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Cofinitely Supplemented Modular Lattices

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Abstract In this paper it is shown that a lattice L is a cofinitely supplemented lattice if and only if every maximal element of L has a supplement in L. If $a/0$ is a cofinitely supplemented sublattice and $1/a$ has no maximal element, then L is cofinitely supplemented. A lattice L is amply cofinitely supplemented if and only if every maximal element of L has ample supplements in L if and only if for every cofinite element a and an element b of L with $a \vee b = 1$ there exists an element c of $b/0$ such that $a \vee c = 1$ where c is the join of finite number of local elements of $b/0$. In particular, a compact lattice L is amply supplemented if and only if every maximal element of L has ample supplements in L.

Keywords Cofinite element · Ample supplement · Amply supplemented lattice · Cofinitely supplemented lattice · Amply cofinitely supplemented lattice

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الملخص

نبين في هذه الورقة البحثية أن الشبكة L تكون مستكملة بشكل منته مرافق إذا وفقط إذا كان لكل عنصر أعظمي في L تكملة في L. إذا كانت $a/0$ شبكة جزئية مستكملة بشكل منته مرافق ولم يكن لـِ $1/a$ عناصر أعظمية، فإن L تكون مستكملة بشكل منته مرافق. تكون الشبكة L مستكملة بشكل واسع بشكل منته مرافق إذا وفقط إذا كانت لكل عنصر أعظمي في L تكملة واسعة في L إذا وفقط إذا وجد لكل عنصر منته مرافق a وعنصر b في L يحقق $a \vee b = 1$ عنصر c في $b/0$ يحقق $a \vee c = 1$ حيث c وصل لعدد منته من العناصر المحلية في $b/0$. بشكل خاص، تكون الشبكة المتراصة L مستكملة بشكل واسع إذا وفقط إذا كانت لكل عنصر أعظمي في L تكملة واسعة في L.

1 Introduction

Throughout L denotes an arbitrary complete modular lattice with smallest element 0 and greatest element 1. A sublattice of the form $b/a = \{x \in L \mid a \leq x \leq b\}$ is called a *quotient sublattice* [3]. An element a of a

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lattice L is said to be *small* if $a \vee b \neq 1$ holds for every $b \neq 1$. It is denoted by $a \ll L$. We will write $a < b$ if $a \leq b$ and $a \neq b$. We have the following properties of small elements:

Lemma 1.1 [3, Lemmas 7.2, 7.3 and 12.4] *Let $a < b$ be elements in a lattice L.*

(1) *If $a \ll b/0$, then $a \vee c \ll (b \vee c)/c$ for every $c \in L$.*
(2) *$b \ll L$ if and only if $a \ll L$ and $b \ll 1/a$.*
(3) *Let $c' \ll c/0$ and $d' \ll d/0$. Then $c' \vee d' \ll (c \vee d)/0$.*

An element a of L is called a *supplement* of an element b if $a \vee b = 1$ and a is minimal with respect to this property. Equivalently, an element a is a supplement of b in L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$ [3, Proposition 12.1]. A lattice L is said to be *supplemented* if every element a of L has a supplement in L. An element a of a lattice L is said to be *cofinite* in L if the quotient sublattice $1/a$ is compact, that is $1 = \bigvee_{i \in I} x_i$ for some elements $x_i \geq a$ implies that $1 = \bigvee_{i \in F} x_i$ for some finite subset F of I. A lattice L is said to be *cofinitely supplemented* if every cofinite element of L has a supplement in L. In Section 2 we prove that a lattice L is cofinitely supplemented if and only if every maximal element of L has a supplement in L. Using this we show that if a lattice L is an arbitrary join of cofinitely supplemented elements a_i with $a_i/0$ cofinitely supplemented, then L is cofinitely supplemented. Also we prove that L is cofinitely supplemented if $a/0$ is a cofinitely supplemented sublattice and $1/a$ has no maximal elements.

