Function Generation with Two Loop Mechanisms Using Decomposition and Correction Method

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Abstract

Method of decomposition has been successfully applied to function generation with multi-loop mechanisms. For a two-loop mechanism, a function y = f(x) can be decomposed into two as w = g(x) and y = h(w) = h(g(x)) = f(x). This study makes use of the method of decomposition for two-loop mechanisms, where the errors from each loop are forced to match each other. In the first loop, which includes the input of the mechanism, the decomposed function (g) is generated and the resulting structural error is determined. Then, for the second loop, the desired output of the function (f) is considered as an input and the structural error of the decomposed function (g) is determined. By matching the obtained structural errors, the final error in the output of the mechanism is reduced. Three different correction methods are proposed. The first method has three precision points per loop, while the second method has four. In the third method, the extrema of the errors from both loops are matched. The methods are applied to a Watt II type planar six-bar linkage for demonstration. Several numerical examples are worked out and the results are compared with the results in the literature.

Keywords: Function generation, decomposition and correction method, Watt II linkage

1. Introduction

There are several methods for the kinematic synthesis of function generating mechanisms. Polynomial approximation methods such as interpolation, least squares and Chebyshev approximation methods are some of the commonly used methods [1]. Among these, interpolation approximation method is the easiest one to implement. On the other hand, least squares approximation method provides more accurate approximation with respect to least squares (or L2) norm and Chebyshev approximation method provides more accurate approximation with respect to Chebyshev (or $L\infty$) norm, provided that these two methods are applied for a single loop function generator mechanism.

The approximation accuracy in function generating mechanism synthesis can be enhanced by increasing the number of design parameters of the mechanism. This can be accommodated by either of the following methods: 1) introducing artificial frames of reference for the input or output of the mechanism [1]; 2) also considering the amount of displacements of the input or the output link [2]; 3) combining linkages with gears [3]; 4) using a mechanism with more design parameters [4]; 5) using additional loops for a selected mechanism [5]. There are several studies on function generating planar six-link mechanisms. Svoboda [6] considered a Watt II type six-bar linkage as a double three-bar linkage and specified nine design parameters (three link lengths per loop – taking scaling into account and three rotation ranges for the three links connected to the ground). He formulized the function generation problem as composition of two functions corresponding to the two loops, which we call the method of decomposition. He proposes alternative uses of the approach. The first use is such that the first loop is designed to approximately generate the desired function and the second loop is used to tune the result. In the second use, the desired function is decomposed into two identical functions such that

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the square root of the function is generated by each of the identical four-bar loops with equal input and output travels. Svoboda calls this latter method as the method of successive approximations. The generalized decomposition method proposed by Alizade et al. [7] allows the decomposition of the function arbitrarily. This enables an extra design parameter for the designer. McLarnan [8] utilized an iterative numerical method for synthesis of planar six-bar linkages for at most 9 precision points with Watt linkages and at most 11 precision points with Stephenson linkages. Rao et al. [9] utilized Burmester theory in order to design six-bar linkages performing function and path generation simultaneously. Several examples of such dual-purpose (combination of function, path, motion generation with the same mechanism) are presented in [10]. Dhingra and Mani [11] derived the input/output (I/O) relationship for Stephenson III and Watt II type planar six-bar linkages and used Newton-Raphson numerical method to solve the function generation synthesis problem with 9 and 11 precision points. Dhingra et al. [12] used homotopy methods for function generation with planar six-bar linkages.

Liu et al. [13] made use of homotopy methods for function generation with six-bar linkages for five precision points. Simionescu and Alexandru [14] worked on the optimal design of Stephenson linkages by increasing the degree-of-freedom to two by removing one of the links. [15] devised a modular approach for design of six-bar function generators. Shiakolas et al. [16] devised a methodology that combines differential evolution and geometric centroid of precision positions technique in order to perform synthesis of Stephenson III type six-bar linkages for dwell and dual-dwell mechanisms with prescribed timing and transmission angle constraints. Kinzel et al. [17] used the so-called geometric constraint programming (GCP) to design a Stephenson III type planar six-bar linkage for function generation with up to 11 precision points. Both graphical and analytical methods are used in GCP and it makes use of commercially available CAD packages to simultaneously meet precision point conditions. Hwang and Chen [18] applied constrained optimization techniques for designing Stephenson II type function generators avoiding order, circuit, and branch defects. Sancibrian [19] made use of an improved version of the generalized reduced gradient optimization method for function generation synthesis of several planar linkages including the Stephenson II, Stephenson III and Watt II type six-bar linkages.

Plecnik and McCarthy [20] also worked on Stephenson II type type of six-bar linkages for function generation with eight precision points. By assuming some of the link lengths, a set of 22 equations with a total degree of 705,432 is obtained. Later on, Plecnik and McCarthy [21] also worked on function generation with a Stephenson II type planar six-bar linkage for 11 precision points. The loop closure equations constitute a set of 70 quadratic equations and the system is reduced to 10 eighth-degree polynomials. The resulting set of equations have a total degree of 1.07×10^9. In both of the last two studies, the equations are solved using continuation method. The latter study resulted in 1,521,037 nonsingular solutions. Agarwal et al. [22] used a genetic-algorithm-based multi-objective optimizer for function generation with a Stephenson-III type planar six-bar linkage. In addition to the structural error defined based on the I/O relationship, the derivative of the structural error is also taken into account, therefore the formulation is called the dual-order formulation. Also, analytical conditions are derived for the identification of the candidate designs which are free of singularities, mobility or branch defects. The numerical examples result in comparable values with the ones that are presented in [20].

All methods mentioned above are based on numerical methods, whereas an analytical formulation is presented in this study. The drawback of the proposed method is that relatively few number of precision points are used. The powerful side of the analytical formulation is that the designer can carry out hundreds of trials in several minutes. Furthermore, the methods proposed in this study allow the designer to tune the design while monitoring several properties such as link length ratios, transmission angle and etc.

The decomposition method is applicable when there are multiple loops in the generator mechanism and is based on decomposition of the function to be generated into as many functions as the number of loops. Maaroof and Dede [23, 24] have worked on application of the interpolation approximation for a Bennett 6R linkage using the decomposition method. Although interpolation approximation seems to result in inferior results compared to the other two approximation methods, Maaroof et al. [25] showed that superiority of one method over the others is lost when the decomposition method is used.

In this study we apply the decomposition method to a Watt II type planar six-bar mechanism. We propose three correction methods to improve the accuracy of function generation. These methods can be easily and conveniently applied to other two-loop mechanisms, as well. The proposed methods are based on correction of the function generation errors of the first loop in the second loop. Interpolation approximation is used as the synthesis method.

2. Description of the Mechanism and Function Synthesis Problem

The Watt II type planar six-bar mechanism is composed of two ternary and four binary links connected to each other by seven revolute joints. In Fig. 1, the input of the mechanism is the ϕ angle and the output is ψ angle. γ angle is the output of the first loop and will be used as an intermediate variable. A Cartesian coordinate frame with origin at A_0 is to be used such that the x-axis passes through B_0. The input angle ϕ and the intermediate angle γ are measured from their respective fixed reference axes which make angles of ϕ^* and γ^*, respectively with the x-axis. The fixed reference axis of the output angle ψ makes an angle of ψ^* with respect to B_0D_0 direction. In general, ϕ^*, γ^* and ψ^* are design parameters to be determined via synthesis. Since the I/O relationship of a mechanism with revolute joints does not change when the mechanism is scaled, the four bar loops A_0ABB_0 and B_0CDD_0 can be independently resized arbitrarily. So, without loss of generality we may assume $|A_0B_0| = |B_0D_0| = 1$ for the fixed link lengths. The other link lengths of the mechanism are denoted as $|A_0A| = a$, $|AB| = b$, $|B_0B| = c$, $\measuredangle BB_0C = \alpha$, $\measuredangle D_0B_0x = \beta$, $|B_0C| = d$, $|CD| = e$ and $|D_0D| = f$. A careful inspection shows that angles α and β are not independent design parameters, but $\alpha + \beta$ is effective in the I/O relationship of four-bar loop B_0CDD_0, because the effective input of the loop is $\measuredangle CB_0D_0 = \gamma + \gamma* - (\alpha + \beta)$. β also contributes to the reference axis angle ψ^* of the output link D_0D, but it just acts as a constant offset to the output angle. Since α and β do not independently effect the I/O relationship, without loss of generality we assume $\measuredangle B_0A_0D_0 = \beta = 0$ during our analyses. If the designer wishes to make use of a nonzero β, it is sufficient to replace α by $\alpha - \beta$. Assigning β a nonzero value is just rotating the B_0CDD_0 by an angle β without affecting the I/O relationship of the loop. Note that neither taking $|A_0B_0| = |B_0D_0| = 1$ or $\beta = 0$ loses generality of the function generation synthesis problem. Also, for computational ease we assume $\gamma^* = \beta$ (= 0 during the computations) and $\psi^* = 0$ in this study. The readers can examine [4] to learn about the tools for handling nonzero intermediate (γ^*) and output (ψ^*) reference angles.

