analyses of the Macfarlane type oscillators were carried out by Shabanov and Rajagopal.[4)–6)] In particular, Shabanov obtained the Hamilton function of a classical deformed oscillator (c-deformed oscillator) on a unit circle from the q-deformed oscillator using path integral formalism.

In a previous paper,[7)] we investigated q-deformed oscillators of the Macfarlane type[1),4),5)] and the Dubna type[3),8)–10)] in a unified way. The aims of this paper are to derive c-deformed oscillators from q-deformed oscillators of the Macfarlane and Dubna types using the WKB method[11)] and to reinvestigate the q-deformed systems using knowledge obtained here regarding the c-deformed systems. Henceforth, we call oscillators of the Macfarlane type "M-oscillators" and oscillators of the Dubna

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type "D-oscillators".

In §2 we outline the unified theory of M- and D-oscillators.[7] The distinction between these two types of oscillators originates in the structure of difference operators [see Eq. (2·7)]. With the aid of basic formulas concerning the difference operators of M- and D-oscillators, we derive explicit forms of the ladder operators of the harmonic oscillators and examine the symmetry structures of the q-deformed systems in §3.

In §4, a new scale parameter, l_q, with the dimension of length[4] is introduced to relate a deformation parameter q with the natural length $l = \sqrt{\hbar/m\omega}$, where m and ω are mass and angular frequency of the oscillator. The Hamilton functions of c-deformed harmonic oscillators are derived from the Hamiltonian of q-deformed harmonic oscillators by keeping l_q finite while taking the limit $\hbar \to 0$. In c-deformed oscillators, the deformation causes the state of the system to be repeated infinitely many times along the coordinate axis. The period of the Hamilton function of the D-oscillator is half of that of the M-oscillator. Both circulation and libration behavior appear in the c-deformed M-system.

The results obtained for the c-deformed systems lead us to reinvestigate the q-deformed systems studied previously.[7] In §5, the Hamiltonians of q-deformed oscillators are proved to be invariant under symmetries of discrete translations with steps that are identical to the periods of the c-deformed oscillators. Propagating waves of the Bloch type are shown to exist in §6 and Appendix A.

§2. Unified description of q-deformed oscillators

The lowering operator $\hat{A}$ and the raising operator $\hat{A}^\dagger$ of q-deformed harmonic oscillators are postulated to satisfy the so-called q-mutator relation[7)–9)]

$$[\hat{A}, \hat{A}^\dagger]_q \equiv q\hat{A}\hat{A}^\dagger - q^{-1}\hat{A}^\dagger\hat{A} = 1, \tag{2·1}$$

where q is the deformation parameter expressed by

$$q = \exp\bigl(s^2 + t^2 + 3st\bigr) \tag{2·2}$$

in terms of the real parameters s and t.[7] The Hamiltonian of the system given by

$$\hat{H}_q = \frac{1}{2}\{\hat{A}, \hat{A}^\dagger\}_q \equiv \frac{1}{2}(q\hat{A}\hat{A}^\dagger + q^{-1}\hat{A}^\dagger\hat{A}) \tag{2·3}$$

possesses the energy eigenvalues

$$E_n = \frac{q + q^{-1} - 2q^{-2n-1}}{2(q - q^{-1})} \tag{2·4}$$

corresponding to the localized eigenstates $(A^\dagger)^n|0\rangle$, where $|0\rangle$ is the lowest energy state characterized by the condition $A|0\rangle = 0$. As shown in Fig. 1, while the energy spectrum is bounded and there exists the upper bound $E = (q + q^{-1})/2(q - q^{-1})$ in the case of the M-oscillator ($q > 1$), the spectrum is unbounded and increases exponentially in the case of the D-oscillator ($q < 1$).[12]

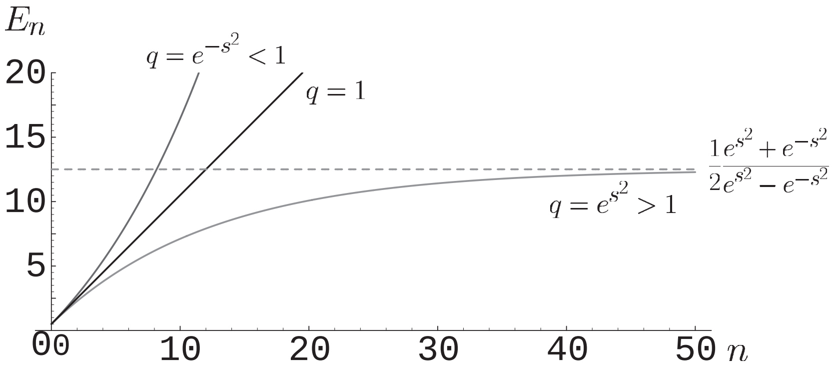

Fig. 1. The energy spectra of the M-oscillator ($q = e^{s^2} > 1$) and the D-oscillator ($q = e^{-s^2} < 1$). The energy spectrum is bounded for the M-oscillator and unbounded for the D-oscillator. The dotted line represents the upper bound of the energy for the M-oscillator. Here we use $s = 0.2$.

The lowering and raising operators in the coordinate representation, $A(x)$ and $A^\dagger(x)$, are postulated to have the separable forms

$$A(x) = \frac{f(x)}{g(x)} \exp[-ih(x)]\, D\left(\frac{1}{i}\frac{d}{dx}\right) \frac{1}{f(x)g(x)} \tag{2·5}$$

and

$$A^\dagger(x) = -\frac{1}{f(x)g(x)} D\left(\frac{1}{i}\frac{d}{dx}\right) \frac{f(x)}{g(x)} \exp[ih(x)], \tag{2·6}$$

where $D\left(\frac{1}{i}\frac{d}{dx}\right)$ is the difference operator defined by

$$D\left(\frac{1}{i}\frac{d}{dx}\right) = \frac{i}{(s-t)}\left[\exp\left(-is\frac{d}{dx}\right) - \exp\left(-it\frac{d}{dx}\right)\right] \tag{2·7}$$

and $f(x)$, $g(x)$ and $h(x)$ are functions that take real values for $x \in \mathbb{R}$. We call these functions "part-functions"[7)] and assume that all of them are continued analytically in the complex x plane. These functions are determined by the conditions that the ladder operators satisfy the q-mutator relation in Eq. (2·1) and reduce to those of the ordinary (non-deformed) harmonic oscillator in the limit $q \to 1$ [see Eqs. (7·1) and (7·2)]. By definition, the function $f(x)$ is intrinsically uncertain by an arbitrary quantity that commutes with the difference operator D, and the sign of the function $g(x)$ also is indeterminate. With the operators $A(x)$ and $A^\dagger(x)$, the x-representation of the Hamiltonian $\hat{H}_q$ is given by

$$H_q(x) = \frac{1}{2}[qA(x)A^\dagger(x) + q^{-1}A^\dagger(x)A(x)]. \tag{2·8}$$

For the q-mutation relation given in Eq. (2·1) to hold, the parameters s and t must satisfy either the conditions

$$s \neq 0 \text{ and } t = 0 \quad (s = 0 \text{ and } t \neq 0) \tag{2·9}$$

or

$$s + t = 0. \tag{2·10}$$

The former and latter conditions correspond, respectively, to the M- and D-oscillators. The part-function $f(x)$ is represented as a superposition or a smooth connection of the Gaussian functions[7)]

$$f_k(x) = \exp\left[-\frac{1}{2}\left(x - \frac{2k\pi}{s}\right)^2\right], \tag{2·11}$$

which are centered at the positions $x = (\pi/s)\times$(even integers).[*)]

Different global structures of the system are realized with different choices of the superposition and connection. As shown in Appendix B, the ladder operators are independent of the global structure of the part-function $f(x)$. Therefore, for brevity, we use the single peak Gaussian factor $f_k(x)$ for the part-function in the arguments given in §§3 and 4.

