

# ABSOLUTE CO-SUPPLEMENT AND ABSOLUTE CO-COCLOSED MODULES

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## Abstract

A module  $M$  is called an *absolute co-coclosed (absolute co-supplement)* module if whenever  $M \cong T/X$  the submodule  $X$  of  $T$  is a coclosed (supplement) submodule of  $T$ . Rings for which all modules are absolute co-coclosed (absolute co-supplement) are precisely determined. We also investigate the rings whose (finitely generated) absolute co-supplement modules are projective. We show that a commutative domain  $R$  is a Dedekind domain if and only if every submodule of an absolute co-supplement  $R$ -module is absolute co-supplement. We also prove that the class *Coclosed* of all short exact sequences  $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$  such that  $A$  is a coclosed submodule of  $B$  is a proper class and every extension of an absolute co-coclosed module by an absolute co-coclosed module is absolute co-coclosed.

**Keywords:** Absolute co-supplement (co-coclosed) module, Supplement (coclosed) submodule.

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## 1. Preliminaries

Throughout this paper unless otherwise stated all rings are associative with identity element and all modules are unitary right  $R$ -modules. A submodule  $N$  of  $M$  is said to be *coclosed* if  $N/K \ll M/K$  implies  $K = N$  for each  $K \leq N$  or equivalently, given any proper submodule  $K$  of  $N$ , there is a submodule  $L$  of  $M$  for which  $N + L = M$  but  $K + L \neq M$ . A submodule  $K$  of  $M$  is said to be *supplement* of  $N$  in  $M$  if  $K$  is minimal with respect to  $K + N = M$  equivalently,  $K + N = M$  and  $K \cap N \ll K$ . A submodule  $L$  of  $M$  is called a *supplement submodule* in  $M$  provided there exists a submodule  $X$  of  $M$  such that  $L$  is a supplement of  $X$  in  $M$ . A module  $M$  is said to be *supplemented*, if every submodule of  $M$  has a supplement in  $M$ . Every supplement submodule of a module  $M$  is

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coclosed (see, for example [3, 20.2]). A module  $M$  is called *absolute supplement* (or *almost injective*) if it is a supplement submodule of every module containing  $M$  (see [2] and [5]). If for all short exact sequences  $0 \longrightarrow X \longrightarrow T \longrightarrow M \longrightarrow 0$  the submodule  $X$  is a supplement submodule of  $T$ , then  $M$  is said to be *absolute co-supplement* (see [5]). Clearly if  $M/N$  is absolute co-supplement, then  $N$  is a supplement submodule of  $M$ .

In this paper we introduce the notion of absolute co-coclosed modules. If for all short exact sequences  $0 \longrightarrow X \longrightarrow T \longrightarrow M \longrightarrow 0$  the submodule  $X$  is coclosed in  $T$ , then  $M$  is called an *absolute co-coclosed* module. Clearly if  $M/N$  is absolute co-coclosed, then  $N$  is a coclosed submodule of  $M$ . Since supplement submodules are coclosed, the following implication hold:

$$\text{absolute co-supplement} \Rightarrow \text{absolute co-coclosed}$$

In section 2 we give a characterization and some properties of absolute co-supplement modules. We also characterize the rings whose modules are absolute co-supplement. For example,  $R$  is semisimple if and only if every  $R$ -module is absolute co-supplement (see Theorem 2.8). We also investigate the rings whose (finitely generated) absolute co-supplement modules are projective (see Theorem 2.14). We prove that if  $R$  is a right hereditary ring, then every absolute co-supplement right  $R$ -module is projective (Theorem 2.15). We show that a module  $M$  is projective if and only if  $M$  is absolute co-supplement and flat, over any ring  $R$  (Theorem 2.18). Let

$$0 \longrightarrow X \longrightarrow P \longrightarrow M \longrightarrow 0$$

be an exact sequence of finitely generated right  $R$ -modules with  $P$  projective and  $J$  the Jacobson radical of  $R$ . Then  $M$  is absolute co-supplement if and only if the induced sequence  $0 \longrightarrow X/XJ \longrightarrow P/PJ \longrightarrow M/MJ \longrightarrow 0$  is split exact if and only if whenever  $Y$  is a right  $R$ -module with  $YJ = 0$ ,  $\text{Ext}_R^1(M, Y) = 0$  (Theorem 2.22). Finally, we prove that a commutative domain  $R$  is a Dedekind domain if and only if every submodule of an absolute co-supplement  $R$ -module is absolute co-supplement (Theorem 2.26).

In section 3 we prove that every right  $R$ -module is absolute co-coclosed if and only if  $R$  is a right  $V$ -ring (Theorem 3.6). We show that the class of absolute co-coclosed modules contains properly the class of absolute co-supplement modules (Example 3.7). We also show that the class of short exact sequences  $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$  with the property that  $A$  is coclosed in  $B$  is a proper class (Theorem 3.11) and so we investigate some properties of absolute co-coclosed modules.

