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Constructing rational and multi-wave solutions to higher order NEEs via the Exp-function method

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In this paper, we present an application of some known generalizations of the Exp-function method to the fifth-order Burgers and to the seventh-order Korteweg de Vries equations for the first time. The two examples show that the Exp-function method can be an effective alternative tool for explicitly constructing rational and multi-wave solutions with arbitrary parameters to higher order nonlinear evolution equations. Being straightforward and concise, as pointed out previously, this procedure does not require the bilinear representation of the equation considered. Copyright © 2011 John Wiley & Sons, Ltd.

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1. Introduction

Nonlinear evolution equations (NEEs) play a crucial role in different branches of the applied sciences. Thus, the study of NEEs is a very active area of research. In the last four decades or so, developing powerful analytic methods for NEEs has become the focus of common concern. Nowadays, some elegant methods are available to tackle NEEs in a straightforward manner. For instance, homogeneous balance method [1], inverse scattering method [2], Hirota's bilinear method [3], Backlund transformation method [4], F-expansion method [5], symmetry method [6], sine-cosine method [7], tanh-coth method [8], first integral method [9], homotopy analysis method [10], homotopy perturbation method [11], (G′/G)-expansion method [12], variational iteration method [13], multi-Exp-function method [14], three-wave method [15] and so forth. Recently, He *et al.* [16] proposed three standard variational iteration algorithms for dealing with differential equations, fractional differential equations, integro-differential equations, fractal differential equations, fractional/fractal differential-difference equations, as well as differential-difference equations arising in applied sciences. However, there is no a single best method that can handle a specialized nonlinear problem.

Since its introduction in 2006, the Exp-function method [17] has gained much popularity because it helps one to obtain exact and explicit solutions for NEEs in a concise manner. This method has been applied to various kinds of nonlinear problems arising in the applied sciences, and lately more attention is paid to its adaptation, generalization, and extension; just to mention a few, multi-dimensional equations [18–20], differential-difference equations [21], coupled NEEs [22], NEEs with variable coefficients [23], stochastic equations [24], n-soliton solutions [25], rational solutions [26], double-wave solutions [27]. The Exp-function method assumes an ansatz, which is based on trying rational combinations of exponential functions, involving unknown parameters to be specified at the stage of solving the problem.

On the other hand, special types of analytic solutions have been important to understand chemical, biological and physical phenomena modeled by NEEs. Among the possible solutions to NEEs, certain special form solutions may depend only on a single combination of variables such as traveling wave variables. Besides, traveling waves of NEEs may be coupled with different frequencies and different velocities. Multi-wave solutions are crucial in the sense that they may sometimes be converted into a single wave of very high energy that propagates over large domains of space without dispersion. Therefore, an extremely destructive wave may be produced. The *tsunami* is an example for this kind of phenomena. Thus, searching exact solutions with multi-velocities and multi-frequencies for NEEs is an important research area in the applied sciences as well.

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As is well known, Hirota's method [3] can be used to find such solutions if these equations can be converted into a bilinear form. However, the bilinear forms may not exist or may not known. The application of the Exp-function method to higher order NEEs for finding multi-wave and rational solutions is still an interesting and important issue. In this paper, we show that such types of solutions can be constructed using the Exp-function method. The main advantage of our procedure is that the bilinear representation for the equation studied becomes redundant. We use two distinct equations to illustrate the effectiveness of the method. The paper is organized as follows: In the next section, we summarize the method to make the paper self-contained. In Sections 3 and 4, we analyze our problems. In Section 5, we provide a brief conclusion.

2. The Exp-function method and its generalizations

In this section, we initiate our study by briefly reviewing the procedure. Let us consider a nonlinear partial differential equation for a function u of two real variables, space x and time t;

$$P(u,u_t,u_x,u_{tt},u_{tx},u_{xx},\ldots)=0, \tag{1}$$

where P is a polynomial in its arguments and subscripts denote partial derivatives. The standard Exp-function method is based on the assumption that a one-wave solution of Equation (1) can be expressed in the form

$$u(x,t)=\frac{\sum_{i=0}^{m} a_i \exp(\mathrm{i}\xi)}{\sum_{j=0}^{n} b_j \exp(\mathrm{j}\xi)},\quad \xi=kx+wt+\delta, \tag{2}$$

where m and n are positive integers to be determined by balancing the highest order terms in Equation (1); a_i, b_i, k and w are arbitrary constants to be specified at the stage of solving Equation (1); δ is the phase shift. To seek for rational and multi-wave solutions to Equation (1), the ansatz (2) can be modified as follows:

For a two-wave solution, one sets

$$u(x,t)=\frac{\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2} a_{i_1 i_2}\exp(i_1\xi_1+i_2\xi_2)}{\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2} b_{j_1 j_2}\exp(j_1\xi_1+j_2\xi_2)},\quad \xi_l=k_l x+w_l t+\delta_l,\quad l=1,2. \tag{3}$$

For a three-wave solution, one considers

$$u(x,t)=\frac{\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}\sum_{i_3=0}^{m_3} a_{i_1 i_2 i_3}\exp(i_1\xi_1+i_2\xi_2+i_3\xi_3)}{\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\sum_{j_3=0}^{n_3} b_{j_1 j_2 j_3}\exp(j_1\xi_1+j_2\xi_2+j_3\xi_3)},\quad \xi_l=k_l x+w_l t+\delta_l,\quad i=1,2,3, \tag{4}$$

and so forth. The ansatze (3) and (4) are due to Marinakis [25].

For a rational solution, one takes

$$u(x,t)=\frac{\sum_{i=0}^{m} a_i(\mu_1\exp(\xi)+\mu_2\xi)^i}{\sum_{j=0}^{n} b_j(\mu_1\exp(\xi)+\mu_2\xi)^j},\quad \xi=kx+wt+\delta, \tag{5}$$

where μ_1 and μ_2 are two embedded constants. The ansatz (5) is due to Zhang [26]. We note that when $\mu_1=1$ and $\mu_2=0$, the ansatz (5) agrees with the ansatz (2).

Finally, substituting the ansatze (2)–(5) into Equation (1) yields nonlinear algebraic systems for the unknown parameters. Solving each resultant system (if possible), one can determine one-wave, two-wave, three-wave, and rational solutions to Equation (1) provided they exist.

3. The fifth-order Burgers equation

First, let us consider the so-called fifth-order Burgers equation which reads

$$u_t+\alpha u_{xxxxx}+10\alpha(u_{xx})^2+15\alpha u_x u_{xxx}+5\alpha u u_{xxxx}+15\alpha(u_x)^3+50\alpha u u_x u_{xx}+10\alpha u^2 u_{xxx}+30\alpha u^2(u_x)^2+10\alpha u^3 u_{xx}+5\alpha u^4 u_x=0, \tag{6}$$

where α is a nonzero constant, and $u=u(x,t)$. As is well known, the study of integrable hierarchies is a significant and interesting topic in wave theory. Equation (6) appears to be a member of Burgers hierarchy in applications. For a one-wave solution, we assume that Equation (6) admits a solution of the form

$$u(x,t)=\frac{a_1\exp(\xi)}{1+b_1\exp(\xi)},\quad \xi=kx+wt+\delta, \tag{7}$$

which is embedded in (2). Substituting (7) into Equation (6) and solving the resultant algebraic system for the unknowns a_1, b_1, k, and w, we obtain the solution set

$$w=-\alpha k^5,\quad a_1=kb_1, \tag{8}$$

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which yields a one-wave solution to Equation (6) as

$$u(x,t)=\frac{kb_1\exp(kx-\alpha k^5t+\delta)}{1+b_1\exp(kx-\alpha k^5t+\delta)}, \tag{9}$$

where k, b_1, and δ remain arbitrary.

3.1. Two-wave solutions

Suppose that Equation (6) admits a solution of the form

$$u(x,t)=\frac{a_{10}\exp(\xi_1)+a_{01}\exp(\xi_2)+a_{11}\exp(\xi_1+\xi_2)}{1+b_{10}\exp(\xi_1)+b_{01}\exp(\xi_2)+b_{11}\exp(\xi_1+\xi_2)}, \quad \xi_l=k_lx+w_lt+\delta_l, \quad l=1,2. \tag{10}$$

