Resonant dispersive Benney and Broer-Kaup systems in 2+1 dimensions

Jyh-Hao Lee[1] and Oktay K Pashaev[2]

[1] Institute of Mathematics, Academia Sinica, Taipei, 10617 Taiwan
[2] Department of Mathematics, Izmir Institute of Technology, Urla-Izmir, 35430 Turkey

E-mail: [1] leejh@math.sinica.edu.tw

E-mail: [2] oktaypashaev@iyte.edu.tr

Abstract. We represent the Benney system of dispersionless hydrodynamic equations as NLS type infinite system of equations with quantum potential. We show that negative dispersive deformation of this system is an integrable system including vector generalization of Resonant NLS and 2+1 dimensional nonlocal Resonant NLS. We obtain bilinear form and soliton solutions in these systems and find the resonant character of soliton interaction. Equivalent vector Broer-Kaup system and non-local 2+1 dimensional nonlocal Broer-Kaup equation are constructed.

1. Introduction

Resonant interaction of solitons in KP-II system has attracted much attention recently. This type of solitons can be constructed from the second and the third flow of the $SL(2, R)$ AKNS hierarchy [1]. The second flow represented by the Reaction-Diffusion (RD) equation has envelope soliton form as the Resonant NLS (RNLS) equation [2]. The resonant interaction of envelope solitons by creation and annihilation processes has been then studied in several nonlinear Schrödinger type models of RNLS type [3]. As was shown in [4], any envelope soliton equation admits the resonant counterpart. This process of recovering dispersion of integrable models from the dispersionless equations always has the resonant option. In the present paper we study dispersionless hydrodynamic system of Benney. Positive dispersive version of this system was considered by Zakharov [5], and soliton solutions was found in [6]. Here we are going to study the negative dispersion equation and resonant phenomena for it.

2. Benney system in the Schrödinger representation

The system of Benney equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} = 0, \tag{1}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta u) = 0, \tag{2}$$

$$h = \int_0^1 \eta(x, \xi, t)\, d\xi, \tag{3}$$

as a descriptive of long shallow water waves has been reduced to an infinite system of two-dimensional hydrodynamic equations [5]. Here the horizontal velocity $u(x, \xi, t)$ is a function of

layers in the z-direction, enumerated by parameter ξ ($0 < \xi < 1$) and the fluid surface shape is $h(x,t)$. The Benney system (1), (2), (3) can be rewritten as nonlinear Schrodinger type wave equation by the Madelung transform to complex wave function

$$\psi(x,\xi,t) = \sqrt{\eta}e^{-i\int_{-\infty}^{x} udx}. \tag{4}$$

Then we get the NLS type infinite system of equations with the quantum potential(QPNLS)

$$i\psi_t = \frac{1}{2}\psi_{xx} - h\psi - \frac{1}{2}\frac{|\psi|_{xx}}{|\psi|}\psi, \tag{5}$$

where

$$h = \int_0^1 |\psi(x,\xi,t)|^2 d\xi. \tag{6}$$

This Madelung transform is some kind of complex Cole-Hopf transformation

$$i\frac{\psi_x}{\psi} = u + iv, \tag{7}$$

where the imaginary part of complex velocity, $v = (\ln\sqrt{\eta})_x$, sometimes is called the quantum velocity. The Benney system rewritten in this wave form is the Hamiltonian system

$$\frac{\partial}{\partial t}\psi(x,\xi,t) = i\frac{\delta H}{\delta\bar{\psi}(x,\xi,t)}, \quad -\frac{\partial}{\partial t}\bar{\psi}(x,\xi,t) = i\frac{\delta H}{\delta\psi(x,\xi,t)}, \tag{8}$$

with

$$H = \frac{1}{2}\int_{-\infty}^{\infty} dx[\int_a^b d\xi\,(\bar{\psi}_x\psi_x - |\psi|_x|\psi|_x) + (\int_a^b d\xi\,|\psi|^2)^2]. \tag{9}$$