An element a of a lattice L has *ample supplements* in L if for every element b of L with $a \vee b = 1$, $b/0$ contains a supplement of a in L. A lattice L is said to be *amply supplemented* if every element a of L has ample supplements in L. In Section 3 we generalize some properties of amply supplemented modules to amply supplemented lattices. Also we study *amply cofinitely supplemented* lattices, that is lattices whose cofinite elements have ample supplements. A lattice L is said to be *local* if the set of elements different from 1 has a largest element. An element l is called a *local element* if the quotient sublattice $l/0$ is local. We show in Theorem 3.9 that a lattice L is amply cofinitely supplemented if and only if every maximal element of L has ample supplements in L. Moreover in this situation for every cofinite element a and an element b of L with $a \vee b = 1$ there exists an element c of $b/0$ such that $a \vee c = 1$ where c is the join of finite number of local elements of $b/0$. In particular, a compact lattice L is amply supplemented if and only if every maximal element of L has ample supplements in L.

We give proofs of the results for lattices when the proofs are different from those in the module case. All definitions and related properties not given here can be found in [3,4].

2 Cofinitely Supplemented Lattices

An element c of L is said to be *compact*, if for every subset $X = \{x_i \mid i \in I\}$ of L with $c \leq \bigvee_{i \in I} x_i$ there exists a finite subset F of I such that $c \leq \bigvee_{i \in F} x_i$. A lattice L is said to be *compact* if 1 is compact and *compactly generated* (or *algebraic*) if each of its elements is a join of compact elements [6]. For compactly generated compact lattices a supplement of an element is compact [3, Proposition 12.2 (2)]. In the following proposition we show that for an arbitrary lattice L a supplement of a cofinite element is compact:

Proposition 2.1 *Let a be a cofinite element of a lattice L and b be a supplement of a. Then $b/0$ is compact.*

Proof Since b is a supplement of a in L, $a \vee b = 1$ and b is minimal with respect to this property. Let $b = \bigvee_{i \in I} b_i$ for some $b_i \leq b$.

$$1 = a \vee b = a \vee \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \vee b_i).$$

Since a is cofinite, $1/a$ is compact, there exists a finite subset F of I such that

$$1 = \bigvee_{i \in F} (a \vee b_i) = a \vee \left(\bigvee_{i \in F} b_i \right).$$

Then $b = \bigvee_{i \in F} b_i$ by minimality of b. So b is compact. □

Recall that a lattice L is said to be *local* if the set of elements different from 1 has a largest element.

Lemma 2.2 [1, Lemma 2.9] *Let* $\{l_i/0\}_{i\in I}$, $I = \{1, \ldots, n\}$ *be a finite collection of local sublattices of a lattice* L *and* a *be an element of* L *such that* $a \vee (\bigvee_{i\in I} l_i)$ *has a supplement* b *in* L. *Then there exists a subset* J *of* I *such that* $b \vee (\bigvee_{i\in J} l_i)$ *is a supplement of* a *in* L.

Proof Induction on n. For $n = 1$, b is a supplement of $a \vee l_1$, i.e. $b \vee (a \vee l_1) = 1$ and $b \wedge (a \vee l_1) \ll b/0$. Put $c = (a \vee b) \wedge l_1$. If $c = l_1$, then $l_1 \leq a \vee b$. So $1 = b \vee (a \vee l_1) = a \vee b$ and $b \wedge a \leq b \wedge (a \vee l_1) \ll b/0$. Thus b is a supplement of a in L. If $c \neq l_1$, then $(a \vee b) \wedge l_1 = c \ll l_1/0$. Therefore, l_1 is a supplement of c in $l_1/0$. By [3, Lemma 12.3] and Lemma 1.1 (3) the following holds:

$$a \wedge (b \vee l_1) \leq [b \wedge (a \vee l_1)] \vee [l_1 \wedge (a \vee b)] \ll (b \vee l_1)/0.$$

So $b \vee l_1$ is a supplement of a. Suppose that $n > 1$ and b is a supplement of $a' \vee (\bigvee_{i=2}^{n} l_i)$ in L where $a' = a \vee l_1$. By the induction hypothesis there is a subset I' of $\{2, \ldots, n\}$ such that $b' = b \vee (\bigvee_{i\in I'} l_i)$ is a supplement of $a' = a \vee l_1$, by the case $n = 1$ either b' or $b' \vee l_1$ is a supplement of a in L. This completes the proof. □

Lemma 2.3 [3, Lemma 12.5 (b)] *In a lattice* L *let* m *be a maximal element. If* l *is a supplement of* m, *then* $l/0$ *is local. Moreover,* $l \wedge m$ *is the largest element of* $l/0$ *different from* l.