It is obvious that the ansatz (10) is embedded in (3). Substituting (10) into Equation (6) and solving the resultant algebraic system for the unknowns a_{10}, a_{01}, a_{11}, b_{10}, b_{01}, b_{11}, k_1, k_2, w_1, and w_2, we get the solution set

$$w_1=-\alpha k_1^5, \quad w_2=-\alpha k_2^5, \quad b_{11}=0, \quad a_{11}=0, \quad a_{01}=k_2b_{01}, \quad a_{10}=k_1b_{10}, \tag{11}$$

which gives a two-wave solution to Equation (6) as

$$u(x,t)=\frac{k_1b_{10}\exp(k_1x-\alpha k_1^5t+\delta_1)+k_2b_{01}\exp(k_2x-\alpha k_2^5t+\delta_2)}{1+b_{10}\exp(k_1x-\alpha k_1^5t+\delta_1)+b_{01}\exp(k_2x-\alpha k_2^5t+\delta_2)}, \tag{12}$$

where b_{01}, b_{10}, k_1, k_2, δ_1, and δ_2 remain arbitrary

3.2. Three-wave solutions

Assume that Equation (6) admits a solution of the form

$$u(x,t)=\frac{\begin{array}{l}a_{100}\exp(\xi_1)+a_{010}\exp(\xi_2)+a_{001}\exp(\xi_2)+a_{110}\exp(\xi_1+\xi_2)+a_{101}\exp(\xi_1+\xi_3)\\ \quad +a_{011}\exp(\xi_2+\xi_3)+a_{111}\exp(\xi_1+\xi_2+\xi_3)\end{array}}{\begin{array}{l}1+b_{100}\exp(\xi_1)+b_{010}\exp(\xi_2)+b_{001}\exp(\xi_2)+b_{110}\exp(\xi_1+\xi_2)+b_{101}\exp(\xi_1+\xi_3)\\ \quad +b_{011}\exp(\xi_2+\xi_3)+b_{111}\exp(\xi_1+\xi_2+\xi_3)\end{array}}, \tag{13}$$

where $\xi_l=k_lx+w_lt+\delta_l$, $l=1,2,3$.

Clearly, the ansatz (13) is embedded in (4). After substituting (13) into Equation (6) and solving the resultant algebraic system for the unknowns a_{100}, a_{010}, a_{001}, a_{110}, a_{101}, a_{011}, a_{111}, b_{100}, b_{010}, b_{001}, b_{110}, b_{101}, b_{011}, b_{111}, k_1, k_2, k_3, w_1, w_2, and w_3, we get the solution set

$$\begin{array}{l}w_1=-\alpha k_1^5, \quad w_2=-\alpha k_2^5, \quad w_3=-\alpha k_3^5, \quad a_{100}=k_1b_{100}, \quad a_{010}=k_2b_{010}, \quad a_{001}=k_3b_{001},\\ b_{101}=0, \quad b_{011}=0, \quad a_{011}=0, \quad a_{101}=0, \quad b_{111}=0, \quad a_{110}=0, \quad b_{110}=0, \quad a_{111}=0,\end{array} \tag{14}$$

which leads a three-wave solution to Equation (6) as

$$u(x,t)=\frac{k_1b_{100}\exp(k_1x-\alpha k_1^5t+\delta_1)+k_2b_{010}\exp(k_2x-\alpha k_2^5t+\delta_2)+k_3b_{001}\exp(k_3x-\alpha k_3^5t+\delta_3)}{1+b_{100}\exp(k_1x-\alpha k_1^5t+\delta_1)+b_{010}\exp(k_2x-\alpha k_2^5t+\delta_2)+b_{001}\exp(k_3x-\alpha k_3^5t+\delta_3)}, \tag{15}$$

where b_{001}, b_{010}, b_{100}, k_1, k_2, k_3, δ_1, δ_2, and δ_3 remain arbitrary.

3.3. Rational solutions

Suppose that Equation (6) admits a solution of the form

$$u(x,t)=\frac{a_1(\mu_1\exp(\xi)+\mu_2\xi)+a_0+a_{-1}(\mu_1\exp(\xi)+\mu_2\xi)^{-1}}{b_1(\mu_1\exp(\xi)+\mu_2\xi)+b_0+b_{-1}(\mu_1\exp(\xi)+\mu_2\xi)^{-1}}, \quad \xi=kx+wt+\delta. \tag{16}$$