The I/O relationships for the loops will be derived separately for different correction methods, because the angle ϕ^* and/or α will be assumed zero in some of the methods.

We want to generate $y = f(x)$ for $x_0 \leq x \leq x_f$ using the Watt II type planar six-bar linkage. We decompose f() function into two as $w = g(x)$ and $y = h(w) = h(g(x)) = f(x)$. g() function may be selected arbitrarily. Initial and final values of w and y are found as $w_0 = g(x_0)$, $w_f = g(x_f)$, $y_0 = f(x_0)$ and $y_f = f(x_f)$. Also let $\Delta x = x_f - x_0$, $\Delta w = w_f - w_0$ and $\Delta y = y_f - y_0$. We associate function variables x, w, y with mechanism variables ϕ, γ, ψ linearly as

$$\frac{\phi-\phi_0}{\Delta\phi}=\frac{x-x_0}{\Delta x}\ ,\ \frac{\gamma-\gamma_0}{\Delta\gamma}=\frac{w-w_0}{\Delta w}\ ,\ \frac{\psi-\psi_0}{\Delta\psi}=\frac{y-y_0}{\Delta y} \tag{1}$$

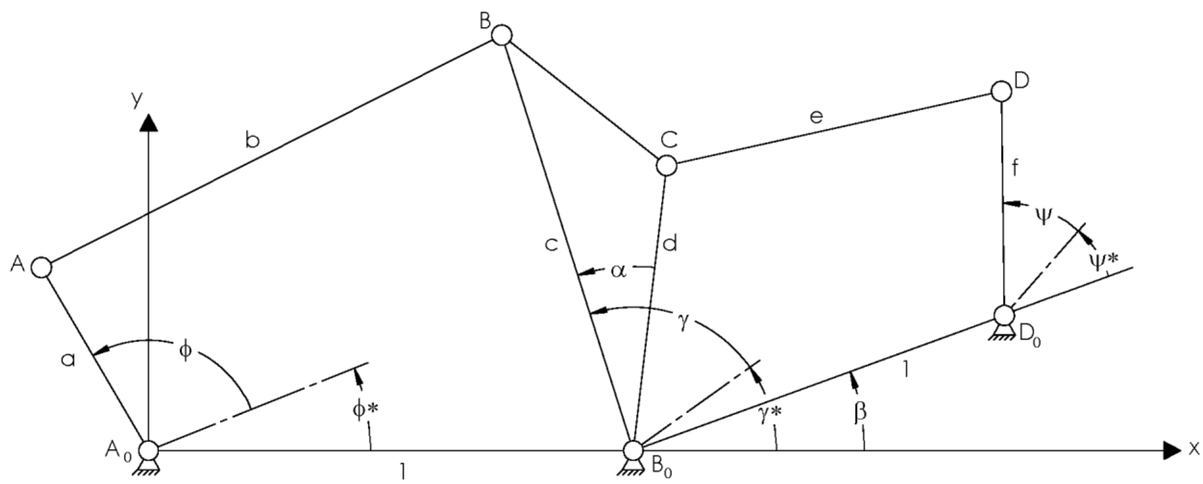

Fig. 1 Kinematic diagram of a Watt II linkage

where $\phi_0 \le \phi \le \phi_f$, $\gamma_0 \le \gamma \le \gamma_f$, $\psi_0 \le \psi \le \psi_f$ and $\Delta\phi = \phi_f - \phi_0$, $\Delta\gamma = \gamma_f - \gamma_0$, $\Delta\psi = \psi_f - \psi_0$. The limits of the angles are chosen arbitrarily. From Eq. (1), desired mechanism variables ϕ, γ, ψ can be determined in terms of function variables x, w, y as follows:

$$\phi = \frac{\Delta\phi}{\Delta x}(x - x_0) + \phi_0 \ , \ \gamma = \frac{\Delta\gamma}{\Delta w}(w - w_0) + \gamma_0 \ , \ \psi = \frac{\Delta\psi}{\Delta y}(y - y_0) + \psi_0 \tag{2}$$

Eq. (2) is used for determining the precision points for interpolation approximation. The precision points can be selected with equal spacing, Chebyshev spacing or any other spacing. Usually Chebyshev spacing gives good results. Conversely, x, w, y can be determined in terms of ϕ, γ, ψ as follows

$$x = \frac{\Delta x}{\Delta\phi}(\phi - \phi_0) + x_0 \ , \ w = \frac{\Delta w}{\Delta\gamma}(\gamma - \gamma_0) + w_0 \ , \ y = \frac{\Delta y}{\Delta\psi}(\psi - \psi_0) + y_0 \tag{3}$$

Eq. (3) is used after the synthesis is performed in order to check the error in between the desired y(x) and the generated $y(\psi)$.

3. Correction Methods

In this section, three correction methods are presented for function generation with a Watt II type planar six-bar linkage. The first correction method involves only the link lengths of the mechanism while the second correction method involves angle references also as extra design parameters. The precision points of the two loops are matched in these two correction methods. In the third correction method, the synthesis of the first loop is the same as the other methods; however precision points of the first loop are not used in the second loop. Instead, the points which correspond to the extrema of the error in the first loop are used for the second loop.

3.1 Correction Method 1

The first correction method is the most basic one. In Fig. 1 assume $\phi^* = 0$ and $\alpha = 0$. For loop A_0ABB_0 the I/O relationship is obtained as follows

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{A_oB} - \overrightarrow{A_oA}\right| \Rightarrow b^2 = \left(1 + c\cos\gamma - a\cos\phi\right)^2 + \left(c\sin\gamma - a\sin\phi\right)^2$$
$$\Rightarrow b^2 - 1 - a^2 - c^2 + 2a\cos\phi + 2ac\cos\left(\gamma - \phi\right) = 2c\cos\gamma \tag{4}$$

Eq. (4) can be written in the polynomial form as

$$\sum_{j=1}^{n} P_j f_j(\mathbf{x}_i) - F(\mathbf{x}_i) = 0 \quad \text{for } i = 1, ..., m \tag{5}$$

where m = n = 3, $\mathbf{x}_i = \{\phi_i, \gamma_i\}$, $P_1 = -\dfrac{1 + a^2 - b^2 + c^2}{2c}$, $P_2 = \dfrac{a}{c}$, $P_3 = a$, $f_1(\mathbf{x_i}) = 1$, $f_2(\mathbf{x_i}) = \cos\phi$, $f_3(\mathbf{x_i}) = \cos(\gamma - \phi)$ and $F(\mathbf{x_i}) = c\gamma$ for Eq. (4). For three precision points $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, Eq. (5) results in a linear set of equations:

$$\begin{array}{l} P_1 + P_2\cos\phi_1 + P_3\cos(\gamma_1 - \phi_1) = \cos\gamma_1 \\ P_1 + P_2\cos\phi_2 + P_3\cos(\gamma_2 - \phi_2) = \cos\gamma_2 \\ P_1 + P_2\cos\phi_3 + P_3\cos(\gamma_3 - \phi_3) = \cos\gamma_3 \end{array} \Rightarrow \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 & \cos\phi_1 & \cos(\gamma_1 - \phi_1) \\ 1 & \cos\phi_2 & \cos(\gamma_2 - \phi_2) \\ 1 & \cos\phi_3 & \cos(\gamma_3 - \phi_3) \end{bmatrix}^{-1} \begin{bmatrix} \cos\gamma_1 \\ \cos\gamma_2 \\ \cos\gamma_3 \end{bmatrix} \tag{6}$$

Once P_1, P_2, P_3 are found from Eq. (6), a, b, c can be uniquely determined as $a = P_3$, $c = a/P_2$ and $b = \sqrt{1 + a^2 + c^2 + 2cP_1}$, provided that b is real. If a gets a negative value, one can add π to the assumed limits (ϕ_0 and ϕ_f) of ϕ in order to make a get a positive value. Similar argument is valid for c and limits (γ_0 and γ_f) of γ. Analyzing the resulting four-bar loop with the designed link length parameters, the generated γ angle values corresponding to the desired ϕ angle values can be calculated from which the generated w values can be determined. One should be careful about the assembly mode of the loop. Let $\delta_1 = w_{desired} - w_{generated1}$ represent the error in loop A_0ABB_0. The error curve δ_1 versus x will be as in Fig. 2.