2.0.1. Discrete layers   In the special case, when the flow is divided into N layers,

$$\eta(x,\xi,t) = \sum_{k=1}^{N} \eta_k(x,t)\delta(\xi - \xi_k) \Rightarrow h(x,t) = \sum_{k=1}^{N} \eta_k(x,t) \tag{10}$$

the Benney system reduces to a multicomponent (vector) hydrodynamic system. This system via Madelung transform $\psi_k = \sqrt{\eta_k}e^{-i\int_{-\infty}^{x} u_k dx}$ to N-complex wave functions $\psi(x,\xi_k,t) \equiv \psi_k(x,t)$, $k = 1, 2, ..., N$, gives the vector NLS equation with quantum potential

$$i\psi_t^a = \frac{1}{2}\psi_{xx}^a - \left(\sum_{b=1}^{n} |\psi^b|^2\right)\psi^a - \frac{1}{2}\frac{|\psi^a|_{xx}}{|\psi^a|}\psi^a. \tag{11}$$

2.0.2. Dispersive deformation and RNLS   It is noted that in (5) the dispersion term and the quantum potential term have equal but opposite sign strength. If this condition is broken then we have more general system

$$i\psi_t = \frac{1}{2}\psi_{xx} - h\psi - \frac{1}{2}(1 - \hbar^2)\frac{|\psi|_{xx}}{|\psi|}\psi, \tag{12}$$

$$h = \int_a^b |\psi(x,\xi,t)|^2 d\xi, \tag{13}$$

where $\hbar$ is deformation parameter. This system is a nonlocal 2+1-dimensional generalization of the Resonant NLS equation [2]. For given ψ in (4) we can relate new wave function

$$\chi(x,\xi,t) = \sqrt{\eta}\, e^{-\frac{i}{\hbar}\int_{-\infty}^{x} u\, dx}, \tag{14}$$

so that the system (12),(13) becomes the infinite system of NLS equations in 2+1 dimensions with non-local self-interaction considered in [5]

$$i\hbar\chi_t = -\frac{\hbar^2}{2}\chi_{xx} - h\chi, \qquad h = \int_a^b |\chi(x,\xi,t)|^2 d\xi. \tag{15}$$

For discrete set of parameter $\xi_1, \xi_2, ..., \xi_N$ the wave functions $\chi^k(x,t) = \chi(x,\xi_k,t)$ satisfy the vector $U(N)$ NLS equation, and for $N \to \infty$, the infinite component vector $U(\infty)$ NLS. Recently in [6] this equation has been considered as nonlocal NLS in 2+1 dimensions with a domain of integration as a whole real line, $-\infty < \xi < \infty$, and the bilinear method of Hirota was applied to solve it for N - soliton solutions localized in plane.

3. Negative dispersion and RNLS

Dispersive deformation described by system (12), (13) is not unique. Indeed the semiclassical expansion of a solution is going in according to parameter $\hbar^2$. Analyticity of this expansion implies that semiclassical series should be analytic in full disk in complex plane with radius $\hbar$. This poses intriguing question of behavior of this expansion inside of the disk. By replacing $\hbar \to i\hbar$ then we get

$$i\psi_t = \frac{1}{2}\psi_{xx} - h\psi - \frac{1}{2}(1+\hbar^2)\frac{|\psi|_{xx}}{|\psi|}\psi, \tag{16}$$

where

$$h = \int_a^b |\psi(x,\xi,t)|^2 d\xi. \tag{17}$$

Both systems (12), (13) and (16), (17) correspond to dispersive Benney system with positive and negative dispersions correspondingly:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} = \pm\frac{\hbar^2}{2}\left(\frac{(\sqrt{\eta})_{xx}}{\sqrt{\eta}}\right)_x, \tag{18}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta u) = 0, \tag{19}$$

$$h = \int_a^b \eta(x,\xi,t)\, d\xi. \tag{20}$$

Positive dispersion case was treated in details in [5]. Here we are going to study the negative dispersion case. We have several motivations for this.

