BOUNDARY VALUE PROBLEMS FOR THE LAPLACE EQUATION USING INTEGRAL EQUATION APPROACH

A Thesis Submitted to
the Graduate School of Engineering and Sciences of
İzmir Institute of Technology
in Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

in Mathematics

by
Gazi Özdemir

July 2016
İZMİR

We approve the thesis of **Gazi Özdemir**

Examining Committee Members:

Assist. Prof. Dr. Olha IVANYSHYN YAMAN
Department of Mathematics, İzmir Institute of Technology

Prof. Dr. Valery YAKHNO
Department of Electrical and Electronics Engineering, Dokuz Eylül University

---

Prof. Dr. Gamze TANOĞLU
Department of Mathematics, İzmir Institute of Technology

27 July 2016

Assist. Prof. Dr. Olha IVANYSHYN YAMAN
Supervisor, Department of Mathematics
İzmir Institute of Technology

Prof. Dr. Engin BÜYÜKAŞIK
Head of the Department of
Mathematics

Prof. Dr. Bilge KARAÇALI
Dean of the Graduate School of
Engineering and Sciences

ACKNOWLEDGMENTS

To begin, I would like to thank my advisor, Asst. Prof. Olha IVANYSHYN YAMAN, who is extremely patient and understanding. From the day I started she has continually provided me scientific guidance, knowledge. No matter how busy she always make time for me to discuss problems. Also I would like to thank Barış ÇİÇEK who helped me with Latex. When I had problem with latex, he was always willing to help me. I own thanks to Damla ISIDICI DEMİREL for her suggestions. I also thank Prof. Dr. Gamze TANOĞLU and Prof. Dr. Valery YAKHNO for being members of my thesis committee. Finally, I am grateful to my parents for their support.

ABSTRACT

BOUNDARY VALUE PROBLEMS FOR THE LAPLACE EQUATION USING INTEGRAL EQUATION APPROACH

The main goal of this thesis is to solve numerically the exterior and interior Robin boundary value problems via a boundary integral equation method, which has an advantage of decreasing the computational dimension of the problem. Representing the solution by a layer potential, we reduce the differential problem in a bounded and an unbounded domain to the Fredholm integral equation of the second kind over the boundary. In the case of exterior problem in two dimension, the fundamental solution to the Laplace equation is logarithmic, and hence additional condition or modification has to be applied that keeps the solution bounded in the unbounded domain. Instead of using a classical single-layer potential and enforcing a condition on the unknown density we propose a modified single layer potential approach. After investigating uniqueness and existence of solution to the obtained integral equations of second kind, we solve the equations numerically by the Nyström method. For the numerical integration of integral operators with continuous kernels the trigonometric quadratures on an equidistant mesh is used. For the numerical integration of weakly singular kernels we first splitt off the logarithmic singularity and apply a special quadrature rule for the improper integrals. The feasibility of the proposed methods, covergence order (super-algebraic for smooth data) is illustrated by numerical examples.

ÖZET

İNTEGRAL DENKLEMİ YAKLAŞIMI KULLANILARAK LAPLACE DENKLEMİ İÇİN SINIR DEĞER PROBLEMLERİ

Bu tezin temel amacı; iç ve dış Robin sınır değer problemlemlerinin, problemin hesaplama boyutunu azaltması avantajına sahip bir yöntem olan sınır integral denklem yötemi ile sayısal olarak çözülmesidir. Çözümün tek katmanlı potansiyel ile gösterilmesiyle; sınırlı ve sınırlı olmayan bölgedeki türevlenebilir problem, sınır üzerinde ikinci tür Fredholm integral denklemine indirgenmiştir. İki boyuttaki dış problem durumunda Laplace denklemin temel çözümü logaritmiktir, ve bundan dolayı sınırlı olmayan bölgedeki çözümü sınırlı tutmak için ek bir şart ya da modifikasyon uygulanmalıdır. Klasik tek katmanlı potansiyel kullanıp, bilinmeyen yoğunluk üzerinde bir şart uygulamak yerine; modifiye edilmiş tek katmanlı potansiyel yaklaşımının kullanılması tercih edilmiştir. Elde edilen ikinci tür integral denklemlerinin çözümünün varlık ve tekliği incelendikten sonra, denklemler sayısal olarak Nyström yöntemi ile çözülmüştür. Sürekli kernela sahip olan integral operatörlerin sayısal integrasyonu için eşit aralıklı meşler üzerinde trigonometrik quadrature kullanılmıştır. Zayıf tekilliği olan kernellerin sayısal integrasyonu için, ilk olarak logaritmik tekilliği ayırılmış ve improper integraller için özel quadrature kuralı uygulanmıştır. Önerilen metodların yapılabilirliği, yakınsama mertebesi sayısal örneklerle açıklanmıştır.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Laplace equation with Robin condition arises in many mathematical and physical phenomena such as physical geodesy, electrostatic, gravitational potential, mathematical physics, fluid mechanics and so on. The Robin boundary conditions model the situation when the boundary absorbs some part of the energy, heat, mass, which is transmitted. It is well-known that the problem has a unique solution [4]. Integral equations are of central importance to elliptic partial differential problems to show existence of the solution. Ivar Fredholm has proved existence of solution to elliptic differential equation with Dirichlet boundary conditions by using single-layer potential with the aid of Fredholm alternative theorem in the late 1800s. In the thesis of Keeler [7], the classical single-layer potential is used to obtain integral equation which is solved numerically via Nyström method based on hybrid Gauss-trapezoidal quadrature. The drawback of using the single-layer potential for the two dimensional exterior problem is its unboundedness at infinity. Hence, the additional condition has to be imposed on the unknown density. In this thesis we propose alternative solution methods to Laplace equation in bounded and unbounded domains with Robin boundary conditions. In the case of the exterior Robin problem, we introduce a modified single-layer potential, inspired by [10], where the modification was considered for investigating an inverse boundary value problem. This representation uses for inverse problems [10]. Introducing appropriate representation which is bounded in unbounded domain for the solution of the exterior problem, we show that the obtained integral equation has a unique solution. We solve numerically the boundary corresponding boundary integral equations of the second kind via Nyström method based on trigonometric quadrature and a special quadrature with logarithmic weight function. Convergence of the proposed method is verified by the numerical examples.

In Chapter 2 we describe basic concepts and potential theory. Representation of the solution of the Laplace equation as an integral is important tool to obtain boundary integral equations. Here classical single- layer potential and double-layer potential are given and also their properties such as jump relation are defined.

In Chapter 3 we give the problem statement. We investigate a solution to the Laplace equation which is bounded in D and $\mathbb{R}^2 \setminus D$ and additionally the solution should satisfy the given Robin boundary conditions. In canonical domains such as circle, rectangle etc, the solution of Laplace equation with boundary conditions can be found analytically by separation of variables [14] but our domains are not canonical domains in this study.

Chapter 4 provides integral equation approach for the solution of the problems. Here we introduce modified single-layer potential which is bounded at infinity and satisfies the Laplace equation. From the boundary condition we derive an integral equation. The corresponding integral operators will be proved to be compact and injective. With the help of Riesz theorem we show uniqueness and existence of the solution of the integral equations we obtain. Since in the proof of the unique solvability of the integral equation corresponding to the interior Robin problem we need additional assumptions on the domain, we give some information about logarithmic capacity of the domain. More details about logarithmic capacity are given in [13]

In Chapter 5 we describe Nyström method for approximate solutions of boundary integral equations. Nyström method is established on quadrature rules. Special quadrature method will be given for singular operators. This chapter includes treating of the weakly singular kernel of the integral operator via splitting off the logarithmic singularity in a special form. Since the trigonometric monomials with this logarithm weight function can be integrated exactly [9, Theorem 8.23], the method is high-order accurate for this type singularity on smooth curve.

In Chapter 6 we present numerical examples which illustrate that the method converges super-algebraically for exterior, interior problems with simply and multiply connected domains.

CHAPTER 2

BASIC CONCEPTS AND POTENTIAL THEORY

In this chapter some basic definitions and some fundamental theorems are given. To study boundary value problem for the Laplace equation we briefly provide general overview of potential theory and finally Green's formula and identities are given in this chapter.

2.1. Basic Definitions

Γ represents boundary of domain $D \subset \mathbb{R}^2$ here. It is defined

$$\Gamma := \{z(t) = (z_1(t), z_2(t)),\ t \in [0, 2\pi]\},\ z(t) : [0, 2\pi] \to \mathbb{R}^2 \text{ and } |z'(t)| \neq 0, \forall t \in [0, 2\pi],$$

if $z(t) = (z_1(t), z_2(t))$ is two times continuously differentiable, that is Γ is of class C^2.

Definition 2.1 *Let X, Y be linear spaces. A mapping $< \cdot, \cdot >: X \times Y \to \mathbb{C}$ is called a bilinear form if*

$$\begin{aligned} < \alpha_1\phi_1 + \alpha_2\phi_2, \psi > &= \alpha_1 < \phi_1, \psi > + \alpha_2 < \phi_2, \psi >, \\ < \phi, \beta_1\psi_1 + \beta_2\psi_2 > &= \beta_1 < \phi, \psi_1 > + \beta_2 < \phi, \psi_2 > \end{aligned}$$

for all $\phi_1, \phi_2, \phi \in X, \psi_1, \psi_2, \psi \in Y$, and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$.

Definition 2.2 *The bilinear form is called nondegenerate if for every $\phi \in X$ with $\phi \neq 0$ there exists $\psi \in Y$ such that $< \phi, \psi > \neq 0$; and for every $\psi \in Y$ with $\psi \neq 0$ there exists $\phi \in X$ such that $< \phi, \psi > \neq 0$.*

Definition 2.3 *Two normed spaces X and Y equipped with a nondegenerate bilinear form $< \cdot, \cdot >: X \times Y \to \mathbb{C}$ are called dual system and denoted by $< X, Y >$.*

Definition 2.4 *Let* $< X_1, Y_1 >$ *and* $< X_2, Y_2 >$ *be two dual systems. Then two operators* $K : X_1 \to X_2,\ K' : Y_2 \to Y_1$ *are called adjoint (with respect to these dual systems) if*

$$< K\phi, \psi >=< \phi, K'\psi >, \tag{2.1}$$

for all $\phi \in X_1, \psi \in Y_2$.

Definition 2.5 (Compact Operator) *Let* X, Y *be normed space, A be linear operator and define* $A : X \longrightarrow Y$ *. If for each bounded sequence* ψ_n *in* X *the sequence* $A\psi_n$ *has a convergent subsequence in Y, A is called compact operator.*

Definition 2.6 (Integral Operator) *Let* $\Gamma \subset \mathbb{R}^2$ *be nonempty domain. The linear integral operator with continuous kernel* $k : \Gamma \times \Gamma \to \mathbb{R}$ *is defined by*

$$\begin{aligned} K :&\ \ C(\Gamma) \longrightarrow C(\Gamma) \\ (K\psi)(x) :&= \int_{\Gamma} k(x, y)\psi(y)ds(y),\ \ x \in \Gamma,\ \ \psi \in C(\Gamma) \end{aligned} \tag{2.2}$$

Definition 2.7 *A function g which is defined on* $D \subset \mathbb{R}^2$ *is called Hölder continuous with* $\alpha \in (0, 1]$ *if*

$$\left|g(x) - g(y)\right| \leq c\left|x - y\right|^{\alpha} \text{for all } x, y \in D, \tag{2.3}$$

where $c > 0$ *is constant independent of x and y. Hölder spaces are denoted by* $C^{0,\alpha}(D)$.