Proceeding as before, we obtain the solution set of the resultant algebraic system as

$$a_0=\frac{a_1b_0}{b_1}+kb_1, \quad a_{-1}=\frac{kb_0}{2}+\frac{a_1b_{-1}}{b_1}\mp\frac{k}{2}\sqrt{b_0^2-4b_{-1}b_1}, \quad w=-\frac{5k\alpha a_1^4}{b_1^4}, \quad \mu_1=0, \quad \mu_2=1 \tag{17}$$

which provide a rational solution to Equation (6) as

$$u^{\mp}(x,t)=\frac{a_1\left(kx-\frac{5k\alpha a_1^4}{b_1^4}t+\delta\right)^2+\left(\frac{a_1b_0}{b_1}+kb_1\right)\left(kx-\frac{5k\alpha a_1^4}{b_1^4}t+\delta\right)+\frac{kb_0}{2}+\frac{a_1b_{-1}}{b_1}\mp\frac{1}{2}k\sqrt{b_0^2-4b_{-1}b_1}}{b_1\left(kx-\frac{5k\alpha a_1^4}{b_1^4}t+\delta\right)^2+b_0\left(kx-\frac{5k\alpha a_1^4}{b_1^4}t+\delta\right)+b_{-1}}, \tag{18}$$

where a_1, b_{-1}, b_0, b_1, k, and δ remain arbitrary.

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4. The seventh-order Korteweg de Vries equation

Second, let us consider the so-called seventh-order Korteweg de Vries equation which reads

$$u_t+14uu_{xxxxx}+70u^2u_{xxx}+42u_xu_{xxxx}+70u_{xx}u_{xxx}+280uu_xu_{xx}+70(u_x)^3+140u^3u_x+u_{xxxxxxx}=0. \tag{19}$$

The significance of Equation (19) in applications is due to the fact that it is a member of the Korteweg de Vries hierarchy which is a well-known family of NEEs. For a one-wave solution, we assume that Equation (19) admits a solution of the form

$$u(x,t)=\frac{a_1\exp(\xi)}{(1+b_1\exp(\xi))^2},\quad \xi=kx+wt+\delta, \tag{20}$$

which is embedded in (2). Substituting (20) into Equation (19) and solving the resultant algebraic system for the unknowns a_1, b_1, k, and w, we obtain the solution set

$$w=-k^7,\quad a_1=2b_1k^2, \tag{21}$$

which gives rise a one-wave solution to Equation (19) as

$$u(x,t)=\frac{2b_1k^2\exp(kx-k^7t+\delta)}{(1+b_1\exp(kx-k^7t+\delta))^2}, \tag{22}$$

where k, b_1, and δ remain arbitrary.

4.1. Two-wave solutions

Suppose that Equation (19) admits a solution of the form

$$u(x,t)=\frac{a_{10}\exp(\xi_1)+a_{01}\exp(\xi_2)+a_{11}\exp(\xi_1+\xi_2)+a_{21}\exp(2\xi_1+\xi_2)+a_{12}\exp(\xi_1+2\xi_2)}{(1+b_{10}\exp(\xi_1)+b_{01}\exp(\xi_2)+b_{11}\exp(\xi_1+\xi_2))^2}, \tag{23}$$

where $\xi_l=k_lx+w_lt+\delta_l$, $l=1,2$.

It is notable that the ansatz (23) is embedded in (3). Substituting (23) into Equation (19) and solving the resultant algebraic system for the unknowns a_{10}, a_{01}, a_{11}, a_{21}, a_{12}, b_{10}, b_{01}, b_{11}, k_1, k_2, w_1, and w_2, we get the solution set

$$\begin{aligned} a_{21}&=\frac{2b_{01}b_{10}^2(k_1-k_2)^2k_2^2}{(k_1+k_2)^2},\quad a_{12}=\frac{2b_{01}^2b_{10}k_1^2(k_1-k_2)^2}{(k_1+k_2)^2},\quad b_{11}=\frac{b_{01}b_{10}(k_1-k_2)^2}{(k_1+k_2)^2},\\ a_{11}&=4b_{01}b_{10}(k_1-k_2)^2,\quad a_{10}=2b_{10}k_1^2,\quad a_{01}=2b_{01}k_2^2,\quad w_1=-k_1^7,\quad w_2=-k_2^7, \end{aligned} \tag{24}$$