The negative dispersion hydrodynamic system (18), (19), (20) for particular case $\eta(x,\xi,t) = \eta(x,t)\delta(\xi - \xi_0)$ of one layer, and with $h = \eta$, describes propagation of long magneto-acoustic waves in cold plasma, moving across the magnetic field, in shallow water approximation [9], [10]. Parameter $\hbar = 2\beta$ then is rescaling parameter for this approximation. Reaction-diffusion form of this system has been derived from the Jackiw-Teitelboim model of low dimensional gravity [11], [12], [2]. It has subsequently been derived in plasma physics and an auto-Bäcklund transformation was constructed in that context in [8]. The resonant NLS equation as a theoretical capillarity model was discussed in [7]. In addition, it turns out that a wide class of NLS equations with underlying Hamiltonian structure may be reduced to consideration of the resonant NLS equation [3].

4. Nonlocal Reaction-Diffusion System

The negative dispersion system (16), (17), can not be reduced to NLS equation and solvability of the model is not evident. However, if instead of Madelung transform (4), we introduce the couple of real functions,

$$e^+(x,\xi,t) = \sqrt{\eta}\, e^{-\int^x u dx'}, \quad e^-(x,\xi,t) = \sqrt{\eta}\, e^{+\int^x u dx'} \tag{21}$$

then the system (18), (19), (20) for negative dispersion case can be rewritten as the couple of nonlocal reaction-diffusion equations

$$e_t^+ = \frac{1}{2}e_{xx}^+ + he^+, \tag{22}$$

$$-e_t^- = \frac{1}{2}e_{xx}^- + he^-, \tag{23}$$

$$h = \int_a^b e^+ e^-\, d\xi. \tag{24}$$

When integration domain is given by the set of discrete points, for $e^\pm(x,\xi_a,t) \equiv e^{\pm(a)}(x,t)$ we are getting the vector reaction-diffusion system

$$e_t^{+(a)} = \frac{1}{2}e_{xx}^{+(a)} + \left(\sum_{b=1}^n e^{+(b)}e^{-(b)}\right)e^{+(a)}, \tag{25}$$

$$-e_t^{-(a)} = \frac{1}{2}e_{xx}^{-(a)} + \left(\sum_{b=1}^n e^{+(b)}e^{-(b)}\right)e^{-(a)}. \tag{26}$$

4.1. Linear Problem

The linear problem in Dirac's notations is

$$\Phi_x = U\Phi, \quad \Phi_t = V\Phi, \tag{27}$$

where

$$\Phi = \begin{pmatrix} \varphi_0 \\ |\varphi> \end{pmatrix}, \tag{28}$$

$$U = \begin{pmatrix} -i\lambda & <e^-| \\ |e^+> & i\lambda I \end{pmatrix}, \tag{29}$$

$$V = \begin{pmatrix} -i\lambda^2 - \frac{i}{2}<e^-|e^+> & \lambda<e^-| + \frac{i}{2}<e_x^-| \\ \lambda|e^+> -\frac{i}{2}|e_x^+> & i\lambda^2 I + \frac{i}{2}|e^+><e^-| \end{pmatrix}. \tag{30}$$

Compatibility condition for this linear system is equivalent to general vector form of the Reaction-Diffusion (RD) system of equations

$$|e_t^+> = \frac{1}{2}|e_{xx}^+> + <e^-|e^+>|e^+>, \tag{31}$$

$$-<e_t^-| = \frac{1}{2}<e_{xx}^-| + <e^-|e^+><e^-|. \tag{32}$$

Here most general real symmetric bilinear form is mixture of discrete and continuous spectrum of ξ:

$$< e^- | e^+ > = \sum_n e_n^+(x,t) e_n^-(x,t) + \int_a^b e^+(x,\xi,t) e^-(x,\xi,t)\, d\xi,$$

corresponding to the unit identity

$$\sum_n |\xi_n >< \xi_n| + \int_a^b |\xi > d\xi < \xi| = I,$$

and giving the vector discrete-continuous system of RD equations.

