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Multi-wave and Rational Solutions for Nonlinear Evolution Equations

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Abstract

Nonlinear evolution equations always admit multi-soliton and rational solutions. The Burgers equation is used as an example, and the exp-function method is used to eluciadte the solution procedure.

Keywords: Exp-function method; Multi-wave solutions; Rational solutions; Burgers equation

1. Introduction

[1]Many modeling problems arising in nonlinear sciences deal with nonlinear evolution equations (NEEs). The investigation of the so-called solitary wave solutions to NEEs has become more and more attractive in soliton theory since they can be widely found in many scientific fields. Over the four decades or so, many analytic methods have been successfuly developed by a diverse groups of scientists; to mention some, Exp-function method [1], first integral method [2], variational iteration method [3], homotopy perturbation method [4], homotopy analysis method [5], (G'/G)-expansion method [6], multi-exp function method [7], three-wave method [8] and so forth. In addition, He et al. [9] proposed three standard variational iteration algorithms for solving differential equations, integro-differential equations, fractional differential equations, fractal differential equations, differential-difference equations and fractional/fractal differential-difference equations.

On the other hand, the Exp-function method has become very popular in the research community nowadays. It allows one to obtain exact and explicit solutions for NEEs in a concise manner. The method has been extensively implemented by the researchers to various kinds of nonlinear problems arising in applied sciences. Lately, more attention has been given to its generalization, adaptation, and extension; for instance, differential-difference equations [10], multi-dimensional equations [11, 12], NEEs with variable coefficients [13], coupled NEEs [14], *n*-soliton solutions [15, 16], stochastic equations [17], rational solutions [18, 19], double-wave solutions [20]. Our main goal in this paper is to demonstrate an application of the Exp-function method to obtain rational and multi-wave solutions for NEEs. We prefer to study the well-known Burgers equation as a test problem.

2. Methodology

Let us consider a nonlinear partial differential equation in the form

$$P\left(u,u_t,u_x,u_{tt},u_{tx},u_{xx},\ldots\right)=0\,, \tag{1}$$

where $u=u(x,t)$ and P is a polynomial in its arguments. The Exp-function method is based on the assumption that the solutions of Eq. (1) can be expressed as

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$$u(x,t)=\sum_{i=0}^{m}a_i e^{i\xi} / \sum_{i=0}^{n}b_i e^{i\xi},\ \xi = kx + wt, \qquad (2)$$

where m and n are positive integers to be specified; a_i, b_i, k and w are unknown constants to be determined. We can determine m and n by substituting the ansatz (2) into Eq. (1) and balancing the highest-order terms.

In order to search for $N(>1)-$wave solutions to Eq. (1), it is reasonable to generalize the ansatz (2) as

$$u(x,t)=\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}a_{ij}e^{i\xi_1+j\xi_2} / \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}b_{ij}e^{i\xi_1+j\xi_2},\ \xi_l = k_l x + w_l t,\ l=1,2, \qquad (3)$$

or

$$u(x,t)=\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\sum_{k=0}^{m_3}a_{ijk}e^{i\xi_1+j\xi_2+k\xi_3} / \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_3}b_{ijk}e^{i\xi_1+j\xi_2+k\xi_3},\ \xi_i = k_i x + w_i t,\ i=1,2,3\,. \qquad (4)$$

Obviously, the ansatze (3) and (4) can reveal 2-soliton and 3-soliton solutions, respectively.

To construct rational solutions to Eq. (1), we consider another modified form of the ansatz (2) as

$$u(x,t)=\sum_{i=0}^{m}a_i\left(\mu_1\exp(\xi)+\mu_2\xi\right)^i / \sum_{i=0}^{n}b_i\left(\mu_1\exp(\xi)+\mu_2\xi\right)^i,\ \xi = kx + wt, \qquad (5)$$

where μ_1 and μ_2 are two embedded constants. It is easy to see that when $\mu_1=1$ and $\mu_2=0$, the ansatz (5) becomes the ansatz (2).

