F(METRIC-AFFINE) GRAVITY:
DISFORMAL AND CROSS-CURVATURE EFFECTS

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ABSTRACT

F(METRIC-AFFINE) GRAVITY: DISFORMAL AND CROSS-CURVATURE EFFECTS

The present thesis consists of two main studies, in the first part, after giving a brief formulation of gravity theories on the metric, affine and metric-affine frameworks, we study the effects of the disformal coupling term $\epsilon\ \mathbb{R}_{\mu\nu}V^{\mu}V^{\nu}$. We track the effects of the disformal term up to field equations, then construct the Einstein tensor $G_{\mu\nu}$ and subsequently identify an effective energy-momentum tensor $T^{eff}_{\mu\nu}$ to extract effective energy density and pressure. We conclude the first part by comparing the results of metric-affine disformal theory with metrical disformal theory. In the second part, we study the cosmological effects of cross-curvature theory with the functional $F(\mathrm{R}, \mathbb{R})$. We derive both Friedmann equations with the general functional $F(\mathrm{R}, \mathbb{R})$ and compare our findings with the known $F(\mathbb{R})$ theory results.

ÖZET

F(METRİK-AFİN) KÜTLE ÇEKİMİ:
DİSFORMAL VE ÇAPRAZ-EĞRİLİK ETKİLERİ

Mevcut tez, iki ana çalışmadan oluşmaktadır, ilk kısımda metrik, afin ve metrik-afin yapılarında kütle çekim teorilerinin kısa bir formulasyonunu verdikten sonra, disformal çiftlenim teriminin $\epsilon\ \mathbb{R}_{\mu\nu}V^{\mu}V^{\nu}$ etkilerini çalışıyoruz. Disformal terimin etkilerini alan denklemlerine kadar izleyerek, sonrasında Einstein tensörünü $G_{\mu\nu}$ oluşturuyoruz ve ardından efektif enerji yoğunluğu ve basınç terimlerini saptamak için efektif enerji-momentum tensörünü belirliyoruz. İlk kısmı, metrik-afin disformal etkilerinin bulgularını, metrik disformal etkilerinin bulguları ile karşılaştırarak sonuçlandırıyoruz. İkinci kısımda, $F(\mathrm{R},\mathbb{R})$ fonksiyoneli ile çapraz-eğrilik teorisinin kozmolojik etkilerini çalışıyoruz. Genel fonksiyonel $F(\mathrm{R},\mathbb{R})$ ile her iki Friedmann denklemlerini de türeterek, bulgularımızı, bilinen $F(\mathbb{R})$ teorisinin sonuçları ile karşılaştırıyoruz.

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CHAPTER 1

INTRODUCTION

Gravity, despite being exposed to physics more than any other phenomenon, people still debate over its formulation. With every new observation that contradicts our understanding of the gravity, we try to adapt ourselves by either modifying the accepted theory or even develop a new one based on the different fundamental grounds.

It was Galileo Galilei who is first to conduct many experiments with falling objects, inclined planes and even with pendulums [1]. In 1665, Isaac Newton gave us the famous ”inverse-square law” to explain the motion of celestial bodies and clear description of motion of projectiles in our everyday life [2]. Although Newton’s gravity managed to explain all of the features of gravity in his time, it failed at some point. With the technological advance of observational astrophysics, Le Verrier observed the famous precession of perihelia of the planed Mercury in 1855. This observed precession was in contradiction with the Newtonian gravity and guided by his previous prediction of Neptune in 1846, he predicted that there must be an another, unobserved ”dark” planet closer to the sun [3]. This dark planet, allegedly responsible for Mercury’s precession was never discovered. Finally, in 1915, Einstein presented his theory of General Relativity (GR) based on different fundamental grounds than Newtonian gravity. In his previous work Special Relativity (SR), he combined space and time into 4-dimensional space-time which was known to be two different things until his clear description. Then he rejected the idea of ”absolute space”, which is the fundamental concept of Newtonian gravity. Unlike Newton’s ”static” absolute space, Einstein’s space-time was ”dynamic”, it can be bend, stretch or even twirl by the matter/energy.

Einstein’s GR, not only explained the precession of perihelia of the Mercury, it predicted new phenomenons such as gravitational deflection of light rays by massive objects which is successfully measured during a solar eclipse by Sir Arthur Eddington in 1919 [4], the gravitational redshift of lights wavelength moving in a gravitational field [5], and even gravitational waves are the distortion of the fabric of space-time’s itself [6].

Once again in the light of cosmological and astrophysical observations that have been conducted in the last few decades, physicists began to question Einstein’s GR whether it is a complete theory or not. Evidence coming from various observations seem to indicate that only %4 of the total energy of our universe comes from ordinary baryonic matter, with the remaining %20 ”dark” matter and %76 is ”dark” energy. Dark matter here refers to an undiscovered type of matter which behaves similarly to ordinary matter and related to velocity of stars rotating around galaxy centers. Both dark matter and ordinary matter satisfies the strong energy condition and they have clustering properties under the effect of gravitation. Dark energy, on the other hand, does not satisfy the strong energy condition, therefore, does not cluster as ordinary baryonic matter and it is related to late-time accelerated expansion of our universe [12].

Most of the experimental tests those GR has passed are held in relatively low curvature environments. GR has never been put the test near a neutron star or black hole where the curvature of space-time is much more higher than the curvature of our solar system.

Even before cosmological observations to question GR as a complete theory of gravity, people investigated its ”special” formulation. GR is formulated in a purely metrical fashion where metric is the only fundamental variable and affine connection of the space-time is chosen to be Levi-Civita of the metric. One must understand that this is a very special way to formulate theory because we are assuming there is a priori relation between affine connection and the metric. There is another formulation of gravity where the affine connection is chosen to be independent of the metric which we call metric-affine theory of gravity. Relaxing the assumption that connection is dependent to metric allows us to formulate gravity based on two independent variables and as we will see in chapter 2 theory ”dynamically” reduces to GR given the same gravitational action. Lastly another formulation of gravity which is partly developed by the very same man who provided the experimental verification of GR by measuring the deflection of light rays through suns gravitational field. Eddington formulated gravity with only the affine connection is being a fundamental variable and no priori given metric. He successfully obtained the Einstein field equations in vacuum.

The present study is organized as follows: In Chapter 2 we give a bried formulation of the three theory of gravitation starting from Einstein’s purely metric formulation then affine formulation and lastly the metric-affine formulation of the gravity. In Chapter 3 we study the disformal effects in metric-affine theory. In Chapter 4 we review the cross-curvature effects in metric-affine theory. In Chapter 5 we conclude our study and summarize the results of both modified actions.

CHAPTER 2

GRAVITY THEORIES

2.1. Metrical Theory

Einstein's general theory of relativity (GR) can be formulated in a classical field theory fashion with the only independent field given as metric $\mathrm{g}_{\mu\nu}$. Due to Hilbert's motivation for writing a generally covariant action from the derivatives of metric, action of the theory commonly known as Einstein-Hilbert action and given as [7, 8]

$$S[\mathrm{g}_{\mu\nu}] = \frac{M_{pl}^2}{2}\int \mathrm{d}^4x\sqrt{-\mathrm{g}}\,\mathrm{R}({}^{\mathrm{g}}\Gamma) + S_M[\mathrm{g},\psi], \tag{2.1}$$