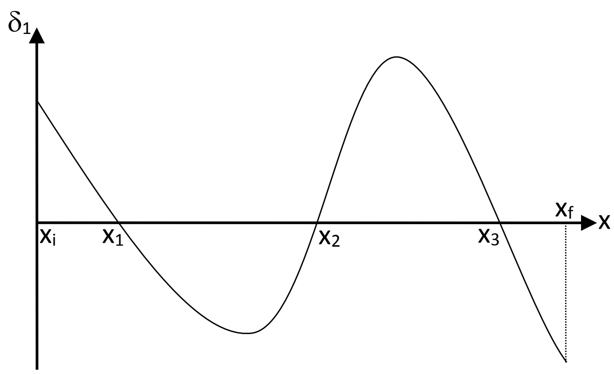

Fig. 2. Error curve for loop A_0ABB_0

Although ψ is the output of the mechanism, for loop B_0CDD_0 we shall assume ψ is given and resulting γ is monitored in order for the errors of the two loops be comparable. Therefore, for loop B_0CDD_0, the error in the intermediate variable w will be monitored as well. Let $\delta_2 = w_{desired} - w_{generated2}$ represent the error value calculated for given desired y and corresponding ψ angle values. For the synthesis we will use the same precision points as the first loop and try to approximately equate δ_2 and δ_1. The I/O relationship for the loop B_0CDD_0 is obtained as

$$\left|\overrightarrow{CD}\right| = \left|\overrightarrow{D_oD} - \overrightarrow{B_oC}\right| \Rightarrow e^2 = \left(1 + f\cos\psi - d\cos\gamma\right)^2 + \left(f\sin\psi - d\sin\gamma\right)^2$$
$$\Rightarrow \frac{1+d^2-e^2+f^2}{2d} + \frac{f}{d}\cos\psi - f\cos\left(\psi-\gamma\right) = \cos\gamma \qquad (7)$$

Eq. (7) can be written in polynomial form (5) with m = n = 3, $\mathbf{x}_i = \{\gamma_i, \psi_i\}$, $P_4 = \frac{1+d^2-e^2+f^2}{2d}$, $P_5 = \frac{f}{d}$, $P_6 = f$, $f_4(\mathbf{x}_i) = 1$, $f_5(\mathbf{x}_i) = c\psi_i$, $f_6(\mathbf{x}_i) = -c(\psi_i - \gamma_i)$ and $F(\mathbf{x}_i) = c\gamma_i$. The three precision points are selected such that the γ angles coincide with those of the precision points of loop A_0ABB_0. The linear equations can be solved as

$$\begin{aligned} P_4 + \cos\psi_1 P_5 - \cos(\psi_1-\gamma_1)P_6 &= \cos\gamma_1 \\ P_4 + \cos\psi_3 P_5 - \cos(\psi_3-\gamma_3)P_6 &= \cos\gamma_3 \\ P_4 + \cos\psi_5 P_5 - \cos(\psi_5-\gamma_5)P_6 &= \cos\gamma_5 \end{aligned} \Rightarrow \begin{bmatrix} P_4 \\ P_5 \\ P_6 \end{bmatrix} = \begin{bmatrix} 1 & \cos\psi_1 & -\cos(\psi_1-\gamma_1) \\ 1 & \cos\psi_3 & -\cos(\psi_3-\gamma_3) \\ 1 & \cos\psi_5 & -\cos(\psi_5-\gamma_5) \end{bmatrix}^{-1} \begin{bmatrix} \cos\gamma_1 \\ \cos\gamma_3 \\ \cos\gamma_5 \end{bmatrix} \qquad (8)$$

After P_4, P_5 and P_6 are found, the design parameters of the loop can be uniquely determined as $f = P_6$, $d = f/P_5$ and $e = \sqrt{1+d^2+f^2-2dP_4}$, provided that e is real. If d receives a negative value, this means that $\alpha = \pi$ instead of $\alpha = 0$, which is acceptable. If f is negative, then one can add π to the assumed limits (ψ_0 and ψ_f) of the output angle ψ to force f to be positive. Analyzing the resulting four-bar loop with the designed link length parameters, the generated γ angle values corresponding to the desired ϕ angle values and hence the generated w values can be determined. Representative error curves δ_1 and δ_2 versus x are illustrated in Fig. 3. As a result of the whole design process, the ψ angle values as the output of the mechanism will result in corresponding y values as the output of the generated function. For given x, and hence corresponding angle ϕ, the error $\delta_y = y_{desired} - y_{generated}$ variance is also depicted in Fig. 3. Definitely $\delta_y = 0$ at the precision points x_1, x_2 and x_3. There may be other points where $\delta_y = 0$ whenever δ_1 curve intersects δ_2, such as the x* point demonstrated in Fig. 3.

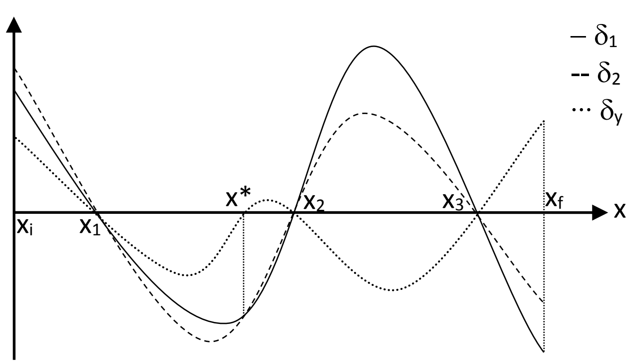

Fig. 3. Error curves for the two loops and function output

The resemblance of δ_1 and δ_2 directly effects δ_y. In order to obtain lower amounts of δ_y, the designer has several parameters to adjust. The limits ϕ_0, ϕ_f, γ_0, γ_f, ψ_0 and ψ_f are free to choose, unless there are some constraints on these parameters. Also, it is possible to adjust the intermediate function g(·) for most of the functions. If a design environment such as Microsoft Excel® is used, the designer can assign spin buttons for the limits of the angles and the intermediate function parameter. By using these spin buttons, the designer can simultaneously see the tendency of change in the variations of δ_1, δ_2 and δ_y. Meanwhile the designer can monitor $|\delta_y|_{max}$ or root mean square of δ_y – whichever is convenient – and minimize it. While performing these adjustments, it is also possible to monitor certain design considerations such as maximum link length to minimum link length, transmission angles, etc.