Definition 2.8 (Weakly Singular) *The kernel* $k(x, y)$ *in* (2.2) *is defined and continuous for all* $x, y \in D \subset \mathbb{R}^m, x \neq y$*, if there exist positive constants* M *independent of* x, y *and* $\alpha \in (0, m]$ *such that*

$$|k(x, y)| \leq M|x - y|^{\alpha - m}, x, y \in D, x \neq y, \tag{2.4}$$

then k is called weakly singular.

Theorem 2.1 *If the kernel of K is continuous or weakly singular, then in the dual system*

$< C(\Gamma), C(\Gamma) >$, the operators are defined by

$$
\begin{aligned}
K : \;& C(\Gamma) \longrightarrow C(\Gamma), \\
(K\psi)(x) : \;& = \int_{\Gamma} k(x,y)\psi(y)ds(y), \;\; x \in \Gamma, \\
K' : \;& C(\Gamma) \longrightarrow C(\Gamma), \\
(K'\psi)(x) : \;& = \int_{\Gamma} k(y,x)\psi(y)ds(y), \;\; x \in \Gamma
\end{aligned}
\tag{2.5}
$$

are adjoint to each other [9, Theorem 4.7].

Theorem 2.2 *The integral operator in* (2.2) *with a weakly singular kernel or continuous kernel is compact operator.*

The proof is given in [9, p.28].

Theorem 2.3 (Riesz Theorem) *Let $B : X \longrightarrow X$ be a compact linear operator on a normed space X. Then $I - B$ is injective if and only if it is surjective. If $I - B$ is injective (and therefore also bijective), the inverse operator $(I - B)^{-1} : X \longrightarrow X$ is bounded [9, Theorem 3.4].*

Corollary 2.1 *Kernel of $K - I = \{0\}$ if and only if $K - I$ is one-to-one.*

2.2. Harmonic Functions

Potential theory alludes to Laplace equation in many fields such as physics, engineering, statics etc. In this chapter we will introduce some properties about Laplace equation and physical interpretation which are associated with Laplace equation and also we will give a overview briefly about some appropriate theories which we will use later. In this work D represents bounded domain.

We start with harmonic functions which are very important for solution of partial differential problems and boundary integral equations. Let Φ be continuously differentiable real-valued function and defined on domain $D \subset \mathbb{R}^m$. We call Φ harmonic function in D if it satisfies Laplace equation

$$\Delta\Phi = 0 \text{ in } D. \tag{2.6}$$

Harmonic functions play a crucial role in many parts of science and engineering. They describe

- Gravitational field
- Electrostatic
- Velocity potentials
- Time independent temperature distributions
- Brownian motion
- Chemical concentration

2.3. Fundamental Solution

The function

$$\Phi(x,y) = \begin{cases} \dfrac{1}{2\pi} \ln \dfrac{1}{|x-y|} & x, y \in \mathbb{R}^2, m = 2, \\ \dfrac{1}{4\pi} \dfrac{1}{|x-y|} & x, y \in \mathbb{R}^3, m = 3, \end{cases} \tag{2.7}$$

where $x \neq y$, is called the fundamental solution of Laplace's equation for all $x, y \in \mathbb{R}^m$. Here we have given the fundamental solution for two and three dimension. For any m dimension it can be written

$$\Delta\Phi(x,y) = \delta(x-y), \quad x, y \in \mathbb{R}^m, \tag{2.8}$$

where δ is Dirac delta distribution .

Lemma 2.1 *Asymptotic behavior of fundamental solution of* $\nabla\Phi(x,y) = \delta(x-y)$ *is*

$$\Phi(x,y) = \begin{cases} \dfrac{1}{2\pi} \ln \dfrac{1}{|x|} + O\left(\dfrac{1}{|x|}\right), & m = 2 \\ O\left(\dfrac{1}{|x|}\right), & m = 3 \end{cases} \tag{2.9}$$

Proof We have this formula by using fundamental solution when $m = 2, 3$. For $m = 2$ the fundamental solution is unbounded when $|x| \to \infty$. The proof also given in [9, p. 83] □

2.4. Properties of Single-Layer Potential

We introduce classical boundary integral representation of the solution of the Laplace equation as a harmonic function in whole space except boundary Γ. In this thesis we will use the single-layer potential to solve interior Robin problem under some conditions. u is a harmonic function which can be represented as a integral and this allows us to represent solution of Laplace equation as integral. It is defined as the following

$$u(x) = \int_\Gamma \psi(y)\Phi(x, y)ds(y),\ x \in \mathbb{R}^m \setminus \Gamma,\ \psi \in C(\Gamma). \tag{2.10}$$

It is called a single-layer potential .

Theorem 2.4 *Let* Γ *be of class* C^2 *and* $\psi \in C(\Gamma)$*. Then the single layer potential* u *is continuous through* $\mathbb{R}^m$*. On the boundary* Γ *we have*

$$u(x) = \int_\Gamma \psi(y)\Phi(x, y)ds(y), \quad x \in \Gamma, \tag{2.11}$$

where the integral exists as an improper integral.

The proof is given in [9, Theorem 6.15].

Theorem 2.5 *Let* Γ *be class of* C^2 *and* ψ *is continuous on* Γ*. Then we have*

$$\frac{\partial u_\pm}{\partial \nu}(x) = \int_\Gamma \psi(y)\frac{\partial \Phi(x, y)}{\partial \nu(x)}ds(y) \mp \frac{1}{2}\psi(x), x \in \Gamma, \tag{2.12}$$

where $\frac{\partial u_\pm}{\partial \nu}(x) := \lim_{h \to +0} \nu(x) \cdot \operatorname{grad} u(x \pm h\nu(x))$*, and the integral* (2.12) *exists as an improper integral.*

Proof This shows normal derivative of a single-layer potential has jump relation on the boundary Γ. The proof is given in [9, Theorem 6.19]. □

2.5. Properties of Double-Layer Potential

Double layer-potential also represents solution to Laplace equation in D and $\mathbb{R}^2 \setminus \bar{D}$. It is defined as

$$\upsilon(x) = \int_\Gamma \psi(y) \frac{\partial \Phi(x,y)}{\partial \nu(x)} ds(y), \; x \in \mathbb{R}^m \setminus \Gamma, \; \psi \in C(\Gamma). \tag{2.13}$$

Theorem 2.6 *Let Γ of class C^2 with $\psi \in C(\Gamma)$. Then Double layer-potential can be continuously extended up to the boundary as follows*

$$\upsilon(x)_\pm = \int_\Gamma \psi(y) \frac{\partial \Phi(x,y)}{\partial \nu(x)} ds(y) \pm \frac{1}{2}\psi(x), x \in \Gamma, \tag{2.14}$$

where

$$\upsilon_\pm(x) := \lim_{h \to +0} \upsilon(x \pm h\nu(x))$$

and the integral exists as an improper integral.

Proof This theorem depicts double-layer potential is discontinuous and it has jump relation on the boundary Γ. Also the proof is given in [9, Theorem 6.18]. □

Theorem 2.7 *Let Γ of class C^2 with $\psi \in C(\Gamma)$. Then double-layer potential υ has property*

$$\lim_{h \to +0} \nu(x) \cdot \{\operatorname{grad} \upsilon(x + h\nu(x)) - \operatorname{grad} \upsilon(x - h\nu(x))\} = 0 \textit{ for all } x \in \Gamma. \tag{2.15}$$

Proof It is taken exterior limit and interior of limit of the normal derivative of the double layer potential. The proof is given in detail in [9, Theorem 6.20]. □

Theorem 2.8 *If $w, u \in C^2(D) \cap C^1(\bar{D})$, then Green's identities*

$$\int_D u \Delta w dx + \int_D (\operatorname{grad} u) \cdot (\operatorname{grad} w) dx = \int_\Gamma u \frac{\partial w}{\partial \nu} ds, \tag{2.16}$$

and

$$\int_D (u\Delta w - w\Delta u)dx = \int_\Gamma (u\frac{\partial w}{\partial \nu} - w\frac{\partial u}{\partial \nu})ds. \tag{2.17}$$

Proof If the divergence theorem is applied for u grad w, then equation (2.16) is obtained and by switching u and w, the equality (2.17) can be established [4, Theorem 1. 26]. □

2.6. Green's Formula

Theorem 2.9 *Let D be bounded domain of class C^1, ν denote the outer unit normal vector to the boundary Γ, u be twice continuously differentiable and u be harmonic in D. Then u can be represented*

$$u(x) = \int_\Gamma \left(\frac{\partial u(y)}{\partial \nu(y)} \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right) ds(y), \ \ x \in D. \tag{2.18}$$

Proof The proof is given in [9, Theorem 6.5]. Green's formula is combination of single-layer and double-layer potential. It is used for direct method to derive integral equation on the boundary. We will use it to derive boundary integral equation for the interior Robin problem. □

Lemma 2.2 *Let D be bounded domain of class C^1, ν denotes the outer unit normal vector to the boundary Γ. If w is harmonic in D, then*

$$\int_D |\text{grad } w|^2 \, dx = \int_\Gamma w \frac{\partial w}{\partial \nu} ds. \tag{2.19}$$

Proof This assertion comes from (2.16) with $w = u$ [4, p. 10]. □

Lemma 2.3 *We assume that u and $\dfrac{\partial u}{\partial \nu}$ satisfy the asymptotic relations at infinity. Namely*

$u = O\left(\frac{1}{r}\right)$ *and* $\frac{\partial u}{\partial \nu} = O\left(\frac{1}{r^2}\right)$ *as* $r \to \infty$*. If u is harmonic in* $\in \mathbb{R}^2 \setminus \bar{D}$*, then*

$$\int_{\mathbb{R}^2 \setminus \bar{D}} \left|\text{grad } u\right|^2 dx = -\int_{\Gamma} u \frac{\partial u}{\partial \nu} ds. \tag{2.20}$$

Proof We take a disk S_R which contains strictly domain D and has sufficiently large radius R. If we apply Lemma 2.2 in $S_R \cap (\mathbb{R}^2 \setminus \bar{D})$, then we have

$$\int_{S_R \cap (\mathbb{R}^2 \setminus \bar{D})} \left|\text{grad } u\right|^2 dx = -\int_{\Gamma} u \frac{\partial u}{\partial \nu} ds + \int_{\partial S_R} u \frac{\partial u}{\partial \nu} ds. \tag{2.21}$$

By assumption, the integral

$$\int_{\partial S_R} u \frac{\partial u}{\partial \nu} ds \to 0 \tag{2.22}$$

when $R \to \infty$. Thus the desired result is obtained [4, p. 10]. □

CHAPTER 3

PROBLEM STATEMENT

In this chapter we will investigate existence of solution to the exterior and the interior problem theoretically and also we will explain what our problems are in this thesis. It is well known that if we have canonical shape, we can find analytic solution via separation of variable method. But if we do not have canonical shape, we can not find analytic solution to partial differential problems. Hence, we need to find solution that approximates exact solution. Before solving the problem analytically or numerically, we need to show that interior and the exterior problem have unique solution. In the following we have explanation of the problems in this thesis. Let D be bounded domain in $\mathbb{R}^2$ with smooth boundary Γ and ν is an outward unit normal to Γ

3.1. Exterior Robin Problem

Given function $f \in C(\Gamma)$, $\lambda \in C(\Gamma)$, $\lambda > 0$, we will find solution $u \in C^2(\mathbb{R}^2 \setminus \bar{D}) \cap C^1(\mathbb{R}^2 \setminus D)$ of the Laplace Equation

$$\Delta u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D} \tag{3.1}$$

which satisfies the impedance boundary conditions

$$\frac{\partial u}{\partial \nu} - \lambda u = f \quad \text{on } \Gamma, \tag{3.2}$$

where ν is an outward unit normal to Γ. Moreover the solution should satisfy the asymptotic behavior at infinity, i.e

$$u = O(1) \quad \text{when } |x| \to \infty. \tag{3.3}$$

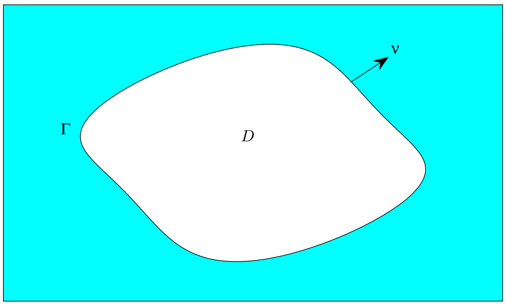

Figure 3.1. Exterior problems

3.2. Interior Robin Problem

Given function $f \in C(\Gamma), \lambda \in C(\Gamma), \lambda > 0$, we will find solution $u \in C^2(D) \cap C^1(\Gamma)$ of the Laplace equation

$$\Delta u = 0 \quad \text{in } D, \tag{3.4}$$

which satisfies the impedance boundary conditions

$$\frac{\partial u}{\partial \nu} + \lambda u = f \quad \text{on } \Gamma, \tag{3.5}$$

where ν is an outward unit normal to Γ.