where S_M represents the action of matter field ψ. $\mathrm{R}({}^{\mathrm{g}}\Gamma) = \mathrm{g}^{\mu\nu}\mathrm{R}_{\mu\nu}$ is the Ricci curvature scalar constructed from Ricci tensor $\mathrm{R}_{\mu\nu}$ and the inverse metric $\mathrm{g}^{\mu\nu}$, $\mathrm{g} = Det[\mathrm{g}_{\mu\nu}]$ is the determinant of the metric $\mathrm{g}_{\mu\nu}$, $M_{pl} = 1/\sqrt{8\pi G_N}$ is the reduced planck mass with Planck's constant h and speed of light c are taken as unity. Ricci tensor $\mathrm{R}_{\mu\nu}$ is defined as the only independent contraction of the Riemann tensor $\mathrm{R}^{\alpha}{}_{\mu\beta\nu}$ with first and third indices contracted. Definition of the Riemann tensor and Levi-Civita connection ${}^{\mathrm{g}}\Gamma^{\lambda}_{\mu\nu}$ respectively [14]

$$\mathrm{R}^{\alpha}{}_{\mu\beta\nu} = \partial_\beta {}^{\mathrm{g}}\Gamma^{\alpha}_{\mu\nu} - \partial_\nu {}^{\mathrm{g}}\Gamma^{\alpha}_{\mu\beta} + {}^{\mathrm{g}}\Gamma^{\alpha}_{\lambda\beta}{}^{\mathrm{g}}\Gamma^{\lambda}_{\mu\nu} - {}^{\mathrm{g}}\Gamma^{\alpha}_{\lambda\nu}{}^{\mathrm{g}}\Gamma^{\lambda}_{\mu\beta}, \tag{2.2}$$

and

$${}^{\mathrm{g}}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\mathrm{g}^{\lambda\sigma}\left(\partial_\mu \mathrm{g}_{\sigma\nu} + \partial_\nu \mathrm{g}_{\sigma\mu} - \partial_\sigma \mathrm{g}_{\mu\nu}\right). \tag{2.3}$$

The relation between Levi-Civita connection ${}^{\mathrm{g}}\Gamma^{\lambda}_{\mu\nu}$ and the metric $\mathrm{g}_{\mu\nu}$ is the direct consequence of two *priori* assumptions, that the affine connection is symmetric under the exchange of lower indices $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{(\mu\nu)}$ and it preserves the metric

$$\nabla_\lambda \mathrm{g}_{\mu\nu} = 0. \tag{2.4}$$

Since metric and inverse metric is related to eachother by the relation $\mathrm{g}^{\mu\nu}\,\mathrm{g}_{\mu\lambda} = \delta^{\nu}_{\lambda}$ and Kronecker delta is invariant under any variation, variations in $\mathrm{g}^{\mu\nu}$ will be equivalent to variations in $\mathrm{g}_{\mu\nu}$. In order to obtain equations of motion, we vary the Einstein-Hilbert action (2.1) with respect to inverse metric, variation of the gravitational part of action can be written as

$$\delta S_G = \frac{M_{pl}^2}{2} \int \mathrm{d}^4x \Big(\sqrt{-\mathrm{g}}\ \mathrm{g}^{\mu\nu} \delta \mathrm{R}_{\mu\nu} + \sqrt{-\mathrm{g}}\ \mathrm{R}_{\mu\nu}\ \delta \mathrm{g}^{\mu\nu} + \mathrm{R}\ \delta\sqrt{-\mathrm{g}} \Big). \tag{2.5}$$

Second term is already in the desired form. Last term contains variation of the determinant of metric, hence we can use the variation of the identity

$$ln(Det\ \mathrm{g}_{\mu\nu}) = Tr(ln\ \mathrm{g}_{\mu\nu}), \tag{2.6}$$

to put it into same form as the second term. With the use of the above identity and variation of Kronecker delta we get

$$\delta\sqrt{-\mathrm{g}} = -\frac{1}{2}\sqrt{-\mathrm{g}}\ \mathrm{g}_{\mu\nu}\ \delta \mathrm{g}^{\mu\nu}. \tag{2.7}$$

The first term in (2.5) contains metric through Levi-Civita connection, we can write the variation of Ricci tensor in terms of variations of Levi-Civita connection as

$$\begin{aligned} \delta S_{G1} &= \frac{M_{pl}^2}{2} \int \mathrm{d}^4x \sqrt{-\mathrm{g}}\ \mathrm{g}^{\mu\nu}\ \delta \mathrm{R}_{\mu\nu} \\ &= \frac{M_{pl}^2}{2} \int \mathrm{d}^4x \sqrt{-\mathrm{g}}\ \mathrm{g}^{\mu\nu} \Big[\nabla_\lambda (\delta \Gamma^\lambda_{\nu\mu}) - \nabla_\nu (\delta \Gamma^\lambda_{\lambda\mu}) \Big] \\ &= \frac{M_{pl}^2}{2} \int \mathrm{d}^4x \sqrt{-\mathrm{g}}\ \nabla_\sigma \Big[\mathrm{g}^{\mu\nu} (\delta \Gamma^\sigma_{\mu\nu}) - \mathrm{g}^{\mu\sigma} (\delta \Gamma^\lambda_{\lambda\mu}) \Big], \end{aligned} \tag{2.8}$$

where we have used the identity

$$\delta \mathrm{R}_{\mu\nu} = \nabla_\lambda (\delta \Gamma^\lambda_{\nu\mu}) - \nabla_\nu (\delta \Gamma^\lambda_{\lambda\mu}), \tag{2.9}$$

relabelled some dummy indices and used the metric compatibility equation (2.4). We still need to express (2.8) in terms of $\delta \mathrm{g}^{\mu\nu}$, taking the variation of (2.4) and using cyclic permutation, we can write $\delta\Gamma^\sigma_{\mu\nu}$ in terms of $\delta \mathrm{g}^{\mu\nu}$ as

$$\delta \Gamma^\sigma_{\mu\nu} = -\frac{1}{2} \Big[\mathrm{g}_{\lambda\mu}\ \nabla_\nu (\delta \mathrm{g}^{\lambda\sigma}) + \mathrm{g}_{\lambda\nu}\ \nabla_\mu (\delta \mathrm{g}^{\lambda\sigma}) - \mathrm{g}_{\mu\alpha} \mathrm{g}_{\nu\beta}\ \nabla^\sigma (\delta \mathrm{g}^{\alpha\beta}) \Big]. \tag{2.10}$$

Putting this into (2.8) we get the following

$$\delta S_{G1} = \frac{M_{pl}^2}{2} \int \mathrm{d}^4x \sqrt{-\mathrm{g}}\, \nabla_\sigma \Big(\mathrm{g}_{\mu\nu}\, \nabla^\sigma(\delta \mathrm{g}^{\mu\nu}) - \nabla_\lambda(\delta \mathrm{g}^{\sigma\lambda}) \Big). \tag{2.11}$$

Since it is a volume integral of the covariant divergence of a vector, we can use Stokes' theorem to write this as a surface term at infinity and the least action principle tells us that the variation of the fields which is metric in this case will be set to zero at the surface. However, our surface term also contains first derivatives of the metric and it is not always zero by default. Therefore in order to get the right field equations from the action (2.1), Einstein-Hilbert action must be supplemented by an additional term to cancel unwanted surface term. This unwanted surface term is not going to be present when we formulate the theory within the metric-affine framework.

Going back to (2.5), we can rewrite variation of the gravitational part of the action as

$$\delta S_G = \frac{M_{pl}^2}{2} \int \mathrm{d}^4x \sqrt{-\mathrm{g}} \left[\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{R}\mathrm{g}_{\mu\nu} \right] \delta \mathrm{g}^{\mu\nu} \tag{2.12}$$

Adding variation of the matter part and equating to zero, we get the Einstein field equations as

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{R}\mathrm{g}_{\mu\nu} = \frac{1}{M_{pl}^2} T_{\mu\nu}, \tag{2.13}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the matter fields defined as

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-\mathrm{g}}} \frac{\delta S_M}{\delta \mathrm{g}^{\mu\nu}}. \tag{2.14}$$

Eq.(2.13) is actually a set of 10 second order differential equations for the field variable $\mathrm{g}_{\mu\nu}$, commonly know as Einstein's field equations. Left-hand side of the field equations (2.13) reflects the dynamics of the geometry via Ricci tensor and Ricci scalar whereas the right-hand side of the equations acts as a source for the curvature [14].