The part-functions $g(x)$ and $h(x)$ of the M- and D-oscillators are expressed in a unified way by using parametric representations as

$$g(x) = \left(\frac{q - q^{-1}}{\ln q}\right)^{\frac{1}{4}} \sqrt{\cos\left(\frac{2st}{s-t}x\right)} \tag{2·12}$$

and

$$h(x) = -2(s+t)x + a_0, \tag{2·13}$$

where a_0 is an arbitrary constant.

The eigenfunction belonging to the eigenvalue E_n of the Hamiltonian $H_q(x)$ is given by

$$\psi_n(x) = K_0 s^n f(x) g(x) \exp\{in[h(x) + (s+t)x]\}$$
$$\times \prod_{j=0}^{n-1}\left\{e^{(s+t)^2}\left[1 - e^{-2s^2(j+1)}\right]\right\}^{-1/2} H_n(x; e^{-s^2}), \tag{2·14}$$

where

$$H_n(x; e^{-s^2}) = \left(\frac{i}{s}\right)^n \sum_{k=0}^{n} (-1)^k \left[\begin{array}{c} n \\ k \end{array}\right]_{e^{-s^2}}$$
$$\times \exp\left\{(2k-n)\left[isx - \frac{1}{2}(s+t)^2\right]\right\}, \tag{2·15}$$

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{e^{-s^2}} = \frac{\displaystyle\prod_{k=0}^{n-1}[1 - e^{-2(k+1)s^2}]}{\displaystyle\prod_{k=0}^{n-m-1}[1 - e^{-2(k+1)s^2}]\prod_{k=0}^{m-1}[1 - e^{-2(k+1)s^2}]} \tag{2·16}$$

*) In §5 and Appendix A, additional Gaussian functions centered at the positions $x = (\pi/s)\times$ (odd integers) are proved to exist for the part-function $f(x)$ in the case of D-oscillator.

and K_0 is a normalization constant. For the D-oscillator, our q-Hermite function $H_n(x;e^{-s^2})$ defined by Eqs. (2·15) and (2·16) is related to the q-Hermite function $H_n(\sin sx \mid e^{-2s^2})$, which was studied by Atakishyeva et al.,[13] as follows:

$$H_n(x;e^{-s^2}) = s^{-n} H_n \left(\sin \left[sx + \frac{i}{2}(s+t)^2 \right] \middle| e^{-2s^2} \right) \Bigg|_{s\neq 0,\, t=0 \text{ or } t=-s} . \quad (2·17)$$

Our q-Hermite function reduces to the ordinary Hermite function in the limit $s \to 0$. While the M-oscillator is not invariant under spatial reflection, the D-oscillator is, and the q-Hermite function of the D-oscillator has definite parity defined by

$$H_n(-x;e^{-s^2}) = (-1)^n H_n(x;e^{-s^2}). \quad (2·18)$$

§3. Common symmetry of q-deformed M- and D-oscillators

In order to examine the symmetry structure of q-deformed systems, it is convenient to use the following formulas concerning the difference operators of the M- and D-oscillators:

$$\exp\left(\frac{n\pi}{s}x\right)\left[\exp\left(-is\frac{d}{dx}\right)-1\right] = \left[(-1)^n \exp\left(-is\frac{d}{dx}\right)-1\right]\exp\left(\frac{n\pi}{s}x\right) \quad (3·1)$$

and

$$\exp\left(\frac{n\pi}{s}x\right)\left[\exp\left(-is\frac{d}{dx}\right)-\exp\left(is\frac{d}{dx}\right)\right]$$
$$= (-1)^n \left[\exp\left(-is\frac{d}{dx}\right)-\exp\left(is\frac{d}{dx}\right)\right]\exp\left(\frac{n\pi}{s}x\right), \quad (3·2)$$

where n is an arbitrary integer. These formulas are readily proved by noting the action of the difference operator on an arbitrary function $F(x)$ of the C^∞ class: $\exp(\pm isd/dx)F(x) = F(x \pm is)$.

With these formulas, we are able to considerably simplify the ladder operators. Let us separate the Gaussian factor centered at the origin from the factor $f_k(x)$ centered at $x = 2k\pi/s$ as

$$f(x) \equiv f_k(x) = \exp\left(-\frac{1}{2}x^2\right) f_*(x), \quad (3·3)$$

where

$$f_*(x) = \exp\left(\frac{2k\pi}{s}x - \frac{2k^2\pi^2}{s^2}\right). \quad (3·4)$$

The formulas given in Eqs. (3·1) and (3·2) enable us to prove that the function $f_*(x)$ commutes with the difference operator D, i.e.,

$$\left[D\left(\frac{1}{i}\frac{d}{dx}\right),\, f_*(x)\right] = 0, \quad (3·5)$$

for both the M- and D-oscillators.

Due to this property expressed by Eq. (3·5), the function $f_*(x)$ does not affect the local structure of the ladder operators $A(x)$ and $A^\dagger(x)$. The Gaussian factor centered at the origin alone determines the ladder operators as follows:

$$A(x)=\sqrt{\frac{\ln q}{q-q^{-1}}}\frac{\exp\left[-\frac{1}{2}x^2+2i(s+t)x-ia_0\right]}{\sqrt{\cos\left[\frac{2st}{(s-t)}x\right]}}D\left(\frac{1}{i}\frac{d}{dx}\right)\frac{\exp\left(\frac{1}{2}x^2\right)}{\sqrt{\cos\left[\frac{2st}{(s-t)}x\right]}} \tag{3·6}$$

and

$$A^\dagger(x)=-\sqrt{\frac{\ln q}{q-q^{-1}}}\frac{\exp\left(\frac{1}{2}x^2\right)}{\sqrt{\cos\left[\frac{2st}{(s-t)}x\right]}}D\left(\frac{1}{i}\frac{d}{dx}\right)\frac{\exp\left[-\frac{1}{2}x^2-2i(s+t)x+ia_0\right]}{\sqrt{\cos\left[\frac{2st}{(s-t)}x\right]}}. \tag{3·7}$$

At this stage, we define an operator imparting a discrete translation by

$$\mathcal{T}: x\rightarrow x+\frac{\pi}{s}. \tag{3·8}$$

Equations (3·1) and (3·2) prove that the ladder operators are invariant under $\mathcal{T}^2$:

$$\mathcal{T}^2A(x)=A\left(x+\frac{2\pi}{s}\right)=A(x),\quad \mathcal{T}^2A^\dagger(x)=A^\dagger\left(x+\frac{2\pi}{s}\right)=A^\dagger(x). \tag{3·9}$$

Therefore, the Hamiltonians of the M- and D-oscillators are both invariant under translations by $2\pi/s$:[*)]

$$\mathcal{T}^2H_q(x)=H_q\left(x+\frac{2\pi}{s}\right)=H_q(x). \tag{3·10}$$

§4. Contraction of q-deformed oscillators

For the sake of mathematical simplicity, all variables and parameters were treated as dimensionless quantities for both the M- and D-oscillators in the preceding sections. Here, we restore the physical dimensions to all such quantities. With the natural units l and $\hbar\omega$, we rewrite the coordinate and Hamiltonian of the q-deformed oscillator as

$$y=lx,\quad H(y)=\hbar\omega H_q\left(\frac{y}{l}\right) \tag{4·1}$$

in dimensional forms. We mainly use the dimensional variable y in this section.

