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BRIEF COMMUNICATIONS

STRONGLY RADICAL SUPPLEMENTED MODULES

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Zöschinger studied modules whose radicals have supplements and called these modules *radical supplemented*. Motivated by this, we call a module *strongly radical supplemented* (briefly *srs*) if every submodule containing the radical has a supplement. We prove that every (finitely generated) left module is an *srs*-module if and only if the ring is left (semi)perfect. Over a local Dedekind domain, *srs*-modules and radical supplemented modules coincide. Over a nonlocal Dedekind domain, an *srs*-module is the sum of its torsion submodule and the radical submodule.

1. Introduction

Throughout this paper, R is an associative ring with identity, and all modules are unital left R-modules. Let M be an R-module. By $N \subseteq M$ we mean that N is a submodule of M. A submodule $L \subseteq M$ is said to be *essential* in M, denoted by $L \trianglelefteq M$, if $L \cap N \neq 0$ for every nonzero submodule $N \subseteq M$. A submodule S of M is called *small* (*in* M), denoted by $S \ll M$, if $M \neq S + L$ for every proper submodule L of M. By $\operatorname{Rad} M$ we denote the sum of all small submodules of M, or, equivalently the intersection of all maximal submodules of M. A module M is called *supplemented* (see [1]) if every submodule N of M has a *supplement*, i.e., a submodule K minimal with respect to $N + K = M$. A submodule K is a supplement of N in M if and only if $N + K = M$ and $N \cap K \ll K$ (see [1]). An R-module M is said to be *radical supplemented* if $\operatorname{Rad} M$ has a supplement in M. Radical supplemented modules were studied by Zöschinger in [2] and [3]. Motivated by this definition, we call a module *strongly radical supplemented* if every submodule containing the radical has a supplement. The *srs*-modules lie between radical supplemented modules and supplemented modules. Some examples are provided to show that these inclusions are proper.

In this paper, among other results, we prove that the *srs*-modules are closed under factor modules and finite sums. Every left R-module is an *srs*-module if and only if R is left perfect. For modules with small radical, the notions of supplemented module and *srs*-module coincide. This implies that every finitely generated R-module is an *srs*-module if and only if R is semiperfect. Over a commutative nonlocal domain, we prove that every reduced *srs*-module M is of the form $M = T(M) + \operatorname{Rad} M$, where $T(M)$ is the torsion submodule of M. A commutative domain is h-local if and only if every finitely generated torsion module is an *srs*-module. Over a local Dedekind domain (i.e., over a DVR), a module is an *srs*-module if and only if it is radical supplemented. Over a nonlocal Dedekind domain, an *srs*-module M is of the form $M = T(M) + \operatorname{Rad} M$.

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2. Strongly Radical Supplemented Modules

First, we show some properties of *srs*-modules.

Proposition 2.1. *Every homomorphic image of an srs-module is an srs-module.*

Proof. Let $L \subseteq N \subseteq M$ and $\operatorname{Rad}(M/L) \subseteq N/L$. Since $(\operatorname{Rad} M + L)/L \subseteq \operatorname{Rad}(M/L)$, we have $\operatorname{Rad} M \subseteq N$. By assumption, N has a supplement, say K, in M. Then, according to [1] (41.1(7)), $(K + L)/L$ is a supplement of N/L in M/L. Hence, M/L is an *srs*-module.

Proposition 2.2. *If M is an srs-module, then $M/\operatorname{Rad} M$ is semisimple.*

Proof. By Proposition 2.1, $M/\operatorname{Rad} M$ is an *srs*-module. We have $\operatorname{Rad}(M/\operatorname{Rad} M) = 0$, and, therefore, $M/\operatorname{Rad} M$ is supplemented. According to [1] (41.2(3)), $M/\operatorname{Rad} M$ is semisimple.