3. Application to the Burgers equation

The famous Burgers equation reads

$$u_t + 2\alpha u u_x + \alpha u_{xx} = 0, \qquad (6)$$

where α is an arbitrary constant. We assume that Eq. (6) admits a solution of the form

$$u(x,t)=e^{\xi}a_1 / \left(1+e^{\xi}b_1\right),\ \xi = kx + wt, \qquad (7)$$

which is embedded in (2). Substituting (7) into Eq. (6), we get the equation

$$\left(1+b_1\exp(\xi)\right)^{-3}\sum_{n=1}^{2}C_n\exp(n\xi)=0, \qquad (8)$$

where $C_1=\alpha k^2 + w$ and $C_2 = 2\alpha a_1 k - \alpha b_1 k^2 + b_1 w$. Thus, solving the system $C_n = 0\ (n=1,2)$ simultaneously, we obtain the solution set

$$\left\{w=-\alpha k^2,\ a_1 = kb_1\right\}, \qquad (9)$$

which yields the 1-soliton solution

$$u_1(x,t)=b_1 k e^{kx-\alpha k^2 t} / \left(1+b_1 e^{kx-\alpha k^2 t}\right), \qquad (10)$$

where k and b_1 are arbitrary constants.

3.1. Double-wave solutions

We assume that Eq. (6) admits a solution of the form

$$u(x,t)=\left(a_{10}e^{\xi_1}+a_{01}e^{\xi_2}+a_{11}e^{\xi_1+\xi_2}\right) / \left(1+b_{10}e^{\xi_1}+b_{01}e^{\xi_2}+b_{11}e^{\xi_1+\xi_2}\right),\ \xi_l = k_l x + w_l t,\ l=1,2\,. \qquad (11)$$

It is clear that the ansatz (11) is embedded in (3). Substituting (11) into Eq. (6), we obtain the relation

$$\left(1+b_{01}e^{\xi_2}+b_{10}e^{\xi_1}+b_{11}e^{\xi_1+\xi_2}\right)^{-3}\sum_{i=0}^{3}\sum_{j=0}^{3}C_{ij}e^{i\xi_1+j\xi_2}=0, \tag{12}$$

where $C_{00}=C_{03}=C_{30}=C_{33}=0$. Hence, solving the system $C_{ij}=0\ (0<i,j\leq 3)$ simultaneously, we obtain the solution set

$$\left\{w_1=-\alpha k_1^2,\ w_2=-\alpha k_2^2,\ b_{11}=0,\ a_{11}=0,\ b_{01}=a_{01}/k_2,\ b_{10}=a_{10}/k_1\right\}, \tag{13}$$

which gives the 2-soliton solution

$$u_2(x,t)=\left(a_{01}e^{k_2x-\alpha k_2^2t}+a_{10}e^{k_1x-\alpha k_1^2t}\right)/\left(1+(a_{10}/k_1)e^{k_1x-\alpha k_1^2t}+(a_{01}/k_2)e^{k_2x-\alpha k_2^2t}\right), \tag{14}$$

where a_{01}, a_{10}, k_1, and k_2 are arbitrary constants.

3.2. Three-wave solutions

We assume that Eq. (6) admits a solution of the form

$$u(x,t)=\frac{a_{100}e^{\xi_1}+a_{010}e^{\xi_2}+a_{001}e^{\xi_3}+a_{110}e^{\xi_1+\xi_2}+a_{101}e^{\xi_1+\xi_3}+a_{011}e^{\xi_2+\xi_3}+a_{111}e^{\xi_1+\xi_2+\xi_3}}{1+b_{100}e^{\xi_1}+b_{010}e^{\xi_2}+b_{001}e^{\xi_3}+b_{110}e^{\xi_1+\xi_2}+b_{101}e^{\xi_1+\xi_3}+b_{011}e^{\xi_2+\xi_3}+b_{111}e^{\xi_1+\xi_2+\xi_3}}, \tag{15}$$

where $\xi_l=k_lx+w_lt\ (l=1,2,3)$.