4.2. Conserved quantities

The mass distribution density along ξ direction, $\mu(\xi) = dM/d\xi$, so that $dM = \mu(\xi)d\xi$, is conserved quantity for arbitrary ξ:

$$\mu(\xi) = \int_{-\infty}^{\infty} dx\, e^+ e^-. \tag{33}$$

Hence the total mass

$$M = \int_a^b d\xi\, \mu(\xi) = \int_a^b d\xi \int_{-\infty}^{\infty} dx\, e^+ e^- \tag{34}$$

is conserved as well. Conserved quantities are also the total momentum and the Hamiltonian function respectively

$$P = \int_a^b d\xi \int_{-\infty}^{\infty} dx\, (e^+ e_x^- - e_x^+ e^-), \tag{35}$$

$$H = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[\int_a^b d\xi e_x^+ e_x^- - \left(\int_a^b d\xi e^+ e^- \right)^2 \right]. \tag{36}$$

4.3. Hamiltonian structure

$$e_t^+ = -\frac{\delta H}{\delta e^-}, \quad e_t^- = +\frac{\delta H}{\delta e^+}, \tag{37}$$

or by Poisson bracket

$$\{A, B\} = \int_{-\infty}^{\infty} \int_a^b \left(\frac{\delta A}{\delta e^+} \frac{\delta B}{\delta e^-} - \frac{\delta A}{\delta e^-} \frac{\delta B}{\delta e^+} \right), \tag{38}$$

$$e_t^+ = \{H, e^+\}, \quad e_t^- = \{H, e^-\}. \tag{39}$$

5. Bilinear Form

For discrete spectrum of ξ we have the vector RD system

$$e_{n_t}^+ = e_{n\,xx}^+ + \left(\sum_{m=1}^{\infty} e_m^+ e_m^- \right) e_n^+, \tag{40}$$

$$-e_{nt}^- = e_{n\,xx}^- + \left(\sum_{m=1}^{\infty} e_m^+ e_m^- \right) e_n^-, \tag{41}$$

where $< \xi_n | e^+ > = e_n^+(x,t)$, $< e^- | \xi_n > = e_n^-(x,t)$. For continuous case we have the nonlocal reaction-diffusion system

$$e_t^+ = e_{xx}^+ + he^+, \tag{42}$$
$$-e_t^- = e_{xx}^- + he^-, \tag{43}$$
$$h = \int_a^b e^+ e^- d\xi. \tag{44}$$

were for convenience to compare with RNLS we drop coefficient $1/2$ in front of dispersion. Here $< \xi | e^+ > = e^+(x,\xi,t)$, $< e^- | \xi > = e^-(x,\xi,t)$. In addition, to treat most general situation we consider integration domain between points a and b, so that in particular case $a = 0$, $b = 1$ we recover the original structure of Benney system. Moreover, for improper integral $a \to -\infty$, $b \to \infty$, we have 2+1 dimensional nonlocal problem similar to the one considered in [6].

We represent two real functions $e^\pm(x,\xi,t)$ in terms of three real functions

$$e^+(x,\xi,t) = \frac{g^+(x,\xi,t)}{f(x,t)}, \quad e^-(x,\xi,t) = \frac{g^-(x,\xi,t)}{f(x,t)}. \tag{45}$$

Then substituting it to the above system (42)-(44) gives bilinear equations

$$(D_t - D_x^2)(g^+ \cdot f) = 0, \tag{46}$$
$$(D_t + D_x^2)(g^- \cdot f) = 0, \tag{47}$$
$$D_x^2(f \cdot f) = \int_a^b g^+ g^- \, d\xi. \tag{48}$$

We have following particular cases of this bilinear form. When g^+ and g^- are independent of ξ the system reduces to the one considered in [2]. If integration region contains only the discrete set of points we have bilinear form for the vector reaction-diffusion system (25), (26)

$$(D_t - D_x^2)(g^{+(a)} \cdot f) = 0, \tag{49}$$
$$(D_t + D_x^2)(g^{-(a)} \cdot f) = 0, \tag{50}$$
$$D_x^2(f \cdot f) = \sum_{a=1}^{n} g^{+(a)} g^{-(a)}. \tag{51}$$

where $a = 1, ..., n$ and

$$e^{+(a)}(x,t) = \frac{g^{+(a)}(x,t)}{f(x,t)}, \quad e^{-(a)}(x,t) = \frac{g^{-(a)}(x,t)}{f(x,t)}. \tag{52}$$