[14] is the general reference for notions of modules not defined in this work.

## 2. Absolute Co-supplement Modules

Let  $\mathcal{P}$  be a class of short exact sequences of  $R$ -modules and  $R$ -module homomorphisms. If a short exact sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

belongs to  $\mathcal{P}$ , then  $f$  is said to be  $\mathcal{P}$ -*monomorphism* and  $g$  is said to be  $\mathcal{P}$ -*epimorphism*. The class  $\mathcal{P}$  is said to be *proper* if it satisfies the following conditions (see [15, Introduction]):

- (P1) If a short exact sequence  $\mathbb{E}$  is in  $\mathcal{P}$ , then  $\mathcal{P}$  contains every short exact sequence isomorphic to  $\mathbb{E}$ .
- (P2)  $\mathcal{P}$  contains all splitting short exact sequences.
- (P3) The composite of two  $\mathcal{P}$ -monomorphisms is a  $\mathcal{P}$ -monomorphism if this composite is defined.

- (P3') The composite of two  $\mathcal{P}$ -epimorphisms is a  $\mathcal{P}$ -epimorphism if this composite is defined.
- (P4) If  $g$  and  $f$  are monomorphisms, and  $g \circ f$  is a  $\mathcal{P}$ -monomorphism, then  $f$  is a  $\mathcal{P}$ -monomorphism.
- (P4') If  $g$  and  $f$  are epimorphisms, and  $g \circ f$  is a  $\mathcal{P}$ -epimorphism, then  $g$  is a  $\mathcal{P}$ -epimorphism.

**2.1. Theorem.** (see [7, Theorem 1] or [5, Theorem 3.1.2]) *The class*

$$\text{Suppl} = \{ \mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \mid A \text{ is a supplement in } B \}$$

*is a proper class.*

A class  $\mathcal{M}$  of modules is said to be *closed under extensions* if  $U, M/U \in \mathcal{M}$  implies  $M \in \mathcal{M}$ . In this case  $M$  is an extension of  $U$  by  $M/U$ . The following proposition shows that the class of absolute co-supplement modules are closed under extensions.

**2.2. Proposition.** (see [5, Proposition 3.2.7]) *For a module  $M$ , if a submodule  $N$  and the quotient module  $M/N$  of  $M$  are absolute co-supplement, then  $M$  is also absolute co-supplement.*

**2.3. Theorem.** (see [5, Proposition 3.2.2]) *For a module  $M$  the following conditions are equivalent:*

- (i)  *$M$  is an absolute co-supplement module, i.e. for all short exact sequences*

$$0 \longrightarrow X \longrightarrow T \longrightarrow M \longrightarrow 0$$

*$X$  is a supplement submodule of  $T$ .*

- (ii) *There exists a short exact sequence*

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$$

*with a projective (or free) module  $P$  such that  $N$  is a supplement submodule of  $P$ .*

Absolute co-supplement and absolute supplement do not imply each other. Clearly every projective module is absolute co-supplement. Therefore  $\mathbb{Z}$  is an absolute co-supplement  $\mathbb{Z}$ -module. Note that since  $\mathbb{Z}$  is not a supplement in  $\mathbb{Q}$ ,  $\mathbb{Z}$  is not absolute supplement.

An arbitrary factor module of an absolute co-supplement (absolute co-coclosed) module need not be absolute co-supplement (absolute co-coclosed) (for example,  $\mathbb{Z}/n\mathbb{Z}$  with  $n$  nonzero), but we have the following:

**2.4. Proposition.** (see [5, Proposition 3.2.6]) *If  $M$  is an absolute co-supplement module and  $N$  is a supplement submodule of  $M$  then  $M/N$  is also absolute co-supplement.*

**2.5. Corollary.** *Let  $M_1, M_2, \dots, M_n$  be modules. Then  $M = M_1 \oplus M_2 \oplus \dots \oplus M_n$  is absolute co-supplement if and only if each  $M_i$  is absolute co-supplement.*

*Proof.* ( $\Rightarrow$ ): By Proposition 2.4,  $M_i \cong M / \bigoplus_{j \neq i}^n M_j$  is absolute co-supplement for each  $i$ .