3.2 Correction Method 2

In the second correction method, ϕ^* and α are assumed nonzero. In this case, the I/O equation for loop A_0ABB_0 becomes

$$\begin{aligned} &\left|\overrightarrow{AB}\right| = \left|\overrightarrow{A_oB} - \overrightarrow{A_oA}\right| \Rightarrow b^2 = \left(1 + c c\gamma - ac\left(\phi + \phi^*\right)\right)^2 + \left(cs\gamma - as\left(\phi + \phi^*\right)\right)^2 \\ &\Rightarrow -\frac{1 + a^2 - b^2 + c^2}{2c} + \frac{a}{c} c\phi^* c\phi - \frac{a}{c} s\phi^* s\phi + ac\phi^* c\left(\gamma - \phi\right) + as\phi^* s\left(\gamma - \phi\right) = c\gamma \end{aligned} \tag{9}$$

Eq. (9) can be written in polynomial form (5) with m = 4, n = 5, $\mathbf{x}_i = \{\phi_i, \gamma_i\}$, $P_1 = -\frac{1 + a^2 - b^2 + c^2}{2c}$, $P_2 = \frac{a}{c} c\phi^*$, $P_3 = \frac{a}{c} s\phi^*$, $P_4 = ac\phi^*$, $P_5 = as\phi^*$, $f_1(\mathbf{x}_i) = 1$, $f_2(\mathbf{x}_i) = c\phi_i$, $f_3(\mathbf{x}_i) = -s\phi_i$, $f_4(\mathbf{x}_i) = c(\gamma_i - \phi_i)$, $f_5(\mathbf{x}_i) = s(\gamma_i - \phi_i)$ and $F(\mathbf{x}_i) = c\gamma_i$. There are four design parameters (a, b, c and ϕ^*), so there should be four precision points: x_1, x_2, x_3 and x_4. However there are five P_j's, hence they cannot be independent of each other. Indeed, $P_3P_4 = P_2P_5$. The problem can be linearized by using a Lagrange's variable. Let $P_5 = \lambda$ and $P_j = m_j + n_j\lambda$ for j = 1, 2, 3, 4, 5. Substituting into the polynomial equations:

$$\begin{aligned} m_1 + n_1\lambda + (m_2 + n_2\lambda)c\phi_1 - (m_3 + n_3\lambda)s\phi_1 + (m_4 + n_4\lambda)c(\gamma_1 - \phi_1) &= c\gamma_1 - s(\gamma_1 - \phi_1)\lambda \\ m_1 + n_1\lambda + (m_2 + n_2\lambda)c\phi_3 - (m_3 + n_3\lambda)s\phi_3 + (m_4 + n_4\lambda)c(\gamma_3 - \phi_3) &= c\gamma_3 - s(\gamma_3 - \phi_3)\lambda \\ m_1 + n_1\lambda + (m_2 + n_2\lambda)c\phi_5 - (m_3 + n_3\lambda)s\phi_5 + (m_4 + n_4\lambda)c(\gamma_5 - \phi_5) &= c\gamma_5 - s(\gamma_5 - \phi_5)\lambda \\ m_1 + n_1\lambda + (m_2 + n_2\lambda)c\phi_7 - (m_3 + n_3\lambda)s\phi_7 + (m_4 + n_4\lambda)c(\gamma_7 - \phi_7) &= c\gamma_7 - s(\gamma_7 - \phi_7)\lambda \end{aligned} \tag{10}$$

In order for Eqs. (10) to be satisfied, the coefficients of λ and the rest of each equation need to be equated to zero. In matrix form:

$$\begin{bmatrix} 1 & c\phi_1 & -s\phi_1 & c(\gamma_1 - \phi_1) \\ 1 & c\phi_3 & -s\phi_3 & c(\gamma_3 - \phi_3) \\ 1 & c\phi_5 & -s\phi_5 & c(\gamma_5 - \phi_5) \\ 1 & c\phi_7 & -s\phi_7 & c(\gamma_7 - \phi_7) \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} c\gamma_1 \\ c\gamma_3 \\ c\gamma_5 \\ c\gamma_7 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & c\phi_1 & -s\phi_1 & c(\gamma_1 - \phi_1) \\ 1 & c\phi_3 & -s\phi_3 & c(\gamma_3 - \phi_3) \\ 1 & c\phi_5 & -s\phi_5 & c(\gamma_5 - \phi_5) \\ 1 & c\phi_7 & -s\phi_7 & c(\gamma_7 - \phi_7) \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{bmatrix} = \begin{bmatrix} -s(\gamma_1 - \phi_1) \\ -s(\gamma_3 - \phi_3) \\ -s(\gamma_5 - \phi_5) \\ -s(\gamma_7 - \phi_7) \end{bmatrix} \tag{11}$$

m_1, m_2, m_3, m_4, n_1, n_2, n_3 and n_4 are solved by matrix inversion. λ is solved as follows:

$$P_3P_4 = P_2P_5 = (m_3 + n_3\lambda)(m_4 + n_4\lambda) = (m_2 + n_2\lambda)\lambda$$
$$\Rightarrow (n_3n_4 - n_2)\lambda^2 + (m_3n_4 + n_3m_4 - m_2)\lambda + m_3m_4 = 0 \tag{12}$$
$$\Rightarrow \lambda = \frac{m_2 - m_3n_4 - n_3m_4 \mp \sqrt{(m_3n_4 + n_3m_4 - m_2)^2 - 4m_3m_4(n_3n_4 - n_2)}}{2(n_3n_4 - n_2)}$$

Next, $P_j = m_j + n_j\lambda$ for j = 1, 2, 3, 4 are evaluated. After determining P_1, P_2, P_3, P_4, P_5, $a = \sqrt{P_4^{\,2} + P_5^{\,2}}$, $\phi^* = \text{atan2}(P_4, P_5)$, $c = ac\phi^*/P_2$ and $b = \sqrt{1 + a^2 + c^2 + 2cP_1}$. All design parameters are uniquely determined provided that b is real. If c is negative, limits of γ should be increased by π. The resulting error variation $\delta_1 = w_{desired} - w_{generated1}$ has at least four precision points (x_1, x_2, x_3 and x_4) if the correct assembly mode of the four bar loop is selected.

For loop B_0CDD_0 the I/O relationship is obtained as

$$\left|\overrightarrow{CD}\right| = \left|\overrightarrow{D_oD} - \overrightarrow{B_oC}\right| \Rightarrow e^2 = \left(1 + fc\psi - dc(\gamma - \alpha)\right)^2 + \left(fs\psi - ds(\gamma - \alpha)\right)^2$$
$$\Rightarrow \frac{1 + d^2 - e^2 + f^2}{2dc\alpha} + \frac{f}{dc\alpha}c\psi - fc(\psi - \gamma) - t\alpha s\gamma + ft\alpha s(\psi - \gamma) = c\gamma \tag{13}$$

Eq. (13) can be written in polynomial form (5) with m = 4, n = 5, $\mathbf{x}_i = \{\gamma_i, \psi_i\}$, $P_6 = \frac{1 + d^2 - e^2 + f^2}{2dc\alpha}$, $P_7 = \frac{f}{dc\alpha}$, $P_8 = f$, $P_9 = t\alpha$, $P_{10} = ft\alpha$, $f_6(\mathbf{x}_i) = 1$, $f_7(\mathbf{x}_i) = c\psi_i$, $f_8(\mathbf{x}_i) = -c(\psi_i - \gamma_i)$, $f_9(\mathbf{x}_i) = -s\gamma_i$, $f_{10}(\mathbf{x}_i) = s(\psi_i - \gamma_i)$ and $F(\mathbf{x}_i) = c\gamma_i$. There are four design parameters (d, e, f and α) and hence four precision points x_1, x_2, x_3 and x_4. The same precision points for loop A_0ABB_0 are used and corresponding γ_i and ψ_i angles for i = 1, 2, 3, 4 are to be used. There are five P_j's, but $P_{10} = P_8P_9$. Let $P_{10} = \lambda$ and $P_j = m_j + n_j\lambda$ for j = 6, 7, 8, 9. Substituting into the polynomial equations:

$$\begin{aligned}
&(m_6 + n_6\lambda) + (m_7 + n_7\lambda)c\psi_1 - (m_8 + n_8\lambda)c(\psi_1 - \gamma_1) - (m_9 + n_9\lambda)s\gamma_1 = c\gamma_1 - s(\psi_1 - \gamma_1)\lambda \\
&(m_6 + n_6\lambda) + (m_7 + n_7\lambda)c\psi_3 - (m_8 + n_8\lambda)c(\psi_3 - \gamma_3) - (m_9 + n_9\lambda)s\gamma_3 = c\gamma_3 - s(\psi_3 - \gamma_3)\lambda \\
&(m_6 + n_6\lambda) + (m_7 + n_7\lambda)c\psi_5 - (m_8 + n_8\lambda)c(\psi_5 - \gamma_5) - (m_9 + n_9\lambda)s\gamma_5 = c\gamma_5 - s(\psi_5 - \gamma_5)\lambda \\
&(m_6 + n_6\lambda) + (m_7 + n_7\lambda)c\psi_7 - (m_8 + n_8\lambda)c(\psi_7 - \gamma_1)_7 - (m_9 + n_9\lambda)s\gamma_7 = c\gamma_7 - s(\psi_7 - \gamma_7)\lambda
\end{aligned} \tag{14}$$