Following Shabanov,[4)] we introduce a new dimensional scale $l_q=\kappa_q^{-1}$ to relate the dimensionless parameter q with the unit of length l. By using the two scales l

*) An additional symmetry is hidden in the D-oscillator. In §5, the ladder operators of the D-oscillator are proved to be invariant under a single operation of $\mathcal{T}$. This additional symmetry, which was not found in the previous investigation,[7)] was discovered only after the analysis of the c-deformed oscillator was carried out.

and l_q, we express s and t appearing in the definition of the deformation parameter q as

$$s = \frac{l}{l_q} s_\star = l\kappa_q s_\star, \quad t = \frac{l}{l_q} t_\star = l\kappa_q t_\star, \tag{4·2}$$

in terms of the new dimensionless real parameters $s_\star$ and $t_\star$. The limiting procedure $q \to 1$ has two physical meanings, viz, the classical limit $l \to 0$ and the non-deformation limit $\kappa_q \to 0$.

It is essential to recognize that the factors $sl/\hbar = s_\star l^2 \kappa_q/\hbar$ and $tl/\hbar = t_\star l^2 \kappa_q/\hbar$ in the difference operator (2·7) and $sx = sy/l = s_\star \kappa_q y$ in the lattice factor $\cos(sx)$ are independent of $\hbar$, because s, t and l are proportional to $\sqrt{\hbar}$. These factors survive in the classical limit $\hbar \to 0$ if κ_q is kept finite. Therefore, the deformation persists in classical systems, and the parameter $\kappa_q = l_q^{-1}$ in q represents the degree of deformation.

4.1. *Contraction to non-deformed quantum oscillators: $\kappa_q \to 0$ with $\hbar$ fixed*

To reduce the Hamiltonians of q-deformed oscillators to those of ordinary oscillators at the quantum level, we express the ladder operators in terms of the dimensional variable y and then take the limit $\kappa_q = l_q^{-1} \to 0$, keeping $\hbar$ finite.

- M-oscillator

  The product of the raising and lowering operators is explicitly given by

$$A^\dagger(y)A(y) = \frac{1}{e^{s^2} - e^{-s^2}} \left\{ e^{-s^2} \exp\left(-2isl\frac{d}{dy}\right) - \left[\exp\left(\frac{is}{l}y + \frac{1}{2}s^2\right) + \exp\left(-\frac{is}{l}y - \frac{1}{2}s^2\right)\right] \exp\left(-isl\frac{d}{dy}\right) + 1 \right\}. \tag{4·3}$$

  A coordinate representation of the Hamiltonian $H(y)$ is obtained by using the q-mutator defined in Eq. (2·1) and substituting Eq. (4·3) into Eq. (2·3). Expanding it around $\kappa_q = 0$, we obtain

$$H(y) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 y^2 + \frac{1}{2}\left(\sqrt{\frac{1}{m^3\hbar\omega}}\hat{p}^3 - 2\sqrt{\frac{\hbar\omega}{m}}\hat{p} + \frac{1}{2}\sqrt{\frac{m\omega^3}{\hbar}}\{\hat{p}, y^2\} \right) l s_\star \kappa_q + O(\kappa_q^2), \tag{4·4}$$

  where $\hat{p} = -i\hbar\frac{d}{dy}$. The lowest terms on the right-hand side of this equation reproduce the Hamiltonian of the ordinary harmonic oscillator. The first-order terms that are not invariant under the parity transformation represent a nontrivial anharmonic interaction.

- D-oscillator

  The product of the raising and lowering operators takes the form

$$A^\dagger(y)A(y) = \frac{1}{4}\frac{1}{e^{s^2} - e^{-s^2}}\frac{1}{\sqrt{\cos\left(\frac{s}{l}y\right)}}$$

$$\times \left[\frac{e^{-s^2}}{\cos\left(\frac{s}{l}y - is^2\right)\sqrt{\cos\left(\frac{s}{l}y - is^2\right)}} \exp\left(-2isl\frac{d}{dy}\right) \right.$$

$$- \frac{\exp\left(s^2 + \frac{2is}{l}y\right)}{\cos\left(\frac{s}{l}y - is^2\right)\sqrt{\cos\left(\frac{s}{l}y\right)}} - \frac{\exp\left(s^2 - \frac{2is}{l}y\right)}{\cos\left(\frac{s}{l}y + is^2\right)\sqrt{\cos\left(\frac{s}{l}y\right)}}$$

$$\left. + \frac{e^{-s^2}}{\cos\left(\frac{s}{l}y + is^2\right)\sqrt{\cos\left(\frac{s}{l}y + 2is^2\right)}} \exp\left(2isl\frac{d}{dy}\right) \right]. \qquad (4{\cdot}5)$$

Expansion of the coordinate representation of the Hamiltonian around $\kappa_q = 0$, which is obtained using the q-mutator in Eq. (2·1) and substituting Eq. (4·5) into Eq. (2·3), results in

$$H(y) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 y^2$$
$$+ \left(\frac{\hat{p}^4}{6m^2\hbar\omega} - \frac{\hat{p}^2}{m} + \frac{\omega}{4\hbar}\{\hat{p}^2, y^2\} + \frac{m^2\omega^3}{3\hbar}y^4 + \frac{3}{4}\hbar\omega \right)(ls_\star\kappa_q)^2$$
$$+ O(\kappa_q^4). \qquad (4{\cdot}6)$$

The lowest-order terms on the right-hand side of this equation are identical to the Hamiltonian of an ordinary harmonic oscillator. The terms second order in κ_q are invariant under the parity transformation and represent anharmonic effects.

4.2. *Contraction to deformed classical oscillators: $\hbar \to 0$ with κ_q fixed*

In the WKB method,[11),14)] the normalized wave function $\psi(y)$ is expressed as

$$\psi(y) = \exp\left[\frac{i}{\hbar}S(y)\right], \qquad (4{\cdot}7)$$

where $S(y)$ reduces to Hamilton's principle function in the limit $\hbar \to 0$. Hence, the classical momentum p is derived from $S(y)$ as follows:

$$p = \frac{d}{dy}\lim_{\hbar\to 0} S(y). \qquad (4{\cdot}8)$$

Operating with $-i\hbar\frac{d}{dy}$ on the wave function $\psi(y)$ and taking the limit $\hbar \to 0$, we obtain the formulas

$$\lim_{\hbar\to 0}\left[\frac{\hbar}{i}\frac{d}{dy}\right]\psi(y) = \lim_{\hbar\to 0}\left\{\frac{dS(y)}{dy}\right\}\lim_{\hbar\to 0}\psi(y) = p\lim_{\hbar\to 0}\psi(y), \qquad (4{\cdot}9)$$

$$\lim_{\hbar\to 0}\left[\frac{\hbar}{i}\frac{d}{dy}\right]^2\psi(y) = \lim_{\hbar\to 0}\left\{\left[\frac{dS(y)}{dy}\right]^2 + \frac{\hbar}{i}\frac{d^2S(y)}{dy^2}\right\}\lim_{\hbar\to 0}\psi(y) = p^2\lim_{\hbar\to 0}\psi(y), \qquad (4{\cdot}10)$$