( $\Leftarrow$ ): By Proposition 2.2 and induction. □

**2.6. Lemma.** *Let  $M$  be any module. If for every small submodule  $N$  of  $M$  the factor module  $M/N$  is absolute co-coclosed, then  $\text{Rad}(M) = 0$ .*

*Proof.* Let  $N$  be a small submodule of  $M$ . Since  $M/N$  is absolute co-coclosed,  $N$  is a coclosed submodule of  $M$ . Thus  $N = 0$ . Therefore  $\text{Rad}(M) = 0$ . □

**2.7. Proposition.** *Let  $M$  be a module. If every factor module of  $M$  is absolute co-supplement, then  $M$  is semisimple.*

*Proof.* In this case every submodule of  $M$  is a supplement. By Lemma 2.6,  $\text{Rad}(M) = 0$ . Hence  $M$  is semisimple.  $\square$

**2.8. Theorem.** *The following are equivalent for a ring  $R$ :*

- (i) *Every right  $R$ -module is absolute co-supplement.*
- (ii) *Every factor module (submodule) of every right  $R$ -module is absolute co-supplement.*
- (iii) *Every factor module of every projective right  $R$ -module is absolute co-supplement.*
- (iv) *Every factor module of every free right  $R$ -module is absolute co-supplement.*
- (v) *Every factor module of  $R_R$  is absolute co-supplement.*
- (vi)  *$R$  is semisimple.*
- (vii) *Every right  $R$ -module is absolute supplement.*
- (viii) *The analogues of the above properties for left  $R$ -modules.*

*Proof.* (i)  $\Leftrightarrow$  (ii) and (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v) are clear.

(v)  $\Rightarrow$  (vi) : By Proposition 2.7.

(vii)  $\Leftrightarrow$  (vi) : By [2, Proposition 2.9].

(vi)  $\Rightarrow$  (i) : Clear.

(vi)  $\Leftrightarrow$  (viii) : By the left right symmetry of semisimple rings.  $\square$

**2.9. Theorem.** (see [11, Theorem 3.7.2]) *Let  $R$  be a ring with  $J = 0$ , where  $J$  is the Jacobson radical of  $R$ . An  $R$ -module  $M$  is projective if and only if  $M$  is absolute co-supplement.*

As a consequence:

**2.10. Corollary.** *Let  $R$  be a regular (or a right  $V$ -)ring. Then any module  $M$  is projective if and only if it is absolute co-supplement.*

**2.11. Lemma.** *Let  $R$  be any ring. Then the following are equivalent:*

- (i) *In every projective right  $R$ -module, every supplement is a direct summand.*
- (ii) *Every absolute co-supplement right  $R$ -module is projective.*

*Proof.* (i)  $\Rightarrow$  (ii) : Let  $M$  be an absolute co-supplement right  $R$ -module. Then there exists a projective module  $P$  such that  $P/X \cong M$  for some supplement submodule  $X$  of  $P$ . By (1),  $X$  is a direct summand of  $P$ . Thus  $M$  is projective.

(ii)  $\Rightarrow$  (i) : Let  $P$  be a projective right  $R$ -module and  $X$  be a supplement submodule of  $P$ . By (ii) and Proposition 2.4,  $P/X$  is projective and so  $X$  is a direct summand of  $P$ .  $\square$

**2.12. Lemma.** *Let  $R$  be any ring. Then the following are equivalent:*

- (i) *In every finitely generated projective right  $R$ -module, every supplement submodule is a direct summand.*
- (ii) *Every finitely generated absolute co-supplement right  $R$ -module is projective.*
- (iii) *The analogues of the above properties for left  $R$ -modules.*

*Proof.* (i)  $\Leftrightarrow$  (ii) : By the same proof as (i)  $\Leftrightarrow$  (ii) of Lemma 2.11.

(i)  $\Leftrightarrow$  (iii) : By [14, Proposition A9] or [18, Corollary of Theorem 2.3].  $\square$

**2.13. Remark.** Let  $R$  be any ring and  $J$  its Jacobson radical. Then the following statements are equivalent (see [13], [14] and [18]):

- (1) If  $M$  is a finitely generated flat right  $R$ -module and  $M/MJ$  is a projective  $R/J$ -module, then  $M$  is a projective right  $R$ -module.

- (2) If  $G$  is a projective right  $R$ -module and  $G/GJ$  is finitely generated, then  $G$  is finitely generated.
- (3) If  $Q$  is a projective right  $R$ -module, then each finitely generated proper submodule is contained in a maximal submodule.
- (4) Every supplement in a finitely generated projective right  $R$ -module is a direct summand.
- (5) Every finitely generated  $J$ -projective right  $R$ -module (a module  $P$  is  $J$ -projective provided whenever  $X \rightarrow Y$  is an epimorphism with  $YJ = 0$ , then  $\text{Hom}(P, X) \rightarrow \text{Hom}(P, Y)$  is an epimorphism) is projective.
- (6) Every finitely presented  $J$ -projective right  $R$ -module is projective.
- (7) If  $M$  is a finitely presented right  $R$ -module such that  $M/MJ$  is projective and  $\text{Tor}_R^1(R/J, M) = 0$ , then  $M$  is projective.
- (8) If  $M$  is a finitely presented right  $R$ -module such that  $\text{Ext}_R^1(M, Y) = 0$  for all modules  $Y$  with  $YJ = 0$ , then  $M$  is projective.
- (9) The analogues of the above properties for left  $R$ -modules.