To prove that the finite sum of *srs*-modules is an *srs*-module, we use the following standard lemma (see [1] (41.2)):

Lemma 2.1. *Let M be an R-module and let M_1 and N be submodules of M with $\operatorname{Rad} M \subseteq N$. If M_1 is an srs-module and $M_1 + N$ has a supplement in M, then N has a supplement in M.*

Proof. Let L be a supplement of $M_1 + N$ in M. Since $\operatorname{Rad} M_1 \subseteq \operatorname{Rad} M \subseteq N$, we have $\operatorname{Rad} M_1 \subseteq (L+N) \cap M_1$. Then $(L+N) \cap M_1$ has a supplement, say K, in M_1 because M_1 is an *srs*-module. Therefore,

$$M = M_1 + N + L = K + [(L + N) \cap M_1] + N + L = (K + N) + L.$$

Since $N + K \subseteq N + M_1$, we conclude that L is also a supplement of $N + K$ in M. Then, according to [4] (Lemma 1.3a), $K + L$ is a supplement of N in M.

Proposition 2.3. *Let $M = M_1 + M_2$, where M_1 and M_2 are srs-modules. Then M is an srs-module.*

Proof. Suppose that $N \subseteq M$ with $\operatorname{Rad} M \subseteq N$. Clearly, $M_1 + M_2 + N$ has the trivial supplement 0 in M, and so, by Lemma 2.1, $M_1 + N$ has a supplement in M. Applying the lemma once again, we obtain a supplement for N in M.

Corollary 2.1. *Every finite sum of srs-modules is an srs-module.*

Lemma 2.2. *Let M be a module with $\operatorname{Rad} M = M$. Then M is an srs-module.*

Proof. Clearly, M has the trivial supplement 0 in M. Since $M = \operatorname{Rad} M$ is the unique submodule containing the radical, we conclude that M is an *srs*-module.

Let M be an R-module. By $P(M)$ we denote the sum of all submodules V of M such that $\operatorname{Rad} V = V$.

Corollary 2.2. *Let M be an R-module. Then $P(M)$ is an srs-module.*

Proof. For any module M, we have $\operatorname{Rad} P(M) = P(M)$. Then, by Lemma 2.2, $P(M)$ is an *srs*-module.

The example below shows that *srs*-modules need not be supplemented.

Example 2.1. Consider the $\mathbb{Z}$-module $M =_{\mathbb{Z}} \mathbb{Q}$. Then M is an *srs*-module because $\operatorname{Rad} \mathbb{Q} = \mathbb{Q}$. On the other hand, M is not supplemented by virtue of [4] (Theorem 3.1).

Proposition 2.4. *Let M be an R-module with* $\operatorname{Rad} M \ll M$. *In this case, M is supplemented if and only if M is an srs-module.*

Proof. In one direction, the statement is obvious. Suppose that M is an *srs*-module. Let N be a submodule of M. Then $N + \operatorname{Rad} M$ has a supplement, say L, in M. Hence,

$$N + \operatorname{Rad} M + L = M \quad \text{and} \quad (N + \operatorname{Rad} M) \cap L \ll L.$$

Since $\operatorname{Rad} M \ll M$, we have

$$N + L = M \quad \text{and} \quad N \cap L \subseteq (N + \operatorname{Rad} M) \cap L \ll L,$$

i.e., $N \cap L \ll L$. Hence, N has a supplement L in M. Thus, M is supplemented.

In [6], a ring R is called left max if every nonzero R-module has a maximal submodule. It is well known that R is a left max ring if and only if $\operatorname{Rad} M \ll M$ for every nonzero left R-module M. By using Proposition 2.4, we obtain the following corollary:

Corollary 2.3. *Every srs-module over a left max ring is supplemented.*

Proposition 2.5. *Let M be an R-module. Suppose that* $\operatorname{Rad} M$ *is supplemented and M is an srs-module. Then M is supplemented.*

Proof. Let N be a submodule of M. By assumption, $\operatorname{Rad} M + N$ has a supplement in M. Since $\operatorname{Rad} M$ is supplemented, N has a supplement in M by virtue of [1] (41.2). Hence, M is supplemented.