2.2. Affine Theory

In the absence of matter, another way of formulating gravitation is taking the affine connection as the only independent variable and building an invariant action only from this connection. Since one only needs an affine connection to differentiate mathematical entities, we can drop the priori given metric and formulate gravitation in terms of affine connection alone. Not long after Einstein formulated his general theory of relativity in a metrical way, Eddington suggested the simplest action you can build from the affine connection alone as [9, 10]

$$S_E[\Gamma^{\lambda}_{\mu\nu}] = \int \mathrm{d}^4x\sqrt{-\mathbb{R}}, \tag{2.15}$$

where $\mathbb{R} = Det[\mathbb{R}_{\mu\nu}]$ is the determinant of the symmetric part of the Ricci tensor $\mathbb{R}_{\mu\nu}(\Gamma)$. Applying the least action principle to (2.15) leads to desired equations of motion for the field variable affine connection

$$\nabla_\lambda\left[\sqrt{-\mathbb{R}}\,(\mathbb{R}^{-1})^{\mu\nu}\right] = 0, \tag{2.16}$$

which can be solved by introducing a new invertible tensor field $\mathrm{g}_{\mu\nu}$ multiplied by a constant Λ such that

$$\mathbb{R}_{\mu\nu} = \Lambda \mathrm{g}_{\mu\nu}. \tag{2.17}$$

If we interpret the new field $\mathrm{g}_{\mu\nu}$ as the metric, we see that Eqn.(2.9) is exactly same as the Einstein field equations in a vacuum with the cosmological constant. There are two important remarks of this approach, firstly we did not impose any priori metric in our formulation of the gravitation but the metric emerged as a dynamical property of the theory. Secondly, similar to dynamically emerged metric, cosmological constant emerges naturally as an integration constant from the Eq.(2.17).

One of the biggest struggles of purely affine formulation is the incorporating matter field into theory. Absence of a priori defined metric in the formulation greatly limits the construction of invariants in the action level, therefore, one has to find other ways to incorporate matter into purely affine theories. One way to include matter into affine theory is, adding the appropriate kinetic terms of matter fields into determinant in (2.15) [15]. Recent study shows that, adding the kinetic structure of scalar field into invariant volume element can even induce inflation, which is the theorized exponential expansion of the very early universe to solve cosmological problems [16, 17]. Another way of including matter in purely affine theory is, extending the invariant volume element in (2.15) with Riemann curvature tensor and identifying an "effective" energy-momentum tensor from the dynamical equations [18].

Although there are some limitations in purely affine theory, it looks fundamental and it can be widened by relaxing some assumptions such as permitting $\Gamma^{\lambda}_{\mu\nu}$ to have an anti-symmetrical part which is called "Torsion". There is increasing literature of affine studies involving the effects of torsion and some other important quantities (such as non-metricity) which is not present in the metric formulation [19–21].

2.3. Metric-Affine Theory

Another approach to formulating gravity is known as the metric-affine gravity, where both the metric $g_{\mu\nu}$ and the affine connection $\Gamma^{\lambda}{}_{\mu\nu}$ is treated as independent variables of the theory [12, 13]. Einstein-Hilbert action (2.1) with the addition of matter action, can be written in terms of independent variables as

$$S[g,\Gamma] = \frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}\,\mathbb{R}(\Gamma) + S_M[g,\psi], \tag{2.18}$$

where now the curvature scalar $\mathbb{R}(\Gamma)$ given as $\mathbb{R}(\Gamma) \equiv g^{\mu\nu}\mathbb{R}_{\mu\nu}(\Gamma)$ and the Ricci tensor $\mathbb{R}_{\mu\nu}(\Gamma)$ is entirely constructed from affine connection $\Gamma^{\lambda}{}_{\mu\nu}$. Following the standart procedure to obtain equations of motion, we variate the action with respect to both variables. Knowing that $\mathbb{R}_{\mu\nu}(\Gamma)$ is independent of metric, variating with respect to $g^{\mu\nu}$ and equating to zero leads to

$$\mathbb{R}_{\mu\nu}(\Gamma) - \frac{1}{2}\mathbb{R}(\Gamma)g_{\mu\nu} = \frac{1}{M_{pl}^2}T_{\mu\nu}. \tag{2.19}$$

Taking the variation of (2.19) with respect to affine connection $\Gamma^{\lambda}_{\mu\nu}$ and using the identity (2.9) we get

$$\begin{aligned}\delta S &= \frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}\;g^{\mu\nu}\delta\mathbb{R}_{\mu\nu} \\ &= \frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}\;g^{\mu\nu}\Big[\nabla_\lambda(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu})\Big],\end{aligned} \tag{2.20}$$

Using integration by parts for the covariant derivatives we find

$$\begin{aligned}\delta S = \frac{M_{pl}^2}{2}\int d^4x\Big[&-\nabla_\lambda(\sqrt{-g}\;g^{\mu\nu})\delta\Gamma^{\lambda}_{\nu\mu} + \nabla_\lambda(\sqrt{-g}\;g^{\mu\nu}\delta\Gamma^{\lambda}_{\nu\mu}) \\ &+ \nabla_\nu(\sqrt{-g}\;g^{\mu\nu})\delta\Gamma^{\lambda}_{\lambda\mu} - \nabla_\nu(\sqrt{-g}\;g^{\mu\nu}\delta\Gamma^{\lambda}_{\lambda\mu})\Big].\end{aligned} \tag{2.21}$$

Again we see that 2nd and 4th terms are covariant divergence of a vector and they can be written as a surface term at infinity, but this time since connection is independent variable, it is fixed at the boundary and variation of the connection $\delta\Gamma^{\lambda}_{\mu\nu}$ vanishes. Neglecting the surface terms and relabeling some dummy indices we can write it as

$$\delta S = \frac{M_{pl}^2}{2} \int \mathrm{d}^4 x \Big[-\nabla_\lambda(\sqrt{-\mathrm{g}}\, \mathrm{g}^{\mu\nu}) + \nabla_\alpha(\sqrt{-\mathrm{g}}\, \mathrm{g}^{\mu\alpha})\delta^\nu_\lambda \Big] \delta\Gamma^{\lambda}_{\nu\mu}. \tag{2.22}$$

Since connection is arbitrary the expression in the brackets must be equal to zero. One final step to simplify above expression is to contract the indices ν and λ then substitute back to expression, finally we have

$$\nabla_\lambda(\sqrt{-\mathrm{g}}\, \mathrm{g}^{\mu\nu}) = 0. \tag{2.23}$$

Above expression is equal to (2.4) which is the relation for metric compatibility. With metric compatibility equation at hand, we can solve for the affine connection in terms of metric and see that it dynamically equals to Levi-Civita connection.

Now we see the remarks of the metric-affine formulation. First, the surface terms vanishing on the boundary means that we don't have to modify our initial action to get right field equations, whereas in the metric formulation we need to add an additional term to our action to cancel unwanted surface terms. Second, we didn't even need to impose metric compatibility, it comes out naturally by the dynamics of the theory.

En passant, it proves useful to mention here also the Palatini formalism [22]. It arises when the matter action is independent of the affine connection. In reality, however, Palatini formalism is hard to realize. The reason is that the fermion kinetic term does always involve affine connection through the spin connection. We will, therefore, always focus on metric-affine theory as the self-consistent formalism based on independent metric and connection variables.