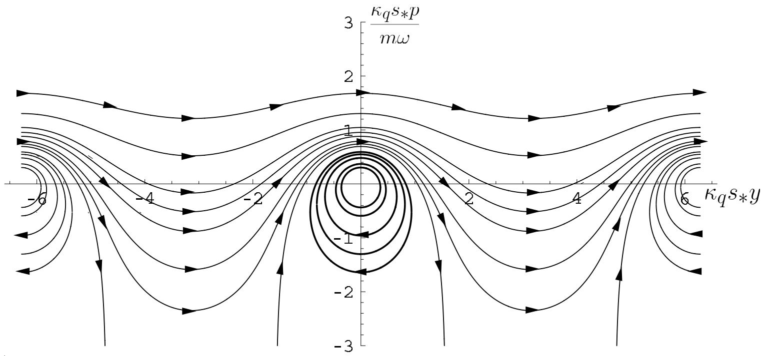

Fig. 2. Orbits of constant energy for the M-oscillator in phase space ($s_\star > 0$). The arrows represent the direction of flow of the vector field. The period of an orbit with respect to $\kappa_q s_\star y$ is 2π. The threshold energy E_c distinguishing closed orbits for libration from open orbits for circulation is $E_c = m\omega^2/(2\kappa_q^2 s_\star^2)$.

and so on. These results imply that the operator $-i\hbar\frac{d}{dy}$ in front of the wave function can be replaced by the ordinary momentum p in the limit $\hbar \to 0$.

Substituting Eqs. (4·3) and (4·5) into the Hamiltonian (2·8) and keeping κ_q finite, we take the limit $\hbar \to 0$ to obtain new Hamilton functions of the c-deformed harmonic oscillators $H_c(y, p)$.

- M-oscillator

The above procedure of the WKB method readily leads to the Hamilton function of the c-deformed oscillator

$$H_c(y,p) = \frac{m\omega^2}{2(\kappa_q s_\star)^2}\left[\exp\left(\frac{2\kappa_q s_\star p}{m\omega}\right) - 2\cos(\kappa_q s_\star y)\exp\left(\frac{\kappa_q s_\star p}{m\omega}\right) + 1\right]. \quad (4·11)$$

This Hamilton function is identical to that obtained by Shabanov in the path integral formalism.[4)] It has no singularity, except in the limit $p \to \infty$, and it tends to infinity in the limit $\omega \to 0$. It is not invariant under the parity transformation. From Hamilton's equation of motion, we have the relation

$$p = \frac{m\omega}{\kappa_q s_\star}\ln\left\{\frac{\cos(\kappa_q s_\star y)}{2} + \sqrt{\left[\frac{\cos(\kappa_q s_\star y)}{2}\right]^2 + \frac{\kappa_q s_\star \dot{y}}{\omega}}\right\}. \quad (4·12)$$

The condition that the momentum is real restricts the motion to the region in which $s_\star \dot{y} \geq -\omega\cos^2(\kappa_q s_\star y)/(4\kappa_q)$. Evidently, the Hamilton function and the momentum have periodic structure of period $2\pi l_q/s_\star = 2\pi l/s$:

$$\mathcal{T}^2 H_c(y,p) = H_c\left(y + \frac{2\pi l_q}{s_\star},\, p\right) = H_c(y,\, p) \quad (4·13)$$

and

$$\mathcal{T}^2 p(\dot{y}, y) = p\left(\dot{y},\, y + \frac{2\pi l_q}{s_\star}\right) = p(\dot{y},\, y). \quad (4·14)$$

As shown in Fig. 2, there are closed and open orbits representing *libration* and *circulation*, respectively, in the phase space. This is, to our knowledge, the first time that this interesting phenomenon has been observed. It is possible for open orbits to appear in a region characterized by large energy, because the height of the potential barrier of the Hamilton function is finite. The threshold energy discriminating closed and open orbits, $E_c = m\omega^2/(2\kappa_q^2 s_\star^2)$, is identical to the maximum value of the energy eigenvalues in Eq. (2·4) in the limit $\hbar \to 0$:

$$\lim_{\hbar\to 0} \hbar\omega \frac{q+q^{-1}}{2(q-q^{-1})} = \frac{m\omega^2}{2\kappa_q^2 s_\star^2}. \tag{4·15}$$

Applying the inverse Legendre transformation, we obtain the c-deformed Lagrange function

$$L(y,\dot{y}) = \frac{m\omega^2}{2(\kappa_q s_\star)^2}\left\{ \cos(\kappa_q s_\star y)\sqrt{\left[\frac{\cos(\kappa_q s_\star y)}{2}\right]^2 + \frac{\kappa_q s_\star \dot{y}}{\omega}} - \frac{\kappa_q s_\star \dot{y}}{\omega} + \frac{\cos^2(\kappa_q s_\star y)}{2} - 1 \right\}$$
$$+ \frac{m\omega}{\kappa_q s_\star}\dot{y}\ln\left\{ \frac{\cos(\kappa_q s_\star y)}{2} + \sqrt{\left[\frac{\cos(\kappa_q s_\star y)}{2}\right]^2 + \frac{\kappa_q s_\star \dot{y}}{\omega}} \right\}. \tag{4·16}$$

- D-oscillator

The Hamilton function of the c-deformed harmonic oscillator is found to be

$$H_c(y,p) = \frac{m\omega^2}{2(\kappa_q s_\star)^2}\left[\frac{1}{\cos^2(\kappa_q s_\star y)}\sinh^2\left(\frac{\kappa_q s_\star}{m\omega}p\right) + \tan^2(\kappa_q s_\star y)\right]. \tag{4·17}$$

It is invariant under the parity transformation and has singularities at all points satisfying the relation $\cos(\kappa_q s_\star y) = 0$, which are related to zero points of the eigenfunction of the D-oscillator given in Eq. (2·14). In the limit $\omega \to 0$, the Hamilton function diverges. From Hamilton's equation of motion, the momentum is given by

$$p = \frac{m\omega}{\kappa_q s_\star}\sinh^{-1}\left[\frac{\kappa_q s_\star \dot{y}}{\omega}\cos^2(\kappa_q s_\star y)\right]. \tag{4·18}$$

Evidently, the Hamilton function and the momentum have periodic structures of period $\pi l_q/s_\star = \pi l/s$:

$$\mathcal{T} H_c(y,p) = H_c\left(y + \frac{\pi l_q}{s_\star}, p\right) = H_c(y,p) \tag{4·19}$$

and

$$\mathcal{T} p(\dot{y},y) = p\left(\dot{y},\, y + \frac{\pi l_q}{s_\star}\right) = p(\dot{y},\, y). \tag{4·20}$$

As shown in Fig. 3, all the orbits in the phase space are closed and periodic with respect to the coordinate y. No classical motion is allowed in the regions

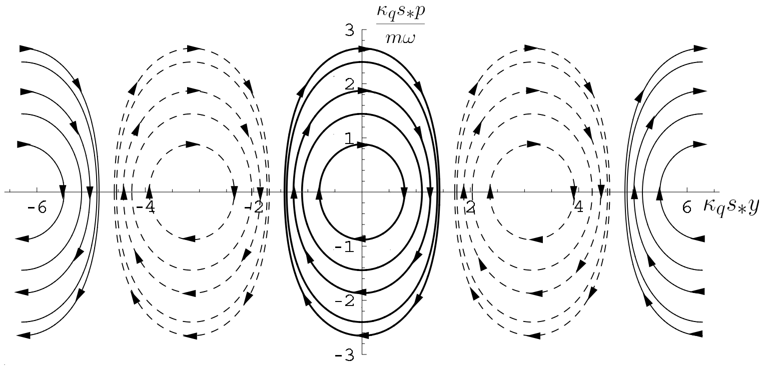

Fig. 3. Orbits of constant energy for the D-oscillator in phase space ($s_\star > 0$). The arrows represent the direction of flow of the vector field. The orbits centered at $\kappa_q s_\star y = \pi\times$(even integers) and $\kappa_q s_\star y = \pi\times$(odd integers) are represented, respectively, by solid and dotted curves.