A submodule $U \subseteq M$ is said to be *cofinite* if M/U is finitely generated. In [5], M is called *cofinitely supplemented* if every cofinite submodule of M has a supplement in M. It is also shown that M is cofinitely supplemented if and only if every maximal submodule of M has a supplement in M (see [5], Theorem 2.8). Since $\operatorname{Rad} M$ is contained in every maximal submodule of M, every *srs*-module is cofinitely supplemented. But the converse need not be true in general, as is shown in the example presented below.

First, we need the following lemma:

Lemma 2.3. *Let M be an R-module and let $U, V \subseteq M$. If V is a supplement of U in M and* $\operatorname{Rad} V \subseteq U$, *then* $\operatorname{Rad} V \ll V$.

Proof. Suppose that $\operatorname{Rad} V + T = V$ for some $T \subseteq V$. Then

$$M = U + V = U + \operatorname{Rad} V + T = U + T.$$

Since V is a supplement and $T \subseteq V$, we have $T = V$. Hence, $\operatorname{Rad} V \ll V$.

Example 2.2. Let $\mathbb{Z}$ be the ring of integers and let p be a prime in $\mathbb{Z}$. Consider the $\mathbb{Z}$-module $M = \bigoplus_{n \geq 1} \mathbb{Z}_{p^n}$, where $\mathbb{Z}_{p^n} = \mathbb{Z}/p^n\mathbb{Z}$. Then M is a torsion module, and it is cofinitely supplemented by virtue of [5] (Corollary 4.7). To see that M is not an *srs*-module, consider the submodule pM of M. Since M/pM is a semisimple module, we have $\operatorname{Rad} M \subseteq pM$. We prove that pM does not have a supplement in M. Assume that pM has a supplement, say N, in M. Then $\operatorname{Rad} N \ll N$ by Lemma 2.3. Since every element of M is annihilated by some power of p, the module M can now be considered as a module over the local ring $\mathbb{Z}_{(p)}$. Then N is a bounded module by virtue of [5] (Lemma 2.1). Therefore, $p^n N = 0$ for some $n \geq 1$. On the other hand, since N is a supplement of pM, we have $M = pM + N$, and so $p^n M = p^{n+1}M + p^n N = p^{n+1}M$. Therefore, $p^n M$ is a divisible module by virtue of [5] (Lemma 4.4). However, M does not have a nonzero divisible submodule. Hence, $p^n M = 0$, a contradiction. Therefore, pM does not have a supplement in M, i.e., M is not an *srs*-module.

Proposition 2.6. *Let R be an arbitrary ring and let M be an R-module. Suppose that $M/\operatorname{Rad} M$ is finitely generated. In this case, M is cofinitely supplemented if and only if it is an srs-module.*

Proof. Let M be an R-module and let N be a submodule of M with $\operatorname{Rad} M \subseteq N$. Note that

$$[M/\operatorname{Rad} M]/[N/\operatorname{Rad} M] \cong M/N$$

is finitely generated, and, thus, N is a cofinite submodule of M. Since M is cofinitely supplemented, N has a supplement in M. Therefore M is an *srs*-module. The converse is obvious.

We now have the following implications on modules:

$$\text{supplemented} \Longrightarrow \textit{srs}\text{-module} \Longrightarrow \text{cofinitely supplemented}.$$

Proposition 2.7. *Let M be an R-module and let $\operatorname{Rad} M \subseteq U \subseteq M$. If V is a supplement of U in M, then $\operatorname{Rad} V \ll V$.*

Proof. Since $\operatorname{Rad} M \subseteq U$, we have $\operatorname{Rad} V \subseteq U$. Then $\operatorname{Rad} V \ll V$ by Lemma 2.3.

Recall from [6] that a submodule L of a module M is called a Rad-*supplement* of a submodule N of M in M if $N + L = M$ and $N \cap L \subseteq \operatorname{Rad} L$. Clearly, every supplement submodule is a Rad-supplement.