All these properties are satisfied for commutative rings, as well as for rings such that every prime factor ring is right (or left) Goldie (in particular, for right or left noetherian rings, and for rings with a polynomial identity).

If  $R$  is a commutative domain or a right noetherian ring, then (4) is valid for every (not necessarily finitely generated) projective  $R$ -module.

Note that Zöschinger called any ring satisfying the condition (2) *right-L-ring* and showed that this notion is left-right symmetric (see [18]).

In the light of Remark 2.13 we have the following theorem:

- 2.14. Theorem.** (1) *Let  $M$  be an absolute co-supplement module. Then  $M$  is projective in each of the following cases for  $R$ :*
- (a)  $R$  is a commutative domain.
  - (b)  $R$  is a right noetherian ring.
- (2) *Let  $M$  be a finitely generated absolute co-supplement module. Then  $M$  is projective in each of the following cases for  $R$ :*
- (a)  $R$  is commutative.
  - (b)  $R$  is a ring such that every prime factor ring is right (or left) Goldie.
  - (c)  $R$  is a right or left noetherian ring.
  - (d)  $R$  is a ring with polynomial identity.

*Proof.* By Remark 2.13 and Lemmas 2.11 and 2.12. □

**2.15. Theorem.** *If  $R$  is a right hereditary ring, then every absolute co-supplement right  $R$ -module is projective.*

*Proof.* Let  $M$  be an absolute co-supplement module. Then there exists a projective module  $P$  such that  $P/X \cong M$  for some supplement submodule  $X$  of  $P$ . By [18, Corollary (1) of Lemma 2.1],  $M$  is projective. □

There exists an absolute co-supplement module which is not projective:

**2.16. Example.** Let  $R$  be a ring with the Jacobson radical  $J$ . Zöschinger proved in [18, Theorem 1.2] that in  $R_R$  every supplement is a direct summand if and only if  $ab = 0$  and  $1 - (a + b) \in J$  imply  $b \cdot \frac{1}{a + b} \cdot a = 0$ . (This means that for any commutative ring  $R$  supplements and direct summands in  $R$  coincide). What Zöschinger observes is that

(\*) if  $v$  is in  $R$  with  $vR = v^2R$  and  $v - v^2 \in J$ , then  $vR$  is a supplement of  $(1 - v)R$ : Since  $vR + (1 - v)R = R$  it follows that  $(v - v^2)R = vR \cap (1 - v)R$ .  $(v - v^2) = (1 - v)v = (1 - v)v^2t = v(v - v^2)t \in vJ$ . Since  $J \ll R$ ,  $(v - v^2)R$  is small in  $vR$ .

Note that if  $t$  is in  $R$  with  $v^2t = v$ , then  $(v - v^2)t = vt - v \in J$ . Moreover  $v(1 - (vt - v)) = v^2$ . So that  $u = 1 - (vt - v)$  satisfies that  $v = v^2u^{-1}$ . Therefore (\*) is equivalent to

(\*\*) If  $v$  is in  $R$  with  $v - v^2 \in J$  such that there exists a unit  $u \in R$  with  $u \in 1 + J$  and  $v = v^2u^{-1}$ , then  $vR$  is a supplement of  $(1 - v)R$ .

When this happens on the right it also happens on the left.

Note that  $v$  is idempotent modulo  $J$ . Having (\*\*) is equivalent to have a projective left module that modulo  $J$  is isomorphic to the projective  $R/J$ -module generated by  $v + J$  which in turn is equivalent to have a projective right module that modulo  $J$  is isomorphic to the projective  $R/J$ -module generated by  $1 - v + J$ .

The Gerasimov-Sakhaev example in [8] was the first showing that for a semilocal ring (\*\*) could be satisfied without  $v$  being idempotent.

Let  $k$  be a field. Take a free algebra on two generators  $k \langle x, y \rangle$ , and let  $\varphi : k \langle x, y \rangle \rightarrow k \times k$  be defined by  $\varphi(x) = (1, 0)$  and  $\varphi(y) = (0, 1)$ . Since  $\varphi(y)\varphi(x) = (0, 0)$  the map can be factorized to the ring  $S = k \langle x, y \rangle / (yx)$ . We call the induced map still  $\varphi$ .

Let  $\Sigma$  denote the set of all squared matrices such that the image via  $\varphi$  is invertible. Then we can invert these matrices to get a new ring  $R = S_\Sigma$  and morphisms  $\lambda : S \rightarrow R$  and  $\varphi' : R \rightarrow k \times k$  such that  $\varphi = \varphi'\lambda$ . Note that this implies that  $\varphi'$  is onto. By the construction,  $\text{Ker}(\varphi') = J$ . So that  $R$  is a semilocal ring and  $R/J \cong k \times k$ .