CHAPTER 3

DISFORMAL METRIC-AFFINE GRAVITY

In general, action for the $F(\mathbb{R}_{\mu\nu}, \mathrm{g}^{\mu\nu})$ theory can be written as

$$S = S_{\mathrm{G}} + S_{\mathrm{M}}, \tag{3.1}$$

where S_{G} and S_{M} are the gravitational and matter parts of the full action, respectively. In the present study we examine the case where matter action is independent of the connection such as $S_{\mathrm{M}} = S_{\mathrm{M}}(\mathrm{g}_{\mu\nu})$. Then variation of the matter action will be

$$\delta S_{\mathrm{M}} = \int \mathrm{d}^4x \, \frac{\delta S_{\mathrm{M}}}{\delta \mathrm{g}^{\mu\nu}} \, \delta \mathrm{g}^{\mu\nu}. \tag{3.2}$$

Corresponding energy-momentum tensor can be defined as

$$T_{\mu\nu} = \frac{-2}{\sqrt{-\mathrm{g}}} \frac{\delta S_{\mathrm{M}}}{\delta \mathrm{g}^{\mu\nu}}. \tag{3.3}$$

Gravitational part of the general $F(R_{\mu\nu}, \mathrm{g}^{\mu\nu})$ theory can be written as

$$S_{\mathrm{G}} = a_1 \int \mathrm{d}^4x \sqrt{-\mathrm{g}} \, F(\mathbb{R}_{\mu\nu}, \mathrm{g}^{\mu\nu}). \tag{3.4}$$

where a_1 is a constant of appropriate dimension. Variation of the gravitational part of the action is given by

$$\begin{aligned} \delta S_{\mathrm{G}} &= a_1 \int \mathrm{d}^4x \, \delta\left(\sqrt{-\mathrm{g}} \, F(\mathbb{R}_{\mu\nu}, \mathrm{g}^{\mu\nu})\right) \\ &= a_1 \int \mathrm{d}^4x \, \sqrt{-\mathrm{g}} \, \left[\left(\frac{\partial F}{\partial \mathrm{g}^{\mu\nu}} - \frac{1}{2} F \mathrm{g}_{\mu\nu}\right) \delta \mathrm{g}^{\mu\nu} + \frac{\partial F}{\partial \mathbb{R}_{\mu\nu}} \, \delta \mathbb{R}_{\mu\nu}\right]. \end{aligned} \tag{3.5}$$

Variation of the Ricci tensor with respect to connection given as

$$\delta \mathbb{R}_{\mu\nu} = \nabla_\alpha \delta \Gamma^\alpha_{\mu\nu} - \nabla_\nu \delta \Gamma^\alpha_{\mu\alpha}. \tag{3.6}$$

Substituting (3.6) in the second line of (3.5) we get

$$\delta S_{\mathrm{G}} = \int \mathrm{d}^4x\,\sqrt{-\mathrm{g}}\,\left(\frac{\partial F}{\partial \mathrm{g}^{\mu\nu}} - \frac{1}{2}F\mathrm{g}_{\mu\nu}\right)\delta\mathrm{g}^{\mu\nu} + \int \mathrm{d}^4x\,\sqrt{-\mathrm{g}}\,\frac{\partial F}{\partial \mathbb{R}_{\mu\nu}}\left(\nabla_\alpha \delta\Gamma^{\alpha}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\alpha}_{\mu\alpha}\right), \tag{3.7}$$

where covariant derivatives are with respect to connection $\Gamma^{\lambda}_{\mu\nu}$. Using integration by parts in the second line we have

$$\delta S_{\mathrm{G}} = \int \mathrm{d}^4x\,\sqrt{-\mathrm{g}}\,\left(\frac{\partial F}{\partial \mathrm{g}^{\mu\nu}} - \frac{1}{2}F\mathrm{g}_{\mu\nu}\right)\delta\mathrm{g}^{\mu\nu} + \int \mathrm{d}^4x\,\left[-\nabla_\lambda\left(\sqrt{-\mathrm{g}}\,\frac{\partial F}{\partial \mathbb{R}_{\mu\nu}}\right) + \nabla_\sigma\left(\sqrt{-\mathrm{g}}\,\frac{\partial F}{\partial \mathbb{R}_{\mu\sigma}}\right)\delta^{\nu}_{\lambda}\right]\delta\Gamma^{\lambda}_{\mu\nu}, \tag{3.8}$$

where we suppressed the surface terms since they are linear in $\delta\Gamma^{\lambda}_{\mu\nu}$ and will vanish at the boundary. We are ready to write variation of the full action and apply least action principle. Corresponding field equations are

$$\frac{\partial F}{\partial \mathrm{g}^{\mu\nu}} - \frac{1}{2}F\mathrm{g}_{\mu\nu} = \frac{1}{2a_1}T_{\mu\nu}, \tag{3.9}$$

and

$$-\nabla_\lambda\left(\sqrt{-\mathrm{g}}\,\frac{\partial F}{\partial \mathbb{R}_{\mu\nu}}\right) + \nabla_\sigma\left(\sqrt{-\mathrm{g}}\,\frac{\partial F}{\partial \mathbb{R}_{\mu\sigma}}\right)\delta^{\nu}_{\lambda} = 0. \tag{3.10}$$

We can go further and take trace of the (3.10) and substitute it back to obtain

$$\nabla_\lambda\left(\sqrt{-\mathrm{g}}\,\frac{\partial F}{\partial \mathbb{R}_{\mu\nu}}\right) = 0. \tag{3.11}$$

From this equation, we can define a dynamical metric $h_{\mu\nu}$

$$\frac{\partial F}{\partial \mathbb{R}^{\mu\nu}} = h_{\mu\nu}, \tag{3.12}$$

where its covariant derivative with respect to general connection will vanish. Since $\mathbb{R}_{\mu\nu}$ is symmetric, $h_{\mu\nu}$ is also symmetric therefore the general connection $\Gamma^{\lambda}_{\mu\nu}$ will be symmetric in the indices μ and ν. Covariant conservation of $h_{\mu\nu}$ allows us to write $\Gamma^{\lambda}_{\mu\nu}$ in terms of the Levi-Civita connection ${}^{\mathrm{g}}\Gamma^{\lambda}_{\mu\nu}$ of the metric $\mathrm{g}_{\mu\nu}$.

$$\Gamma^{\lambda}_{\mu\nu} = {}^{\mathrm{g}}\Gamma^{\lambda}_{\mu\nu} + \tfrac{1}{2}(h^{-1})^{\lambda\rho}\left({}^{\mathrm{g}}\nabla_\mu h_{\nu\rho} + {}^{\mathrm{g}}\nabla_\nu h_{\rho\mu} - {}^{\mathrm{g}}\nabla_\rho h_{\mu\nu}\right), \tag{3.13}$$

where covariant derivatives ${}^{\rm g}\nabla_\mu$ corresponds to Levi-Civita connection ${}^{\rm g}\Gamma^\lambda_{\mu\nu}$. The Ricci tensor of the general connection $\mathbb{R}_{\mu\nu}(\Gamma)$ can also be written in terms of ${}^{\rm g}{\rm R}_{\mu\nu}({\rm g})$ as