Proof l is a supplement of m if and only if $l \vee m = 1$ and $l \wedge m \ll l/0$. Let $x \in l/0$ and $x \neq l$. If $x \leq m$, then $x \leq l \wedge m$. If $x \not\leq m$ ($x \not\leq l \wedge m$), then since m is maximal $x \vee m = 1$.

$$l = l \wedge 1 = l \wedge (x \vee m) = x \vee (l \wedge m).$$

Since $l \wedge m \ll l/0$, $x = l$. This is a contradiction. Thus $l \wedge m$ is the largest element ($\neq l$) of $l/0$. □

Theorem 2.4 [5, Theorem 5.3.33] *A lattice* L *is a cofinitely supplemented lattice if and only if every maximal element of* L *has a supplement in* L.

Proof (⇒) Let m be a maximal element of L. Then there are only two elements of $1/m$: $1, m$. So m is cofinite. Since L is cofinitely supplemented, m has a supplement in L.

(⇐) Let the join of local elements of L be denoted by $\mathrm{Loc}(L)$. Let m be a maximal element in $1/\mathrm{Loc}(L)$. Then m is a maximal element of L. By assumption m has a supplement b in L. By Lemma 2.3, $b/0$ is a local sublattice; therefore, b is a local element. Then $b \leq \mathrm{Loc}(L) \leq m$ and so $1 = m \vee b = m$. This is a contradiction. So there is no maximal element in $1/\mathrm{Loc}(L)$. Let a be a cofinite element of L. Then $a \vee \mathrm{Loc}(L)$ is cofinite in L. Since there is no maximal element in $1/\mathrm{Loc}(L)$, $1/(a \vee \mathrm{Loc}(L))$ has no maximal element, but by [3, Lemma 2.4], if $a \vee \mathrm{Loc}(L) \neq 1$, then $1/(a \vee \mathrm{Loc}(L))$ has at least one maximal element ($\neq 1$). So $a \vee \mathrm{Loc}(L) = 1$. Since $1/a$ is compact for some local elements $l_1, \ldots, l_n$ of L

$$a \vee (l_1 \vee \cdots \vee l_n) = 1.$$

0 is a supplement of $a \vee (l_1 \vee \cdots \vee l_n) = 1$ in L. Thus by Lemma 2.2, a has a supplement in L. □

Using Theorem 2.4 we prove that if a lattice L is an arbitrary join of cofinitely supplemented principal ideals, then L is cofinitely supplemented.

Theorem 2.5 *Let* $\{a_i/0\}_{i\in I}$ *be a collection of cofinitely supplemented sublattices of* L *with* $1 = \bigvee_{i\in I} a_i$. *Then* L *is a cofinitely supplemented lattice.*

Proof Let m be any maximal element of L. If $a_i \leq m$ for all $i \in I$, then $1 = \bigvee_{i\in I} a_i \leq m$ which is a contradiction. So there exists a $j \in I$ such that $a_j \not\leq m$. Then $1 = a_j \vee m$. Since $a_j/(a_j \wedge m) \cong (a_j \vee m)/m = 1/m$, the element $a_j \wedge m$ is maximal in $a_j/0$. By hypothesis there is a supplement c of $a_j \wedge m$ in $a_j/0$, i.e. $(a_j \wedge m) \vee c = a_j$ and $a_j \wedge m \wedge c \ll c/0$. If $c \leq m$, then $a_j = (a_j \wedge m) \vee c \leq m$, a contradiction. So $c \not\leq m$. Therefore, $1 = m \vee c$ and $m \wedge c = a_j \wedge m \wedge c \ll c/0$. Thus c is a supplement of m in L. By Theorem 2.4, L is a cofinitely supplemented lattice. □

Theorem 2.4 is also used in the proof of the following theorem which gives a new result for modules:

Theorem 2.6 *If* $a/0$ *is a cofinitely supplemented sublattice of* L *and* $1/a$ *has no maximal element, then* L *is also a cofinitely supplemented lattice.*