3.3. Existence and Uniqueness

Existence of the solution to elliptic partial differential problems are equivalent to the existence of the solution to integral equation. They have relationship with each other reciprocally. Existence of the solution of the exterior and interior problems by using Fredholm alternative theorem can be shown. The following theorems are given to show existence and uniqueness of the elliptic partial differential problems.

Theorem 3.1 *Each of exterior boundary value problem and interior boundary value problem has at most one solution.*

Proof Let u be the difference of any two solutions of the interior Robin boundary value problem. Then

$$\frac{\partial u}{\partial \nu}(x) + (\lambda u)(x) = 0, \quad \text{where } x \in \Gamma \text{ and } \lambda > 0. \tag{3.6}$$

This can be written $\frac{\partial u}{\partial \nu}(x) = -(\lambda u)(x)$. From Lemma 2.2

$$\int_D |\text{grad}\, u|^2 + \int_\Gamma \lambda u^2 ds = 0, x \in \Gamma. \tag{3.7}$$

By reason of the fact that λ is positive, from the equation (3.7) it follows that $u(x) = 0, x \in \Gamma$. In what follows, similar arguments can be done for the exterior Robin boundary value problem by using the Lemma 2.3. This shows it is unique [4, p. 13]. The existence is given in the Chapter 4. □

CHAPTER 4

INTEGRAL EQUATION METHOD

In this chapter we will find boundary integral equation equivalent to the interior and exterior Robin problems. Thanks to this we will solve the problem on the boundary instead of the domain. Moreover, we introduce boundary representation of u as modified single-layer potential $u(x)$, $x \in \mathbb{R}^2 \setminus \bar{D}$ which satisfies Laplace equation and that is bounded at infinity. The reason why we introduce modified single-layer potential is that it satisfies asymptotic behavior at infinity but single-layer potential does not. Finally, we will prove the uniqueness and the existence of the solution to the boundary integral equation.

4.1. Integral Equation Method For The Exterior Problem

There are some modified forms of single-layer potential in [4]. Here we will give simpler representation of solution to Laplace equation for the exterior Robin problem. In order to identify the solution to the exterior Robin problem, we will propose a solution $u(x)$, $x \in \mathbb{R}^2 \setminus \bar{D}$ as modified single layer potential.

Theorem 4.1 *Let $D \subset \mathbb{R}^2$ with smooth boundary Γ and $(0,0) \in D$ then*

$$u(x) = \int_\Gamma \psi(y)\Phi(x,y)ds(y) + \int_\Gamma (1 - \Phi(x,0))\psi(y)ds(y), \quad x \in \mathbb{R}^2 \setminus \bar{D} \qquad (4.1)$$

satisfies Laplace equation and asymptotic behavior at infinity, i.e $u = O(1)$ when $|x| \longrightarrow \infty$.

Proof We prove that modified single layer potential satisfies

- Laplace equation
- Asymptotic behavior at infinity

By the fact that the function (4.1) is composed of a single layer potential and constant function, the function (4.1) satisfies Laplace equation. Secondly, by substituting fundamental solution into the representation (4.1), we get

$$u(x) = \frac{1}{2\pi}\int_\Gamma \psi(y)\ln\frac{1}{|x-y|}ds(y) - \int_\Gamma \psi(y)\ln\frac{1}{|x|}ds(y) + \int_\Gamma \psi(y)ds(y), \quad x \in \mathbb{R}^2 \setminus \bar{D}. \tag{4.2}$$

The equation (4.2) can be written as

$$u(x) = \frac{1}{2\pi}\int_\Gamma \psi(y)\ln\frac{|x|}{|x-y|}ds(y) + \int_\Gamma \psi(y)ds(y), \quad x \in \mathbb{R}^2 \setminus \bar{D}. \tag{4.3}$$

Since $x \in \mathbb{R}^2 \setminus \bar{D}$, there exists $y^* \in \Gamma$ such that

$$\left|u(x)\right| \leq \left|\frac{1}{2\pi}\ln\frac{|x|}{|x-y^*|}\int_\Gamma \psi(y)ds(y) + \int_\Gamma \psi(y)ds(y)\right|. \tag{4.4}$$

From property of integral

$$\left|\frac{1}{2\pi}\ln\frac{|x|}{|x-y^*|}\int_\Gamma \psi(y)ds(y)\right| + \left|\int_\Gamma \psi(y)ds(y)\right| \leq \left|\frac{1}{2\pi}\ln\frac{|x|}{|x-y^*|}\right|\int_\Gamma \left|\psi(y)\right|ds(y) + \int_\Gamma \left|\psi(y)\right|ds(y).$$

Then when $|x| \to \infty$, $\left|u(x)\right|$ is bounded, i.e $|u(x)| \leq c$. We have shown that when $x \to \infty$, the modified single-layer potential satisfies asymptotic behavior. □

Theorem 4.2 *Let $D \subset \mathbb{R}^2$ be bounded and closed domain with $\lambda, f \in C(\Gamma)$. If $\psi \in C(\Gamma)$ is solution to*

$$\begin{aligned} &2\int_\Gamma \psi(y)\frac{\partial\Phi(x,y)}{\partial\nu(x)}ds(y) - 2\lambda(x)\int_\Gamma \psi(y)\Phi(x,y)ds(y) - \psi(x) \\ &-2\int_\Gamma \psi(y)\frac{\partial\Phi(x,0)}{\partial\nu(x)}ds(y) - 2\lambda(x)\int_\Gamma (1-\Phi(x,0))\psi(y)ds(y) = 2f(x),\ x \in \Gamma. \end{aligned} \tag{4.5}$$

Then

$$u(x) = \int_\Gamma \psi(y)\Phi(x,y)ds(y) + \int_\Gamma (1-\Phi(x,0))\psi(y)ds(y),\ x \in \mathbb{R}^2 \setminus \bar{D} \tag{4.6}$$

is solution to the exterior problem

$$
\begin{aligned}
\Delta u &= 0 \; in \; \mathbb{R}^2 \setminus \bar{D}, & (4.7)\\
\frac{\partial u}{\partial \nu} - \lambda u &= f \quad on \; \Gamma, & (4.8)\\
\left|u(x)\right| &= O(1) \; when \; |x| \longrightarrow \infty. & (4.9)
\end{aligned}
$$

Proof From Theorem 4.1, u satisfies Laplace equation and asymptotic behavior at infinity. The remaining part is to show u satisfies the Robin boundary condition (4.8). Using the jump relation from Theorem 2.5, we obtain

$$
\begin{aligned}
\frac{\partial u}{\partial \nu}(x) - \lambda(x)u(x) = \int_\Gamma \psi(y)\frac{\partial \Phi(x,y)}{\partial \nu(x)}ds(y) - \lambda(x)\int_\Gamma \psi(y)\Phi(x,y)ds(y) \qquad (4.10)\\
- \frac{1}{2}\psi(x) - \int_\Gamma \psi(y)\frac{\partial \Phi(x,0)}{\partial \nu(x)}ds(y) - \lambda(x)\int_\Gamma (1-\Phi(x,0))\psi(y)ds(y), \; x \in \Gamma.
\end{aligned}
$$

□

By assumption ψ is the solution to the (4.5), therefore u satisfies the (4.8). We have the Fredholm integral equation of second kind in the (4.10) which can be represented in abstract form. We introduce operators

$$
\begin{aligned}
(K'\psi)(x) &:= 2\int_\Gamma \psi(y)\frac{\partial \Phi(x,y)}{\partial \nu(x)}ds(y), & (S\psi)(x) &:= 2\int_\Gamma \psi(y)\Phi(x,y)ds(y), \quad (4.11)\\
(H\psi)(x) &:= 2\int_\Gamma \psi(y)\frac{\partial \Phi(x,0)}{\partial \nu(x)}ds(y), & (T\psi)(x) &:= 2\int_\Gamma (1-\Phi(x,0))\psi(y)ds(y),\\
(K\psi)(x) &:= \int_\Gamma \psi(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}ds(y), & (S_\lambda\psi)(x) &:= \int_\Gamma \lambda(y)\psi(y)\Phi(x,y)ds(y).
\end{aligned}
$$

Linearity for these operators (4.11) are obvious with these notations. Boundary integral equation (4.10) can be represented by

$$
(K' - \lambda S - I)\psi - (H + \lambda T)\psi = 2f \qquad (4.12)
$$

which is boundary integral equation of Fredholm of second kind.

