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The first integral method for constructing exact and explicit solutions to nonlinear evolution equations

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Problems that are modeled by nonlinear evolution equations occur in many areas of applied sciences. In the present study, we deal with the negative order KdV equation and the generalized Zakharov system and derive some further results using the so-called first integral method. By means of the established first integrals, some exact traveling wave solutions are obtained in a concise manner. Copyright © 2012 John Wiley & Sons, Ltd.

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1. Introduction

Applied sciences rely on processes that are usually modeled by nonlinear evolution equations (NEEs), especially because of their frequent applications in various research fields, such as mathematical biology, mathematical finance, industrial mathematics, etc. In the past decades there has been a growing interest in the problem of finding exact and explicit solutions of NEEs because they play an important role in science and engineering, and their importance will certainly grow in the future. Usually, it is difficult to obtain exact solutions for these models. Many different methods have been developed for tackling NEEs from both analytical and numerical point of views. Some recent ones include: homogeneous balance method [1], tanh function method [2], (G′/G)-expansion method [3], Exp-function method [4], F-expansion method [5], homotopy perturbation method [6–10], the solitary wave ansatz method [11], multiple Exp-function method [12], Adomian–Pade technique [13], further improved F-expansion method [14], and variational iteration method [15–18], etc. However, different methods are required for different targets because each mathematical method has its own features.

The study of finding first integrals for nonlinear ordinary differential equations (NODEs) has also been an important subject because they allow us to get the general solution by means of quadratures. However, there is no systematic approach that can provide us a way of finding the first integrals of NODEs, nor is there a logical way for telling us what these first integrals are. Not long ago, via the ring theory of commutative algebra, Feng [19, 20] coined an algebraic curve method for solving NEEs in terms of the so-called traveling waves. The approach is currently known as the first integral method. The method has found several applications in the applied sciences (see, for instance, [21–28] and the references therein). The main idea behind the method is to construct a polynomial first integral (with polynomial coefficients) to an autonomous planar system which is equivalent to the equation to be solved. Through the established first integrals, exact solutions can be obtained under some parameter conditions.

However, the application of the first integral method to distinct types of equations is still an interesting and important research problem. Every nonlinear equation has its own physically rich and complicated structure that is worth to be analyzed by innovative new methods. The focus of the present paper is to further extend the first integral method to the negative order KdV equation and the generalized Zakharov system for the first time. The outline of the paper is as follows: in Section 2 we explain our method that allows us to derive the desired results in Sections 3 and 4. We end our analysis with a conclusion in Section 5.

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2. The first integral method

Let us consider a partial differential equation for a function $u\,(x,t)$ in the form

$$P\,(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0, \tag{1}$$

where P is a polynomial in its arguments while subscripts denote partial derivatives. By the wave transformation $u(x,t) = U(\xi)$, $\xi = kx - wt + \xi_0$, where k, w, and ξ_0 are arbitrary constants, Equation (1) can be reduced to an ODE of the form

$$P(U, -wU', kU', w^2U'', -kwU'', k^2U'', \ldots) = 0, \tag{2}$$

where $U = U(\xi)$ and the primes denote ordinary derivatives with respect to ξ. Introducing the new variables

$$X(\xi) = U(\xi), \quad Y(\xi) = U_\xi(\xi), \tag{3}$$

we assume that Equation (2) can be reduced to a two-dimensional autonomous system of the form

$$\begin{aligned} X_\xi(\xi) &= Y(\xi), \\ Y_\xi(\xi) &= Q\,(X(\xi), Y(\xi))\,, \end{aligned} \tag{4}$$

where the subscript denotes ordinary derivative with respect to ξ. In general, solving a planar autonomous system of ODEs of the form (4) is a challenging and difficult task. Hence, based on the qualitative theory of ODEs [29], if one can derive a single first integral for the system (4), then one may be able to reduce Equation (2) to a first-order integrable ODE. Then, a class of exact solutions may be obtained by solving the resulting first-order ODE by a quadrature. At this stage, the following Division Theorem will play the central role in our analysis:

Division Theorem. *Suppose that $P\,(w,z)$ and $Q\,(w,z)$ are polynomials in $\mathbb{C}\,[w,z]$ and $P\,(w,z)$ is irreducible in $\mathbb{C}\,[w,z]$. If $Q\,(w,z)$ vanishes at all zero points of $P\,(w,z)$, then there exist a polynomial $G\,(w,z)$ in $\mathbb{C}\,[w,z]$ such that $Q\,(w,z) = P\,(w,z)\,G\,(w,z)$.*

Here, $\mathbb{C}\,[w,z]$ denotes the complex domain. We shall not prove this result here, but refer the interested reader to Refs. [30, 31].