Equating coefficients of λ and the rest to zero in Eqs. (14) and writing in matrix form:

$$\begin{bmatrix} 1 & c\psi_1 & -c(\psi_1 - \gamma_1) & -s\gamma_1 \\ 1 & c\psi_3 & -c(\psi_3 - \gamma_3) & -s\gamma_3 \\ 1 & c\psi_5 & -c(\psi_5 - \gamma_5) & -s\gamma_5 \\ 1 & c\psi_7 & -c(\psi_7 - \gamma_7) & -s\gamma_7 \end{bmatrix} \begin{bmatrix} m_6 \\ m_7 \\ m_8 \\ m_9 \end{bmatrix} = \begin{bmatrix} c\gamma_1 \\ c\gamma_3 \\ c\gamma_5 \\ c\gamma_7 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & c\psi_1 & -c(\psi_1 - \gamma_1) & -s\gamma_1 \\ 1 & c\psi_3 & -c(\psi_3 - \gamma_3) & -s\gamma_3 \\ 1 & c\psi_5 & -c(\psi_5 - \gamma_5) & -s\gamma_5 \\ 1 & c\psi_7 & -c(\psi_7 - \gamma_7) & -s\gamma_7 \end{bmatrix} \begin{bmatrix} n_6 \\ n_7 \\ n_8 \\ n_9 \end{bmatrix} = \begin{bmatrix} -s(\psi_1 - \gamma_1) \\ -s(\psi_3 - \gamma_3) \\ -s(\psi_5 - \gamma_5) \\ -s(\psi_7 - \gamma_7) \end{bmatrix} \tag{15}$$

m_6, m_7, m_8, m_9, n_6, n_7, n_8 and n_9 are solved by matrix inversion. λ is solved as follows:

$$\lambda = P_{10} = P_8 P_9 = \left(m_8 + n_8\lambda\right)\left(m_9 + n_9\lambda\right)$$
$$\Rightarrow n_8 n_9 \lambda^2 + \left(m_8 n_9 + n_8 m_9 - 1\right)\lambda + m_8 m_9 = 0 \qquad (16)$$
$$\Rightarrow \lambda = \frac{1 - m_8 n_9 - n_8 m_9 \mp \sqrt{\left(m_8 n_9 + n_8 m_9 - 1\right)^2 - 4 m_8 m_9 n_8 n_9}}{2 n_8 n_9}$$

Next, $P_j = m_j + n_j\lambda$ for j = 6, 7, 8, 9 are evaluated. Then, $f = P_8$, $\alpha = \tan^{-1} P_9$, $d = f/\left(P_7 c\alpha\right)$ and $e = \sqrt{1 + d^2 + f^2 - 2dc\alpha P_6}$. Solution results in a mechanism if e is real. In this case, the solution is not unique, because $\alpha = \tan^{-1} P_9 + \pi$ is also a feasible solution. If d is positive for $\alpha = \tan^{-1} P_9$, α is kept as it is. Otherwise, π is added to α. If f is negative, π can be added to limits of ψ. The error variation $\delta_2 = w_{desired} - w_{generated2}$ is determined with at least four precision points with the right assembly mode of the loop. As in the first correction method, $\delta_y = y_{desired} - y_{generated}$ is obtained and minimized by adjusting the angles and the intermediate function parameter while monitoring the necessary design considerations.

3.3 Correction Method 3

In the first two correction methods explained above, numerical examples showed that mostly the maximum error occurs due to the difference in the local extrema of δ_1 and δ_2. A third correction method is developed to equate local extrema rather than the precision points. In this last correction method, $\phi^* = 0$, but α is nonzero. In this case, the I/O equation for loop A_0ABB_0 is given by Eq. (4) and the I/O equation for loop B_0CDD_0 is given by Eq. (13). Synthesis is performed for loop A_0ABB_0 with three precision points x_1, x_2 and x_3. The resulting error variation $\delta_1 = w_{desired} - w_{generated1}$ is given once again in Fig. 4, however in this case the two local extrema inside the domain and the corresponding x_4 and x_5 locations are also shown. (x_4, $\delta(x_4)$) and (x_5, $\delta(x_5)$) points in the error curve in Fig. 4 will be used for the design of loop B_0CDD_0.

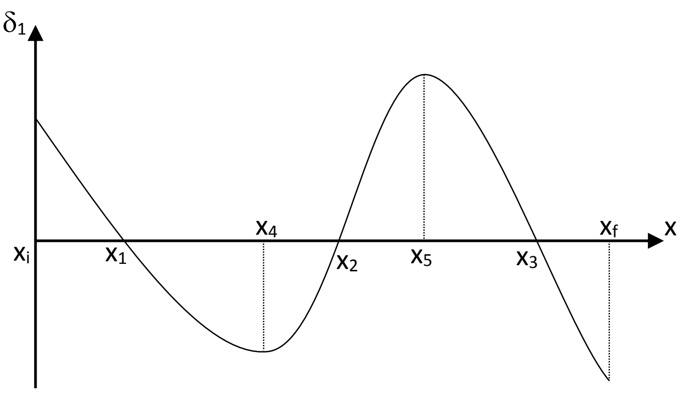

Fig. 4. Error curve for loop A_0ABB_0

x_4 and x_5 are analytically determined from $\frac{\partial \delta_1}{\partial x} = 0$ as follows:

$$\frac{\partial\delta_1}{\partial x}=\frac{\partial}{\partial x}\left(w_d-w_g\right)=\frac{\partial w_d}{\partial x}-\frac{\partial w_g}{\partial\gamma}\frac{\partial\gamma}{\partial\phi}\frac{\partial\phi}{\partial x}=\frac{\partial w_d}{\partial x}-\frac{\Delta w_g}{\Delta\gamma}\frac{\Delta\phi}{\Delta x}\frac{\partial\gamma}{\partial\phi} \tag{17}$$

where w_d is $w_{desired}$ and w_g is $w_{generated1}$. $\partial\gamma/\partial\phi$ is to be evaluated by differentiating the I/O equation of loop A_0ABB_0 with respect to ϕ:

$$\begin{gathered}P_1+P_2c\phi+P_3c\left(\gamma-\phi\right)=c\gamma\Rightarrow P_2s\phi+P_3s\left(\gamma-\phi\right)\left(\frac{\partial\gamma}{\partial\phi}-1\right)=s\gamma\frac{\partial\gamma}{\partial\phi}\\ \Rightarrow\frac{\partial\gamma}{\partial\phi}=\frac{P_2s\phi-P_3s\left(\gamma-\phi\right)}{s\gamma-P_3s\left(\gamma-\phi\right)}\end{gathered} \tag{18}$$

Substituting in $\partial\delta_1/\partial x$ in Eq. (17) and equating to zero:

$$\frac{\partial w_d}{\partial x}-\frac{\Delta w_g}{\Delta\gamma}\frac{\Delta\phi}{\Delta x}\frac{P_2s\phi-P_3s\left(\gamma-\phi\right)}{s\gamma-P_3s\left(\gamma-\phi\right)}=0\Rightarrow\frac{\partial w_d}{\partial x}s\gamma-\frac{\Delta w_g}{\Delta\gamma}\frac{\Delta\phi}{\Delta x}s\phi P_2+\left(\frac{\Delta w_g}{\Delta\gamma}\frac{\Delta\phi}{\Delta x}-\frac{\partial w_d}{\partial x}\right)s\left(\gamma-\phi\right)P_3=0 \tag{19}$$

Eq. (19) is a nonlinear equation in x and can be solved for x using a numerical method. There should be two solutions corresponding to the local extrema inside the domain of x. In Excel, the built-in function "Goal Seek" can be used to find the roots. The "Goal Seek" function of Excel is based on the Newton-Raphson root finding algorithm. Also a simple macro can be written inside the spin button codes so that the Goal Seek function is automatically employed when the function limits or the intermediate function parameter are altered. When the initial value for the Goal Seek is set to the previous value of the roots, the convergence is immediate, because the initial value will be very close to the solution.