around $y = (\pi l_q/2s_\star)\times$(odd integers), at which the Hamilton function is divergent. Accordingly, all orbits become isolated and closed for the D-oscillator, in contrast to the M-oscillator whose Hamilton function has no potential singularity. The c-deformed Lagrange function is found to be

$$L(y,\dot{y}) = -\frac{m\omega^2}{2(\kappa_q s_\star)^2 \cos^2(\kappa_q s_\star y)}\sqrt{1+\left[\frac{\kappa_q s_\star \dot{y}}{\omega}\cos^2(\kappa_q s_\star y)\right]^2} + \frac{m\omega\dot{y}}{2\kappa_q s_\star}\sinh^{-1}\left[\frac{\kappa_q s_\star \dot{y}}{\omega}\cos^2(\kappa_q s_\star y)\right] - \frac{m\omega^2}{2(\kappa_q s_\star)^2}\tan^2(\kappa_q s_\star y). \quad (4{\cdot}21)$$

The orbits plotted in Figs. 2 and 3 show explicitly that the deformation has the effects of creating infinite duplications in the coordinate space. The M- and D-oscillators have closed orbits centered at $y = (\pi l_q/s_\star)\times$(even integers). In addition, there exists another sequence of closed orbits centered at $y = (\pi l_q/s_\star)\times$(odd integers) for the D-oscillator, which is represented by dotted curves in Fig. 3. Henceforth, we call the former an *even sequence* and the latter an *odd sequence*.

We obtain an ordinary harmonic oscillator in the limit $\kappa_q \to 0$ of a c-deformed oscillator. In the $\kappa_q \to 0$ limit, all orbits, except for the closed orbits centered at the origin and depicted by thick solid curves move off to infinity.

§5. Additional symmetry of the q-deformed D-oscillator

For classical oscillators, the effect of deformation causes the appearance of infinite repetitions of orbits in the phase space. In Figs. 2 and 3, the closed orbit centered at the origin $y = 0$ is accompanied by infinitely many duplications. The c-deformed M- and D-oscillators both have an even sequence of closed orbits, and the D-oscillator has an additional odd sequence.

It is natural to interpret even sequences of orbits of the M- and D-oscillators as classical manifestations of the invariance of the q-deformed Hamiltonian H_q in Eq. (3·10) under the group action generated by the operator $\mathcal{T}^2$. Then, what sort of symmetry for the q-deformed oscillator is that which appears as the additional odd sequence only for the c-deformed oscillator of the Dubna type?

Using the formula in Eq. (3·2) with $n = 1$ and accounting for the sign change of the factor $\cos(sx)$, we find the relation

$$\mathcal{T} A(x) = A\left(x + \frac{\pi}{s}\right) = A(x), \tag{5·1}$$

which results in

$$\mathcal{T} H_q(x) = H_q\left(x + \frac{\pi}{s}\right) = H_q(x) \tag{5·2}$$

for the D-oscillator. In contrast, considering the formula in Eq. (3·1) with $n = 1$, we see that a similar relation does not hold for the M-oscillator. Therefore, while the D-oscillator is invariant under the group action generated by $\mathcal{T}$, the M-oscillator is invariant under $\mathcal{T}^2$ only. These results enable us to interpret the odd sequence in the c-deformed D-oscillator as a reflection of the higher symmetry represented by $\mathcal{T}$ of the q-deformed D-oscillator.

Now, we must examine effects of the discrete translation $\mathcal{T}$ on the wave function. An increase in symmetry leads inevitably to an increase in the degeneracy of quantum states. Here we look for eigenfunctions that are independent of $\psi_n(x)$ in Eq. (2·14). Invariance under the translation $\mathcal{T}$ in Eq. (5·1) requires introduction of a new series of Gaussian functions,

$$f_{k+1/2}(x) \equiv f_k\left(x - \frac{\pi}{s}\right) = \exp\left\{-\frac{1}{2}\left[x - \frac{(2k+1)\pi}{s}\right]^2\right\}. \tag{5·3}$$

In parallel to the arguments made in §§2 and 3, here we introduce the part-function consisting of $f_{k+1/2}(x)$ as

$$f'(x) \equiv f_{k+1/2}(x) = \exp\left(-\frac{1}{2}x^2\right) f'_*(x), \tag{5·4}$$

where the Gaussian factor centered at the origin is separated, and

$$f'_*(x) = \exp\left[\frac{(2k+1)\pi}{s}x - \frac{(2k+1)^2\pi^2}{2s^2}\right]. \tag{5·5}$$

The formula in Eq. (3·2) allows us to prove that the factor $f'_*(x)$ anti-commutes with the difference operator of the D-oscillator, i.e.,

$$\left\{D\left(\frac{1}{i}\frac{d}{dx}\right), f'_*(x)\right\} = 0. \tag{5·6}$$

Replacing the part-function $f(x)$ by the new part-function $f'(x)$ in Eqs. (2·5) and (2·6), we define new ladder operators. Then, as seen from the anti-commutation relation in Eq. (5·6), the factor $f'_*(x)$ is eliminated in these ladder operators. In fact, we obtain the expressions in Eqs. (3·6) and (3·7), with reversed signs for the operators $A(x)$ and $A^\dagger(x)$. The sign of the ladder operators can be adjusted by appropriately choosing the constant a_0 in the part-function $h(x)$. The q-mutator and the Hamiltonian $H_q(x)$ in Eq. (2·8) take exactly the same forms for both of the part-functions $f(x)$ and $f'(x)$. Therefore, all results in §4 hold for the D-oscillators whether $f(x)$ or $f'(x)$ is used as the part-function.

With the new part-function $f'(x)$, we obtain the additional eigenfunctions

$$\psi'_n(x) = K'_0 s^n f'(x) g(x) \exp\{in[h(x)]\} \times \prod_{j=0}^{n-1} \left[1 - e^{-2s^2(j+1)}\right]^{-1/2} H_n(x; e^{-s^2}), \tag{5·7}$$

where K'_0 is a normalization constant. The eigenfunctions $\psi'_n(x)$ and $\psi_n(x)$ have the same eigenvalue E_n and are orthogonal with each other. To clarify this unique characteristic of the D-oscillator, let us examine the effect of the factor $\exp(\pi x/s)$ that results from the translation $f_k(x) \to f_k(x + \pi/s)$.

Application of the relation in Eq. (3·2) to the Hamiltonian $H_q(x)$ shows that the factor $\exp(\pi x/s)$ commutes with $H_q(x)$:

$$\left[\exp\left(\frac{\pi x}{s}\right), H_q(x)\right] = 0. \tag{5·8}$$

This implies that eigenstates of the Hamiltonian are necessarily degenerate for the D-oscillator, since an eigenfunction multiplied by the factor $\exp(\pi x/s)$ is also an eigenfunction of $H_q(x)$.