which provides a two-wave solution to Equation (19) as

$$u(x,t)=\frac{\begin{array}{c}2b_{10}k_1^2\exp(\xi_1)+2b_1k_2^2\exp(\xi_2)+4b_1b_{10}(k_1-k_2)^2\exp(\xi_1+\xi_2)\\ +\dfrac{2b_1b_{10}^2(k_1-k_2)^2k_2^2}{(k_1+k_2)^2}\exp(2\xi_1+\xi_2)+\dfrac{2b_1^2b_{10}k_1^2(k_1-k_2)^2}{(k_1+k_2)^2}\exp(\xi_1+2\xi_2)\end{array}}{\left(1+b_{10}\exp(\xi_1)+b_{01}\exp(\xi_2)+\dfrac{b_{01}b_{10}(k_1-k_2)^2}{(k_1+k_2)^2}\exp(\xi_1+\xi_2)\right)^2}, \tag{25}$$

where $\xi_1=k_1x-k_1^7t+\delta_1$, $\xi_2=k_2x-k_2^7t+\delta_2$ and b_{01}, b_{10}, k_1, k_2, δ_1, δ_2 remain arbitrary

4.2. Three-wave solutions

Assume that Equation (19) admits a solution of the form

$$u(x,t)=\frac{v_1(\xi_1,\xi_2,\xi_3)}{v_2(\xi_1,\xi_2,\xi_3)}, \tag{26}$$

where $\xi_l=k_lx+w_lt+\delta_l$, $l=1,2,3$, and

$$\begin{aligned} v_1(\xi_1,\xi_2,\xi_3)=\;&a_{100}\exp(\xi_1)+a_{010}\exp(\xi_2)+a_{001}\exp(\xi_3)+a_{110}\exp(\xi_1+\xi_2)+a_{101}\exp(\xi_1+\xi_3)\\ &+a_{011}\exp(\xi_2+\xi_3)+a_{120}\exp(\xi_1+2\xi_2)+a_{102}\exp(\xi_1+2\xi_3)+a_{012}\exp(\xi_2+2\xi_3)\\ &+a_{210}\exp(2\xi_1+\xi_2)+a_{201}\exp(2\xi_1+\xi_3)+a_{021}\exp(2\xi_2+\xi_3)+a_{111}\exp(\xi_1+\xi_2+\xi_3)\\ &+a_{211}\exp(2\xi_1+\xi_2+\xi_3)+a_{121}\exp(\xi_1+2\xi_2+\xi_3)+a_{112}\exp(\xi_1+\xi_2+2\xi_3)\\ &+a_{221}\exp(2\xi_1+2\xi_2+\xi_3)+a_{122}\exp(\xi_1+2\xi_2+2\xi_3)+a_{212}\exp(2\xi_1+\xi_2+2\xi_3),\end{aligned}$$

$$v_2(\xi_1,\xi_2,\xi_3)=\begin{pmatrix}1+b_{100}\exp(\xi_1)+b_{010}\exp(\xi_2)+b_{001}\exp(\xi_3)+b_{110}\exp(\xi_1+\xi_2)+b_{101}\exp(\xi_1+\xi_3)\\ +b_{011}\exp(\xi_2+\xi_3)+b_{111}\exp(\xi_1+\xi_2+\xi_3)\end{pmatrix}^2.$$

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It is obvious that the ansatz (26) is embedded in (4). After substituting (26) into Equation (19) and proceeding as before, we get the solution set of the resultant algebraic system as:

$$w_1=-k_1^7,\quad w_2=-k_2^7,\quad w_3=-k_3^7,\quad a_{100}=2b_{100}k_1^2,\quad a_{010}=2b_{010}k_2^2,\quad a_{001}=2b_{001}k_3^2, \tag{27}$$

$$a_{011}=4b_{001}b_{010}(k_2-k_3)^2,\quad a_{101}=4b_{001}b_{100}(k_1-k_3)^2,\quad a_{110}=4b_{010}b_{100}(k_1-k_2)^2, \tag{28}$$

$$a_{102}=\frac{2b_{001}^2b_{100}k_1^2(k_1-k_3)^2}{(k_1+k_3)^2},\quad a_{012}=\frac{2b_{001}^2b_{010}k_2^2(k_2-k_3)^2}{(k_2+k_3)^2},\quad b_{011}=\frac{b_{001}b_{010}(k_2-k_3)^2}{(k_2+k_3)^2}, \tag{29}$$