From (48) by dividing on f^2 we have identity

$$\int_a^b e^+(x,\xi,t) \, e^-(x,\xi,t) \, d\xi = 2(\ln f)_{xx}. \tag{53}$$

For the scalar case it gives

$$e^+(x,t) \, e^-(x,t) = 2(\ln f)_{xx}, \tag{54}$$

and for the vector case

$$\sum_{a=1}^{n} e^{+(a)}(x,t) \, e^{-(a)}(x,t) = 2(\ln f)_{xx}. \tag{55}$$

5.1. One Soliton Solution

By Hirota perturbation $g^\pm = \epsilon g_1^\pm + \epsilon^3 g_3^\pm + ...$, $f = 1 + \epsilon^2 f_2 + \epsilon^4 f_4 + ...$ we get one dissipaton solution

$$g_1^\pm = a_1^\pm(\xi)\, e^{\eta_1^\pm}, \quad f = 1 + \frac{e^{\eta_1^+ + \eta_1^-}}{(k_1^+ + k_1^-)^2} \int_a^b a_1^+ a_1^-\, d\xi, \tag{56}$$

where $\eta_1^\pm = k_1^\pm x \pm (k_1^\pm)^2 t + \eta_1^{\pm(0)}$. We notice that constants $\eta_1^{\pm(0)}$ can be absorbed to functions $a_1^\pm(\xi)$, but for comparison our results with RNLS case it would be convenient to use it in the above form. Then particular choice of $a_1^\pm(\xi) = 1$ gives reduction to the RNLS case. For one dissipaton solution we have

$$e^+ = \frac{a_1^+(\xi)\, e^{\eta_1^+}}{1 + \frac{e^{\eta_1^+ + \eta_1^-}}{(k_1^+ + k_1^-)^2} \int_a^b a_1^+ a_1^-\, d\xi}, \quad e^- = \frac{a_1^-(\xi)\, e^{\eta_1^-}}{1 + \frac{e^{\eta_1^+ + \eta_1^-}}{(k_1^+ + k_1^-)^2} \int_a^b a_1^+ a_1^-\, d\xi}, \tag{57}$$

with

$$e^+ e^- = \frac{a_1^+(\xi) a_1^-(\xi)}{4\gamma_{11}} \frac{(k_1^+ + k_1^-)^2}{\cosh^2 \frac{\eta_1^+ + \eta_1^- + \phi_{11} + \ln \gamma_{11}}{2}}, \tag{58}$$

where $\gamma_{11} = \int_a^b a_1^+(\xi) a_1^-(\xi) d\xi$ and $e^{\phi_{11}} = (k_1^+ + k_1^-)^{-2}$. In terms of amplitude and velocity variables $k = \frac{k_1^+ + k_1^-}{2}$, $v = k_1^- - k_1^+$, we have soliton

$$e^+ e^- = \frac{a_1^+(\xi) a_1^-(\xi)}{\gamma_{11}} \frac{k^2}{\cosh^2 k(x - vt - x_0)}, \tag{59}$$

where $x_0 = -(\phi_{11} + \ln \gamma_{11})/2k$.

5.1.1. Scalar Dissipaton In particular case when $a_1^\pm$ are independent of ξ, we can choose $a_1^\pm = 1 \to \gamma_{11} = 1$, and we get one soliton solution of RNLS [2].

5.1.2. Vector Dissipaton For N discrete values of parameter ξ: $\xi_1, ..., \xi_N$ we have solution

$$g_1^{\pm(a)}(x,t) = a_1^{\pm(a)} e^{\eta_1^\pm}, \quad (a = 1, ..., N), \tag{60}$$

$$f = 1 + \frac{e^{\eta_1^+ + \eta_1^-}}{(k_1^+ + k_1^-)^2} \sum_{a=1}^N a_1^{+(a)} a_1^{-(a)}, \tag{61}$$

where $a_1^{\pm(a)} \equiv a_1^\pm(\xi_a)$, $(a = 1, ..., n)$. It gives the vector dissipaton

$$e^{\pm(a)} = \frac{a_1^{\pm(a)} e^{\eta_1^\pm}}{1 + \frac{e^{\eta_1^+ + \eta_1^-}}{(k_1^+ + k_1^-)^2} \sum_{b=1}^N a_1^{+(b)} a_1^{-(b)}}, \tag{62}$$