$$
\begin{aligned}
\mathbb{R}_{\mu\nu} = {}^{\rm g}{\rm R}_{\mu\nu} + \frac{1}{4}(h^{-1})^{\alpha\beta}\Big[& -2\Big({}^{\rm g}\nabla_\beta\, {}^{\rm g}\nabla_\alpha h_{\mu\nu} - {}^{\rm g}\nabla_\beta\, {}^{\rm g}\nabla_\mu h_{\nu\alpha} \\
& - {}^{\rm g}\nabla_\beta\, {}^{\rm g}\nabla_\nu h_{\mu\alpha} + {}^{\rm g}\nabla_\mu\, {}^{\rm g}\nabla_\nu h_{\alpha\beta}\Big) + (h^{-1})^{\kappa\lambda}\Big(-2\, {}^{\rm g}\nabla_\beta h_{\nu\lambda}\, {}^{\rm g}\nabla_\kappa h_{\mu\alpha} \\
& - {}^{\rm g}\nabla_\alpha h_{\mu\nu}({}^{\rm g}\nabla_\beta h_{\kappa\lambda} - 2\, {}^{\rm g}\nabla_\lambda h_{\beta\kappa}) + 2\, {}^{\rm g}\nabla_\kappa h_{\mu\alpha}\, {}^{\rm g}\nabla_\lambda h_{\nu\beta} \\
& + {}^{\rm g}\nabla_\beta h_{\kappa\lambda}\, {}^{\rm g}\nabla_\mu h_{\nu\alpha} - 2\, {}^{\rm g}\nabla_\lambda h_{\beta\kappa}\, {}^{\rm g}\nabla_\mu h_{\nu\alpha} + {}^{\rm g}\nabla_\mu h_{\alpha\kappa}\, {}^{\rm g}\nabla_\nu h_{\beta\lambda} \\
& + {}^{\rm g}\nabla_\beta h_{\kappa\lambda}\, {}^{\rm g}\nabla_\nu h_{\mu\alpha} - 2\, {}^{\rm g}\nabla_\lambda h_{\beta\kappa}\, {}^{\rm g}\nabla_\nu h_{\mu\alpha}\Big)\Big].
\end{aligned}
\tag{3.14}
$$

From here, one can go further and contract this equation with ${\rm g}^{\mu\nu}$ and find a relation between $\mathbb{R}(\Gamma)$ and ${}^{\rm g}{\rm R}({\rm g})$.

Up until now, we have dealt with a general function $F(\mathbb{R}_{\mu\nu}, {\rm g}^{\mu\nu})$. It is, however, useful to focus on a realistic functional structure to reveal salient features of such extended gravity theories. To this end, we consider the case

$$F(\mathbb{R}_{\mu\nu}, {\rm g}^{\mu\nu}) = \mathbb{R}_{\mu\nu}{\rm g}^{\mu\nu} + \epsilon \mathbb{R}_{\mu\nu} V^\mu V^\nu, \tag{3.15}$$

where ϵ is a dimensionless parameter, it represents the coupling strength of V^μ to curvature $\mathbb{R}_{\mu\nu}$. Here, V^μ can be taken as either as an Abelian vector field or a generic cosmological 4-velocity field ("background velocity field"). Each option has its own physics implications. Cosmological implications of the special case where $\epsilon = 0$ is studied in [36] since theory can be written in terms of $F(\mathbb{R})$ in that case.

Before going into cosmological implications, we note that, with the specific form in (3.15), the dynamical metric $h_{\mu\nu}$ in (3.12) takes the form

$$h_{\mu\nu} = {\rm g}_{\mu\nu} + \epsilon\, V_\mu V_\nu, \tag{3.16}$$

and the corresponding inverse metric $(h^{-1})^{\mu\nu}$ becomes

$$(h^{-1})^{\mu\nu} = {\rm g}^{\mu\nu} - \frac{\epsilon}{1-\epsilon}\, V^\mu V^\nu. \tag{3.17}$$

Existence of the second term in (3.16) breaks the conformal relation between dynamical and gravitational metric, hence this kind of relation between $h_{\mu\nu}$ and ${\rm g}_{\mu\nu}$ is called the disformal transformation [23]. Disformal transformations alter the angles between geodesics and may change the causal structure of the given manifold. In order to preserve causal behavior of the particles exists in the manifold, square of the infinitesimal line element on the trajectory of a physical particles must have a time-like separation such as

$$\begin{aligned} ds^2 &= h_{\mu\nu}\, dx^\mu dx^\nu < 0 \\ &= (g_{\mu\nu} dx^\mu dx^\nu + \epsilon\, V_\mu V_\nu dx^\mu dx^\nu) < 0. \end{aligned} \tag{3.18}$$

Namely, ϵ must be negative [23, 24]. Another important property of the metric is the signature of $g_{\mu\nu}$. Since we can always define a local Lorentz coordinates with a lorentzian signature, disformally transformed metric $h_{\mu\nu}$ must have the same metric signature as the minkowski metric $\eta_{\mu\nu}$.

Having revealed the disformal structure, we now turn to the same dynamical equations in a cosmological setting. With the given functional form of $F\,(\mathbb{R}_{\mu\nu}, g^{\mu\nu})$, field equations Eq.(3.9) becomes

$$\mathbb{R}_{\mu\nu} = \frac{1}{2a_1} T_{\mu\nu} + \frac{1}{2}\mathbb{R}\; g_{\mu\nu} + \frac{1}{2}\epsilon \langle \mathbb{R} \rangle g_{\mu\nu}, \tag{3.19}$$

where $\langle \mathbb{R} \rangle$ is the shorthand notation defined as $\langle \mathbb{R} \rangle \equiv \mathbb{R}_{\mu\nu} v^\mu v^\nu$. We can go further and eliminate curvature terms in the right-hand side of the (3.19) and express $\mathbb{R}_{\mu\nu}(\Gamma)$ solely in terms of energy-momentum tensor $T_{\mu\nu}$. multiplying the above equation with $V^\mu V^\nu$ we have

$$\langle \mathbb{R} \rangle = \frac{1}{2+\epsilon} \Big(\frac{1}{a_1} \langle T \rangle - \mathbb{R} \Big), \tag{3.20}$$

where we have used the rest-frame normalization condition $g_{\mu\nu} V^\mu V^\nu = -1$ for the 4-velocity field and $\langle T \rangle$ is again the shorthand notation defined as $\langle T \rangle \equiv T_{\mu\nu} V^\mu V^\nu$ to simplfy equations. Taking the trace of (3.19) we find

$$\mathbb{R} = -\frac{1}{2a_1} T - 2\epsilon \langle \mathbb{R} \rangle. \tag{3.21}$$

Substituting both (3.20) and (3.21) to (3.19) leads to

$$\mathbb{R}_{\mu\nu} = \frac{1}{2a_1} \Big[T_{\mu\nu} - \frac{1}{2-\epsilon} \Big(T - \epsilon \langle T \rangle \Big) g_{\mu\nu} \Big]. \tag{3.22}$$

Eq.(3.22) expresses Ricci tensor $\mathbb{R}_{\mu\nu}(\Gamma)$ in terms of energy-momentum tensor $T_{\mu\nu}$ and 4-velocity field V_μ. Turning back to (3.14), we can use the explicit form of our dynamical metric (3.16) to find relation between $\mathbb{R}_{\mu\nu}$ and ${}^{g}R_{\mu\nu}$. Keeping in mind that all the covariant derivatives in (3.14) are with respect to Levi-Civita connection, we get