3. The negative order KdV equation

First, let us consider the negative order KdV equation, which reads

$$uu_{xxt} - u_t u_{xx} + 2u^3 u_x = 0, \tag{5}$$

where $u = u\,(x,t)$. In fact, Equation (5) can be rewritten in compact form

$$\left(\frac{u_{xx}}{u}\right)_t + 2uu_x = 0, \tag{6}$$

which belongs to the negative KdV hierarchy and leads to the Camassa–Holm equation via a hodograph transformation [32]. Now, assume that Equation (6) admits a traveling wave solution of the form

$$u\,(x,t) = U\,(\xi)\,,\ \xi = kx - wt + \xi_0, \tag{7}$$

where k and w are arbitrary constants to be specified, while ξ_0 denotes an arbitrary phase shift. Substituting (7) into Equation (6) and integrating the resulting equation once yield

$$U'' = cU + \frac{1}{kw}U^3, \tag{8}$$

where $U = U\,(\xi)$, the primes denote derivatives with respect to ξ, and c is an integration constant. Letting $z = U$ and $y = U'$, Equation (8) can be rewritten as the plane autonomous system

$$\begin{cases} \frac{\mathrm{d}z}{\mathrm{d}\xi} = y, \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} = cz + \frac{1}{kw}z^3. \end{cases} \tag{9}$$

Now, suppose that $z = z(\xi)$ and $y = y(\xi)$ are nontrivial solutions of (9). Also, assume that $q(z,y) = \sum_{i=0}^{m} A_i(z)y^i$ is an irreducible polynomial in the complex domain $\mathbb{C}$ such that

$$q(z(\xi), y(\xi)) = \sum_{i=0}^{m} A_i(z)y^i = 0, \tag{10}$$

where the polynomials $A_i(z)$ $(i = 0, 1, \ldots, m)$ are relatively prime in $\mathbb{C}$ with $A_m(z) \not\equiv 0$. Equation (10) is called a first integral of Equation (9). We note that $\mathrm{d}q/\mathrm{d}\xi$ is a polynomial in z and y. Thus, $q(z(\xi), y(\xi)) = 0$ implies that $\mathrm{d}q/\mathrm{d}\xi = 0$. Then, by the Division Theorem, there exists a polynomial $B(z) + C(z)\,y$ in the complex domain $\mathbb{C}$ such that

$$\frac{\mathrm{d}q}{\mathrm{d}\xi} = \frac{\partial q}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}\xi} + \frac{\partial q}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}\xi} = (B(z) + C(z)\,y)\left[\sum_{i=0}^{m} A_i(z)y^i\right]. \tag{11}$$

We consider the case $m = 2$ of (10). Hence, taking Equations (9) and (11) into account, we get

$$\sum_{i=0}^{2}\left[A_i'(z)\,y^{i+1}\right] + \sum_{i=0}^{2}\left[iA_i(z)\,y^{i-1}\left(cz + \frac{1}{kw}z^3\right)\right] = [B(z) + C(z)\,y]\left[\sum_{i=0}^{2} A_i(z)\,y^i\right]. \tag{12}$$

Equating the coefficients of y^i $(0 \le i \le 3)$ in Equation (12) leads to the system

$$y^3: A_2'(z) = C(z)\,A_2(z)\,, \tag{13}$$

$$y^2: A_1'(z) = C(z)\,A_1(z) + B(z)\,A_2(z)\,, \tag{14}$$

$$y^1: A_0'(z) = C(z)\,A_0(z) + B(z)\,A_1(z) - 2\left(cz + \frac{1}{kw}z^3\right)A_2(z)\,, \tag{15}$$

$$y^0: B(z)\,A_0(z) = A_1(z)\left(cz + \frac{1}{kw}z^3\right). \tag{16}$$

Because $A_2(z)$ and $C(z)$ are polynomials, from Equation (13) we deduce that $C(z) = 0$ and $A_2(z)$ must be a constant. For simplicity, we can take $A_2(z) = 1$. Balancing the degrees of $A_0(z)$, $A_1(z)$, and $B(z)$, we conclude that $\deg B(z) = 1$ only. Suppose that $B(z) = b_1 z + b_0$ $(b_1 \neq 0)$. Then, from (14) and (15), we get