with soliton shape of the partial mass density

$$e^{+(a)} e^{-(a)} = \frac{a_1^{+(a)} a_1^{-(a)}}{\gamma_{11}} \frac{k^2}{\cosh^2 k(x - vt - x_0)}, \tag{63}$$

where $x_0 = -(\phi_{11} + \ln \sum_{a=1}^n a_1^{+(a)} a_1^{-(a)})/2k$, $\gamma_{11} \equiv \sum_{a=1}^n a_1^{+(a)} a_1^{-(a)}$ and the soliton shape of the total mass density

$$\sum_{a=1}^n e^{+(a)} e^{-(a)} = \frac{k^2}{\cosh^2 k(x - vt - x_0)}. \tag{64}$$

5.2. Two Soliton Solution

Continuing Hirota's perturbation we have two dissipaton solution

$$g^+ = a_1^+(\xi)e^{\eta_1^+} + a_2^+(\xi)e^{\eta_2^+}$$

$$+(k_1^+ - k_2^+)\left[\frac{a_1^+(\xi)\gamma_{21}}{k_{11}^{+-}(k_{21}^{+-})^2} - \frac{a_2^+(\xi)\gamma_{11}}{k_{21}^{+-}(k_{11}^{+-})^2}\right]e^{\eta_1^+ + \eta_1^- + \eta_2^+}$$

$$+(k_1^+ - k_2^+)\left[\frac{a_1^+(\xi)\gamma_{22}}{k_{12}^{+-}(k_{22}^{+-})^2} - \frac{a_2^+(\xi)\gamma_{12}}{k_{22}^{+-}(k_{12}^{+-})^2}\right]e^{\eta_2^+ + \eta_2^- + \eta_1^+}, \tag{65}$$

$$g^- = a_1^-(\xi)e^{\eta_1^-} + a_2^-(\xi)e^{\eta_2^-}$$

$$+(k_1^- - k_2^-)\left[\frac{a_1^-(\xi)\gamma_{12}}{k_{11}^{+-}(k_{12}^{+-})^2} - \frac{a_2^-(\xi)\gamma_{11}}{k_{12}^{+-}(k_{11}^{+-})^2}\right]e^{\eta_1^+ + \eta_1^- + \eta_2^-}$$

$$+(k_1^- - k_2^-)\left[\frac{a_1^-(\xi)\gamma_{22}}{k_{21}^{+-}(k_{22}^{+-})^2} - \frac{a_2^-(\xi)\gamma_{21}}{k_{22}^{+-}(k_{21}^{+-})^2}\right]e^{\eta_2^+ + \eta_2^- + \eta_1^-}, \tag{66}$$

and

$$f = 1 + \gamma_{11}\frac{e^{\eta_1^+ + \eta_1^-}}{(k_{11}^{+-})^2} + \gamma_{12}\frac{e^{\eta_1^+ + \eta_2^-}}{(k_{12}^{+-})^2} + \gamma_{21}\frac{e^{\eta_2^+ + \eta_1^-}}{(k_{21}^{+-})^2} + \gamma_{22}\frac{e^{\eta_2^+ + \eta_2^-}}{(k_{22}^{+-})^2}$$

$$+\frac{(k_{11}^{+-}k_{22}^{+-} - k_{12}^{+-}k_{21}^{+-})(\gamma_{12}\gamma_{21}k_{11}^{+-}k_{22}^{+-} - \gamma_{11}\gamma_{22}k_{12}^{+-}k_{21}^{+-})}{(k_{11}^{+-}k_{12}^{+-}k_{21}^{+-}k_{22}^{+-})^2}e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-}, \tag{67}$$

where $\eta_i^\pm = k_i^\pm x \pm (k_i^\pm)^2 t + \eta_i^{\pm(0)}$, and constants $k_{ij}^{+-} \equiv k_i^+ + k_j^-$, $i,j = 1,2$ and $\gamma_{ij} \equiv \int_a^b a_i^+(\xi)a_j^-(\xi)\,d\xi$.