For loop B_0CDD_0 we impose $\delta_2(x_4)=\delta_1(x_4)$, $\delta_2(x_5)=\delta_1(x_5)$ and $\partial\delta_2/\partial x=0$ at both of x_4 and x_5.The first two conditions are generated from the I/O relationship of loop B_0CDD_0 by using generated γ values from loop A_0ABB_0 and desired ψ corresponding to x_4 and x_5:

$$\begin{gathered}P_4+c\psi_4P_5-c\left(\psi_4-\gamma_4\right)P_6-s\gamma_4P_7+s\left(\psi_4-\gamma_4\right)P_8=c\gamma_4\\ P_4+c\psi_5P_5-c\left(\psi_5-\gamma_5\right)P_6-s\gamma_5P_7+s\left(\psi_5-\gamma_5\right)P_8=c\gamma_5\end{gathered} \tag{20}$$

We need to take derivative of δ_2 and equate to zero at x_4 and x_5 for the extrema conditions:

$$\frac{\partial\delta_2}{\partial x}=\frac{\partial}{\partial x}\left(w_d-w_g\right)=\frac{\partial w}{\partial x}-\frac{\partial w}{\partial\gamma}\frac{\partial\gamma}{\partial\psi}\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial x}=\frac{\partial w}{\partial x}-\frac{\Delta w}{\Delta\gamma}\frac{\Delta\psi}{\Delta y}\frac{\partial y}{\partial x}\frac{\partial\gamma}{\partial\psi} \tag{21}$$

where w_d is $w_{desired}$ and w_g is $w_{generated2}$. $\partial\gamma/\partial\psi$ is to be evaluated by differentiating the I/O equation of loop B_0CDD_0 with respect to ψ:

$$\begin{gathered}P_4+P_5c\psi-P_6c\left(\psi-\gamma\right)-P_7s\gamma+P_8s\left(\psi-\gamma\right)=c\gamma\\ \Rightarrow-P_5s\psi+P_6s\left(\psi-\gamma\right)\left(1-\frac{\partial\gamma}{\partial\psi}\right)-P_7c\gamma\frac{\partial\gamma}{\partial\psi}+P_8c\left(\psi-\gamma\right)\left(1-\frac{\partial\gamma}{\partial\psi}\right)=-s\gamma\frac{\partial\gamma}{\partial\psi}\\ \Rightarrow\frac{\partial\gamma}{\partial\psi}=\frac{P_5s\psi-P_6s\left(\psi-\gamma\right)-P_8c\left(\psi-\gamma\right)}{s\gamma-P_6s\left(\psi-\gamma\right)-P_7c\gamma-P_8c\left(\psi-\gamma\right)}\end{gathered} \tag{22}$$

Substituting in $\partial\delta_2/\partial x$ in Eq. (21) and equating to zero:

$$\frac{\partial\delta_2}{\partial x}=\frac{\partial w}{\partial x}-\frac{\Delta w}{\Delta\gamma}\frac{\Delta\psi}{\Delta y}\frac{\partial y}{\partial x}\frac{P_5 s\psi-P_6 s(\psi-\gamma)-P_8 c(\psi-\gamma)}{s\gamma-P_6 s(\psi-\gamma)-P_7 c\gamma-P_8 c(\psi-\gamma)}=0$$

$$\Rightarrow \Delta w\Delta\psi s\psi\frac{\partial y}{\partial x}P_5+\left[\Delta y\Delta\gamma\frac{\partial w}{\partial x}-\Delta w\Delta\psi\frac{\partial y}{\partial x}\right]s(\psi-\gamma)P_6+\ldots \tag{23}$$

$$\ldots+\Delta y\Delta\gamma c\gamma\frac{\partial w}{\partial x}P_7+\left[\Delta y\Delta\gamma\frac{\partial w}{\partial x}-\Delta w\Delta\psi\frac{\partial y}{\partial x}\right]c(\psi-\gamma)P_8=\Delta y\Delta\gamma s\gamma\frac{\partial w}{\partial x}$$

Eq. (23) is linear in P_5, P_6, P_7 and P_8. Eq. (20) and Eq. (23) written for x_4 and x_5 constitute four equations in 5 unknowns (P_4, P_5, P_6, P_7, P_8). However, $P_8 = P_6P_7$, so not all P_j's are independent. Let $P_8 = \lambda$ and $P_j = m_j + n_j\lambda$ for j = 4, 5, 6, 7. Substituting in Eq. (20) and Eq. (23):

- $(m_4+n_4\lambda)+c\psi_4(m_5+n_5\lambda)-c(\psi_4-\gamma_4)(m_6+n_6\lambda)-s\gamma_4(m_7+n_7\lambda)=c\gamma_4-s(\psi_4-\gamma_4)\lambda$
- $(m_4+n_4\lambda)+c\psi_5(m_5+n_5\lambda)-c(\psi_5-\gamma_5)(m_6+n_6\lambda)-s\gamma_5(m_7+n_7\lambda)=c\gamma_5-s(\psi_5-\gamma_5)\lambda$
- $\Delta w\Delta\psi s\psi_4\left.\frac{\partial y}{\partial x}\right|_{x_4}(m_5+n_5\lambda)+s(\psi_4-\gamma_4)\left[\Delta y\Delta\gamma\left.\frac{\partial w}{\partial x}\right|_{x_4}-\Delta w\Delta\psi\left.\frac{\partial y}{\partial x}\right|_{x_4}\right](m_6+n_6\lambda)+\ldots$

$$\ldots+\Delta y\Delta\gamma c\gamma_4\left.\frac{\partial w_d}{\partial x}\right|_{x_4}(m_7+n_7\lambda)=\Delta y\Delta\gamma s\gamma_4\left.\frac{\partial w}{\partial x}\right|_{x_4}-c(\psi_4-\gamma_4)\left[\Delta y\Delta\gamma\left.\frac{\partial w}{\partial x}\right|_{x_4}-\Delta w\Delta\psi\left.\frac{\partial y}{\partial x}\right|_{x_4}\right]\lambda \tag{24}$$

- $\Delta w\Delta\psi s\psi_5\left.\frac{\partial y}{\partial x}\right|_{x_5}(m_5+n_5\lambda)+s(\psi_5-\gamma_5)\left[\Delta y\Delta\gamma\left.\frac{\partial w}{\partial x}\right|_{x_5}-\Delta w\Delta\psi\left.\frac{\partial y}{\partial x}\right|_{x_5}\right](m_6+n_6\lambda)+\ldots$

$$\ldots+\Delta y\Delta\gamma c\gamma_5\left.\frac{\partial w_d}{\partial x}\right|_{x_5}(m_7+n_7\lambda)=\Delta y\Delta\gamma s\gamma_5\left.\frac{\partial w}{\partial x}\right|_{x_5}-c(\psi_5-\gamma_5)\left[\Delta y\Delta\gamma\left.\frac{\partial w}{\partial x}\right|_{x_5}-\Delta w\Delta\psi\left.\frac{\partial y}{\partial x}\right|_{x_5}\right]\lambda$$