$$A_1(z) = \frac{b_1}{2}z^2 + b_0 z + e, \tag{17}$$

$$A_0(z) = \left(\frac{1}{8}b_1^2 - \frac{1}{2kw}\right)z^4 + \frac{1}{2}b_1 b_0 z^3 + \left(\frac{eb_1 + b_0^2}{2} - c\right)z^2 + eb_0 z + f, \tag{18}$$

where e and f are integration constants. Substituting $A_0(z)$, $A_1(z)$, and $B(z)$ into (16) and setting the coefficients of z^i $(0 \le i \le 5)$ to zero, we derive a system of nonlinear algebraic equations for b_0, b_1, e, f, k and w. Solving the resultant system simultaneously, we get the solution set

$$f = \frac{kwc^2}{2},\ e = \mp c\sqrt{2kw},\ b_0 = 0,\ b_1 = \mp 2\sqrt{\frac{2}{kw}}, \tag{19}$$

where all other constants remain arbitrary. Using the relation (19) in (10), we get

$$y = \pm\frac{z^2 + kwc}{\sqrt{2kw}}. \tag{20}$$

Combining the first Equation of (9) with Equation (20), solving the resulting equations by quadratures, and changing to the original variables, we obtain

Observation 1
The negative order KdV Equation (5) *admits traveling wave solutions of the form*

$$u_1^{\pm}(x,t) = \pm\sqrt{ckw}\tan\left(\sqrt{\frac{c}{2}}\,(kx - wt + \xi_0)\right),\ c > 0,\ kw > 0, \tag{21}$$

$$u_2^{\pm}(x,t) = \pm\sqrt{-ckw}\tanh\left(\sqrt{\frac{-c}{2}}\,(kx - wt + \xi_0)\right),\ c < 0,\ kw > 0, \tag{22}$$

where all involved constants remain arbitrary.

Our results can be compared with those of [32].

4. The generalized Zakharov system

Second, let us consider the so-called generalized Zakharov system [33], which reads

$$\begin{aligned} &u_{tt} - \gamma^2 u_{xx} - \beta\left(|v|^2\right)_{xx} = 0, \\ &iv_t + \alpha v_{xx} - \delta_1 uv + \delta_2 |v|^2 v + \delta_3 |v|^4 v = 0, \end{aligned} \tag{23}$$

where the real unknown function $u = u(x,t)$ is the fluctuation in the ion density about its equilibrium value, and the complex unknown function $v = v(x,t)$ is the slowly varying envelope of highly oscillatory electron field, i is the imaginary unit, the parameters α, β, γ, δ_1, δ_2, and δ_3 are real numbers, while γ is proportional to the ion acoustic speed (or electron sound speed). Now, we assume that Equation (23) admits a solution of the form

$$v(x,t) = \psi(\xi) \times \exp\left(i(\theta x - \varepsilon t)\right), \; u(x,t) = u(\xi), \; \xi = kx - wt + \xi_0, \tag{24}$$

where $\psi = \psi(\xi)$ is an undetermined real function, θ, ε, k and w are arbitrary constants to be specified, and ξ_0 is an arbitrary phase shift. Then, the system (23) leads to

$$\begin{aligned} &\left(w^2 - k^2\gamma^2\right) u'' - \beta k^2 \left(\psi^2\right)'' = 0, \\ &\left(\epsilon - \alpha\theta^2 - \delta_1 u\right)\psi + \delta_2\psi^3 + \delta_3\psi^5 + k^2\alpha\psi'' + i\left(2k\alpha\theta - w\right)\psi' = 0. \end{aligned} \tag{25}$$

Setting the imaginary part of (25) to zero, we get the relation $w = 2k\alpha\theta$. Integrating the first equation of (25) and taking the constants of integration as zero give

$$u = \frac{\beta}{4\alpha^2\theta^2 - \gamma^2}\psi^2, \; \gamma \neq \mp 2\alpha\theta. \tag{26}$$

Substituting (26) into the second equation of (25)

$$\psi'' + \frac{\epsilon - \alpha\theta^2}{k^2\alpha}\psi + \left(\frac{\beta\delta_1}{k^2\alpha\left(\gamma^2 - 4\alpha^2\theta^2\right)} + \frac{\delta_2}{k^2\alpha}\right)\psi^3 + \frac{\delta_3}{k^2\alpha}\psi^5 = 0, \; \alpha \neq 0, \tag{27}$$

where the primes denote derivatives with respect to ξ. Letting $z = \psi$ and $y = \psi'$, Equation (27) can be rewritten as the plane autonomous system

$$\begin{cases} \frac{dz}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{\alpha\theta^2 - \epsilon}{k^2\alpha}z - \left(\frac{\beta\delta_1}{k^2\alpha(\gamma^2 - 4\alpha^2\theta^2)} + \frac{\delta_2}{k^2\alpha}\right)z^3 - \frac{\delta_3}{k^2\alpha}z^5, \end{cases} \tag{28}$$