$$\begin{aligned}\mathbb{R}_{\mu\nu} = {}^{g}\mathrm{R}_{\mu\nu} + \frac{\epsilon}{2}\Bigg[&- V_\nu\, {}^{g}\nabla_\alpha {}^{g}\nabla^\alpha V_\mu - V_\mu\, {}^{g}\nabla_\alpha {}^{g}\nabla^\alpha V_\nu + V_\nu\, {}^{g}\nabla_\alpha {}^{g}\nabla_\mu V^\alpha \\ &+ V_\mu\, {}^{g}\nabla_\alpha {}^{g}\nabla_\nu V^\alpha + {}^{g}\nabla^\alpha V_\nu\, {}^{g}\nabla_\mu V_\alpha + {}^{g}\nabla^\alpha V_\mu\, {}^{g}\nabla_\nu V_\alpha + \epsilon\Big(V_\mu V_\nu\, {}^{g}\nabla_\beta V_\alpha\, {}^{g}\nabla^\beta V^\alpha \\ &- V_\mu V_\nu\, {}^{g}\nabla_\beta V_\alpha\, {}^{g}\nabla^\alpha V^\beta \Big) + \frac{1}{1-\epsilon}\Big(V^{\alpha g}\nabla_\alpha\, {}^{g}\nabla_\mu V_\nu + V^{\alpha g}\nabla_\alpha\, {}^{g}\nabla_\nu V_\mu + {}^{g}\nabla_\alpha V^\alpha\, {}^{g}\nabla_\mu V_\nu \\ &+ {}^{g}\nabla_\alpha V^\alpha\, {}^{g}\nabla_\nu V_\mu + \epsilon\Big(V^\alpha V_\nu\, {}^{g}\nabla_\alpha V_\mu\, \mathrm{g}\nabla_\beta V^\beta + V^\alpha V_\mu\, {}^{g}\nabla_\alpha V_\nu\, \mathrm{g}\nabla_\beta V^\beta \\ &+ V^\alpha V^\beta\, {}^{g}\nabla_\alpha V_\mu\, {}^{g}\nabla_\beta V_\nu + V^\alpha V^\beta V_\nu\, {}^{g}\nabla_\beta\, {}^{g}\nabla_\alpha V_\mu + V^\alpha V^\beta V_\mu\, {}^{g}\nabla_\beta\, {}^{g}\nabla_\alpha V_\nu \\ &+ {}^{g}\nabla_\mu V^\alpha\, {}^{g}\nabla_\beta V_\alpha \Big) + (\epsilon - 2)\, {}^{g}\nabla_\alpha V_\nu\, {}^{g}\nabla^\alpha V_\mu - \epsilon^2 V^\alpha V^\beta V_\mu V_\nu\, {}^{g}\nabla_\alpha V^\sigma\, {}^{g}\nabla_\beta V_\sigma \Big) \Bigg]. \end{aligned} \tag{3.23}$$

Eq.(3.23) gives us the "geometrical" relation between the affine Ricci tensor $\mathbb{R}_{\mu\nu}(\Gamma)$ and the "metrical" Ricci tensor ${}^{g}\mathrm{R}_{\mu\nu}$ and it is sourced from the connection field equations whereas Eq.(3.22) comes from the metrical field equations and as we said before it relates affine geometry to energy-momentum tensor of matter fields. Combining both equations and solving for ${}^{g}\mathrm{R}_{\mu\nu}$ will give us a chance to represent theory in a metrical fashion where we can make a clear comparison to GR and ultimately pinpoint the effects of the disformal coupling. Substituting (3.23) to (3.22) and solving for ${}^{g}\mathrm{R}_{\mu\nu}$ leads to

$$\begin{aligned}{}^{g}\mathrm{R}_{\mu\nu} = \frac{1}{2a_1}\Big[T_{\mu\nu} - \frac{1}{2-\epsilon}\Big(T - \epsilon\langle T\rangle\Big)\mathrm{g}_{\mu\nu}\Big] - \frac{\epsilon}{2}\Bigg[&- V_\nu\, {}^{g}\nabla_\alpha {}^{g}\nabla^\alpha V_\mu \\ - V_\mu\, {}^{g}\nabla_\alpha {}^{g}\nabla^\alpha V_\nu + V_\nu\, {}^{g}\nabla_\alpha {}^{g}\nabla_\mu V^\alpha + V_\mu\, {}^{g}\nabla_\alpha {}^{g}\nabla_\nu V^\alpha &+ {}^{g}\nabla^\alpha V_\nu\, {}^{g}\nabla_\mu V_\alpha \\ + {}^{g}\nabla^\alpha V_\mu\, {}^{g}\nabla_\nu V_\alpha + \epsilon\Big(V_\mu V_\nu\, {}^{g}\nabla_\beta V_\alpha\, {}^{g}\nabla^\beta V^\alpha &- V_\mu V_\nu\, {}^{g}\nabla_\beta V_\alpha\, {}^{g}\nabla^\alpha V^\beta \Big) \\ + \frac{1}{1-\epsilon}\Big(V^\alpha\, {}^{g}\nabla_\alpha\, {}^{g}\nabla_\mu V_\nu + V^\alpha\, {}^{g}\nabla_\alpha\, {}^{g}\nabla_\nu V_\mu &+ {}^{g}\nabla_\alpha V^\alpha\, {}^{g}\nabla_\mu V_\nu \\ + {}^{g}\nabla_\alpha V^\alpha\, {}^{g}\nabla_\nu V_\mu + \epsilon\Big(V^\alpha V_\nu\, {}^{g}\nabla_\alpha V_\mu\, \mathrm{g}\nabla_\beta V^\beta &+ V^\alpha V_\mu\, {}^{g}\nabla_\alpha V_\nu\, \mathrm{g}\nabla_\beta V^\beta \\ + V^\alpha V^\beta\, {}^{g}\nabla_\alpha V_\mu\, {}^{g}\nabla_\beta V_\nu + V^\alpha V^\beta V_\nu\, {}^{g}\nabla_\beta\, {}^{g}\nabla_\alpha V_\mu &+ V^\alpha V^\beta V_\mu\, {}^{g}\nabla_\beta\, {}^{g}\nabla_\alpha V_\nu \\ + {}^{g}\nabla_\mu V^\alpha\, {}^{g}\nabla_\beta V_\alpha \Big) + (\epsilon - 2)\, {}^{g}\nabla_\alpha V_\nu\, {}^{g}\nabla^\alpha V_\mu &- \epsilon^2 V^\alpha V^\beta V_\mu V_\nu\, {}^{g}\nabla_\alpha V^\sigma\, {}^{g}\nabla_\beta V_\sigma \Big) \Bigg], \end{aligned} \tag{3.24}$$

and the corresponding curvature scalar ${}^{\mathrm{g}}\mathrm{R}$ can be easily found by taking the trace of Eq.(3.24), giving

$$
\begin{aligned}
{}^{\mathrm{g}}\mathrm{R} = & \frac{1}{2a_1}\Big[T - \frac{4}{2-\epsilon}\Big(T - \epsilon\langle T\rangle\Big)\Big] \\
& - \epsilon\Big[V^\alpha\, {}^{\mathrm{g}}\nabla_\beta\, {}^{\mathrm{g}}\nabla_\alpha V^\beta + \frac{1}{2}(2+\epsilon)\, {}^{\mathrm{g}}\nabla_\alpha V_\beta\, {}^{\mathrm{g}}\nabla^\beta V^\alpha \\
& - \frac{\epsilon}{2}\Big({}^{\mathrm{g}}\nabla_\beta V_\alpha\, {}^{\mathrm{g}}\nabla^\beta V^\alpha + V^\alpha V^\beta\, {}^{\mathrm{g}}\nabla_\alpha V^\sigma\, {}^{\mathrm{g}}\nabla_\beta V_\sigma\Big) \\
& + \frac{1}{1-\epsilon}\Big(V^\alpha\, {}^{\mathrm{g}}\nabla_\alpha {}^{\mathrm{g}}\nabla_\beta V^\beta + {}^{\mathrm{g}}\nabla_\alpha V^\alpha\, {}^{\mathrm{g}}\nabla_\beta V^\beta\Big)\Big].
\end{aligned}
\tag{3.25}
$$

Last step is the construct left-hand side of (2.5) which is commonly known as Einstein tensor $G_{\mu\nu}$ defined as

$$
G_{\mu\nu} = {}^{\mathrm{g}}\mathrm{R}_{\mu\nu} - \frac{1}{2}\, {}^{\mathrm{g}}\mathrm{R}\; \mathrm{g}_{\mu\nu}. \tag{3.26}
$$