Corollary 2.4. *Let M be an R-module and let $N \subseteq M$ be such that $\operatorname{Rad} M \subseteq N$. Suppose that $N + L = M$ for some $L \subseteq M$. In this case, L is a supplement of N in M if and only if L is a* Rad-*supplement of N and* $\operatorname{Rad} L \ll L$.

In the proposition below, we characterize supplements of the radical of a module over semilocal rings.

Proposition 2.8. *Let R be a semilocal ring and let M be an R-module. A submodule $N \subseteq M$ is a supplement of $\operatorname{Rad} M$ in M if and only if N is coatomic, M/N does not have maximal submodules, and* $\operatorname{Rad} N = N \cap \operatorname{Rad} M$.

Proof. ($\Rightarrow$) Let N be a supplement of $\operatorname{Rad} M$ in M. Then, according to [1] (41.1(5)), we have $\operatorname{Rad} N = N \cap \operatorname{Rad} M$. If $N = M$, then, clearly, $\operatorname{Rad} M \ll M$. Since R is semilocal, $M/\operatorname{Rad} M$ is semisimple. Therefore, every proper submodule of M is contained in a maximal submodule, i.e., M is coatomic. Assume that N is a proper submodule of M. If K is a maximal submodule of M with $N \subseteq K$, then $M = \operatorname{Rad} M + N \subseteq K$, a contradiction. Therefore, N is not contained in any maximal submodule of M, i.e., M/N does not have maximal submodules. By Proposition 2.7, we have $\operatorname{Rad} N \ll N$. Since $N/\operatorname{Rad} N$ is semisimple, N is coatomic.

($\Leftarrow$) Suppose that $N + \operatorname{Rad} M \neq M$. Then $(N + \operatorname{Rad} M)/\operatorname{Rad} M \subsetneqq M/\operatorname{Rad} M$. Since R is semilocal, we conclude that $M/\operatorname{Rad} M$ is semisimple, and so there exists a maximal submodule $K/\operatorname{Rad} M$ of $M/\operatorname{Rad} M$ such that $(N + \operatorname{Rad} M)/\operatorname{Rad} M \subseteq K/\operatorname{Rad} M$. Hence, $N + \operatorname{Rad} M \subseteq K$, which implies that $N \subseteq K$. Therefore, K/N is a maximal submodule of M/N, a contradiction. Consequently. $N + \operatorname{Rad} M = M$. By assumption, $N \cap \operatorname{Rad} M = \operatorname{Rad} N \ll N$. Hence, N is a supplement of $\operatorname{Rad} M$ in M.

We now characterize the rings over which all (finitely generated) modules are *srs*-modules.

Corollary 2.5. *For a ring R, the following statements are equivalent:*

(1) R is semiperfect;

(2) ${}_RR$ is an srs-module;

(3) every finitely generated left R-module is an srs-module.

Proof. For every finitely generated module M, we have $\operatorname{Rad} M \ll M$. On the other hand, according to [1] (42.6), R is semiperfect if and only if every finitely generated R-module is supplemented. In view of this fact and Proposition 2.4, the implications $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ are obvious.

Corollary 2.6. *For a ring R, the following statements are equivalent:*

(1) R is left perfect;

(2) the left R-module $R^{(\mathbb{N})}$ is an srs-module;

(3) every left R-module is an srs-module.

Proof. The implications $(1) \Rightarrow (3)$ and $(3) \Rightarrow (2)$ are obvious.

$(2) \Rightarrow (1)$. According to Proposition 2.1, ${}_RR$ is an *srs*-module. Hence, R is semilocal by virtue of Proposition 2.2. Since $R^{(\mathbb{N})}$ is an *srs*-module, $\operatorname{Rad} R^{(\mathbb{N})}$ has a (weak) supplement in $R^{(\mathbb{N})}$. Therefore, R is left perfect by virtue of [7] (Theorem 1).