According to the first integral method, we consider the case $m = 2$ of (10). Then, by equating the coefficients of y^i $(0 \leq i \leq 3)$ on both sides of (11) with the consideration of (28), we have

$$y^3 : A_2'(z) = C(z) A_2(z), \tag{29}$$

$$y^2 : A_1'(z) = B(z) A_2(z) + C(z) A_1(z), \tag{30}$$

$$y^1 : A_0'(z) = B(z) A_1(z) - 2\left(\frac{\alpha\theta^2 - \epsilon}{k^2\alpha}z - \left(\frac{\beta\delta_1}{k^2\alpha\left(\gamma^2 - 4\alpha^2\theta^2\right)} + \frac{\delta_2}{k^2\alpha}\right)z^3 - \frac{\delta_3}{k^2\alpha}z^5\right) A_2(z) + C(z) A_0(z), \tag{31}$$

$$y^0 : B(z) A_0(z) = \left(\frac{\alpha\theta^2 - \epsilon}{k^2\alpha}z - \left(\frac{\beta\delta_1}{k^2\alpha\left(\gamma^2 - 4\alpha^2\theta^2\right)} + \frac{\delta_2}{k^2\alpha}\right)z^3 - \frac{\delta_3}{k^2\alpha}z^5\right) A_1(z). \tag{32}$$

From Equation (29), we obtain $A_2(z) = c_0 \exp\left(\int C(z) dz\right)$, where c_0 is an integration constant. Because $A_2(z)$ and $C(z)$ are polynomials, we deduce that $C(z) = 0$ and $A_2(z)$ must be a constant. For simplicity, we can take $A_2(z) = 1$. Then, Equations (30) and (31) reduce to the following equations:

$$A_1'(z) = B(z), \tag{33}$$

$$A_0'(z) = B(z) A_1(z) - 2\left(\frac{\alpha\theta^2 - \epsilon}{k^2\alpha}z - \left(\frac{\beta\delta_1}{k^2\alpha\left(\gamma^2 - 4\alpha^2\theta^2\right)} + \frac{\delta_2}{k^2\alpha}\right)z^3 - \frac{\delta_3}{k^2\alpha}z^5\right). \tag{34}$$

Case 1
deg $A_1(z) = 0$.

In this case, from (32) and (33), we conclude that $A_1(z) \equiv 0$. Then, (34) gives

$$A_0(z) = \frac{\delta_3}{3k^2\alpha}z^6 + \frac{1}{2}\left(\frac{\beta\delta_1}{k^2\alpha\left(\gamma^2 - 4\alpha^2\theta^2\right)} + \frac{\delta_2}{k^2\alpha}\right)z^4 - \frac{\alpha\theta^2 - \epsilon}{k^2\alpha}z^2, \tag{35}$$

where we set the integration constant to zero for simplicity. Hence, Equation (10) becomes

$$\frac{\delta_3}{3k^2\alpha}z^6 + \frac{1}{2}\left(\frac{\beta\delta_1}{k^2\alpha\left(\gamma^2 - 4\alpha^2\theta^2\right)} + \frac{\delta_2}{k^2\alpha}\right)z^4 - \frac{\alpha\theta^2 - \epsilon}{k^2\alpha}z^2 + y^2 = 0, \tag{36}$$

which is a first integral of Equation (28). Solving Equation (36) for y, we get

$$y = \pm\left(\frac{\alpha\theta^2 - \epsilon}{k^2\alpha}z^2 - \frac{1}{2}\left(\frac{\beta\delta_1}{k^2\alpha\left(\gamma^2 - 4\alpha^2\theta^2\right)} + \frac{\delta_2}{k^2\alpha}\right)z^4 - \frac{\delta_3}{3k^2\alpha}z^6\right)^{1/2}. \tag{37}$$

Here and henceforth, the signs ($\pm$) or ($\mp$) should be taken into account in a vertical order. Combining the first equation of (28) with Equation (37), solving the resulting equation by a quadrature, and changing to the original variables, we obtain