Obviously, the ansatz (15) is embedded in (4). After substituting (15) into Eq. (6) and making similar manipulations, we get the solution set of the resulted algebraic system as

$$\left\{\begin{matrix}w_1=-\alpha k_1^2,\ w_2=-\alpha k_2^2,\ w_3=-\alpha k_3^2,\ b_{100}=a_{100}/k_1,\ b_{010}=a_{010}/k_2,\ b_{001}=a_{001}/k_3,\\ b_{101}=0,\ b_{011}=0,\ a_{011}=0,\ a_{101}=0,\ b_{111}=0,\ a_{110}=0,\ b_{110}=0,\ a_{111}=0\end{matrix}\right\}, \tag{16}$$

which leads to the 3-soliton solution

$$u_3(x,t)=\left(a_{001}e^{k_3x-\alpha k_3^2t}+a_{010}e^{k_2x-\alpha k_2^2t}+a_{100}e^{k_1x-\alpha k_1^2t}\right)/\left(1+(a_{100}/k_1)e^{k_1x-\alpha k_1^2t}+(a_{010}/k_2)e^{k_2x-\alpha k_2^2t}+(a_{001}/k_3)e^{k_3x-\alpha k_3^2t}\right), \tag{17}$$

where a_{001}, a_{010}, a_{100}, k_1, k_2, and k_3 are arbitrary constants.

The higher level multi-soliton solutions, for $N>4$, can be obtained in a parallel manner.

3.3. Rational solutions

We assume that Eq. (6) admits a solution of the form

$$u(x,t)=\frac{a_1(\mu_1\exp(\xi)+\mu_2\xi)+a_0+a_{-1}(\mu_1\exp(\xi)+\mu_2\xi)^{-1}}{b_1(\mu_1\exp(\xi)+\mu_2\xi)+b_0+b_{-1}(\mu_1\exp(\xi)+\mu_2\xi)^{-1}},\ \xi=kx+wt. \tag{18}$$

By the same procedure, we get the following solution sets of the algebraic system

$$\{w=-2k\alpha a_1/b_1,\ a_0=kb_1,\ a_{-1}=0,\ b_0=0,\ b_{-1}=0,\ \mu_1=0,\ \mu_2=1\}, \tag{19}$$

$$\left\{\begin{matrix} w=k\alpha\left(-2a_{-1}+k\left(b_0+\sqrt{b_0^2-4b_{-1}b_1}\right)\right)/b_{-1},\ a_1=b_1\left(2a_{-1}-k\left(b_0+\sqrt{b_0^2-4b_{-1}b_1}\right)\right)/2b_{-1}, \\ a_0=2a_{-1}b_0+k\left(2b_{-1}b_1-b_0\left(b_0+\sqrt{b_0^2-4b_{-1}b_1}\right)\right)/2b_{-1},\ \mu_1=0,\ \mu_2=1 \end{matrix}\right\}, \tag{20}$$

which lead to the rational solutions

$$u_4(x,t)=\frac{a_1}{b_1}+\frac{b_1}{b_1x-2\alpha a_1t}, \tag{21}$$

where a_1 and b_1 are arbitrary constants and

$$u_5^{\pm}(x,t)=\frac{\begin{matrix}2a_{-1}b_{-1}+\left(2a_{-1}b_0-kb_0^2+2kb_{-1}b_1\pm kb_0\sqrt{b_0^2-4b_{-1}b_1}\right)(kx+wt)\\ +\left(2a_{-1}b_1-kb_0b_1\pm kb_1\sqrt{b_0^2-4b_{-1}b_1}\right)(kx+wt)^2\end{matrix}}{2b_{-1}^2+2b_{-1}b_0(kx+wt)+2b_{-1}b_1(kx+wt)^2}, \tag{22}$$

where $w=k\alpha\left(-2a_{-1}+k\left(b_0\mp\sqrt{b_0^2-4b_{-1}b_1}\right)\right)/b_{-1}$, and a_{-1}, b_{-1}, b_0, b_1, k are arbitrary constants.

4. Conclusion

Multi-soliton solutions are crucial since they reveal the interactions between the inner-waves and the various frequency and velocity components. In this study, we have successfully shown that the Exp-function method can be a straightforward and effective mathemaical tool for constructing rational and multi-wave solutions to many integrable or non-integrable NEEs. We have checked the correctness of the obtained results, with the aid of MATHEMATICA, by putting them back into the original equation.

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