Theorem 4.3 *The operators $I - K$ and $I - K'$ have trivial nullspaces [9, Theorem 6.21].*

Theorem 4.4 $(K' - \lambda S - I)\psi - A\psi = 0$ *has only a trivial solution.*

Proof Now we analyze the integral operators (4.12) to check the injectivity which is one of conditions to have unique solution for integral equation. Let ψ be solution to the homogeneous integral equation.

$$(K' - \lambda S - I)\psi - (H + \lambda T)\psi = 0. \tag{4.13}$$

We demonstrate that $\psi = 0$ to prove injectivity. We recall the definition of the function u, (4.6). From the equation (4.13), we obtain that $\frac{\partial u}{\partial \nu} - \lambda u = 0$ in $\mathbb{R}^2 \setminus \bar{D}$. From the Theorem 3.1 exterior Robin problem has unique solution. It follows that $u = 0$, and therefore

$$\frac{1}{2\pi}\int_\Gamma \psi(y) \ln \frac{|x|}{|x-y|} ds(y) + \int_\Gamma \psi(y) ds(y) = 0, \; x \in \mathbb{R}^2 \setminus \bar{D}. \tag{4.14}$$

When $x \longrightarrow \infty$, because of asymptotic behavior of modified single-layer potential the (4.14) we obtain

$$\int_\Gamma \psi(y) ds(y) = 0. \tag{4.15}$$

By using the equation (4.6) and (4.15) with $\frac{\partial u}{\partial \nu}\Big|_\Gamma = 0$, eventually we have

$$(K' - I)\psi = 0. \tag{4.16}$$

From the Theorem 4.3, the homogeneous integral equation (4.16) has trivial nullspaces. Namely $\psi = 0$, thus the integral operator in (4.12) is injective . □

What we have done until now is not enough to have unique solution for Fredholm integral equation of second kind (4.5). For this reason, also the integral operators in the (4.5) must be compact so we describe parametrized form of integral operators to show compactness. Assume that Γ has a C^2-smooth and 2π-periodic representation

$$\Gamma := \{z(t) = (z_1(t), z_2(t)), \quad |z'(t)| > 0 \quad \text{for} \quad t \in [0, 2\pi]\}.$$

We introduce the parametrized operators

$$\tilde{K}' : C_{2\pi} \longrightarrow C_{2\pi}$$

$$(\tilde{K}'\psi)(t) = 2\int_0^{2\pi} \frac{\partial \Phi(z(t), z(\tau))}{\partial \nu(z(t))} \psi(z(\tau))|z'(\tau)|d\tau, \ t \in [0.2\pi]$$

$$\tilde{S} : C_{2\pi} \longrightarrow C_{2\pi}$$

$$(\tilde{S}\psi)(t) = 2\int_0^{2\pi} \Phi(z(t), z(\tau))\psi(z(\tau))|z'(\tau)|d\tau, \ t \in [0.2\pi]$$

$$\tilde{H} : C_{2\pi} \longrightarrow C_{2\pi}$$

$$(\tilde{H}\psi)(t) = 2\int_0^{2\pi} \frac{\partial \Phi(z(t), 0)}{\partial \nu(z(t))} \psi(z(\tau))|z'(\tau)|d\tau, \ t \in [0.2\pi]$$

$$\tilde{T} : C_{2\pi} \longrightarrow C_{2\pi}$$

$$(\tilde{T}\psi)(t) = 2\int_0^{2\pi} (1 - \Phi(z(t), 0))\psi(z(\tau))|z'(\tau)|d\tau, \ t \in [0.2\pi]$$

where $\nu(z(t)) = \dfrac{[z_2'(t), -z_1'(t)]}{|z'(t)|} = \dfrac{[z'(t)]^{\perp}}{|z'(t)|}$ which is unit normal directed outside of the domain D. Having described the parametrization of our integral operators and given appropriate theory, now we turn to show compactness of integral operators in the equation (4.12) for exterior Robin problem.

Operator $\tilde{K}'$

The kernel of the operator $\tilde{K}'$ is given by

$$k(t, \tau) = 2\frac{\partial \Phi(z(t), z(\tau))}{\partial \nu(z(t))}|z'(\tau)|. \tag{4.17}$$

By taking derivative of fundamental solution with respect to $z(t)$, we have

$$\operatorname{grad}_{z(t)} \Phi(z(t), z(\tau)) = \frac{(z(t) - z(\tau))}{|z(t) - z(\tau)|^2}, \tag{4.18}$$

and by substituting expression (4.18) and ν in the kernel (4.17), we obtain

$$\begin{aligned} 2\frac{\partial \Phi(z(t), z(\tau))}{\partial \nu(z(t))}|z'(\tau)| &= 2|z'(\tau)|\nu(z(t)) \cdot \operatorname{grad}_{z(t)} \Phi(z(t), z(\tau)), \\ &= -\frac{1}{\pi}\frac{[z'(t)]^{\perp}}{|z'(t|} \cdot \frac{(z(t) - z(\tau))}{|z(t) - z(\tau)|^2}|z'(\tau)|. \end{aligned} \tag{4.19}$$

By taylor expansion of function z around the point t, we have

$$z(\tau) - z(t) = (\tau - t)z'(t) + (t-\tau)^2 \int_0^1 (1-\lambda)z''(\tau + \lambda(t-\tau))d\lambda. \tag{4.20}$$

Substituting the expression (4.20) into the expression (4.19) and taking limit as $t \to \tau$, we get

$$\frac{1}{2\pi}\frac{z''(\tau)\cdot[z'(\tau)]^{\perp}}{|z'(\tau)|^2}. \tag{4.21}$$

Eventually we have proven that

$$2\frac{\partial\Phi(z(t), z(\tau))}{\partial\nu(z(t))} = \begin{cases} -\dfrac{1}{\pi}\dfrac{[z'(t)]^{\perp}}{|z'(t)|}\cdot\dfrac{(z(t)-z(\tau))}{|z(t)-z(\tau)|^2}|z'(\tau)|, & \text{if } t \neq \tau, \\ \dfrac{1}{2\pi}\dfrac{z''(t)\cdot[z'(t)]^{\perp}}{|z'(t)|^2}, & \text{if } t = \tau. \end{cases} \tag{4.22}$$

It follows that the kernel (4.17) is continuous and hence by the Theorem 2.2 the operator $\tilde{K}'$ is compact.

Operator $\tilde{S}$

Operator $\tilde{S}$ is defined as

$$(\tilde{S}\psi)(t) = \frac{1}{\pi}\int_0^{2\pi} \ln\Big(\frac{1}{|z(t)-z(\tau)|}\Big)\psi(z(\tau))|z'(\tau)|d\tau, \qquad t \in [0, 2\pi]. \tag{4.23}$$

It can be found $\exists M > 0, 0 < \alpha \leq 1$ such that

$$\left|\ln\frac{1}{|z(t)-z(\tau)|}\right| \leq \frac{M}{|z(t)-z(\tau)|^{1-\alpha}} \leq \widetilde{M}|t-\tau|^{\alpha-1}. \tag{4.24}$$

From Theorem 2.2, the kernel is weakly singular, and therefore the operator $\tilde{S}$ is compact.

Operator $\tilde{H}$

Recalling the definition of the operator $\tilde{H}$

$$(\tilde{H}\psi)(t) = 2\int_0^{2\pi} \frac{\partial\Phi(z(t),0)}{\partial\nu(z(t))}\psi(z(\tau))|z'(\tau)|, \tag{4.25}$$

it is clear that the kernel of $\tilde{H}$ is continuous sincce $(0,0) \in D$. For this reason, the operator $\tilde{H}$ is compact by Theorem 2.2.

Operator $\tilde{T}$

By the same reason as operator $\tilde{H}$, the operator $\tilde{T}$ is compact. We have shown that the linear boundary integral operators in (4.12) are compact and the homogenous equation has only a trivial solution, so Riesz theorem can be applied. According to Riesz theorem and Theorem 4.3, the boundary integral equation (4.5) has solution and it is unique.

4.2. Integral Equation Method for Interior Problem

We have two methods to obtain boundary integral equation for interior Robin boundary value problem. One way is to use single-layer potential as representation of solution of the Laplace equation under some conditions. This method is called indirect method. Another method is direct method which is based on Green's formula. By using two ways, we obtain Fredholm integral equation of second kind. In order to have unique solution, we need to put some restrictions on domain D.

Theorem 4.5 *In two dimensions suppose there exists $x_0 \in D$ such that $|x - x_0| \neq 1$ for all $x \in \Gamma$. Then the single layer operator $S : C(\Gamma) \longrightarrow C(\Gamma)$ is injective.*

Proof The proof is given [9, Theorem 7.38]. □

We can give a counter example when the assumptions of the theorem are not fulfilled. Consider a unit circle and $\lambda = 1$, $\psi = \frac{1}{||\Gamma||}$, where $||\Gamma||$ represents the length of the curve. Then by using polar coordinates, the operator

$$S\psi = \int_\Gamma \Phi(x,y)\psi(y)ds(y) = -\frac{1}{||\Gamma||}\int_0^{2\pi} R\ln R d\theta = -\ln 1 = 0. \tag{4.26}$$

Instead of using the assumption in Theorem 4.5 one can show the injectivity of a single layer potential operator by introducing the concept of logarithmic capacity.

Theorem 4.6 *There is a unique nonzero $\psi \in C^{0,\alpha}(\Gamma)$ and a unique constant ω such that*

$$\int_\Gamma \Phi(x,y)\psi(y)ds(y) = \omega, \quad \int_\Gamma \psi(y)ds(y) = 1, \tag{4.27}$$

where Γ is class of C^2 and $\alpha \in (0,1)$.

Proof For details we refer to [4, p. 21]. □

Definition 4.1 *The numbers $2\pi\omega$ and $e^{-2\pi\omega}$ are called Robins's constant and logarithmic capacity of Γ.*

Theorem 4.7 *If the logarithmic capacity of Γ is 1, then the single layer operator S coincides with $(K' + \frac{1}{2}I)$. In all other cases, this space consists of zero alone.*

Proof We will not give the proof. The proof is given in [4, p. 22]. □

4.2.1. Using Singe-Layer Potential

Here we will use indirect method which is based on single-layer potential to derive our Fredholm integral equation of second kind. By substituting the single-layer (2.10) instead of u in boundary condition (3.5) and using the Theorem 2.5, we get

$$2\int_\Gamma \psi(y)\frac{\partial \Phi(x,y)}{\partial \nu(x)}ds(y) + \psi(x) + 2\lambda(x)\int_\Gamma \psi(y)\Phi(x,y)ds(y) = 2f(x), \; x \in \Gamma. \tag{4.28}$$

Recalling the operators (4.11), the boundary integral equation (4.28) can be written as

$$(K' + \lambda S + I)\psi = 2f. \tag{4.29}$$

Theorem 4.8 *Assume that $K : C(\Gamma) \longrightarrow C(\Gamma)$, $S : C(\Gamma) \longrightarrow C(\Gamma)$ and $f, \lambda \in C(\Gamma)$, there exists $x_0 \in D$ such that $|x - x_0| \neq 1$ such that for all $x \in \Gamma$. Then $(K' + \lambda S + I)\,\psi = 2f$ has at most one solution.*

Proof We assume that $(K' + \lambda S + I)\psi = 0$, and define

$$u(x) = \int_\Gamma \Phi(x,y)\psi(y)ds(y). \tag{4.30}$$

By using representation (4.30) and $(K' + \lambda S + I)\psi = 0$, we obtain

$$\frac{\partial u}{\partial \nu} + \lambda u = 0. \tag{4.31}$$

From the uniqueness of the interior Robin problem, $u = 0$ in D, by analycity $u|_\Gamma = 0$, and hence $\lambda S\psi = 0$. From the Theorem 4.5, we conclude that $\psi = 0$. More the detail about this approach is given in [9]. □

We define $\tilde{\lambda}(t) = \lambda(z(t)),\ \tilde{f}(t) = f(z(t))$.

Theorem 4.9 *Assume there exists* $x_0 \in D, |x - x_0| \neq 1,\ \forall x \in \Gamma$. *Then*

$$(\tilde{K}' + \tilde{\lambda}\tilde{S} + I) = 2\tilde{f} \tag{4.32}$$

has a unique solution.