Observation 2
The generalized Zakharov system (23) admits a traveling wave solution of the form

$$u(x,t) = \frac{\beta}{4\alpha^2\theta^2 - \gamma^2}\psi(\xi)^2,\; v(x,t) = \psi(\xi) \times \exp\left(i\left(\theta x - \varepsilon t\right)\right),\; \xi = kx - 2k\alpha\theta t + \xi_0,\; \gamma \neq \mp 2\alpha\theta, \tag{38}$$

where

$$\psi(\xi) = \pm\frac{4\sqrt{3a}}{\left(\sqrt{6}\left(6b^2 - 32ac + 1\right)\cosh\left(2\sqrt{a}\xi\right) + \sqrt{6}\left(6b^2 - 32ac - 1\right)\sinh\left(2\sqrt{a}\xi\right) - 12b\right)^{1/2}}, \tag{39}$$

in which $a = \frac{\alpha\theta^2 - \epsilon}{k^2\alpha}$, $b = -\frac{\beta\delta_1}{k^2\alpha(\gamma^2 - 4\alpha^2\theta^2)} - \frac{\delta_2}{k^2\alpha}$, *and* $c = -\frac{\delta_3}{k^2\alpha}$ *,while all involved constants remain arbitrary.*

Remark 1
We observed that the cases deg $A_1(z) = 1$ or deg $A_1(z) = 2$ provides the same first integral as (36). We omit to present the details for brevity.

Case 2
deg $A_1(z) = 3$.

In this case, from (32) and (33), we conclude that deg $B(z) = 2$ and deg $A_0(z) = 6$. Assuming $A_1(z) = a_3z^3 + a_2z^2 + a_1z + a_0$ $(a_3 \neq 0)$ and $B(z) = b_2z^2 + b_1z + b_0$ $(b_2 \neq 0)$ in (33), we get $b_2 = 3a_3$, $b_1 = 2a_2$, and $b_0 = a_1$. Substituting $A_1(z)$ and $B(z)$ into Equation (34) and integrating the resulting equation leads to

$$\begin{aligned} A_0(z) = {} & \left(\frac{a_3^2}{2} + \frac{\delta_3}{3k^2\alpha}\right)z^6 + a_2a_3z^5 + \left(\frac{a_2^2 + 2a_1a_3}{2} + \frac{\beta\delta_1}{2k^2\alpha\left(\gamma^2 - 4\alpha^2\theta^2\right)} + \frac{\delta_2}{2k^2\alpha}\right)z^4 + \left(a_1a_2 + a_0a_3\right)z^3 \\ & + \left(\frac{a_1^2 + 2a_0a_2}{2} - \frac{\alpha\theta^2 - \epsilon}{k^2\alpha}\right)z^2 + a_0a_1z + d, \end{aligned} \tag{40}$$

where d denotes an integration constant. Then, substituting $A_0(z)$, $A_1(z)$, and $B(z)$ into Equation (32), equating the coefficients of z^i $(0 \leq i \leq 8)$ to zero, and solving the resulting system of nonlinear algebraic equations simultaneously, we get the relations

$$d = 0,\; a_3 = -\frac{2}{k}\sqrt{-\frac{\delta_3}{3\alpha}},\; a_1 = \mp\frac{2}{k}\sqrt{\frac{\alpha\theta^2 - \epsilon}{\alpha}},\; 3\left(\delta_2 + \frac{\beta\delta_1}{\gamma^2 - 4\alpha^2\theta^2}\right)^2 = 16\delta_3\left(\epsilon - \alpha\theta^2\right),\; a_0 = 0,\; a_2 = 0, \tag{41}$$

$$d = 0,\; a_3 = \frac{2}{k}\sqrt{-\frac{\delta_3}{3\alpha}},\; a_1 = \mp\frac{2}{k}\sqrt{\frac{\alpha\theta^2 - \epsilon}{\alpha}},\; 3\left(\delta_2 + \frac{\beta\delta_1}{\gamma^2 - 4\alpha^2\theta^2}\right)^2 = 16\delta_3\left(\epsilon - \alpha\theta^2\right),\; a_0 = 0,\; a_2 = 0, \tag{42}$$

where all other constants remain arbitrary. Hence, Equation (10) becomes

$$y^2+\left(\mp\frac{2}{k}\sqrt{\frac{\alpha\theta^2-\epsilon}{\alpha}}z-\frac{2}{k}\sqrt{-\frac{\delta_3}{3\alpha}}z^3\right)y-\frac{\delta_3}{3k^2\alpha}z^6\pm\frac{2\sqrt{\delta_3\left(\epsilon-\alpha\theta^2\right)}}{\sqrt{3}k^2\alpha}z^4+\frac{\alpha\theta^2-\epsilon}{k^2\alpha}z^2=0, \tag{43}$$