By abuse of notation (which is better not to do), we call  $x, y \in R$  to the images of  $x + (yx)$  and  $y + (yx)$  via  $\lambda$ . Since  $\varphi(x + y + (yx)) = (1, 1)$  which is a  $1 \times 1$  invertible matrix over  $k \times k$  we deduce that  $u = x + y$  is invertible in  $S$ . Note that  $y(x + y) + (yx) = y^2 + (yx)$  so that  $y = y^2u$  in  $R$ . Note also that  $y - y^2 + (yx)$  is in the kernel of  $\varphi$  so that  $\varphi'(y - y^2) = 0$  and so  $y - y^2 \in J$ . Therefore,  $yR$  is a supplement of  $(1 - y)R$  in  $R$ . Note that in [8], Gerasimov and Sakhaev show that  $yR$  is not a direct summand of  $R_R$ . Therefore the  $R$ -module  $R/yR$  is absolute co-supplement but not projective over the ring  $R$ .

**2.17. Example.** It is well-known that  $\mathbb{Q}_z$  is flat but not projective, and so it is not absolute co-supplement by Theorem 2.9.

Therefore we can give the following theorem:

**2.18. Theorem.** *Let  $M$  be a module. Then  $M$  is absolute co-supplement and flat if and only if  $M$  is projective.*

*Proof.* We only need to prove the necessity. Since  $M$  is absolute co-supplement, there exists a projective module  $P$  such that  $P/X \cong M$  for some supplement submodule  $X$  of  $P$ . Since  $P/X$  is flat,  $X$  is a pure submodule of  $P$  by [1, Exercise 19(11) or Lemma 19.18]. Now by [18, Corollary (2) of Lemma 2.1],  $X$  is a direct summand of  $P$ . Hence  $M$  is projective.  $\square$

**2.19. Proposition.** *If  $M$  is a finitely generated absolute co-supplement module, then  $M$  is finitely presented.*

*Proof.* Since  $M$  is absolute co-supplement and finitely generated, there is a finitely generated projective module  $P$  such that  $M \cong P/X$  where  $X$  is a supplement submodule of  $P$ . It is well-known that  $X$  is finitely generated since  $P$  is finitely generated. Therefore  $M$  is finitely presented.  $\square$

**2.20. Corollary.** *Let  $R$  be a serial ring. Then every finitely generated absolute co-supplement module is a finite direct sum of local modules.*

*Proof.* By [17, Corollary 3.4] and Proposition 2.19,  $M$  is a finite direct sum of local modules.  $\square$

Any finitely presented module need not be absolute co-supplement (for example,  $\mathbb{Z}/n\mathbb{Z}$  with  $n$  nonzero). Now we have the following corollary:

**2.21. Corollary.** *Let  $M$  be a finitely generated flat module. Then the following are equivalent:*

- (i)  $M$  is projective.
- (ii)  $M$  is absolute co-supplement.
- (iii)  $M$  is finitely presented.

*Proof.* (i)  $\Rightarrow$  (ii) : Clear.

(ii)  $\Rightarrow$  (i) : By Theorem 2.18.

(ii)  $\Rightarrow$  (iii) : By Proposition 2.19.

(iii)  $\Rightarrow$  (i) : By [10, Theorem 4.30] or [9, Corollary 5.3],  $M$  is projective.  $\square$

**2.22. Theorem.** *Let  $0 \longrightarrow X \longrightarrow P \longrightarrow M \longrightarrow 0$  be an exact sequence of finitely generated right  $R$ -modules with  $P$  projective (for example if  $M$  is finitely presented) and  $J$  the Jacobson radical of  $R$ . Then the following are equivalent:*

- (i)  $M$  is absolute co-supplement.
- (ii) The induced sequence  $0 \longrightarrow X/XJ \longrightarrow P/PJ \longrightarrow M/MJ \longrightarrow 0$  is split exact.
- (iii) If  $Y$  is a right  $R$ -module with  $YJ = 0$ , then  $\text{Ext}_R^1(M, Y) = 0$ .

*Proof.* By [13, Theorem 2.6].  $\square$

On the other hand, we have:

**2.23. Proposition.** *Let  $M$  be an absolute co-supplement module. Then  $\text{Ext}_R^1(M, K) = 0$  for all modules  $K$  with  $\text{Rad}(K) = 0$ .*

*Proof.* Let  $0 \longrightarrow K \longrightarrow A \longrightarrow M \longrightarrow 0$  be a short exact sequence. Since  $M$  is absolute co-supplement, there exists a submodule  $L$  of  $A$  such that  $A = K + L$  and  $K \cap L \ll K$ . So  $A = K \oplus L$ . Hence  $\text{Ext}_R^1(M, K) = 0$ .  $\square$

It is well known that for a nonzero finitely generated  $\mathbb{Z}$ -module  $M$ ,  $\text{Ext}_{\mathbb{Z}}^1(M, \mathbb{Z}) = T(M)$ , where  $T(M)$  is the torsion submodule of  $M$ . Now Proposition 2.23 gives us a well known result for nonzero finitely generated projective  $\mathbb{Z}$ -modules: Let  $M$  be a nonzero finitely generated projective  $\mathbb{Z}$ -module. Then  $M$  is torsion-free.