Equating coefficients of λ and the rest to zero in Eq. (24) and writing in matrix form:

$$[A]\begin{bmatrix}m_4\\m_5\\m_6\\m_7\end{bmatrix}=\begin{bmatrix}c\gamma_4\\c\gamma_5\\\Delta y\Delta\gamma s\gamma_4\left.\frac{\partial w}{\partial x}\right|_{x_4}\\\Delta y\Delta\gamma s\gamma_5\left.\frac{\partial w}{\partial x}\right|_{x_5}\end{bmatrix}\text{ and }[A]\begin{bmatrix}n_4\\n_5\\n_6\\n_7\end{bmatrix}=\begin{bmatrix}-s(\psi_4-\gamma_4)\\-s(\psi_5-\gamma_5)\\-c(\psi_4-\gamma_4)\left(\Delta y\Delta\gamma\left.\frac{\partial w}{\partial x}\right|_{x_4}-\Delta w\Delta\psi\left.\frac{\partial y}{\partial x}\right|_{x_4}\right)\\-c(\psi_5-\gamma_5)\left(\Delta y\Delta\gamma\left.\frac{\partial w}{\partial x}\right|_{x_5}-\Delta w\Delta\psi\left.\frac{\partial y}{\partial x}\right|_{x_5}\right)\end{bmatrix} \tag{25}$$

$$\text{where }[A]=\begin{bmatrix}1 & c\psi_4 & -c(\psi_4-\gamma_4) & -s\gamma_4\\1 & c\psi_5 & -c(\psi_5-\gamma_5) & -s\gamma_5\\0 & \Delta w\Delta\psi s\psi_4\left.\frac{\partial y}{\partial x}\right|_{x_4} & s(\psi_4-\gamma_4)\left(\Delta y\Delta\gamma\left.\frac{\partial w}{\partial x}\right|_{x_4}-\Delta w\Delta\psi\left.\frac{\partial y}{\partial x}\right|_{x_4}\right) & \Delta y\Delta\gamma c\gamma_4\left.\frac{\partial w_d}{\partial x}\right|_{x_4}\\0 & \Delta w\Delta\psi s\psi_5\left.\frac{\partial y}{\partial x}\right|_{x_5} & s(\psi_5-\gamma_5)\left(\Delta y\Delta\gamma\left.\frac{\partial w}{\partial x}\right|_{x_5}-\Delta w\Delta\psi\left.\frac{\partial y}{\partial x}\right|_{x_5}\right) & \Delta y\Delta\gamma c\gamma_5\left.\frac{\partial w_d}{\partial x}\right|_{x_5}\end{bmatrix}.$$

m_4, m_5, m_6, m_7, n_4, n_5, n_6 and n_7 are solved by inverting [A]. λ is solved as follows:

$$\lambda = P_8 = P_6 P_7 = (m_6 + n_6\lambda)(m_7 + n_7\lambda) \Rightarrow n_6 n_7 \lambda^2 + (m_6 n_7 + n_6 m_7 - 1)\lambda + m_6 m_7 = 0$$

$$\Rightarrow \lambda = \frac{1 - m_6 n_7 - n_6 m_7 \mp \sqrt{(m_6 n_7 + n_6 m_7 - 1)^2 - 4 m_6 m_7 n_6 n_7}}{2 n_6 n_7} \tag{26}$$

Next, $P_j = m_j + n_j\lambda$ for j = 4, 5, 6, 7 are evaluated. Then d, e and f are found as $f = P_6$, $\alpha = \tan^{-1} P_7$, $d = \frac{f}{P_5 c\alpha}$ and $e = \sqrt{1 + d^2 + f^2 - 2dc\alpha P_4}$. Once again e should be real, if d is positive for $\alpha = \tan^{-1} P_9$, otherwise, $\alpha = \tan^{-1} P_9 + \pi$ and if f is negative, π can be added to limits of ψ. The error variation $\delta_2 = w_{desired} - w_{generated2}$ is determined with at least three precision points with the right assembly mode of the loop. $\delta_y = y_{desired} - y_{generated}$ is obtained and minimized by adjusting the angles and the intermediate function parameter while monitoring the necessary design considerations.

4. Numerical Examples

All formulations in the Section 3 are implemented in Excel and several different function synthesis tasks are performed. In this section, we present the application of each of the three correction methods for generation of a power function, an exponential function and a trigonometric function.

First function worked on is $y = x^2$ for $1 \le x \le 5$. $y = x^2$ is decomposed as $w = x^k$ and $y = w^{2/k}$. Second function is $y = e^{0.5x}$ for $1 \le x \le 5$ and the function is decomposed as $w = e^{0.5x/k}$ and $y = w^k$. For both of the functions, k is the intermediate function parameter. k is a design parameter which can be selected arbitrarily by the designer and can be adjusted to minimize the function generation error. The third function is $y = \sin(x)$ for $0 \le x \le \pi/2$ and the function is decomposed as $w = \tan(x/2)$ and $y = 2w/(1+w^2)$. In this case, there is no intermediate function parameter.

For the three correction methods, the mechanism angle limits and, if exists, the intermediate function parameter are varied to minimize the maximum absolute error $|\delta_y|_{max}$ subject to the conditions that the maximum to minimum link length ratio (r_{ll}) of the mechanism is less than 10 and $\Delta\phi$, $\Delta\gamma$ or $\Delta\psi$ is not less than 20°. Recall that the fixed link lengths are $|A_0B_0| = |B_0D_0| = 1$, but the loops can be independently scaled as one wishes. Therefore, r_{ll} is monitored independently for the two loops, but the relative sizes of the two loops is also considered. Since there are 6 angle limits and 1 intermediate function parameter for the first two functions and no intermediate parameter for the last function, there is a 7 or 6 dimensional optimization space. Using the spin buttons in Excel, several trials can be performed in a short time to find an optimum solution. Use of complicated numerical optimization techniques is not necessary. After several trials for the three functions with the three correction methods, the resulting designed link lengths are provided in Tables 1, 2 and 3 for $y = x^2$, $y = e^{0.5x}$ and $y = \sin(x)$, respectively.

Table 1. Intermediate function parameter, angle limits and designed link lengths of the 6-bar linkage for generation of $y = x^2$ for $1 \le x \le 5$

Method	k	ϕ_0	ϕ_f	γ_0	γ_f	ψ_0	ψ_f	ϕ^*	a	b	c	α	d	e	f	r_{ll}	$\lvert\delta_y\rvert_{max}$
1	1.2	155°	33°	99°	44°	230°	309°	0°	0.119	1.090	0.259	0°	0.379	1.052	0.303	9.177	6.91×10^{-2}
2	1.2	80°	0°	110°	58°	82°	59°	73.4°	0.780	1.536	1.338	239.1°	1.873	4.534	2.347	4.534	2.97×10^{-4}
3	1.3	247°	315°	170°	140°	60°	120°	0°	0.341	0.693	0.585	169.2°	0.459	0.635	0.160	6.244	2.15×10^{-3}

Table 2. Intermediate function parameter, angle limits and designed link lengths of the 6-bar linkage for generation of $y = e^{0.5x}$ for $1 \le x \le 5$

Method	k	ϕ_0	ϕ_f	γ_0	γ_f	ψ_0	ψ_f	ϕ^*	a	b	c	α	d	e	f	r_{ll}	$\lvert\delta_y\rvert_{max}$
1	2	159°	265°	250°	317°	205°	231°	0°	0.649	1.680	0.856	0°	5.477	4.711	1.571	5.477	4.08×10^{-2}
2	2	273°	230°	120°	54°	133°	107°	148.2°	1.840	0.716	1.570	229°	1.893	2.903	1.423	2.903	1.81×10^{-3}
3	2	159°	268°	248°	323°	205°	275°	0°	0.742	1.752	0.898	-61.0°	8.750	8.503	1.471	8.750	2.48×10^{-2}

Table 3. Angle limits and designed link lengths of the 6-bar linkage for generation of $y = \sin(x)$ for $0 \le x \le \pi/2$

Method	ϕ_0	ϕ_f	γ_0	γ_f	ψ_0	ψ_f	ϕ^*	a	b	c	α	d	e	f	r_{ll}	$\lvert\delta_y\rvert_{max}$
1	213°	75°	150°	45°	57°	105°	0°	1.577	1.973	1.994	180°	0.329	1.447	0.823	4.398	1.99×10^{-3}
2	180°	96°	150°	270°	265°	185°	244.1°	0.684	0.422	0.514	177.3°	0.594	0.678	0.854	1.682	3.00×10^{-3}
3	183°	90°	148°	80°	42°	90°	0°	0.705	1.128	0.816	255.2°	0.147	1.154	0.117	9.837	1.39×10^{-3}

All error curves of the worked out nine examples will not be presented here. To illustrate, the error curves for generation of $y = x^2$ with correction method 2 is presented in Fig. 5.