$$y^2+\left(\mp\frac{2}{k}\sqrt{\frac{\alpha\theta^2-\epsilon}{\alpha}}z+\frac{2}{k}\sqrt{-\frac{\delta_3}{3\alpha}}z^3\right)y-\frac{\delta_3}{3k^2\alpha}z^6\mp\frac{2\sqrt{\delta_3\left(\epsilon-\alpha\theta^2\right)}}{\sqrt{3}k^2\alpha}z^4+\frac{\alpha\theta^2-\epsilon}{k^2\alpha}z^2=0, \tag{44}$$

which are first integrals of Equation (28). Solving Equations (43) and (44) for y, respectively, we get

$$y=\frac{1}{\sqrt{3}k}\sqrt{\frac{-\delta_3}{\alpha}}z^3\pm\frac{1}{k}\sqrt{\frac{\alpha\theta^2-\epsilon}{\alpha}}z, \tag{45}$$

$$y=-\frac{1}{\sqrt{3}k}\sqrt{\frac{-\delta_3}{\alpha}}z^3\pm\frac{1}{k}\sqrt{\frac{\alpha\theta^2-\epsilon}{\alpha}}z. \tag{46}$$

Combining the first equation of (28) with Equations (45) and (46), solving the resulting equations by quadratures, and changing to the original variables, we obtain

Observation 3
The generalized Zakharov system (23) admits traveling wave solutions of the form

$$u(x,t)=\frac{\beta}{4\alpha^2\theta^2-\gamma^2}\psi\left(\xi\right)^2,\; v(x,t)=\psi\left(\xi\right)\times\exp\left(i\left(\theta x-\varepsilon t\right)\right),\; \xi=kx-2k\alpha\theta t+\xi_0,\; \gamma\neq\mp 2\alpha\theta, \tag{47}$$

where

$$\psi\left(\xi\right)=\mp\left(\frac{\frac{3}{k}\sqrt{\theta^2-\frac{\epsilon}{\alpha}}}{\cosh\left(\frac{2}{k}\sqrt{\theta^2-\frac{\epsilon}{\alpha}}\xi\right)-\sinh\left(\frac{2}{k}\sqrt{\theta^2-\frac{\epsilon}{\alpha}}\xi\right)-\frac{1}{k}\sqrt{-\frac{3\delta_3}{\alpha}}}\right)^{1/2}, \tag{48}$$

or

$$\psi\left(\xi\right)=\mp\left(\frac{\frac{3}{k}\sqrt{\theta^2-\frac{\epsilon}{\alpha}}}{\cosh\left(\frac{2}{k}\sqrt{\theta^2-\frac{\epsilon}{\alpha}}\xi\right)+\sinh\left(\frac{2}{k}\sqrt{\theta^2-\frac{\epsilon}{\alpha}}\xi\right)-\frac{1}{k}\sqrt{-\frac{3\delta_3}{\alpha}}}\right)^{1/2}, \tag{49}$$

in which $3\left(\beta\delta_1+\left(\gamma^2-4\alpha^2\theta^2\right)\delta_2\right)^2=16\left(\epsilon-\alpha\theta^2\right)\left(\gamma^2-4\alpha^2\theta^2\right)^2\delta_3$ *, while all involved constants remain arbitrary.*

Remark 2
Other cases are not possible for if $\deg A_1(z)=k>3$, then we deduce that $\deg B(z)=k-1$ and $\deg A_0(z)=2k$ because Equations (33) and (34). However, the degree of the polynomial on the left side of Equation (32) is $3k-1$ while the degree of the polynomial on the right side of Equation (32) is $k+5$, which is a contradiction.

5. Conclusion

The exact solutions when they exist can help one to well understand the mechanism of the complicated physical phenomena modeled by NEEs. In this paper, we presented two possible new applications of the first integral method for such types of equations, namely, the negative order KdV equation and the generalized Zakharov system. First integrals are of great importance in the study of NODEs because they reduce the problem to quadratures. It is observed that the first integral method works for NEEs, which can be converted to a second-order ODE through a suitable transformation. The method is straightforward, concise, and its applications to other types of NEEs are promising. It will be interesting to apply our method to other types of problems such as the ones presented in [34–36]. This will be our future research task.

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