If $\frac{\partial}{\partial\xi}e^\pm = 0$, we have reduction to the reaction-diffusion system and the above two-dissipaton solution reduces to the one obtained in [2]. But as was shown in that paper, interaction of two dissipatons has resonant character. This is why now we can describe the resonant soliton dynamics for the 2+1 dimensional model in the continuous case, and for vector RNLS in the discrete case. For vector reduction in above formulas we substitute $a_i^\pm(\xi_a) = a_i^{\pm(a)}$, $(i = 1, 2)$, $(a = 1, ..., n)$, and $\gamma_{ij} = \sum_{a=1}^n a_i^{+(a)} a_j^{-(a)}$.

6. Soliton as non-relativistic particle

For one soliton solution (57) we have the mass density

$$\mu(\xi) = \frac{a_1^+(\xi)a_1^-(\xi)}{\gamma_{11}}2\,|k| \tag{68}$$

and the momentum density

$$\pi(\xi) = \mu(\xi)\,v, \tag{69}$$

where $\pi(\xi) = dP/d\xi$ and $dP = \pi(\xi)d\xi$. The total mass and total momentum of the soliton are

$$M = 2\,|k|, \quad P = M\,v. \tag{70}$$

Then for energy $E = -4H$ of one soliton we have

$$E = \frac{Mv^2}{2} + \frac{1}{6}M^3. \tag{71}$$

6.0.1. Resonance conditions The positive rest energy of soliton $E_0 = \frac{1}{6}M^3$, is the reason why the resonant soliton behavior takes place for this system. Decay of a soliton at rest on the pair of solitons with positive energies is allowed only if the rest energy is positive. Due to inequality $\frac{1}{6}M^3 = \frac{1}{6}(M_1 + M_2)^3 > \frac{1}{6}M_1^3 + \frac{1}{6}M_2^3$ it allows creation of two solitons. The first three conservation laws for the decaying process are

$$M = M_1 + M_2, \quad P = P_1 + P_2, \quad E = E_1 + E_2. \tag{72}$$

By substituting asymptotic form of created solitons, we find that the defect of mass vanishes $\Delta M = M - (M_1 + M_2) = 0$ and velocity of the decaying soliton is just the center-of-mass velocity $v = (M_1 v_1 + M_2 v_2)/(M_1 + M_2)$. From energy conservation law we find resonance constraint on velocities as

$$|v_1 - v_2| = M_1 + M_2. \tag{73}$$

As an example, let us illustrate decay of soliton in a rest to two solitons. Then $v_2 = -v_1 M_1/M_2$ and $v_1^2 = M_2^2$, $v_2^2 = M_1^2$. In particular case of equal solitons $|v_1| = |v_2|$, $M_1 = M_2$ it gives $k_1^+ = k_2^+$ and $k_1^- = k_2^-$. It is easy to see that under this constraint the last term in (67) vanishes and two soliton solution reduces to the one soliton.

7. Distribution in ξ direction

Functions $a_{1,2}^{\pm}(\xi)$ are arbitrary integrable functions determining the mass distribution density $\mu(\xi)$ along the ξ direction according to (33). By choosing these functions we have different situations. If for one soliton solution (68) we consider the discrete set of mass distribution

$$\mu(\xi) = \sum_{k=1}^{N} \mu_k \delta(\xi - \xi_k), \tag{74}$$

then one soliton solution becomes vector soliton. We can also describe continuous and localized distribution of soliton mass by considering delta sequence $\delta_n(\xi)$, $\lim_{n \to \infty} \delta_n(\xi) = \delta(\xi)$:

$$\int_a^b \delta_n(\xi) \, d\xi = 1. \tag{75}$$

Below we list some examples (with $\alpha = const.$, $(a, b) = (-\infty, \infty)$):

$$a^{\pm}(\xi) = \sqrt{\gamma_{11} \frac{n}{2}} \frac{e^{\pm \alpha}}{\cosh n\xi} \rightarrow \delta_n(\xi) = \frac{n}{2 \cosh^2 n\xi}; \tag{76}$$

$$a_1^{\pm} = \sqrt{\gamma_{11} \frac{n}{\pi}} \frac{1}{\sqrt{1 + n^2 \xi^2}} \rightarrow \delta_n(\xi) = \frac{n}{\pi} \frac{1}{1 + n^2 \xi^2}; \tag{77}$$

$$a_1^{\pm} = \sqrt{\gamma_{11} \frac{n}{\sqrt{\pi}}} e^{-n^2 \xi^2/2} \rightarrow \delta_n(\xi) = \frac{n}{\sqrt{\pi}} e^{-n^2 \xi^2}. \tag{78}$$