Putting (3.24) and (3.25) into (3.26) we find the left-hand side of Einstein field equations as

$$
\begin{aligned}
G_{\mu\nu} = & \frac{1}{2a_1}\Bigg\{T_{\mu\nu} + \frac{\epsilon}{2-\epsilon}\Big(\frac{T}{2} + \langle T\rangle\Big)\mathrm{g}_{\mu\nu} + a_1\epsilon\Bigg[\Big(\frac{V^\alpha}{1-\epsilon}\, {}^{\mathrm{g}}\nabla_\alpha {}^{\mathrm{g}}\nabla_\beta V^\beta \\
& + \frac{1}{1-\epsilon}\, {}^{\mathrm{g}}\nabla_\alpha V^\alpha\, {}^{\mathrm{g}}\nabla_\beta V^\beta - \frac{\epsilon}{2} V^\alpha V^\beta\, {}^{\mathrm{g}}\nabla_\alpha V^\sigma\, {}^{\mathrm{g}}\nabla_\beta V_\sigma + V^\alpha\, {}^{\mathrm{g}}\nabla_\beta {}^{\mathrm{g}}\nabla_\alpha V^\beta \\
& - \frac{2+\epsilon}{2}\, {}^{\mathrm{g}}\nabla_\alpha V_\beta\, {}^{\mathrm{g}}\nabla^\beta V^\alpha - \frac{\epsilon}{2}\, {}^{\mathrm{g}}\nabla_\beta V_\alpha\, {}^{\mathrm{g}}\nabla^\beta V^\alpha\Big)\mathrm{g}_{\mu\nu} - \frac{1}{1-\epsilon}\Big(V^\alpha\, {}^{\mathrm{g}}\nabla_\alpha {}^{\mathrm{g}}\nabla_\mu V_\nu \\
& + V^\alpha\, {}^{\mathrm{g}}\nabla_\alpha {}^{\mathrm{g}}\nabla_\nu V_\mu - (2-\epsilon){}^{\mathrm{g}}\nabla_\alpha V_\nu\, {}^{\mathrm{g}}\nabla^\alpha V_\mu + \epsilon V^\alpha V^\beta\, {}^{\mathrm{g}}\nabla_\alpha V_\mu\, {}^{\mathrm{g}}\nabla_\beta V_\nu\Big) \\
& - V_\mu\Big(- {}^{\mathrm{g}}\nabla_\alpha {}^{\mathrm{g}}\nabla^\alpha V_\nu + {}^{\mathrm{g}}\nabla_\alpha {}^{\mathrm{g}}\nabla_\nu V^\alpha + \frac{\epsilon}{1-\epsilon} V^\alpha\, {}^{\mathrm{g}}\nabla_\alpha V_\nu\, {}^{\mathrm{g}}\nabla_\beta V^\beta \\
& + \frac{\epsilon}{1-\epsilon} V^\alpha V^\beta\, {}^{\mathrm{g}}\nabla_\beta {}^{\mathrm{g}}\nabla_\alpha V_\nu\Big) - V_\nu\Big(- {}^{\mathrm{g}}\nabla_\alpha {}^{\mathrm{g}}\nabla^\alpha V_\mu + {}^{\mathrm{g}}\nabla_\alpha {}^{\mathrm{g}}\nabla_\mu V^\alpha \\
& + \frac{\epsilon}{1-\epsilon} V^\alpha\, {}^{\mathrm{g}}\nabla_\alpha V_\mu\, {}^{\mathrm{g}}\nabla_\beta V^\beta + \frac{\epsilon}{1-\epsilon} V^\alpha V^\beta\, {}^{\mathrm{g}}\nabla_\beta {}^{\mathrm{g}}\nabla_\alpha V_\mu\Big) \\
& + \frac{\epsilon^2}{1-\epsilon} V^\alpha V^\beta V_\nu V_\mu\, {}^{\mathrm{g}}\nabla_\alpha V^\sigma\, {}^{\mathrm{g}}\nabla_\beta V_\sigma + \epsilon V_\nu V_\mu\, {}^{\mathrm{g}}\nabla^\beta V^\alpha\Big({}^{\mathrm{g}}\nabla_\alpha V_\beta - {}^{\mathrm{g}}\nabla_\beta V_\alpha\Big) \\
& - {}^{\mathrm{g}}\nabla^\alpha V_\nu\, {}^{\mathrm{g}}\nabla_\mu V_\alpha - {}^{\mathrm{g}}\nabla^\alpha V_\mu\, {}^{\mathrm{g}}\nabla_\nu V_\alpha - \frac{1}{1-\epsilon}\, {}^{\mathrm{g}}\nabla_\alpha V^\alpha\Big({}^{\mathrm{g}}\nabla_\mu V_\nu + {}^{\mathrm{g}}\nabla_\nu V_\mu\Big) \\
& - \frac{\epsilon}{1-\epsilon}\, {}^{\mathrm{g}}\nabla_\mu V^\alpha\, {}^{\mathrm{g}}\nabla_\nu V_\alpha\Bigg]\Bigg\}
\end{aligned}
\tag{3.27}
$$

Now we are ready to compare Eq.(3.27) to Eq.(2.5), remembering that a_1 is the constant with the appropriate dimensions to construct action and corresponds to $M_{pl}^2/2$ in (2.5), we see that the expression in the curly brackets acts as a "effective" energy-momentum tensor $T_{\mu\nu}^{eff}$ identified as

$$\begin{aligned} T_{\mu\nu}^{eff} = \Bigg\{ & T_{\mu\nu} + \frac{\epsilon}{2-\epsilon}\Big(\frac{T}{2} + \langle T\rangle\Big)\mathrm{g}_{\mu\nu} + a_1\epsilon\Bigg[\Big(\frac{1}{1-\epsilon}\,(\dot{\theta}+\theta^2) \\ & - \frac{\epsilon}{2}\,a^\alpha a_\alpha + \nabla_\alpha a^\alpha - 2\,\nabla_\alpha V_\beta\,\nabla^\beta V^\alpha - \epsilon\,\nabla^\beta V^\alpha\,\nabla_{(\beta}V_{\alpha)}\Big)\mathrm{g}_{\mu\nu} \\ & - \frac{1}{1-\epsilon}\Big(2\,\theta\,\nabla_{(\mu}V_{\nu)} + 2(\nabla_{(\mu}\dot{V}_{\nu)}) - (2-\epsilon)\nabla_\alpha V_\nu\,\nabla^\alpha V_\mu + \epsilon\,a_\mu a_\nu\Big) \\ & - V_\mu\Big(-\Box V_\nu + \nabla_\alpha\nabla_\nu V^\alpha + \frac{\epsilon}{1-\epsilon}a_\nu\,\theta + \frac{\epsilon}{1-\epsilon}V^\alpha\,(\nabla_\alpha \dot{V}_\nu)\Big) \\ & - V_\nu\Big(-\Box V_\mu + \nabla_\alpha\nabla_\mu V^\alpha + \frac{\epsilon}{1-\epsilon}a_\mu\,\theta + \frac{\epsilon}{1-\epsilon}V^\alpha\,(\nabla_\alpha \dot{V}_\mu)\Big) \\ & + \frac{\epsilon^2}{1-\epsilon}V_\nu V_\mu\,a^\alpha a_\alpha + \epsilon\,V_\nu V_\mu\,\nabla^\beta V^\alpha\Big(\nabla_\alpha V_\beta - \nabla_\beta V_\alpha\Big) \\ & - 2\,\nabla^\alpha V_{(\nu}\,\nabla_{\mu)}V_\alpha - \frac{2}{1-\epsilon}\,\theta\,\nabla_{(\mu}V_{\nu)} - \frac{\epsilon}{1-\epsilon}\,{}^{\mathrm{g}}\nabla_\mu V^\alpha\,{}^{\mathrm{g}}\nabla_\nu V_\alpha\Bigg]\Bigg\}, \end{aligned} \tag{3.28}$$

where we have dropped the superscript g for covariant derivatives since all the derivatives are with respect to Levi-Civita connection and defined further simplifications commonly used in the literature. Box operator is the d'Alembertian for 4-dimensional space-time defined as $\Box \equiv \nabla_\alpha\nabla^\alpha$. Dot represents the derivative with respect to proper time and given as $(\dot{..}) \equiv V^\alpha\nabla_\alpha(..) = \frac{\mathrm{d}(..)}{\mathrm{d}\tau}$. θ is known as the expansion of the congruence of geodesics, it represents the change in volume of the small sphere of test particles with respect to their central geodesic and it is one of the key parameters of Raychaudhuri's equation [25–27]. 4-acceleration vector a_μ is just derivative of 4-velocity V_μ with respect to proper time τ defined as $a_\mu \equiv V^\alpha\nabla_\alpha V_\mu = \frac{\mathrm{d}V_\mu}{\mathrm{d}\tau}$.