To examine this degeneracy further, it is effectual to introduce the operator[8)]

$$T = \frac{1}{g(x)} \cosh\left(is\frac{d}{dx}\right) \frac{1}{g(x)} \tag{5·9}$$

which can be expressed by a linear combination of ladder operators as

$$T = \frac{s}{2\sin sx}\left\{\frac{1}{\sqrt{q}}\exp[ih(x)]A + \sqrt{q}\exp\left[-ih(x)\right] A^\dagger\right\}. \tag{5·10}$$

The operator T^2 and the Hamiltonian satisfy the linear relation

$$T^2 = s^2\left(H_q + \frac{1}{2}\frac{1+q^2}{1-q^2}\right). \tag{5·11}$$

Therefore, the operator T^2 and the Hamiltonian have the same eigenfunctions, with

$$T^2\psi_n(x) = s^2\frac{q^{-2n}}{1-q^2}\psi_n(x),\ T^2\psi'_n(x) = s^2\frac{q^{-2n}}{1-q^2}\psi'_n(x). \tag{5·12}$$

As can be shown using the relation in Eq. (3·2), the operator T anti-commutes with the factor $\exp(\pi x/s)$:

$$\left\{\exp\left(\frac{\pi x}{s}\right), T\right\} = 0. \tag{5·13}$$

Equation (5·3) shows that the two functions $f_{k+1/2}(x)$ and $\exp(\pi x/s) f_k(x)$ are linearly dependent. Therefore, the wave function $\psi'_n(x)$ can be obtained by multiplying

$\psi_n(x)$ by the factor $\exp(\pi x/s)$ and adjusting a multiplicative constant. Therefore, the eigenvalue problem for the operator T is solved as follows:

$$T\psi_n(x) = s\left(\frac{q^{-2n}}{1-q^2}\right)^{\frac{1}{2}} \psi_n(x) \tag{5·14}$$

and

$$T\psi'_n(x) = -s\left(\frac{q^{-2n}}{1-q^2}\right)^{\frac{1}{2}} \psi'_n(x). \tag{5·15}$$

The wave functions $\psi_n(x)$ and $\psi'_n(x)$, which possess the same energy eigenvalue E_n and different eigenvalues of T, are orthogonal with each other.

For the M-oscillator, whose Hamiltonian is not invariant under the translation $\mathcal{T}$, no localized states appear other than the eigenfunctions ψ_n. Note that the part-function $f'(x)$ with the Gaussian factor $f_{k+1/2}(x)$ fails to reproduce the representations in Eqs. (3·6) and (3·7) for the ladder operators.

§6. Propagating wave functions of the Bloch type

Discovery of the open orbits for the M-oscillator, which is one of important outcomes of our numerical investigation of the c-deformed oscillator, implies the existence of propagating wave functions of the Bloch type[15] in the q-deformed system. We use the Fourier series analysis to find such solutions of the energy eigenvalue problem that correspond to classical motion of the open orbits.

6.1. *q-Deformed M-oscillator* ($q = e^{s^2} > 1$)

Let us consider the stationary state solution $\psi_\Omega(x,t)$ of the Schrödinger equation with energy $E = \hbar\Omega$. Because the Hamiltonian of the q-deformed M-oscillator is invariant under the translations generated by $\mathcal{T}^2$, the Bloch theorem[15] tells us that the wave function satisfies the relation

$$\mathcal{T}^2\psi_\Omega(x,t) = \psi_\Omega\left(x+\frac{2\pi}{s}\right) = e^{2k\pi/s}\psi_\Omega(x,t). \tag{6·1}$$

Therefore, it is possible to represent the stationary wave function using a modulatory plane wave as

$$\psi_\Omega(x,t) = u_\Omega(x)e^{i(kx-\Omega t)}, \tag{6·2}$$

with the $\mathcal{T}^2$-invariant amplitude

$$u_\Omega(x) = \sum_{n=-\infty}^{\infty} a_n e^{insx}. \tag{6·3}$$

The energy eigenvalue E and the Fourier coefficients c_n are determined by solving the stationary state Schrödinger equation

$$H_q u_\Omega(x)e^{ikx} = Eu_\Omega(x)e^{ikx}, \tag{6·4}$$

which is reduced to the recursion formula

$$q^{2n-2}e^{2ks}a_n - q^{n-\frac{1}{2}}e^{ks}a_{n+1} - q^{n-\frac{3}{2}}e^{ks}a_{n-1} + \frac{1}{2}(q+q^{-1})a_n = (q-q^{-1})Ea_n \quad (6\cdot5)$$

for three successive coefficients $a_{n\pm1}$ and a_n. By setting

$$a_n = q^{-\frac{1}{2}n}b_n \quad (6\cdot6)$$

and

$$\lambda = \frac{1}{2}(q+q^{-1}) - (q-q^{-1})E = \cosh s^2 - 2E\sinh s^2, \quad (6\cdot7)$$

we can simplify the above recursion formula into the form

$$b_{n+1} + b_{n-1} = -d_n b_n, \quad (6\cdot8)$$

where

$$d_n = -e^{sk}q^{n-1} - \lambda e^{-sk}q^{-n+1}. \quad (6\cdot9)$$

An investigation of the recursive relation in Eq. (6·8) is carried out in Appendix A. The condition that Eq. (6·8) has a non-trivial solution leads to the following secular equation in continued fraction form:

$$d_0 - \cfrac{1}{d_1 - \cfrac{1}{d_2 - \cfrac{1}{d_3 - \cdots}}} - \cfrac{1}{d_{-1} - \cfrac{1}{d_{-2} - \cfrac{1}{d_{-3} - \cdots}}} = 0. \quad (6\cdot10)$$

Solutions of this equation determine dispersion relations that relate the wave number k and the energy $E = \hbar\Omega$. For each energy eigenvalue, the Bloch wave function is determined by solving the recursion relations in Eq. (6·8) for the Fourier coefficients b_n.

In general, it is not simple to solve the secular equation analytically. Here we describe only the solution of Eqs. (6·10) in the region of sufficiently large q, where $|d_n| \gg 1$ for $n \geq 2$ and $n \leq -1$. In such a case the secular equation can be approximated as

$$d_0 \approx \frac{1}{d_1}, \quad (6\cdot11)$$

which yields the physically admissible solution

$$\lambda \approx -e^{2sk} - \frac{1}{4q}e^{-2sk}. \quad (6\cdot12)$$

Then, we obtain the dispersion relation (energy band) of one of the Bloch waves for the M-oscillator as

$$\Omega \approx \frac{1}{2}\omega\left(\coth s^2 + \frac{1}{\sinh s^2}e^{2sk} + \frac{1}{4\sinh s^2}e^{-s^2-2sk}\right). \quad (6\cdot13)$$

The corresponding energy eigenvalue, $E = \hbar\Omega$, is larger than the maximal value of the spectrum given Eq. (2·4) for a localized state with eigenfunction given in Eq. (2·14).

6.2. *q-Deformed D-oscillator* ($q = e^{-s^2} < 1$)

In contrast to the M-oscillator, the Hamilton function of the c-deformed D-oscillator is infinite at $x = (\pi/2s)\times$(odd integers). These singularities cause all classical orbits to be closed and forbid the existence of open orbits. In quantum theory, however, there is the possibility for the D-oscillator to have propagating or stationary waves of the Bloch type as a result of tunneling effects.