Now we close this section with the following elementary observations:

**2.24. Lemma.** *Let  $R$  be a ring. If  $R$  is right hereditary, then every submodule of an absolute co-supplement right  $R$ -module is absolute co-supplement.*

*Proof.* By Theorem 2.15.  $\square$

**2.25. Theorem.** *Let  $R$  be a right noetherian ring. Then  $R$  is right hereditary if and only if every submodule of an absolute co-supplement right  $R$ -module is absolute co-supplement. In this case  $R$  is left semihereditary.*

*Proof.* By Lemma 2.24, Theorem 2.14(1)(b) and [10, Corollary 7.65].  $\square$

**2.26. Theorem.** *Let  $R$  be a commutative domain. Then  $R$  is a Dedekind domain if and only if every submodule of an absolute co-supplement  $R$ -module is absolute co-supplement.*

*Proof.*  $(\Rightarrow)$  : By Theorem 2.25.

$(\Leftarrow)$  : By Theorem 2.14(1)(a),  $R$  is hereditary. Thus  $R$  is a Dedekind domain.  $\square$

### 3. Absolute Co-coclosed Modules

**3.1. Lemma.** *Let  $M$  be an absolute co-coclosed module such that it has a projective cover. Then  $M$  is projective.*

*Proof.* There exists a projective module  $P$  and a submodule  $K$  of  $P$  such that  $K$  is small in  $P$  and  $M \cong P/K$ . Since  $M$  is absolute co-coclosed,  $K$  is a coclosed submodule of  $P$ . Thus  $K = 0$ . Hence  $M$  is projective.  $\square$

**3.2. Proposition.** *Let  $R$  be a right perfect (semiperfect) ring. Then the following are equivalent for a (finitely generated) module  $M$ :*

- (i)  $M$  is projective.
- (ii)  $M$  is absolute co-supplement.
- (iii)  $M$  is absolute co-coclosed.

*Proof.* By Lemma 3.1.  $\square$

Recall that any module  $M$  is co-semisimple (or a  $V$ -module) if every simple  $R$ -module is  $M$ -injective. Equivalently,  $\text{Rad}(M/T) = 0$  for every submodule  $T$  of  $M$  (see [4, 2.13]).

**3.3. Lemma.** (see [3, 3.8]) *Let  $M$  be any module. Then  $M$  is a  $V$ -module if and only if every submodule of  $M$  is coclosed.*

**3.4. Proposition.** *Let  $M$  be a module. If every factor module of  $M$  is absolute co-coclosed, then  $M$  is a  $V$ -module.*

*Proof.* In this case every submodule of  $M$  is coclosed. By Lemma 3.3,  $M$  is a  $V$ -module.  $\square$

By [4, 2.13], we know that any ring  $R$  is a right  $V$ -ring if and only if every right  $R$ -module is a  $V$ -module. Thus we have the following lemma:

**3.5. Lemma.** (see [6, Proposition 2.1]) *Let  $R$  be a ring.  $R$  is a right  $V$ -ring if and only if for any  $R$ -module  $M$ , every submodule of  $M$  is coclosed in  $M$ .*

**3.6. Theorem.** *Let  $R$  be a ring. The following are equivalent:*

- (i) Every right  $R$ -module is absolute co-coclosed.
- (ii) Every factor module (submodule) of every right  $R$ -module is absolute co-coclosed.
- (iii) Every factor module of every projective right  $R$ -module is absolute co-coclosed.
- (iv) Every factor module of every free right  $R$ -module is absolute co-coclosed.
- (v) Every factor module of  $R_R$  is absolute co-coclosed.
- (vi)  $R$  is a right  $V$ -ring.

*Proof.* (i)  $\Leftrightarrow$  (ii) and (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v) are clear.

(v)  $\Rightarrow$  (vi) : By Proposition 3.4.

(vi)  $\Rightarrow$  (i) : Assume that  $R$  is a right  $V$ -ring. Let  $M$  be any  $R$ -module. Let  $M \cong T/X$  for a module  $T$  and a submodule  $X$  of  $T$ . By Lemma 3.5,  $X$  is coclosed in  $T$ . Thus  $M$  is absolute co-coclosed.  $\square$

**3.7. Example.** There exists an absolute co-coclosed module  $M$  which is not absolute co-supplement over a right  $V$ -ring which is not semisimple by Theorems 3.6 and 2.8. For example, let  $K$  be a field and let  $R = \prod_{n \geq 1} K_n$  with  $K_n = K$  for all  $n \geq 1$ . Then the ring  $R$  is a commutative von Neumann regular ring (namely it is a  $V$ -ring) which is not

semisimple. Note that  $A = \bigoplus_{n \geq 1} K_n$  is not a direct summand of  $R$ . By Theorem 3.6,  $R/A$  is an absolute co-coclosed  $R$ -module. By Corollary 2.10 it is not absolute co-supplement.