Proof Let m be a maximal element of L. If $a \leq m$, then m is a maximal element of $1/a$, but $1/a$ has no maximal element. So $a \nleq m$; therefore $a \vee m = 1$ and $a/(a \wedge m) \cong (a \vee m)/m = 1/m$. Since m is a maximal element of L, $a \wedge m$ is a maximal and therefore a cofinite element of $a/0$. Then there is a supplement c of $a \wedge m$ in $a/0$, that is $(a \wedge m) \vee c = a$ and $(a \wedge m) \wedge c \ll c/0$. Since c is in $a/0$, $c \wedge m = c \wedge (a \wedge m) \ll c/0$. $c \vee m = c \vee (a \wedge m) \vee m = a \vee m = 1$. So c is a supplement of m in L. By Theorem 2.4, L is a cofinitely supplemented lattice. □

For a module K over a ring the *radical* Rad K of K is the intersection of all maximal submodules of K, so Rad $K = K$ if K has no maximal submodules.

Corollary 2.7 *Let M be a module, N be a cofinitely supplemented submodule of M. If* $\mathrm{Rad}(M/N) = M/N$, *then M is cofinitely supplemented.*

3 Amply Supplemented Lattices

A homomorphic image of a small element under a lattice morphism need not be small unlike the module case [2, Example 2.1]. Nevertheless, we will show that the quotient sublattices $1/a$ of an amply supplemented lattice is amply supplemented by using properties of small elements given in Lemma 1.1.

Proposition 3.1 *If a lattice L is amply supplemented, then for every element a of L the quotient sublattice $1/a$ is amply supplemented.*

Proof Let x be an element of $1/a$. If $x \vee y = 1$ for some $y \in 1/a$, then x has a supplement $y' \leq y$ in L because L is amply supplemented, i.e. $x \vee y' = 1$ and $x \wedge y' \ll y'/0$. Then $x \vee (y' \vee a) = 1$. By modular law, $x \wedge (y' \vee a) = a \vee (x \wedge y')$ and since $x \wedge y' \ll y'/0$, $a \vee (x \wedge y') \ll (y' \vee a)/a$ by Lemma 1.1 (1). So $y' \vee a$ is a supplement of x in $1/a$ with $y' \vee a \leq y$. □

Proofs of Proposition 3.2 and 3.4 are similar to the proofs of [7, 41.7(1)] and [7, 41.8].

Proposition 3.2 *If L is an amply supplemented lattice, then for every supplement a of an element of L, $a/0$ is amply supplemented.*

If $a \vee a' = 1$ and $a \wedge a' = 0$ for elements a and a' of L, then we use the notation $a \oplus a' = 1$ and call this a *direct sum*. In this case a and a' are called *direct summands* of 1.

Corollary 3.3 [7, 41.7(2)] *If L is amply supplemented, then for a direct summand a of L, the quotient sublattice $a/0$ is amply supplemented.*

Proposition 3.4 *Let $a, b \in L$ with $a \vee b = 1$. If a and b have ample supplements in L, then $a \wedge b$ has also ample supplements in L.*

Given elements $a \leq b$ of L, the inequality $a \leq b$ is called *cosmall* in L if $b \ll 1/a$. One can easily modify the proofs of [8, Proposition 2.1] and [4, 20.24] to prove the following proposition:

Proposition 3.5 *The following are equivalent for a lattice L:*

(*a*) *L is amply supplemented.*
(*b*) *Every element a of L is of the form $a = x \vee y$ with $x/0$ supplemented and $y \ll L$.*
(*c*) *For every element a of L, there is an element $x \leq a$ such that the quotient sublattice $x/0$ is supplemented with the inequality $x \leq a$ cosmall in L.*

Corollary 3.6 *If the quotient sublattice $a/0$ is supplemented for every element a of a lattice L, then L is amply supplemented.*

Proposition 3.7 *Let L be an amply cofinitely supplemented lattice. Then for every $a \in L$, $1/a$ is amply cofinitely supplemented.*

Proof Let b be a cofinite element of $1/a$. Then $1/b$ is compact. So b is a cofinite element of L. Suppose $b \vee c = 1$ for some $c \in 1/a$. Since L is amply cofinitely supplemented, $c/0$ contains a supplement x of b in L, i.e. $b \vee x = 1$ and $b \wedge x \ll x/0$. Then $b \vee (x \vee a) = 1 \vee a = 1$ and by Lemma 1.1 (1),

$$(a \vee x) \wedge b = (b \wedge x) \vee a \ll (x \vee a)/a,$$

i.e. $x \vee a$ is a supplement of b in $1/a$. Since $a \leq c$ and $x \leq c$, we have $a \vee x \leq c$. Hence, $1/a$ is amply cofinitely supplemented. □