Proof We have shown that the operators $\tilde{K}'$ and $\tilde{S}$ are compact so from the Theorem 2.3 and the Theorem 4.5, the integral equation (4.32) has unique solution. □

4.2.2. Using Green's Formula

Here we use direct method which is based on Green's representation formula to obtain integral equation of second kind. We substitute $\frac{\partial u}{\partial \nu} = -\lambda u + f$ into formula (2.18) and then we have

$$\begin{aligned} u(x) = -\int_\Gamma \lambda(y)u(y)\Phi(x,y)ds(y) \ &- \int_\Gamma u(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}ds(y) \\ &+ \int_\Gamma f(y)\Phi(x,y)ds(y),\ x \in D. \end{aligned} \tag{4.33}$$

Introducing $\psi = u|_\Gamma$ and $g(x) = \int_\Gamma f(y)\Phi(x,y)ds(y)$, we derive a boundary integral equation

$$(K + S_\lambda + I)\psi = g. \tag{4.34}$$

Theorem 4.10 *Suppose that $K : C(\Gamma) \longrightarrow C(\Gamma)$, $S : C(\Gamma) \longrightarrow C(\Gamma)$ and $f, \lambda \in C(\Gamma)$, and there exists $x_0 \in D$ such that $|x - x_0| \neq 1$ such that for all $x \in \Gamma$. Then*

$$(K + S_\lambda + I)\psi = g \tag{4.35}$$

has a unique solution where $u|_\Gamma = \psi$ and $g(x) = \int_\Gamma f(y)\Phi(x, y)ds(y),\ x \in \Gamma$.

Proof Let $\psi \in C(\Gamma)$ be solution to $(I + K + S_\lambda)\psi = 0$. We define

$$u(x) = \int_\Gamma \frac{\partial u}{\partial \nu}(y)\Phi(x, y)ds(y) - \int_\Gamma u(y)\frac{\partial \Phi(x, y)}{\partial \nu(y)}ds(y) \text{ in } D. \tag{4.36}$$

By using $(I + K + S_\lambda)\psi = 0$ and (4.36), we obtain

$$\frac{\partial u}{\partial \nu} + \lambda u = 0 \text{ on } \Gamma. \tag{4.37}$$

From the equation (4.37) and the representation (4.33), we find that

$$u|_\Gamma = \psi \text{ and } \frac{\partial u}{\partial \nu} = -\lambda\psi. \tag{4.38}$$

Then by Lemma 2.2, we have

$$\int_\Gamma u\frac{\partial u}{\partial \nu}ds = -\int_\Gamma \psi^2\lambda ds = \int_D |\nabla u|^2\, dx. \tag{4.39}$$

This yields $\psi = 0$ since $\lambda > 0$. Thus the operator $(I + K + S_\lambda)$ is injective. Compactness is known for operator S_λ and K. From the Theorem 2.3, it follows that (4.35) has solution which is unique □

This work also can be extended to another types of assumption on the domain D by considering the Hölder space $C^{0,\alpha}$. This is given in detail in [4].

CHAPTER 5

NUMERICAL SOLUTION TO INTEGRAL EQUATIONS

In this chapter we focus on Nyström method for the numerical solution of integral equations. Besides we give the error estimate for Nyström method based on the quadrature rules for 2π-periodic functions. Then we focus on treating singularity by splitting the kernel into two parts. We have two types of kernels which are 2π-periodic. For continuous kernel, the integrals are calculated by trapezoidal rule which is high-order accurate method for 2π periodic functions. For the singular kernel, we use the way of splitting off the singularity and calculating the resulting singular part by a special quadrature rule introduced by R. Kress and K. Atkinson. Although there are some high -order accurate methods for calculating integrals with weakly singular kernel [2, 6], the method based on splitting off the singularity is superior since it is based on the exact integration. The general overview of these methods can be found in [5].

5.1. Quadrature Rule

For given weights w_k and quadrature nodes y_k, for $k \in \{1, 2, \ldots n\}$,

$$Q(\psi) = \int_0^{2\pi} \psi(y)dy \tag{5.1}$$

is approximated by quadrature formula

$$Q_n(\psi) = \sum_{k=1}^{n} w_k \psi_k. \tag{5.2}$$

To guarantee the convergence $Q_n(\psi) \to Q(\psi)$, we have theorem below.

Theorem 5.1 *(Steklov) Assume* $Q_n(1) \to Q(1)$ *as* $n \to \infty$ *and quadrature weights are* *all positive. Then the quadrature rules* (Q_n) *converge if and only if* $Q_n(\psi) \to Q(\psi)$*, as* $n \to \infty$*, for all* ψ *in some dense subset* $M \subset C(G)$*,* $G \subset \mathbb{R}^m$*.*

Proof The proof is given in [9, p. 222]. □

The classical quadrature rules are inappropriate for improper integral, and therefore we introduce some special quadrature formulas for improper integral whose integrand is 2π periodic functions with logarithmic singularity.

$$(A\psi)(t) := \frac{1}{2\pi}\int_0^{2\pi} \ln\left(4\sin^2\left(\frac{t-\tau}{2}\right)\right) k(t,\tau)\psi(\tau)d\tau,\ t \in [0,2\pi], \psi \in C[0,2\pi], \tag{5.3}$$

where $k \in C([0,2\pi] \times [0,2\pi])$.

We assume that n is the number of quadrature nodes. By substituting trigonometric interpolation polynomial instead of 2π periodic continuous function ψ and using Lagrange basis for interpolation, we obtain

$$(A_n\psi)(t) := \sum_{j=0}^{2n-1} R_j^{(n)}(t)k(t,t_j)\psi(t_j), \tag{5.4}$$

with the equidistant points $t_j = \dfrac{j\pi}{n}$ and the weights

$$R_j^{(n)}(t) = -\frac{1}{n}\left\{\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t-t_j) + \frac{1}{2n}\cos n(t-t_j)\right\}, \tag{5.5}$$

for $j = 0,\ldots,2n-1$. The special quadrature rule converges uniformly for all trigonometric polynomials. The proof is given in [9, p. 208].

5.2. Treatment of Singularity

The operator $\tilde{S}$ has singularity so it needs to be treated. That is why we split the kernel of operator $\tilde{S}$ into two parts. To do this, one needs to simplify kernel of the operator. We add and subtract $\dfrac{1}{2\pi}|z'(\tau)|\ln\left(4\sin^2\dfrac{(t-\tau)}{2}\right)$ to the kernel of operator $\tilde{S}$.

Then we have

$$\frac{1}{2\pi}|z'(\tau)|\ln\left(\frac{1}{|z(t)-z(\tau)|}\right)^2 \pm \frac{1}{2\pi}|z'(\tau)|\ln\left(4\sin^2\frac{(t-\tau)}{2}\right). \tag{5.6}$$

By the Taylor expansion (4.20), the kernel of the operator $\tilde{S}$ can be written as

$$P(z(t), z(\tau)) - \frac{1}{2\pi}|z'(\tau)|\ln\left(4\sin^2\frac{(t-\tau)}{2}\right), \tag{5.7}$$

where

$$P(z(t)), z(\tau)) = \begin{cases} \frac{1}{2\pi}|z'(\tau)|\ln\left(\frac{4\sin^2\frac{t-\tau}{2}}{|z(t)-z(\tau)|^2}\right), & \tau \neq t, \\ \frac{1}{2\pi}|z'(\tau)|\ln\left(\frac{1}{|z'(\tau)|^2}\right), & \tau = t. \end{cases} \tag{5.8}$$

We have obtained two distinct parts of the kernels of the operator. One of them is continuous and another one is discontinuous part. We have treated singularity in the kernel of operator and constructed some simple kernels and consequently the resulting kernel can be integrated by quadrature formula (5.4) for improper integral.

5.3. Error Estimate for 2π-Periodic Functions

We will give a error estimate of numerical integration for periodic analytic functions. This estimate demonstrates that for periodic analytic functions quadrature rules converge exponentially. Also we provide error estimate for continuously differentiable functions.

Theorem 5.2 *Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be analytic and 2π-periodic . Then the error*

$$R_T(g) := \frac{1}{2\pi}\int_0^{2\pi} g(x)dx - \frac{1}{2n}\sum_{i=0}^{2n-1} g\left(\frac{i\pi}{n}\right), \tag{5.9}$$

for composite trapezoidal rule can be estimated by

$$\left|R_T(g)\right| \leq Ce^{-2ns}, \tag{5.10}$$

where C and s are positive constants depending on g.

Proof The proof is given in [9, p. 201]. □

Theorem 5.3 *Assume* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *is m-times continuously differentiable and* 2π*-periodic function. The error for the trapezoidal rule*

$$R_T(g) := \frac{1}{2\pi}\int_0^{2\pi} g(x)dx - \frac{1}{2n}\sum_{i=0}^{2n-1} g\left(\frac{i\pi}{n}\right), \tag{5.11}$$

can be estimated as follows

$$\left|R_T(g)\right| \leq \frac{C}{n^m}\left\|g^{(m)}\right\|_\infty. \tag{5.12}$$

Proof The proof is given in [9]. The main idea of the proof in the following. □

Let $f : [a, b] \rightarrow \mathbb{R}$ be m-times continuously differentiable for $m \geq 2$ and recall definition of the trapezoidal sum with $h = \frac{b-a}{n}$,

$$T_h(f) := h\left[\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + f(x_n)\frac{1}{2}\right], \text{ for } f \in C[a, b]. \tag{5.13}$$

By Euler-Maclaurin expansion

$$\begin{aligned}\int_a^b f(x)dx &= T_h(f) - \sum_{j=1}^{[\frac{m}{2}]} \frac{b_{2j}h^{2j}}{(2j)\,!}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right] \\ &+(-1)^m h^m \int_a^b \bar{B}_m\left(\frac{x-a}{h}\right) f^{(m)}(x)dx,\end{aligned} \tag{5.14}$$

where $\left[\frac{m}{2}\right]$ denotes the largest integer smaller than or equal to $\frac{m}{2}$.

$$\overline{B_{2m}}(x) = 2(-1)^{(m-1)} \sum_{k=1}^{\infty} \frac{\cos 2\pi kx}{(2\pi k)^{2m}} \text{ and } \overline{B_{2m-1}}(x) = 2(-1)^{(m)} \sum_{k=1}^{\infty} \frac{\sin 2\pi kx}{(2\pi k)^{2m-1}} \tag{5.15}$$

denotes the periodic extension of the Bernoulli polynomial. When we have 2π-periodic continuous functions f, the trapezoidal rule coincides with the rectangular rule

$$\int_0^{2\pi} f(x)dx \approx \frac{2\pi}{n} \sum_{k=1}^{n} f\left(\frac{2\pi k}{n}\right). \tag{5.16}$$

The equality 5.14 illustrates why the trapezoidal rule for periodic functions is superior to any other quadrature rule by (5.14). For more details we refer to [8].

5.4. Nyström Method

Assume that a quadrature rule is convergent. We consider the Fredholm equation of second kind

$$\psi(x) = \int_0^{2\pi} k(x,y)\psi(y)dy + g(x) \quad \text{represented } \psi = K\psi + g, \; y \in [0, 2\pi]. \tag{5.17}$$

Approximating operator K via the quadrature rule, we have

$$\psi_n(x) = \sum_{k=1}^{N} w_k k(x, y_k)\psi_n(y_k) + g(x) \text{ represented by } \psi_n = K_n\psi_n + g, \tag{5.18}$$

which is known as Nyström interpolation. By virtue of the fact that the density function in the quadrature node points satisfies semi-discretized expression (5.18), the linear system of equations

$$\psi(x_k) = \sum_{k=1}^{N} w_k k(x_k, y_k)\psi(y_k) + g(x_k) \tag{5.19}$$

are obtained. Generally the matrix (5.19) gives us some useful information about stability which is related to condition number. These are some advantages of Nyström method. In the following we will give error analysis of Nyström method.

5.5. Error and Convergence of Nyström Method

Convergence of Nyström method depends on the quadrature formula we choose. In order to show complete error analysis of Nyström method, we give the following theorem.