As an easy observation we can give the following lemma:

**3.8. Lemma.** *Let  $M$  be an absolute co-coclosed module. Then every epimorphism from  $A$  to  $M$  with small kernel is an isomorphism for any module  $A$ .*

**3.9. Proposition.** *Let  $M$  be a quasi-discrete module. If  $M$  is absolute co-coclosed, then  $M$  is discrete.*

*Proof.* By Lemma 3.8 and [14, Lemma 5.1]. □

Note that any discrete module need not be absolute co-coclosed ( $\mathbb{Z}/p\mathbb{Z}$ , where  $p$  is any prime).

We announce that the following lemma was proved by Zöschinger in [19]. In this note we give his proof for the completeness.

**3.10. Lemma.** (see [19, Lemma A4]) *Let  $U \leq V \leq M$  be submodules of  $M$ . If  $U$  is coclosed in  $M$  and  $V/U$  is coclosed in  $M/U$ , then  $V$  is coclosed in  $M$ .*

*Proof.* Let  $X \leq V$  and  $V/X \ll M/X$ . Firstly, we will prove that  $U/(X \cap U) \ll M/(X \cap U)$ . Let  $X \cap U \leq W \leq M$  with  $W + U = M$ :

Step 1:  $V/[U + (W \cap X)] \ll M/[U + (W \cap X)]$ : Let  $U + (W \cap X) \leq Z \leq M$  and  $Z + V = M$ . Then  $(Z \cap W) + V = M$ . Since  $V/X$  is small in  $M/X$ ,  $(Z \cap W) + X = M$  and since  $(Z \cap W) + (X \cap W) = W$ ,  $W \leq Z$ . Finally since  $Z + W = M$ ,  $Z = M$ .

Step 2: Since  $V/U$  is coclosed in  $M/U$ ,  $U + (W \cap X) = V$ . Now  $(U \cap X) + (W \cap X) = X$ , and  $X \leq W$ . Since  $V/X \ll M/X$ ,  $W = M$ . □

By using Lemma 3.10 the following Theorem 3.11 can be obtained easily.

**3.11. Theorem.** *The class*

$$\text{Coclosed} = \{ \mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \mid A \text{ is coclosed in } B \}$$

*is a proper class.*

A module  $C$  is said to be *coprojective relative to a proper class  $\mathcal{P}$  (or  $\mathcal{P}$ -coprojective)* if every epimorphism  $B \rightarrow C$  is a  $\mathcal{P}$ -epimorphism. With the help of (P3) it can be shown that if  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a short exact sequence in  $\mathcal{P}$  with the modules  $A$  and  $C$  both  $\mathcal{P}$ -coprojective, then so is  $B$  [12, Proposition 1.9]. A module  $C$  is a  $\mathcal{P}$ -coprojective module if and only if there is a  $\mathcal{P}$ -epimorphism of a projective module  $B$  onto  $C$  [12, Proposition 1.12]. When (P3') is taken into account it follows that the image of a  $\mathcal{P}$ -coprojective module under a  $\mathcal{P}$ -epimorphism is always  $\mathcal{P}$ -coprojective [12, Proposition 1.13]. In this paper *Coclosed-coprojective (Suppl-coprojective)* modules have been called absolute co-coclosed (absolute co-supplement) modules. As a result of above information Proposition 3.12, Theorem 3.13 and Corollary 3.14 are obtained immediately. Here we give their proofs for completeness. For similar results on absolute co-supplement modules see [5].

**3.12. Proposition.** *Every extension of an absolute co-coclosed module by an absolute co-coclosed module is absolute co-coclosed.*

*Proof.* Let  $N \leq M$  such that  $N$  and  $M/N$  are absolute co-coclosed. We want to show that  $M$  is absolute co-coclosed. Let  $0 \rightarrow X \rightarrow T \rightarrow M \rightarrow 0$  be any short

exact sequence. We have the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & X & = & X & & \\
 & & \downarrow f & & \downarrow g \circ f & & \\
 0 & \longrightarrow & K & \xrightarrow{g} & T & \longrightarrow & M/N \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

Since  $N$  is absolute co-coclosed,  $f$  is a *Coclosed*-monomorphism and since  $M/N$  is absolute co-coclosed,  $g$  is a *Coclosed*-monomorphism. Therefore  $g \circ f$  is a *Coclosed*-monomorphism by Theorem 3.11. Thus  $M$  is an absolute co-coclosed module.  $\square$

**3.13. Theorem.** *For a module  $M$  the following conditions are equivalent:*

- (i)  $M$  is an absolute co-coclosed module.
- (ii) There exists a projective (or free) module  $P$  with  $M \cong P/N$  such that  $N$  is coclosed in  $P$ .