For the set of solutions localized in ξ direction we can take also superposition of these distributions. Since the mass density $\mu(\xi)$ is conserved quantity, the mass distribution will not change with time. However, as we have seen above, only total momentum and energy are conserved. So it opens possibility to have momentum and energy exchange between localized distributions in ξ direction. For discrete distribution it would leads to the exchange between vector soliton components. Combination of this phenomena with resonant character of soliton interaction in our model promises quite interesting structure and is under investigation.

8. Non-Madelung Fluid Form

In addition to the Madelung fluid representation (18), (19), (20) the system (42)-(44) can be rewritten in terms of another hydrodynamical variables. The density of fluid is the same

$$\rho(x, \xi, t) = e^+ e^-, \tag{79}$$

but the velocity field called the drift velocity is derived by the Cole-Hopf type formula

$$v(x, \xi, t) = \frac{e_x^+}{e^+}. \tag{80}$$

In terms of these variables we have nonlocal 2+1 dimensional version of the Broer-Kaup system

$$v_t = (v_x + v^2 + h)_x, \tag{81}$$

$$\rho_t + \rho_{xx} = (\rho v)_x, \tag{82}$$

$$h = \int_a^b \rho \, d\xi. \tag{83}$$

Dispersionless limit of this hydrodynamic system is the Benney system (1), (2), (3). Moreover, in particular case when ρ is independent of ξ, so that $h(x, t) = \rho(x, t)$, the system reduces to the Broer-Kaup system

$$v_t = (v_x + v^2 + \rho)_x, \tag{84}$$

$$\rho_t + \rho_{xx} = (\rho v)_x. \tag{85}$$

For one soliton solution, velocity

$$v = (\ln e^+)_x = \frac{k_1^+ - k_1^- e^{\eta_1^+ + \eta_1^- + \phi_{11} + \ln \gamma_{11}}}{1 + e^{\eta_1^+ + \eta_1^- + \phi_{11} + \ln \gamma_{11}}} \tag{86}$$

is shock-soliton

$$v = -\frac{v}{2} - k \tanh k(x - vt - x_0), \tag{87}$$

with soliton profile for the density

$$\rho(x, \xi, t) = \frac{a_1^+(\xi) a_1^-(\xi)}{\gamma_{11}} \frac{k^2}{\cosh^2 k(x - vt - x_0)}. \tag{88}$$

By choosing proper distribution in ξ direction we can localize this soliton in (x, ξ) plane.

8.1. Acknowledgments

References

[1] Pashaev O K and Francisco M L Y 2005 *Theor Math Phys* **144** 1022-29
[2] Pashaev O K and Lee J H 2002 *Mod. Phys.Lett.* A **17** 1601-19
[3] Pashaev O K, Lee J H and Rogers C, 2008 *J. Phys. A. Math Theor* **41** 452001-9
[4] Pashaev O K 2012 *Theor Math Phys* **172** 1147-59
[5] Zakharov V E 1981 *Physica* D **3** 193-202
[6] Maruno K I and Y. Ohta Y 2008 *Phys. Lett.* A **372** 4446
[7] Rogers C and Schief W K 1999 *Nuovo Cimento* **114** 1409-12
[8] Lee J H, Pashaev O K, Rogers C and Schief W K, 2007 *J. Plasma Physics* **73** 257-72
[9] Gurevich A V and Meshcherkin A P 1984 *Soviet Phys. JETP* **60** 73240
[10] El G A, Khodorovskii V V and Tyurina A V 2004 *Phys. Lett.* A **333** 33440
[11] Pashaev O K 1997 *Nucl. Phys. B Proc. Suppl.* **57** 33841
[12] Martina L, Pashaev O K and Soliani G 1997 *Class. Quantum Grav.* **14** 317986