Theorem 5.4 *Assume that X is a Banach space and S, T are bounded operators from X to X. Let S be compact. Also assume $I - T : X \longrightarrow X$ is bijective which shows $(I - T)^{-1}$ exists as a bounded operator from X to X. Assume that*

$$\left\|(T - S)S\right\| < \frac{1}{\left\|(I - T)^{-1}\right\|}. \tag{5.20}$$

Then $(I - S)^{-1}$ exists and is bounded from X to X, accompanying

$$\left\|(I - S)^{-1}\right\| \leq \frac{1 + \left\|(I - T)^{-1}\right\| \|S\|}{1 - \left\|(I - T)^{-1}\right\| \left\|(T - S)S\right\|}. \tag{5.21}$$

If $(I - T)u = g$ and $(I - S)v = g$, then

$$\|u - v\| \leq \left\|(I - S)^{-1}\right\| \|Tu - Sv\|. \tag{5.22}$$

Proof The proof is given in detail in [3, Theorem 12.4.3]. □

Theorem 5.4 yields the following convergence result for the Nyström method.

Theorem 5.5 *Let $k(x, y)$ be continuous for all $x, y \in [0, 2\pi]$. Assume quadrature rule is convergent for all continuous functions on $[0, 2\pi]$. Moreover, assume that the integral equation (5.17) is uniquely solvable for given $g \in C([0, 2\pi])$. Then for all sufficiently large n, say $n \geq N$, the approximate inverses $(I - K_n)^{-1}$ exist and are uniformly bounded,*

$$\left\|(I - K_n)^{-1}\right\| \leq \frac{1 + \left\|(I - K)^{-1}\right\| \|K_n\|}{1 - \left\|(I - K)^{-1}\right\| \left\|(K - K_n)K_n\right\|} \leq C_y, \ n \geq N, \tag{5.23}$$

with a appropriate constant $C_y < \infty$. For the integral equations $(I - K)v = g$ and $(I - K_n)v_n = g$, we have

$$\left\|\psi - \psi_n\right\|_\infty \leq \left\|(I - K_n)^{-1}\right\| \left\|(K - K_n)\psi\right\|_\infty \leq C_y \left\|(K - K_n)\psi\right\|_\infty . \tag{5.24}$$

Proof We substitute $S = K$ and $T = K_n$ in Theorem 5.4. Then we get the desired result about the convergence of the Nyström method. From the inequality (5.24), $\left\|\psi - \psi_n\right\|_\infty$ is bounded by

$$\left\|(K - K_n)\psi\right\|_\infty = \max_{x \in [0,2\pi]} \left| \int_0^{2\pi} k(x, y)\psi(y)dy - \sum_{i=1}^{n} w_i k(x, x_i)\psi(x_i) \right| . \tag{5.25}$$

It can be written

$$(I - K_n)(\psi - \psi_n) = (K - K_n)\psi. \tag{5.26}$$

From (5.26), it is obtained

$$\left\|(K - K_n)\psi\right\|_\infty \leq \|I - K_n\| \left\|\psi - \psi_n\right\|_\infty . \tag{5.27}$$

Also from (5.26), it can be obtained

$$\left\|\psi - \psi_n\right\|_\infty \leq \left\|(I - K_n)^{-1}\right\| \left\|(K - K_n)\psi\right\|_\infty . \tag{5.28}$$

This depicts $\left\|(K - K_n)\psi\right\|_\infty$ and $\left\|\psi - \psi_n\right\|_\infty$ converge to 0 as $n \to 0$ with the same speed. The proof is given in details in [3, Theorem 12.4.4]. □

CHAPTER 6

NUMERICAL EXAMPLES

Having given the formulation of the integral equations and information about some of the suitable theory, we now turn to numerical implementation which yields numerical approximations of the solutions to the exterior and interior boundary problem.

6.1. Numerical Examples For the Exterior Problem

First, we present the exact solution, then we demonstrate the table of convergence for the solution to the exterior Robin problem and finally, numerical solution is illustrated by the total field for a monopole and dipole. Our domains which we will use are

$$\begin{aligned} \Gamma_1 &:= \left\{(4\cos t + \sin 2t, 4\sin t + 2\sin^2 t + 7),\ t \in [0, 2\pi]\right\} \\ \Gamma_2 &:= \{(2\cos t - \sin 3t, 2\sin t - \sin 3t),\ t \in [0, 2\pi]\} \end{aligned}$$

which are smooth and regular. By taking $\lambda = 0$ in the condition (3.2), the Robin condition reduces to the Neumann condition. It is well known that exterior Neumann problem has unique solution but interior Neumann is not uniquely solvable. To implement the method, we have started with the exterior Neumann condition and then extended the code to the impedance conditions. Here we will give the numerical results for the exterior Robin problem for

$\lambda(x) = -2\sin(|x|) + 4.5.$

Example 6.1.

$$\Delta u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}, \tag{6.1}$$

$$f = \frac{\partial u}{\partial \nu} - \lambda u \quad \text{on } \Gamma, \tag{6.2}$$

$$f(x) = \frac{\partial\left(\Phi(x_1, x) - \Phi(x_2, x)\right)}{\partial \nu(x)} - \lambda(x)\left(\Phi(x_1, x) - \Phi((x_2, x)\right), \quad x \in \Gamma, x_1,\ x_2 \in D, \tag{6.3}$$

$$u = O(1) \quad \text{when } |x| \to \infty. \tag{6.4}$$

The exact solution $u_E(x) = \Phi(|x - x_1|) - \Phi(|x - x_2|)$ with $x_1 = (0, 0)$, $x_2 = (0.1, 1) \in D$ and $x \in \mathbb{R}^2 \setminus \bar{D}$ satisfies equations above. Besides, u_E is bounded at infinity while the fundamental solution (2.7) is not bounded in two dimension.

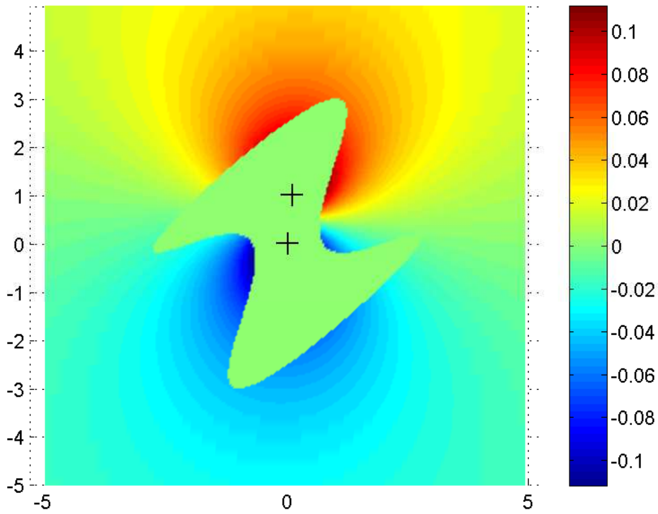

Figure 6.1. Approximate solution

As illustrated in Fig. 6.1, there are two fixed points inside domain because of form of the exact solution. Fig. 6.1 demonstrates the solution in exterior of domain. Now we present the table of convergence for the exterior Robin problem. For convergence of the exterior Robin problem, we consider modified single layer potential as approximate solution.

Table 6.1. Convergence of exterior Robin problem for one domain

n	Cond. Number	Error at (4,5)	Error at (-4,-2)
8	90.04	0.001104075430519	0.000230184056030
16	114.70	0.000099629624383	0.000042541648225
32	131.20	0.000000440955601	0.000000051071255
64	138.28	0.000000000018878	0.000000000003923
128	142.73	0.000000000000000	0.000000000000000

This table exhibits the error estimate at the point (4,5) and (-4,-2) respectively. As illustrated in the Table (6.1), correct digits almost double so it converges super-algebraically and condition number is uniformly bounded due to the Nyström method and the operators in the equation (4.29). Additionally in the Table 6.1 the convergence agrees with the

Theorem 5.3. This demonstrates the method converges for exterior Robin boundary value problem. We consider a case of a monopole, that is we put one source Φ at the point $x^\star$.

Example 6.2.

Mathematically, we consider the solution to the exterior Robin problem with the function

$$f(x) = -\left(\frac{\partial \Phi(x^\star, x)}{\partial \nu(x)} - \lambda(x)\Phi(x^\star, x)\right), \; x \in \Gamma, \; x^\star \in \mathbb{R}^2 \setminus \bar{D}, \tag{6.5}$$

where $x^\star = (2, 3)$. The total field is defined as following

$$u^T(x) \quad = \Phi(x^\star, x) + u(x), \quad x^\star, \; x \in \mathbb{R}^2 \setminus \bar{D}, \tag{6.6}$$

where $u(x)$ is approximate solution.

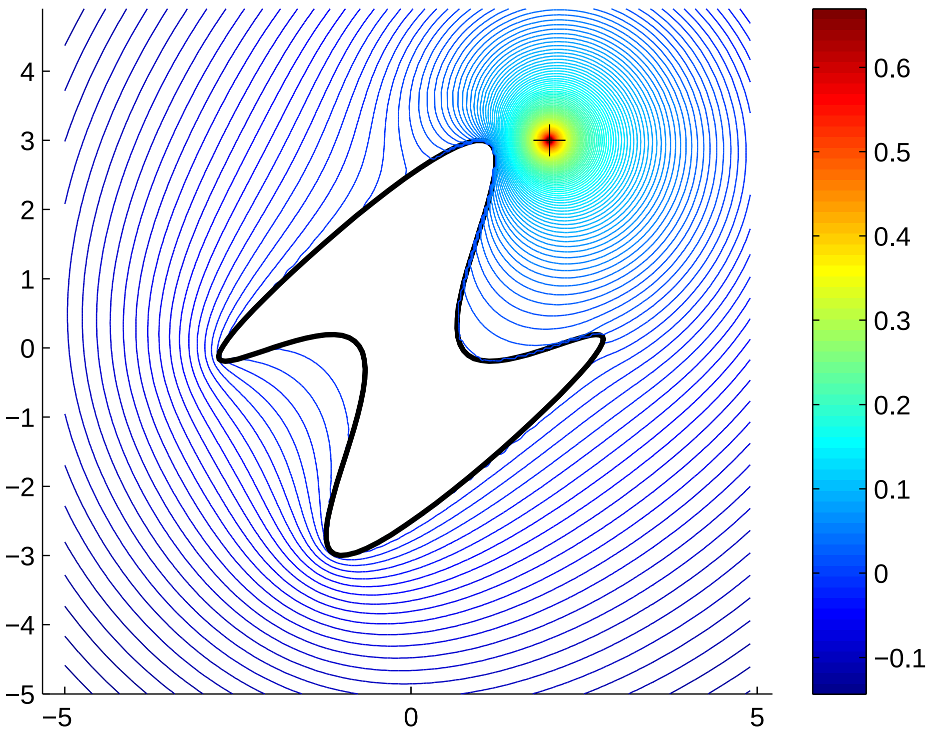

Figure 6.2. Equipotential lines of total field for a monopole for one domain

In Fig. 6.2 the total field is illustrated for a monopole for one domain. This figure can be interpreted as total field for a unit charge.

Example 6.3.

We have two sources $x_1^\star = (-3,-2)$ and $x_2^\star = (2,3)$ for the exterior problem. Total field u^T and f are defined in the following

$$u^T(x) = \Phi(x_1^\star, x) - \Phi(x_2^\star, x) + u(x), \quad x_1^\star, x_2^\star, \tag{6.7}$$

where $x \in \mathbb{R}^2 \setminus \bar{D}$, $u(x)$ is approximate solution.

$$f(x) = -\left(\frac{\partial\left(\Phi(x_1^\star, x) - \Phi(x_2^\star, x)\right)}{\partial\nu(x)} - \lambda(x)\left(\Phi(x_1^\star, x) - \Phi((x_2^\star, x)\right)\right), \tag{6.8}$$

where $x \in \Gamma$.