$$a_{021}=\frac{2b_{001}b_{010}^2(k_2-k_3)^2k_3^2}{(k_2+k_3)^2},\quad b_{101}=\frac{b_{001}b_{100}(k_1-k_3)^2}{(k_1+k_3)^2},\quad a_{201}=\frac{2b_{001}b_{100}^2(k_1-k_3)^2k_3^2}{(k_1+k_3)^2}, \tag{30}$$

$$a_{120}=\frac{2b_{010}^2b_{100}k_1^2(k_1-k_2)^2}{(k_1+k_2)^2},\quad a_{112}=\frac{4b_{001}^2b_{010}b_{100}(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_3)^2(k_2+k_3)^2}, \tag{31}$$

$$b_{110}=\frac{b_{010}b_{100}(k_1-k_2)^2}{(k_1+k_2)^2},\quad a_{221}=\frac{2b_{001}b_{010}^2b_{100}^2(k_1-k_2)^4(k_1-k_3)^2(k_2-k_3)^2k_3^2}{(k_1+k_2)^4(k_1+k_3)^2(k_2+k_3)^2}, \tag{32}$$

$$a_{210}=\frac{2b_{010}b_{100}^2(k_1-k_2)^2k_2^2}{(k_1+k_2)^2},\quad a_{122}=\frac{2b_{001}^2b_{010}^2b_{100}k_1^2(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^4}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^4}, \tag{33}$$

$$a_{121}=\frac{4b_{001}b_{010}^2b_{100}(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_2+k_3)^2},$$

$$b_{111}=\frac{b_{001}b_{010}b_{100}(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}, \tag{34}$$

$$a_{111}=\frac{8b_{001}b_{010}b_{100}(k_2^2k_3^2(k_2^2-k_3^2)^2+k_1^6(k_2^2+k_3^2)-2k_1^4(k_2^4+k_3^4)+k_1^2(k_2^6+k_3^6))}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}, \tag{35}$$

$$a_{212}=\frac{2b_{001}^2b_{010}b_{100}^2(k_1-k_2)^2k_2^2(k_1-k_3)^4(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^4(k_2+k_3)^2}, \tag{36}$$

$$a_{211}=\frac{4b_{001}b_{010}b_{100}^2(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2}. \tag{37}$$

Finally, employing the determined coefficients (27)–(37) to (26), we derive a three-wave solution to Equation (19), where b_{100}, b_{010}, b_{001}, k_1, k_2, k_3, δ_1, δ_2, and δ_3 remain arbitrary.

4.3. Rational solutions

Assuming that Equation (19) admits a solution of the form (16), we obtain the solution set of the resultant algebraic system as

$$w=-\frac{140ka_1^3}{b_1^3},\quad a_0=\frac{a_1b_0}{b_1},\quad a_{-1}=\frac{a_1b_0^2-8k^2b_1^3}{4b_1^2},\quad b_{-1}=\frac{b_0^2}{4b_1},\quad \mu_1=0,\quad \mu_2=1, \tag{38}$$

which leads to a rational solution to Equation (19) as

$$u(x,t)=\frac{4a_1b_1^2\left(kx-\frac{140ka_1^3}{b_1^3}t+\delta\right)^2+4a_1b_0b_1\left(kx-\frac{140ka_1^3}{b_1^3}t+\delta\right)+a_1b_0^2-8k^2b_1^3}{4b_1^3\left(kx-\frac{140ka_1^3}{b_1^3}t+\delta\right)^2+4b_0b_1^2\left(kx-\frac{140ka_1^3}{b_1^3}t+\delta\right)+b_0^2b_1}, \tag{39}$$

where a_1, b_1, b_0, k, and δ remain arbitrary.

5. Conclusion

Seeking exact and explicit solutions with multi-velocities and multi-frequencies for NEEs is an important research area in the applied sciences. In this study, the Exp-function method is used to explicitly construct one-, two-, and three-wave solutions, as well as rational solutions, of completely integrable NEEs. As pointed out in [25, 26], our method has the advantage in the sense that the bilinear forms of the equations are no longer needed. We used two different kinds of nonlinear equations to demonstrate the power

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of the method. The correctness of the obtained results is tested by substitution into the original equations; this provides an extra measure of confidence in the results.

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