The Schrödinger equation is assumed to have a stationary state solution of the form given in Eq. (6·2). In place of Eq. (6·4), it is sufficient to solve the eigenvalue problem of the T operator in Eq. (5·10) as follows:

$$Tu_\Omega(x)e^{ikx} = \tau u_\Omega(x)e^{ikx}. \tag{6·14}$$

The eigenvalue τ here is related to the energy eigenvalue $E = \hbar\Omega$ as

$$\tau^2 = s^2\left(E + \frac{1}{2}\frac{1+q^2}{1-q^2}\right). \tag{6·15}$$

Here, it is natural to postulate that the modulatory amplitude of the Bloch wave function has the form

$$u_\Omega(x) = \sqrt{\cos(sx)}\sum_{n=-\infty}^{\infty} c_n e^{insx}. \tag{6·16}$$

Then, the eigenvalue problem in Eq. (6·14) is reduced to

$$\frac{1}{2}\sqrt{\frac{s^2}{q^{-1}-q}}(q^n e^{-sk}+q^{-n}e^{sk})\sum_{n=-\infty}^{\infty} c_n e^{insx}e^{ikx} = \tau\cos(sx)\sum_{n=-\infty}^{\infty} c_n e^{insx}e^{ikx}, \tag{6·17}$$

which results directly in the recursion formula

$$c_{n+1} + c_{n-1} = -d_n c_n, \tag{6·18}$$

where

$$d_n = -\frac{1}{\tau}\sqrt{\frac{s^2}{q^{-1}-q}}(q^n e^{-sk} + q^{-n}e^{sk}). \tag{6·19}$$

This recursion formula for the coefficients c_n is the same as Eq. (6·8). Therefore, the condition that this formula has a non-trivial solution implies a secular equation identical to Eq. (6·10). The Bloch wave functions $\psi^\pm_\Omega(x,t) = u^\pm_\Omega(x)\exp(kx - \Omega t)$ are determined by solving the recursion formula in Eq. (6·18), into which the eigenvalues $\pm\tau$ obtained from the secular equation are substituted.

In the case of sufficiently large q, for which $|d_n| \gg 1$ for $n \geq 2$ and $n \leq -2$, we approximate the secular equation as

$$d_0 \approx \frac{1}{d_1} + \frac{1}{d_{-1}} \tag{6·20}$$

which leads to

$$\tau^2 \approx s^2\left(\frac{q^{-2}+q^2}{q^{-2}-q^2} + \frac{e^{2sk}+e^{-2sk}}{q^{-2}-q^2}\right). \tag{6·21}$$

Then, substituting this result into Eq. (6·15), we obtain the dispersion relation

$$\Omega \approx \frac{1}{2}\omega\left(\tanh s^2 + \frac{2}{\sinh 2s^2}\cosh 2sk\right) \tag{6·22}$$

for the Bloch wave of the D-oscillator. Note that by substituting the eigenvalues $\pm\tau$ into the recursion formula in Eq. (6·18), we obtain two independent eigenstates of the Bloch type.

§7. Discussion

It is natural and appropriate in physics to formulate one-dimensional harmonic oscillators, with or without deformation, on *the straight x-axis*. It turns out that such a deformation creates rich structure of discrete translational symmetry both in the quantum and classical systems for the M- and D-oscillators. In the following, we summarize the results of our analyses of the c- and q-deformed oscillators.

∗ *c-Deformed oscillators*
Each of the infinitely repeated closed orbits in Figs. 2 and 3 is isolated and equivalent. Note that there is no principle to select one specific orbit among them.

- M-oscillator
  The Hamilton function and the momentum are periodic functions with period $2\pi l_q/s_\star = 2\pi l/s$. In Fig. 2, which depicts orbits of constant energy of the oscillator, localized oscillation is represented by one of the isolated closed orbits in the even sequence selected spontaneously in the low energy region ($E < E_c$) and non-localized motion of circulation is represented by one of the open orbits in the high energy region ($E > E_c$).
- D-oscillator
  The Hamilton function and the momentum is periodic functions with period $\pi l_q/s_\star$. All classical states are oscillatory and described by one of the closed orbits in the even and odd sequences selected spontaneously for all energies.

∗ *q-Deformed oscillators*
The Hamiltonians and Hamilton functions for the M- and D-oscillators have the same discrete translational symmetries. Owing to these symmetries, the q-deformed systems have propagating waves of the Bloch type in addition to localized states.

- M-oscillator
  In this case, the Hamiltonian is invariant under the discrete group generated by the translation operator $\mathcal{T}^2$. The localized eigenstate and the propagating state are given, respectively, by ψ_n and the Bloch wave function $\psi_\Omega(x,t)$.
- D-oscillator
  Here, the Hamiltonian possesses the discrete symmetry group generated by the operator $\mathcal{T}$. Both the localized eigenstates and the propagating states are doubly degenerate and given, respectively, by ψ_n and ψ'_n and the Bloch wave functions $\psi^{\pm}_\Omega(x,t)$.

Mathematically it is appealing to consider the mapping from the entire interval

$I_\infty = (-\infty, \infty)$ onto the unit circle $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Description of the q- and c-deformed oscillators on the unit circle is economical in the sense that the duplicated states appearing along the coordinate axis are all represented by a single state. However, we must be careful to formulate such a mapping consistently even in the case that there is no deformation, because a fictitious force must be included in order to constrain the motion of oscillators on this curved space.

The limit of zero deformation, $\kappa_q \to 0$, reduces the ladder operators $A(x)$ and $A(x)^\dagger$ of the q-deformed oscillators in Eqs. (3·6) and (3·7) to

$$A(x) \to \pm e^{-ia_0} \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + x \right) \tag{7·1}$$

and

$$A^\dagger(x) \to \mp e^{ia_0} \frac{1}{\sqrt{2}} \left(\frac{d}{dx} - x \right), \tag{7·2}$$

where the upper and lower signs, respectively, correspond to the part-functions $f(x)$ and $f'(x)$. Accordingly, we are able to obtain Dirac-Hall-Infeld representations for both of the ladder operators by choosing $a_0 = 1$ and $a_0 = \pi$, respectively, for the part-functions $f(x)$ and $f'(x)$.

Finally, we point out that the Dubna group carried out a dimensional analysis with the unit of length defined by

$$\kappa_q = l_q^{-1} = \frac{\omega}{c}, \tag{7·3}$$

which was derived through analysis of a q-deformed oscillator in one-dimensional momentum space with constant curvature.[10)] With this choice of the unit l_q, it is possible to take the limit $\omega \to 0$, i.e., the free particle limit, of Eqs. (4·11) and (4·17).

We have found that the q- and c-deformations lead naturally to the appearance of periodic structures and cause peculiar energy spectra in oscillator systems. An interesting application of the q-deformation is presented in a recent work by Naka et al.,[12)] in which the mass structure of a particle in a 5-dimensional spacetime of the Randall-Sundrum type[17)] with a q-deformed extra dimension is investigated, and the propagator of particle are shown to acquire a natural ultraviolet-cutoff effect.

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Appendix A
—— *Secular Equation for Propagating Wave Functions of Bloch Type* ——

The recursion formulas in Eqs. (6·8) and (6·18) for the Fourier coefficients of the Bloch wave functions correspond to continued fractions of the same form. Accordingly, the condition[16)] that each of these recursion formulas has a non-trivial solution leads to a secular equation of the following form

$$\Delta(E) = \begin{vmatrix} \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \ddots & d_n & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & d_{n-1} & 1 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 1 & d_0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & d_{-n+1} & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & d_{-n} & \ddots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{vmatrix} = 0, \qquad (\text{A}\cdot 1)$$

where the quantities d_n in the diagonal entries are the functions given in Eqs. (6·9) and (6·19).