*Proof.* (i)  $\Rightarrow$  (ii) : For every module  $M$ , there is a projective (or free) module  $P$  and an epimorphism  $f$  from  $P$  to  $M$ . So  $M \cong P/\text{Ker } f$  and by the definition of an absolute co-coclosed module  $\text{Ker } f$  is coclosed in  $P$ .

(ii)  $\Rightarrow$  (i) : Let  $0 \longrightarrow X \longrightarrow T \xrightarrow{\alpha} M \longrightarrow 0$  be a short exact sequence. By (ii), there is a short exact sequence  $0 \longrightarrow N \longrightarrow P \xrightarrow{g} M \longrightarrow 0$  with a projective module  $P$  such that  $N$  is coclosed in  $P$ . So we have the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & N & = & N & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & X & \longrightarrow & Y & \xrightarrow{f} & P \longrightarrow 0 \\
 & & \parallel & & \downarrow \beta & & \downarrow g \\
 0 & \longrightarrow & X & \longrightarrow & T & \xrightarrow{\alpha} & M \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

where  $Y$  together with homomorphisms  $f$  and  $\beta$  is a pullback of the pair of homomorphisms  $g$  and  $\alpha$ . Since  $P$  is projective,  $f$  is a splitting epimorphism. So  $\alpha \circ \beta = g \circ f$  is a *Coclosed*-epimorphism. Then  $\alpha$  is a *Coclosed*-epimorphism by Theorem 3.11. Therefore  $M$  is an absolute co-coclosed module.  $\square$

**3.14. Corollary.** *If  $N$  is a coclosed submodule of an absolute co-coclosed module  $M$ , then  $M/N$  is also absolute co-coclosed.*

*Proof.* Since  $N$  is coclosed in  $M$ , the short exact sequence

$$0 \longrightarrow N \longrightarrow M \xrightarrow{\sigma} M/N \longrightarrow 0$$

is in the class *Coclosed*. Since  $M$  is an absolute co-coclosed module, there exists a short exact sequence

$$0 \longrightarrow K \longrightarrow P \xrightarrow{f} M \longrightarrow 0$$

with a projective module  $P$  such that  $K$  is coclosed in  $P$  by Theorem 3.13. Then we have the following diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & K & = & K & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & P & \xrightarrow{\sigma \circ f} & M/N \longrightarrow 0 \\ & & \downarrow & & \downarrow f & & \parallel \\ 0 & \longrightarrow & N & \longrightarrow & M & \xrightarrow{\sigma} & M/N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where  $T = \text{Ker}(\sigma \circ f)$ . Now  $f$  and  $\sigma$  are *Coclosed*-epimorphisms, therefore  $\sigma \circ f$  is also a *Coclosed*-epimorphism by Theorem 3.11. So by Theorem 3.13,  $M/N$  is absolute co-coclosed.  $\square$

Let  $M$  be any module. Talebi and Vanaja define

$$\overline{Z}(M) = \bigcap \{ \text{Ker } g \mid g \in \text{Hom}(M, L), L \text{ is a small module} \}.$$

They call  $M$  a *non-cosingular* module if  $\overline{Z}(M) = M$  (see [16]).

**3.15. Corollary.** *Let  $M$  be an absolute co-coclosed module and  $N \leq M$ .*

- (1) *If  $N$  is a non-cosingular submodule of an absolute co-coclosed module  $M$ , then  $M/N$  is also absolute co-coclosed.*
- (2) *If  $M$  is, in addition, non-cosingular then the following are equivalent:*
  - (i)  *$N$  is non-cosingular.*
  - (ii)  *$M/N$  is absolute co-coclosed.*
  - (iii)  *$N$  is coclosed in  $M$ .*

*Proof.* (1) Clear by Corollary 3.14 and [16, Lemma 2.3 (2)].

(2) (i)  $\Rightarrow$  (ii) : By (1).

(ii)  $\Rightarrow$  (iii) : By definition.

(iii)  $\Rightarrow$  (i) : By [16, Lemma 2.3 (3)].

$\square$

Finally we prove the following:

**3.16. Proposition.** *Let  $M = M_1 + M_2$  such that every factor modules of  $M_1$  and  $M_2$  are absolute co-supplement (absolute co-coclosed). Then every factor module of  $M$  is absolute co-supplement (absolute co-coclosed).*

*Proof.* Let  $N$  be a submodule of  $M$ . Then  $(M_1 + N)/N \cong M_1/(N \cap M_1)$  is absolute co-supplement (absolute co-coclosed). Since

$$\frac{M/N}{(M_1 + N)/N} \cong \frac{M_2}{M_2 \cap (M_1 + N)}$$

is absolute co-supplement (absolute co-coclosed), by Proposition 2.2 (Proposition 3.12),  $M/N$  is absolute co-supplement (absolute co-coclosed).  $\square$

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