The statement below is a slight modification of Lemma 1.3 (Folgerung) in [4].

Proposition 2.9. *Let M be an R-module and let K be a submodule of M. If K and M/K are srs-modules and K has a supplement L in P for every submodule P with $K \subseteq P \subseteq M$, then M is an srs-module.*

Proof. Let N be a submodule of M with $\operatorname{Rad} M \subseteq N$. It follows from [4] (Lemma 1.1(d)) that we can write

$$\operatorname{Rad}(M/K) = (\operatorname{Rad} M + K)/K \subseteq (N + K)/K.$$

Since M/K is an *srs*-module, $(N+K)/K$ has a supplement in M/K. This means that there exists a submodule V/K of M/K such that $(N+K)/K + V/K = M/K$ and $[(N+K)/K] \cap [V/K] \ll V/K$. Since $K \subseteq V$, we conclude that K has a supplement in V. Therefore, $V = K + L$ and $K \cap L \ll L$ for some $L \subseteq V$. We now have

$$M = N + V = N + (K + L) = (N + K) + L.$$

Suppose that $M = (N+K) + L'$ for some $L' \subseteq L$. Then $M/K = (N+K)/K + (L'+K)/K$. However, V/K is a supplement of $(N+K)/K$ in M/K and $(L'+K)/K \subseteq V/K$. By virtue of the minimality of V/K, we obtain $(L'+K)/K = V/K$. Then $V = L' + K$. Since L is a supplement of K in V, we have $L' = L$. Therefore, L is a supplement of $N + K$ in M. By virtue of Lemma 2.1, N has a supplement in M. Hence, M is an *srs*-module.

The corollary below is a direct consequence of Proposition 2.9.

Corollary 2.7. *Let M be an R-module that contains an Artinian submodule K. In this case, M is an srs-module if and only if M/K is an srs-module.*

Proof. In one direction, the statement follows from Proposition 2.1. Conversely, suppose that M/K is an *srs*-module. By assumption, K is supplemented, and so it is an *srs*-module. It follows from [3] that K has a supplement in every P with $K \subseteq P \subseteq M$. Therefore, M is an *srs*-module by Proposition 2.9.

3. *srs*-Modules over Dedekind Domains

Throughout this section, unless otherwise stated, we consider commutative rings. The result below is due to Zöschinger.

Lemma 3.1 [3] (Satz 3.1)**.** *For a module over a discrete valuation ring (DVR), the following statements are equivalent:*

(1) M is radical supplemented;

(2) $M = T(M) \oplus X$, where the reduced part of $T(M)$ is bounded and $X/\operatorname{Rad} X$ is finitely generated.

We now prove that radical supplemented modules and *srs*-modules coincide over discrete valuation rings. First, we need the following lemma:

Lemma 3.2. *Let R be a local ring and let M be an R-module. If $M/\operatorname{Rad} M$ is finitely generated, then M is an srs-module.*

Proof. Let N be a submodule of M such that $\operatorname{Rad} M \subseteq N$. Then M/N is finitely generated, and so $M = N + L$ for some finitely generated submodule L of M. Since ${}_R R$ is supplemented, L is also supplemented because it is finitely generated. Thus, N has a supplement in M by Lemma 2.1.

Proposition 3.1. *Let R be a DVR and let M be an R-module. In this case, M is an srs-module if and only if M is radical supplemented.*

Proof. In one direction, the statement is clear. Suppose that M is radical supplemented. Then $M = T(M) \oplus X$ as in Lemma 3.1. Since $T(M)$ is bounded, it is supplemented by virtue of [4] (Theorem 2.4). According to Lemma 3.2, X is an *srs*-module. Therefore, M is an *srs*-module by Corollary 2.1.

Note that, according to Example 2.2, Proposition 3.1 is not true in general for modules over Dedekind domains that are not DVR.