Lemma 3.8 *If a is a cofinite element of a lattice L, then there exists a maximal element m of L such that $a \leq m$.*

Proof Since a is a cofinite element of L, by [3, Lemma 2.4] there is a maximal element m in $1/a$. So m is a maximal element of L containing a. □

The following theorem generalizes [1, Theorem 2.10] to lattices:

Theorem 3.9 *The following are equivalent for a lattice L:*

(*a*) *L is amply cofinitely supplemented.*
(*b*) *Every maximal element of L has ample supplements in L.*
(*c*) *For every cofinite element a and an element b of L with $a \vee b = 1$ there exists an element $c = \bigvee_{i \in F} l_i$ where F is a finite set and each l_i is a local element of $b/0$ such that $a \vee c = 1$.*

Proof $(a) \Rightarrow (b)$ Clear since every maximal element m is cofinite.
$(b) \Rightarrow (c)$ Let a be a cofinite element of L, $b \in L$ and $a \vee b = 1$. Let

$$C = \left\{ c \in b/0 \mid c = \bigvee_{i \in F} l_i, \ F \text{ is finite, } l_i \text{ is local} \right\}.$$

Suppose that $a \vee c \neq 1$ for every $c \in C$. Let Ω denote the collection of elements x of L such that $a \leq x$ and $x \vee c \neq 1$ for every $c \in C$. Let $\Gamma = \{x_\lambda \mid \lambda \in \Lambda\}$ be a chain in Ω and $x = \bigvee_{\lambda \in \Lambda} x_\lambda$. Since $\forall \lambda, \ x_\lambda \in \Omega, a \leq x_\lambda$. Then $a \leq \bigvee_{\lambda \in \Lambda} x_\lambda = x$. So x is an upper bound for Γ. Suppose that $(\bigvee_{\lambda \in \Lambda} x_\lambda) \vee c = 1$ for some $c \in C$. Since $1/a$ is compact, there exists a finite subset F of Λ such that $x = (\bigvee_{\lambda \in F} x_\lambda) \vee c = 1$. Then $\bigvee_{i \in F} x_\lambda = x_{\lambda_0}$ for some $\lambda_0 \in F$, so $x_{\lambda_0} \vee c = 1$. This is a contradiction. Thus $x = \bigvee_{\lambda \in \Lambda} x_\lambda \in \Omega$. By Zorn's Lemma Ω contains a maximal element $u \neq 1$. Since a is cofinite, $1/a$ is compact and since $a \leq u$, $1/u$ is compact, i.e. u is cofinite. By Lemma 3.8, there exists a maximal element m of L such that $u \leq m$ and $m \vee b = 1$. Since m has ample supplements, there exists $y \in b/0$ such that y is a supplement of m in L. By Lemma 2.3, $y/0$ is local. Since $y \nleq m$, $y \nleq u$. Therefore, $u \neq u \vee y$. By maximality of u, there exists an element $v \in C$ such that $1 = (u \vee y) \vee v$. Since $y \vee v \leq b$ and $y \vee v$ is a finite join of local elements, $1 = u \vee (y \vee v)$, therefore $u \notin \Omega$. This is a contradiction.
$(c) \Rightarrow (a)$ Suppose that a is a cofinite element and $a \vee b = 1$ for some $b \in L$. We want to show that a has ample supplements in $b/0$. By hypothesis there is an element $c = \bigvee_{i \in F} l_i$ where F is a finite set and each l_i is a local element of $b/0$ such that $a \vee c = 1$. Since 0 is a supplement of $1 = a \vee c$, there exists a subset $J \subseteq F$ such that $\bigvee_{i \in J} l_i$ is a supplement of a in $b/0$ by Lemma 2.2. □

Clearly, every element of a compact lattice is a cofinite element. So we have the following corollary:

Corollary 3.10 *A compact lattice L is amply supplemented if and only if every maximal element has ample supplements in L.*

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