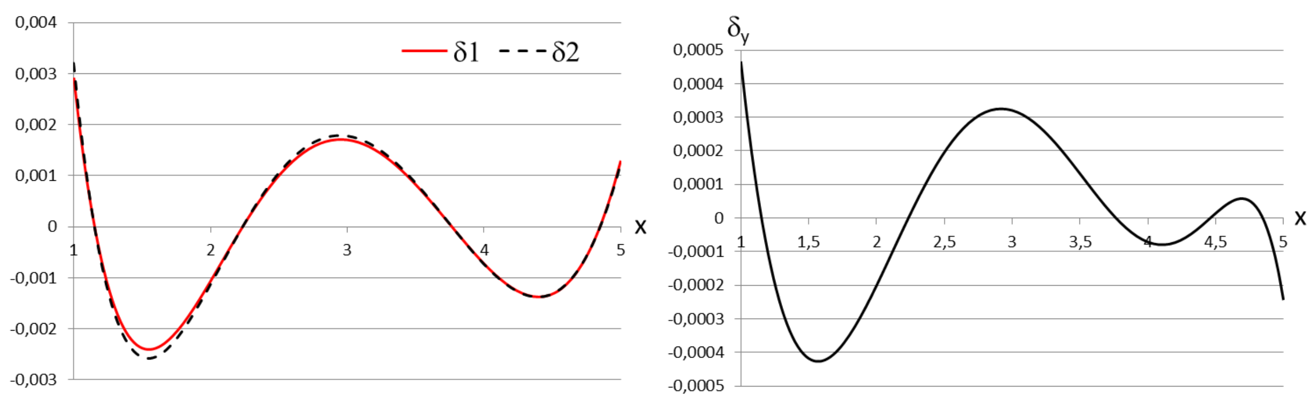

Fig. 5. Error curves for $y = x^2$ with correction method 2

5. Performance Evaluation

For $y = x^2$ and $y = e^{0.5x}$, the solutions found for correction method 2 are superior than the solutions found for correction method 3 and correction method 1 yields the worst results. It is expectable that correction method 2 with four precision points is better than the correction method 1 with three precision points. On the other hand, we were expecting to get the best results with correction method 3, which makes use of the first order derivatives. Unlike the other two functions, for $y = \sin(x)$, correction method 3 gives the best results and correction method 2 gives the worst results. Once again, we do not claim that the results cannot be improved, however, the obtained numerical results indicate that any of the correction methods may be better than the others depending on the function to be generated and the domain of the independent variable.

Different types of mechanisms, different functions and different independent variable domains are used in the examples in the literature. Still, for comparing our results with the previously published results, we can take a look at the order of magnitudes of the errors. As a measure to be compared let us use the percentage error defined as

$$\%\varepsilon = 100 \cdot |\text{Maximum error}|/|\text{Range of the output}|$$

For our examples the best results are as follows: %ε = 0.00124% for $y = x^2$ with $1 \le x \le 5$, %ε = 0.0172% for $y = e^{0.5x}$ with $1 \le x \le 5$ and %ε = 0.139% for $y = \sin(x)$ with $0 \le x \le \pi/2$. The maximum percentage error we get is 0.139%.

Kinzel et al. [17] designed a Stephenson III type six-bar linkage for generation of $y = \log_{10}(x)$ for $1 \le x \le 2$ with eleven precision points. The maximum error in the output of the function can be evaluated from the link lengths provided in the paper as 1.62×10^{-5}, so %ε = 0.0054%. For comparison, we also worked out $y = \log_{10}(x)$ for $1 \le x \le 2$ with Correction method 2, i.e. for four precision points for both loops. A good solution is obtained for the angle limits listed in Table 4. The percentage error is evaluated as %ε = 0.0018% which is better than the error of Kinzel et al. [17]. But, note that the linkages utilized are of different type.

Table 4. Angle limits and designed link lengths of the 6-bar linkage for generation of $y = \log_{10}(x)$ for $1 \le x \le 2$

Method	ϕ_0	ϕ_f	γ_0	γ_f	ψ_0	ψ_f	ϕ^*	a	b	c	α	d	e	f	r_{ll}	$\lvert\delta_y\rvert_{max}$
2	40°	164°	60°	131°	1°	67°	45.03°	1.432	8.181	8.846	-32.5°	1.126	3.250	2.012	8.846	5.47×10^{-6}

Hwang and Chen [18] designed a Stephenson II type function generator as an example for generation of $y = x^2$ for $-1 \le x \le 1$. The result is a maximum error of 0.498° for an output range of $\Delta\theta$ = 60°. This corresponds to a percentage error of %ε = 0.83%. Sancibrian [19] presented several design examples with different types of linkages for several different functions. The best result with the smallest magnitude of error is obtained with a Stephenson III type six-bar linkage for generation of a rise-dwell-return type simple harmonic motion for 24° oscillation of the output link. When the mechanism is analyzed, we found that the maximum amount of absolute error is 0.411° within the range of 24°, which corresponds to a percentage error of %ε = 1.71%. These results are summarized in Table 5.

Table 5. Comparison of designs

Author	**Function**	**Input domain**	**Output range**	**Linkage**	**%ε**
Kinzel et al. [17]	$y = \log_{10}(x)$	$1 \le x \le 2$	$0 \le y \le 0.301$	Stephenson III	0.0054%
Hwang and Chen [18]	$y = x^2$	$-1 \le x \le 1$	$0 \le y \le 1$	Stephenson II	0.83%
Sancibrian [19]	$y = \begin{cases} \pi[1-\cos(2x)]/30 & \text{for } 0 \le x \le \pi/2 \\ \pi/15 & \text{for } \pi/2 \le x \le 3\pi/2 \\ \pi[1-\cos(2x)]/30 & \text{for } 3\pi/2 \le x \le 2\pi \end{cases}$	$0 \le x \le 2\pi$	$0 \le y \le 2\pi/15$	Stephenson III	1.71%
Kiper et al.	$y = \log_{10}(x)$	$1 \le x \le 2$	$0 \le y \le 0.301$	Watt II	0.0018%
Kiper et al.	$y = x^2$	$1 \le x \le 5$	$1 \le y \le 25$	Watt II	0.00124%
Kiper et al.	$y = e^{0.5x}$	$1 \le x \le 5$	$1.65 \le y \le 12.2$	Watt II	0.0172%
Kiper et al.	$y = \sin(x)$	$0 \le x \le \pi/2$	$0 \le y \le 1$	Watt II	0.139%

6. Conclusions

In this study, the method of decomposition is successfully applied to a Watt II type planar six-bar linkage. The method also can be easily adapted for the Stephenson III type planar six-bar linkage, as well, provided that the input link of the mechanism is a binary link connected to the ternary links. The three types of proposed correction methods are applied for three different functions: a power function, an exponential function and a trigonometric function.

Using a two-loop function generator mechanism instead of a single-loop mechanism has the obvious advantage of reduced generation error. This can be seen by comparing the two plots in Fig. 5. The left plot for the single-loop mechanisms has errors in the order of 0.001s, while the right plot for the two-loop mechanism has errors in the order of 0.0001s. This is of course just an example, however it is natural to expect the superiority of a two-loop mechanism over a single-loop mechanism since the two-loop mechanism has more number of design parameters in total, regardless of it has been decomposed into two single-loops or not. Also via the examples presented in Section 5, we have demonstrated the power of the proposed methods in this study compared to the other methods presented in the literature for two-loop planar 6-bar mechanisms.

The application of the synthesis methods enclosed herein require specification of the six mechanism angle limits and, if exists, the intermediate function parameter by the designer. This gives the designer a flexibility in design. Although the selection of these six or seven design parameters is done manually in this study, it is also possible to run a numerical optimization technique with these free design parameters. We do not prefer to use such an optimization technique, because we want to see the changing behavior of the error variation as the free parameters are continuously varied. However, with this manual operation, we can only obtain limited number of feasible solutions and cannot guarantee that there is no better solution with less amount of error. Still, we can speculate on the results of the numerical examples in order to compare the three correction methods for the generation of the three different functions and conclude that superiority of the methods with respect to each other depends on the function to be generated.

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