Proposition 3.2. *Let R be a nonlocal domain and let M be a reduced R-module. If M is an srs-module, then $M = T(M) + \operatorname{Rad} M$.*

Proof. Suppose that $T(M) + \operatorname{Rad} M \neq M$. Since $\operatorname{Rad} M \subseteq T(M) + \operatorname{Rad} M$, we conclude that $T(M) + \operatorname{Rad} M$ has a supplement, say L, in M. Then L has a maximal submodule K because M is reduced. Let $K' = T(M) + \operatorname{Rad} M + K$. It is easy to see that K' is a maximal submodule of M. Then K' has a supplement V in M. According to [1] (41.1(3)), V is local, and so $V \cong R/I$ for some nonzero $I \subseteq R$. Therefore, V is a torsion one, and so $V \subseteq T(M)$. We get

$$M = K' + V = T(M) + \operatorname{Rad} M + K + V = T(M) + \operatorname{Rad} M + K = K',$$

a contradiction. Hence, $M = T(M) + \operatorname{Rad} M$.

We now prove that the converse of Proposition 3.2 is true under a certain condition.

Proposition 3.3. *Let R be a domain and let M be an R-module. Suppose that $M = T(M) + \operatorname{Rad} M$ and $T(M)$ is supplemented. Then M is an srs-module.*

Proof. Let N be a submodule of M such that $\operatorname{Rad} M \subseteq N$. Then

$$N = N \cap T(M) + \operatorname{Rad} M = T(N) + \operatorname{Rad} M.$$

Let L be a supplement of $T(N)$ in $T(M)$. Then $T(N) + L = T(M)$ and $T(N) \cap L \ll L$. Hence,

$$M = T(M) + \operatorname{Rad} M = T(N) + L + \operatorname{Rad} M \subseteq N + L,$$

and so $M = N + L$. Since L is a torsion one, we have $N \cap L = T(N) \cap L$. Therefore, L is a supplement of N in M.

Let R be a Dedekind domain and let M be an R-module. Since R is a Dedekind domain, $P(M)$ is the *divisible part* of M. According to [5] (Lemma 4.4), $P(M)$ is (divisible) injective, and so there exists a submodule N of M such that $M = P(M) \oplus N$. Here, N is called the *reduced part* of M. Note that $P(M) \subseteq \operatorname{Rad} M$. By Corollary 2.2, we know that $P(M)$ is an *srs*-module. Using these facts, we obtain the following result:

Proposition 3.4. *Let R be a Dedekind domain and let M be an R-module. In this case, M is an srs-module if and only if the reduced part N of M is an srs-module.*

Proof. According to Proposition 2.1, N is an *srs*-module as a homomorphic image of M. The converse follows from Proposition 2.3.

Proposition 3.5. *Let R be a nonlocal Dedekind domain and let M be an srs-module. Then $M = T(M) + \operatorname{Rad} M$.*

Proof. Let $M = P(M) \oplus N$ with N reduced. Then N is an *srs*-module as a direct summand of M. By Proposition 3.2, we have $N = T(N) + \operatorname{Rad} N$. Therefore,

$$M = P(M) \oplus N = P(M) + T(N) + \operatorname{Rad} N \subseteq T(M) + \operatorname{Rad} M.$$

Hence, $M = T(M) + \operatorname{Rad} M$.

Recall from [5] that a commutative domain R is called *h-local* if every nonzero nonunit of R belongs to only finitely many maximal ideals and R/P is a local ring for every prime ideal P of R. It is also proved that a commutative domain R is h-local if and only if R/I is a semiperfect ring for every nonzero ideal I of R (see [5], Lemma 4.5). It is proved in [5] that R is h-local if and only if every finitely generated torsion R-module is supplemented. Since, for finitely generated modules, supplemented modules and *srs*-modules coincide, we obtain the following statement:

Proposition 3.6. *Let R be a commutative domain. In this case, R is h-local if and only if every finitely generated torsion R-module is an srs-module.*

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