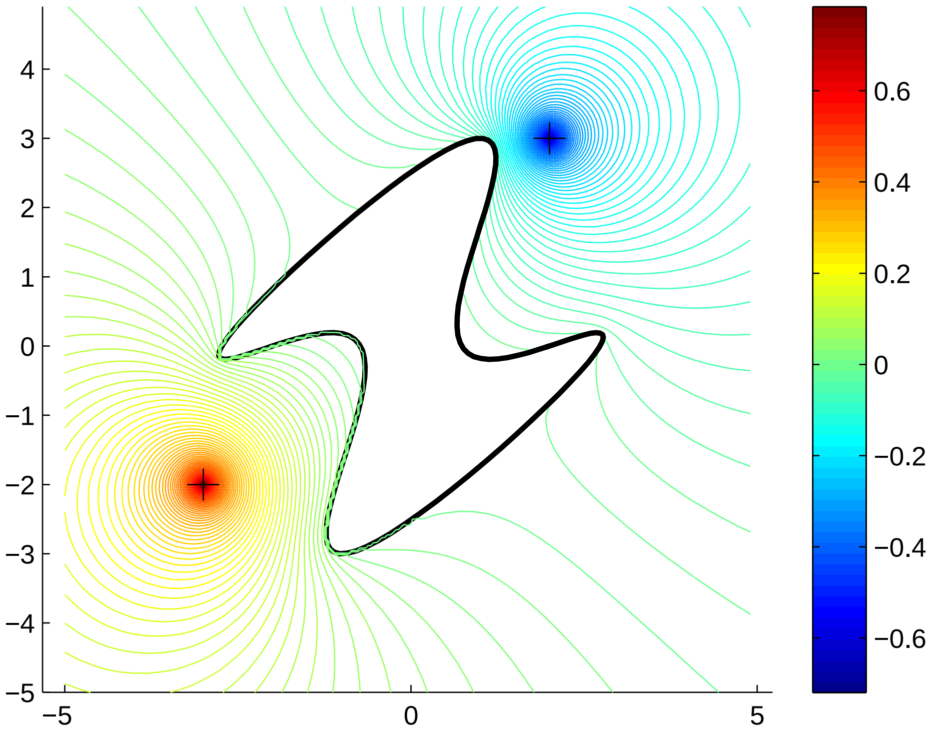

Figure 6.3. Equipotential lines of total field for a dipole for one domain

Fig. 6.3 demonstrates the total field for a dipole for the exterior Robin problem with one domain. This figure can be interpreted as total field for one positive and one negative charge unit charges.

Example 6.4.

In this example we take into consideration the exterior Robin problem for a doubly connected domain. We have one point source $x^\star = (2, 3)$ and so we construct u^T and f with one source point in the following.

$$u^T(x) = \Phi(x^\star, x) + u(x), \; x^\star, x \in \mathbb{R}^2 \setminus \bar{D}, \tag{6.9}$$

$$f(x) = -\left(\frac{\partial \Phi(x^\star, x)}{\partial \nu(x)} - \lambda(x)\Phi(x^\star, x)\right), \; x \in \Gamma, x^\star \in \mathbb{R}^2 \setminus \bar{D}. \tag{6.10}$$

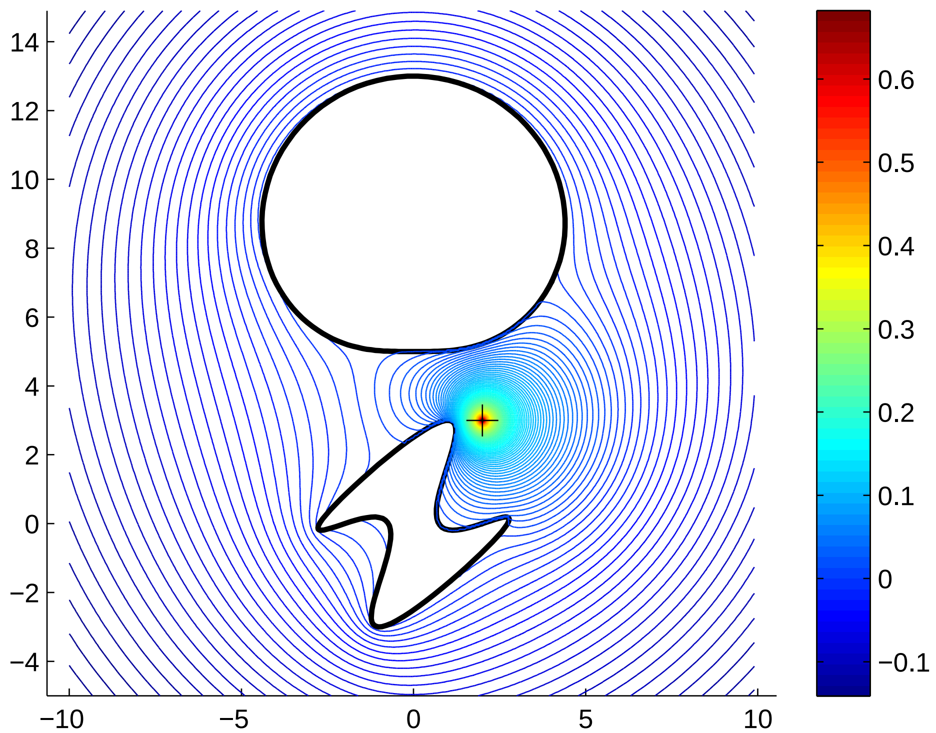

Figure 6.4. Equipotential lines of total field for a monopole

In Fig. 6.4 u^T for a monopole with multiply connected domain is illustrated. Also this shows total field for a unit charge for multiply connected domains.

Example 6.5.

In this example we consider the exterior Robin problem. We have two source points $x_1^\star = (-3, -2)$, $x_2^\star = (2, 3)$ for multiply connected domain also u^T and f are constructed as follows

$$u^T(x) = \Phi(x_1^\star, x) - \Phi(x_2^\star, x) + u(x), \tag{6.11}$$

where $x_1^\star, x_2^\star, x \in \mathbb{R}^2 \setminus \bar{D}$ and $u(x)$ is approximate solution.

$$f(x) = \frac{\partial\left(\Phi(x_1^\star, x) - \Phi(x_2^\star, x)\right)}{\partial\nu(x)} - \lambda(x)\left(\Phi(x_1^\star, x) - \Phi((x_2^\star, x)\right), \tag{6.12}$$

where $x \in \Gamma, x_1^\star, x_2^\star \in \mathbb{R}^2 \setminus \bar{D}$.

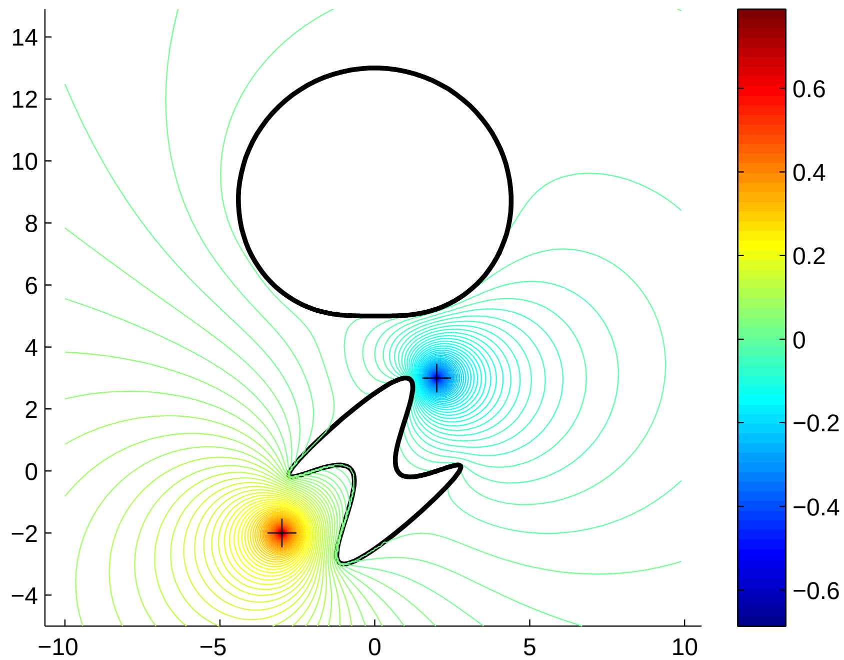

Figure 6.5. Equiptential lines of the total field for a dipole

Fig. 6.5 illustrates equipotential lines of the total field for a dipole with multiply connected domains. This can be interpreted as total field for positive and negative unit charges for multiply connected domains.

6.2. Numerical Examples For the Interior Problem

In this section we present some numerical examples for interior Robin problem with $\lambda(x) = -\cos(|x|) + 4.5$. The numerical approach of interior Robin problem is based on single-layer potential from section 4.8. Our domains are defined as

$$\Gamma_1 := \left\{(4\cos t + \sin 2t, 4\sin t + 2\sin^2 t + 7),\ t \in [0, 2\pi]\right\}$$
$$\Gamma_2 := \{(2\cos t - \sin 3t, 2\sin t - \sin 3t,\ t \in [0, 2\pi]\}$$

Example 6.1. For convergence test we have

$$\Delta u = 0 \text{ in } D, \tag{6.13}$$
$$\frac{\partial u}{\partial \nu} + \lambda u = f \text{ on } \Gamma, \tag{6.14}$$
$$f(x) = \frac{\partial \Phi(x^\star, x)}{\partial \nu} + \lambda(x)\Phi(x^\star, x)\ x \in \Gamma, x^\star \in \mathbb{R}^2 \setminus \bar{D}, \tag{6.15}$$
$$u_E = \Phi(x^\star, x),\ \ x^\star \in \mathbb{R}^2 \setminus \bar{D}, x \in D. \tag{6.16}$$

The exact solution u_E is defined by 6.16 with $x^\star = (5, 5) \in \mathbb{R}^2 \setminus \bar{D}$.

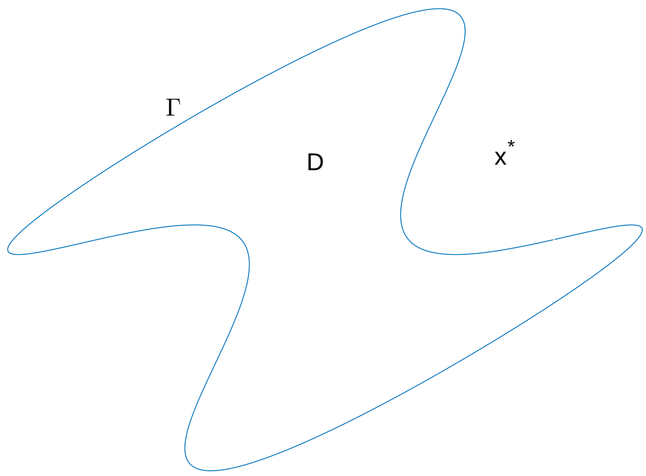

Figure 6.6. Domain for convergence test for the interior problem

Table 6.2. Convergence of interior Robin problem for one domain

n	Cond. Number	Error at (0,0)	Error at (-1,1)
8	20.27	0.000266178205197	0.005788897607039
16	25.82	0.000009527392591	0.000727624242802
32	27.67	0.000000004951059	0.000016070916386
64	29.36	0.000000000000525	0.000000006042646
128	30.73	0.000000000000000	0.000000000000120
256	31.39	0.000000000000000	0.000000000000000

As illustrated in Table 6.2, the error decreases and it converges super-algebraically. Also this convergence agrees with the Theorem 5.3. This shows this numerical method converges for interior Robin problem. We consider the approximate solution to the interior Robin problem in this example.