To solve this equation without ambiguity, we introduce the following three kinds of finite determinants:

$$\Delta^n(E) = \begin{vmatrix} d_n & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & d_{n-1} & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & d_0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & d_{-n+1} & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & d_{-n} \end{vmatrix}, \qquad (\text{A}\cdot 2)$$

$$\Delta^n_\nu(E) = \begin{vmatrix} d_n & 1 & 0 & 0 & 0 \\ 1 & d_{n-1} & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & d_{\nu+1} & 1 \\ 0 & 0 & 0 & 1 & d_\nu \end{vmatrix} \qquad (\text{A}\cdot 3)$$

and

$$\Delta^n_{-\nu}(E) = \begin{vmatrix} d_{-\nu} & 1 & 0 & 0 & 0 \\ 1 & d_{-\nu-1} & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 1 & d_{-n+1} & 1 \\ 0 & 0 & 0 & 1 & d_{-n} \end{vmatrix}, \qquad (\text{A}\cdot 4)$$

where n and ν are non-negative integers with $n \geq \nu$.

The Laplace expansion of the $(2n+1)\times(2n+1)$ determinant $\Delta^n(E)$ with respect to the row including d_0 gives rise to the identity

$$\Delta^n(E) = d_0(E) - \frac{\Delta^n_2(E)}{\Delta^n_1(E)} - \frac{\Delta^n_{-2}(E)}{\Delta^n_{-1}(E)}. \qquad (\text{A}\cdot 5)$$

Using similar expansions, the determinants $\Delta_\nu^n(E)$ and $\Delta_{-\nu}^n(E)$ are proved to satisfy the recursion formulas

$$\Delta_\nu^n(E) = d_\nu(E)\Delta_{\nu+1}^n(E) - \Delta_{\nu+2}^n(E) \tag{A·6}$$

and

$$\Delta_{-\nu}^n(E) = d_{-\nu}(E)\Delta_{-\nu-1}^n(E) - \Delta_{-\nu-2}^n(E). \tag{A·7}$$

Here, we postulate that the limit $n \to \infty$ of the finite determinant is equal to the infinite-dimensional determinant $\Delta(E)$,

$$\lim_{n\to\infty} \Delta^n(E) = \Delta(E), \tag{A·8}$$

and that the finite determinants $\Delta_{\pm\nu}^n(E)$ have the limits

$$\lim_{n\to\infty} \Delta_{\pm\nu}^n(E) = \Delta_{\pm\nu}(E), \tag{A·9}$$

where

$$\Delta_\nu(E) = \begin{vmatrix} \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \ddots & d_n & 1 & 0 & 0 & 0 \\ \cdots & 1 & d_{n-1} & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \cdots & 0 & 0 & 1 & d_{\nu+1} & 1 \\ \cdots & 0 & 0 & 0 & 1 & d_\nu \end{vmatrix} \tag{A·10}$$

and

$$\Delta_{-\nu}(E) = \begin{vmatrix} d_{-\nu} & 1 & 0 & 0 & 0 & \cdots \\ 1 & d_{-\nu-1} & 1 & 0 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \cdots \\ 0 & 0 & 1 & d_{-n+1} & 1 & \cdots \\ 0 & 0 & 0 & 1 & d_{-n} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{vmatrix}. \tag{A·11}$$

Then, the secular equation is cast in the form

$$d_0(E) - \frac{\Delta_2(E)}{\Delta_1(E)} - \frac{\Delta_{-2}(E)}{\Delta_{-1}(E)} = 0. \tag{A·12}$$

Therefore, using the limiting forms of the recursion formulas

$$\Delta_\nu(E) = d_\nu(E)\Delta_{\nu+1}(E) - \Delta_{\nu+2}(E) \tag{A·13}$$

and

$$\Delta_{-\nu}(E) = d_{-\nu}(E)\Delta_{-\nu-1}(E) - \Delta_{-\nu-2}(E), \tag{A·14}$$

we finally obtain the secular equation given in Eq. (6·10).

Appendix B
—— *General Forms of the Part-Function $f(x)$* ——

The compact representations of the ladder operators given in Eqs. (3·6) and (3·7) were derived using the part-function $f(x)$ consisting solely of either $f_k(x)$ or $f_{k+1/2}(x)$. It is possible to obtain the same representations from more general forms of the part-function $f(x)$.

Let us construct the part-functions $f(x)$ and $f'(x)$ by superposing $f_k(x)$ and $f_{k+1/2}(x)$, respectively, as

$$f(x) = \sum_{k=-\infty}^{\infty} c_k f_k(x) \tag{B·1}$$

and

$$f'(x) = \sum_{k=-\infty}^{\infty} c'_k f_{k+1/2}(x), \tag{B·2}$$

with arbitrary coefficients c_k and c'_k. We separate the Gaussian factor centered at the origin from $f(x)$ and $f'(x)$ as

$$f(x) = \exp\left(-\frac{1}{2}x^2\right) f_*(x) \tag{B·3}$$

and

$$f'(x) = \exp\left(-\frac{1}{2}x^2\right) f'_*(x), \tag{B·4}$$

where

$$f_*(y) = \sum_{k=-\infty}^{\infty} c_k \exp\left[\frac{(2k)\pi}{s}x - \frac{(2k)^2\pi^2}{2s^2}\right] \tag{B·5}$$

and

$$f'_*(y) = \sum_{k=-\infty}^{\infty} c'_k \exp\left[\frac{(2k+1)\pi}{s}x - \frac{(2k+1)^2\pi^2}{2s^2}\right]. \tag{B·6}$$

The formulas in Eqs. (3·1) and (3·2) with $n = 2k$ prove that the function $f_*(x)$ commutes with the difference operator D for both the M- and D-oscillators. Similarly, the formula in Eq. (3·2) with $n = 2k+1$ enables us to prove that the factor $f'_*(y)$ anti-commutes with the difference operator of the D-oscillator. Hence, the commutation relation in Eq. (3·5) and the anticommutation relation in Eq. (5·6) hold, respectively, for the general functions $f_*(x)$ and $f'_*(x)$ in this section.

Here, it should be noted that the part-functions $f(x)$ and $f'(x)$ consist exclusively of either $f_k(x)$ or $f_{k+1/2}(x)$. It can be shown that any mixed superposition including both $f_k(x)$ and $f_{k+1/2}(x)$ cannot be decomposed into a product of a Gaussian function centered at the origin and a factor that commutes or anti-commutes with the difference operator D.

The energy eigenstates ψ_n given in Eq. (2·14) [ψ'_n in Eq. (5·7)] were assumed to have only one non-vanishing Gaussian factor centered at the point $2k\pi/s$ [$(2k+1)\pi/s$]

for a certain fixed integer k. It is natural to interpret such a state as corresponding to the closed orbit of the c-oscillator centered at $2k\pi/s$ $[(2k+1)\pi/s]$.

In quantum theory, general eigenstates of the ψ_n and ψ'_n types, including the general part-functions in Eqs. (B·1) and (B·2), are not forbidden. Note, however, that there is no principle that determines the coefficients of superpositions other than the normalization condition of the eigenfunctions. Therefore, the coefficients c_k and c'_k must be interpreted as being spontaneously selected.

The arguments made here are also applicable to a system with globally periodic structure,[7] where the functions $f_k(x)$ defined and normalized on the finite interval $I_k = [(2k-1)\pi/s, (2k+1)\pi/s]$ are smoothly connected over the entire interval $\cup_{k=-\infty}^{\infty} I_k$. The expressions for the ladder operators given in Eqs. (3·6) and (3·7) are valid also in such a system.

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