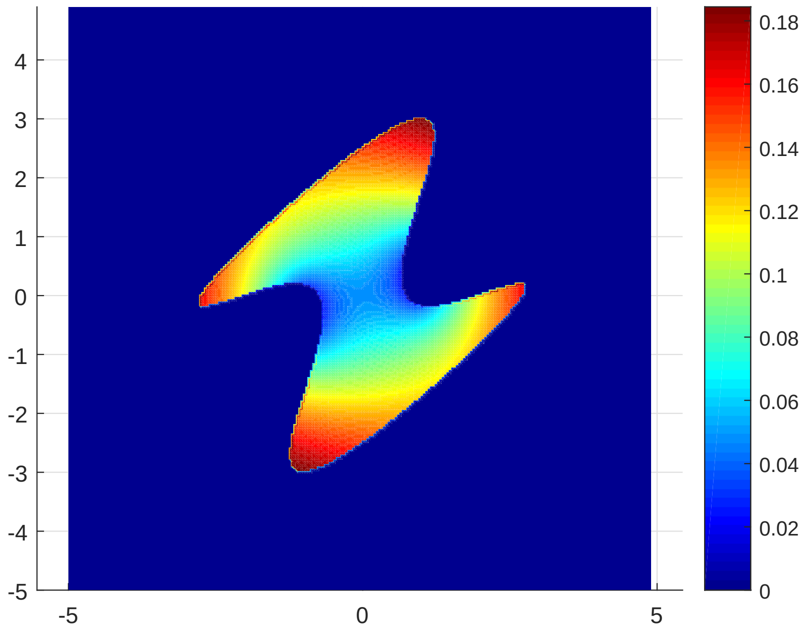

Figure 6.7. Approximate solution for interior Robin boundary value problem

The solution is illustrated for interior Robin problem in the Fig. 6.7

Example 6.2.

In this example we focus on the interior Robin problem for total field for a simply connected domain and f is defined by

$$f(x) = -\left(\frac{\partial \Phi(x^\star, x)}{\partial \nu} + \lambda(x)\Phi(x^\star, x)\right), \quad x \in \Gamma, x^\star \in D, \tag{6.17}$$

where $x^\star = (0,0)$. The total field is given by

$$u^T(x) \quad = \Phi(x^\star, x) + u(x) \quad x^\star, x \in D. \tag{6.18}$$

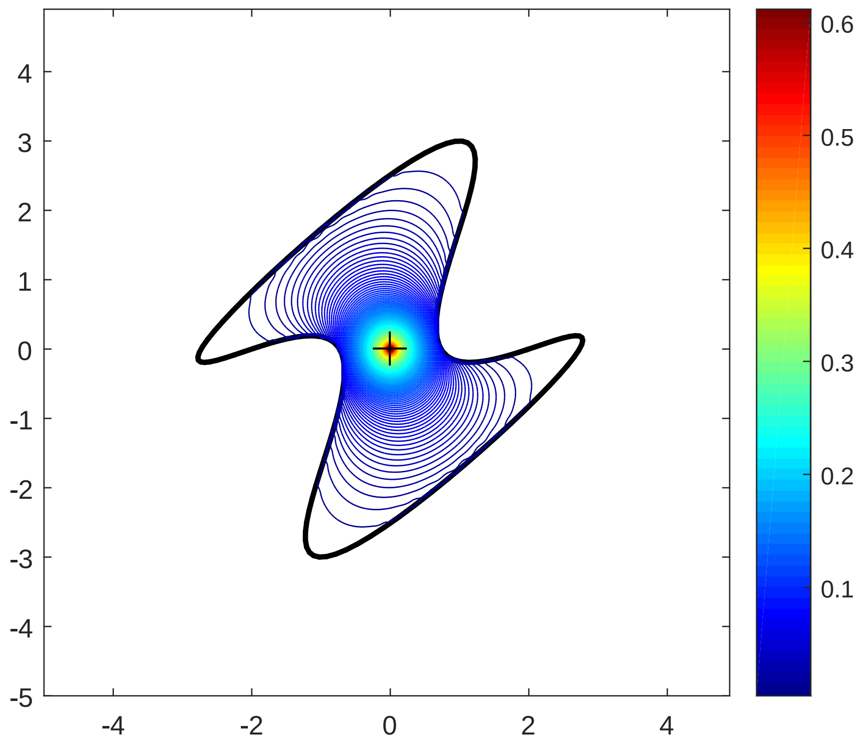

Figure 6.8. Equipotential lines of total field for a monopole

As illustrated in the Fig. 6.8, we plot total field for a monopole. We have one source in the Fig. 6.8. This can be interpreted as total field for unit charge.

Example 6.3.

We have two source $x_1^\star = (0,0), x_2^\star = (0.1,1)$ in this example. u^T and f are defined in the following way

$$f(x) = -\left(\frac{\partial\left(\Phi(x_1^\star, x) - \Phi(x_2^\star, x)\right)}{\partial\nu(x)} + \lambda(x)\left(\Phi(x_1^\star, x) - \Phi((x_2^\star, x)\right)\right), \tag{6.19}$$

where $x \in \Gamma, x_1^\star, x_2^\star \in D$.

$$u^T(x) = \Phi(x_1^\star, x) - \Phi(x_2^\star, x) + u(x), \tag{6.20}$$

where $u(x)$ is approximate solution and $x, x_1^\star, x_2^\star \in D$.

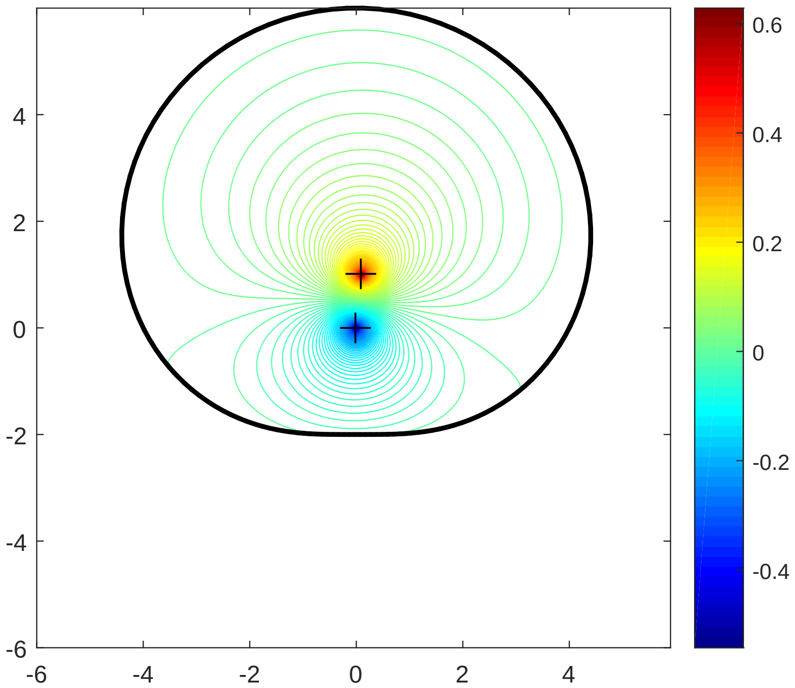

Figure 6.9. Equipotential lines of total field for a dipole

Fig. 6.9 illustrates equipotential lines of total field for a dipole in a simply connected domain. This figure can be interpreted as total field for one positive and one negative unit charges.

CHAPTER 7

CONCLUSION

In this thesis integral representations of solutions to the planar Robin boundary value problems for the Laplacian are considered which lead to the Fredholm integral equations of second kind. The uniqueness and existence of the solutions to the boundary integral equations are investigated. For the exterior Robin boundary value problem a special modification of the single layer potential was introduced. This representation has an advantage of avoiding an additional condition on the unknown solution of the integral equation. The constructed integral equation were solved numerically by Nyström method which is based on quadrature methods. Some of the integral operators have singular kernels so the singularity is treated by splitting weakly singular kernel into two parts. For calculation of singular parts we have used special quadrature rule which is extremely accurate for the weakly singular kernel. The provided numerical examples demonstrate convergence and high accuracy for the solutions of the exterior and interior problems for simply and multiply connected domains. As expected by the theory and validated by the numerical results, the proposed numerical approaches converge super-algebraic for smooth data. As a result, integral equation method has been demonstrated to be effective for the exterior and the interior problem Robin boundary value problem for the Laplace equation.

Our study can be extended to three dimensions but the main drawback of three dimensions includes many calculations, [1]. We solved the problems for smooth boundaries, but the Robin boundary value problems can be also solved for domains with piecewise smooth boundary, [11]. Additionally, this study can be applied to inverse problems, [12]. Future work includes solution method of the Robin boundary value problem in the half plane by using Green's function, [14].

REFERENCES

[1] J. F. Ahner and H. W. Wiener, *On an exterior Laplace equation problem with Robin boundary condition*, Journal of mathematical analysis and applications, 157 (1991), pp. 127–146.

[2] B. K. Alpert, *Hybrid Gauss-trapezoidal quadrature rules*, SIAM Journal on Scientific Computing, 20 (1999), pp. 1551–1584.

[3] K. Atkinson and W. Han, *Theoretical numerical analysis*, vol. 39, Springer, 2005.

[4] C. Constanda, *Direct and indirect boundary integral equation methods*, Chapman & Hall, 2000.

[5] S. Hao, A. Barnett, P.-G. Martinsson, and P. Young, *High-order accurate methods for Nyström discretization of integral equations on smooth curves in the plane*, Advances in Computational Mathematics, 40 (2014), pp. 245–272.

[6] S. Kapur and V. Rokhlin, *High-order corrected trapezoidal quadrature rules for singular functions*, SIAM Journal on Numerical Analysis, 34 (1997), pp. 1331–1356.

[7] T. Keeler, *An integral equation method for solving Laplace's equation with Robin boundary conditions*, master's thesis, Simon Fraser University, Department of Mathematics, 2011.

[8] R. Kress, *Numerical analysis, volume 181 of graduate texts in mathematics*, 1998.

[9] R. Kress, *Linear Integral Equations (Applied Mathematical Sciences) (Vol 82)*, Springer, 3nd ed., 4 2014.

[10] R. Kress and W. Rundell, *Nonlinear integral equations and the iterative solution for an inverse boundary value problem*, Inverse Problems, 21 (2005), pp. 1207–1223.

[11] D. Medková, *The third boundary value problem in potential theory for domains with a piecewise smooth boundary*, Czechoslovak Mathematical Journal, 47 (1997), pp. 651–679.

[12] S. Mottin, *The analytical solution of the Laplace equation with the Robin boundary conditions on a sphere: Applications to some inverse problems*, arXiv preprint arXiv:1503.05476, (2015).

[13] T. Ransford, *Computation of logarithmic capacity*, Computational Methods and Function Theory, 10 (2011), pp. 555–578.

[14] W. A. Strauss, *Partial differential equations*, vol. 92, Wiley New York, 1992.