Looking at the Eq.(3.28) we see that standard energy-momentum tensor $T_{\mu\nu}$ gets multiple contributions from disformal coupling term, contributions comes in various orders of ϵ up to ϵ^3. Again we see that canceling the disformal term ($\epsilon = 0$) gives back the standard energy-momentum tensor as expected. Assuming both $T_{\mu\nu}^{eff}$ and $T_{\mu\nu}$ are in the form of perfect fluid such as

$$T_{\mu\nu} = (\rho + p)V_\mu V_\nu + p\mathrm{g}_{\mu\nu}, \tag{3.29}$$

where ρ, p and V_μ are the rest-frame energy density, pressure and 4-velocity of the fluid, we can further identify the effective energy density ρ_{eff} and effective pressure p_{eff} of $T^{eff}_{\mu\nu}$ in terms of ρ, p and contributions from disformal coupling. Remembering the normalization condition for V_μ, energy density ρ can be found by multiplying energy-momentum tensor $T_{\mu\nu}$ with $V^\mu V^\nu$

$$\rho = T_{\mu\nu} V^\mu V^\nu. \tag{3.30}$$

Similarly, pressure p can be found by multiplying $T_{\mu\nu}$ with the projection operator $H^{\mu\nu}$ defined as

$$H^{\mu\nu} = \mathrm{g}^{\mu\nu} + V^\mu V^\nu. \tag{3.31}$$

$H^{\mu\nu}$ projects the multiplied expression onto subspace that is orthogonal to V^μ and satisfies the relation $H^{\mu\nu} V_\mu = 0$ as it can be easily verified using the normalization condition. With the help of projection operator, pressure p can be calculated as

$$p = T_{\mu\nu} \frac{H^{\mu\nu}}{3}. \tag{3.32}$$

Using Eq.(3.30) and (3.32), we find the corresponding ρ_{eff} and p_{eff} as

$$\begin{aligned}\rho_{eff} = \rho - \frac{\epsilon}{2(2-\epsilon)}\Big(\rho + 3p\Big) + a_1\epsilon\Bigg[&\frac{-(\dot{\theta} + \theta^2)}{1-\epsilon} + 2a^\alpha a_\alpha \\ &- \nabla^\beta V^\alpha\Big(\nabla_\alpha V_\beta + 2\nabla_\beta V_\alpha\Big) + V^\alpha \nabla_\beta \nabla_\alpha V^\beta \\ &+ \frac{\epsilon}{2}\Big(- a^\alpha a_\alpha + \nabla^\beta V^\alpha(\nabla_\alpha V_\beta - \nabla_\beta V_\alpha)\Big)\Bigg],\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}p_{eff} = p + \frac{\epsilon}{2(2-\epsilon)}\Big(\rho + 3p\Big) + a_1\epsilon\Bigg[&\frac{\dot{\theta} + \theta^2}{3(1-\epsilon)} + \frac{2}{3}a^\alpha a_\alpha \\ &+ \frac{2}{3}\nabla_\beta V_\alpha \nabla^\beta V^\alpha + \frac{1}{3}\nabla_\alpha V_\beta \nabla^\beta V^\alpha + V^\alpha \nabla_\beta \nabla_\alpha V^\beta \\ &+ \frac{\epsilon}{2}\Big(- a^\alpha a_\alpha + \nabla^\beta V^\alpha(\nabla_\alpha V_\beta - \nabla_\beta V_\alpha)\Big)\Bigg].\end{aligned} \tag{3.34}$$

Before concluding this chapter, we give a brief study of the disformal coupling effects in the metric formulation and show the differences between both formulation even with same coupling term.

3.1. Disformal Metric Gravity

Similar to metric-affine action (3.4), gravitational part of the action can be written as

$$S_{\mathrm{G}} = a_1 \int \mathrm{d}^4x\sqrt{-\mathrm{g}}\, F\left(\mathrm{R}_{\mu\nu}, \mathrm{g}^{\mu\nu}\right), \tag{3.35}$$

where the functional $F(\mathrm{R}_{\mu\nu}, \mathrm{g}^{\mu\nu})$ is now constructed from the metrical Ricci tensor $\mathrm{R}_{\mu\nu}$. Open form of the functional $F(\mathrm{R}_{\mu\nu}, \mathrm{g}^{\mu\nu})$ is given as

$$F(\mathrm{R}_{\mu\nu}, \mathrm{g}^{\mu\nu}) = \mathrm{R}_{\mu\nu}\mathrm{g}^{\mu\nu} + \epsilon\, \mathrm{R}_{\mu\nu}V^\mu V^\nu. \tag{3.36}$$

Adding the matter action to (3.35) and finding the stationary points where the action is stationary, we find the following field equations as

$$\begin{aligned} G_{\mu\nu} = \frac{1}{2a_1}\Bigg\{ & T_{\mu\nu} + \frac{\epsilon}{2-\epsilon}\Big(\langle T\rangle + \frac{T}{2}\Big)\mathrm{g}_{\mu\nu} + a_1\,\epsilon\Big[2\nabla_\beta\nabla_\nu(V_\mu V^\beta) - \Box(V_\mu V_\nu) \\ & - \mathrm{g}_{\mu\nu}\nabla_\alpha\nabla_\beta(V^\alpha V^\beta) + \frac{\epsilon}{2-\epsilon}\mathrm{g}_{\mu\nu}\Big(2V^\sigma V^\alpha\nabla_\beta\nabla_\sigma(V_\alpha V^\beta) - V^\sigma V^\alpha\Box(V_\sigma V_\alpha)\Big)\Big]\Bigg\}, \end{aligned} \tag{3.37}$$

where the expression inside curly brakects is the $T^{eff}_{\mu\nu}$ for the metrical theory. Assuming $T^{eff}_{\mu\nu}$ and $T_{\mu\nu}$ is in the form of perfect fluid, we apply the same procedure as before and find the corresponding $(\rho_{eff})_{GR}$ and $(p_{eff})_{GR}$ as

$$\begin{aligned} (\rho_{eff})_{GR} = \rho - \frac{\epsilon}{2(2-\epsilon)}\Big(\rho + 3p\Big) + a_1\epsilon\Big[\theta^2 + \dot{\theta} + \nabla_\alpha a^\alpha \\ + \frac{4(1-\epsilon)}{2-\epsilon}\Big(- V^\alpha\, \nabla_\beta\nabla_\alpha V^\beta - a^\alpha a_\alpha - \nabla^\alpha V^\beta\, \nabla_\alpha V_\beta\Big)\Big], \end{aligned} \tag{3.38}$$

and

$$(p_{eff})_{GR} = p + \frac{\epsilon}{2(2-\epsilon)}\Big(\rho + 3p\Big) + a_1\epsilon\Bigg[\frac{5}{9}\Big(\theta^2 + \dot{\theta} + \nabla_\alpha a^\alpha\Big) + \frac{4(1-\epsilon)}{3(2-\epsilon)}\Big(- a^\alpha a_\alpha - V^\alpha\,\nabla_\beta\nabla_\alpha V^\beta - \nabla^\alpha V^\beta\,\nabla_\alpha V_\beta\Big)\Bigg]. \qquad (3.39)$$

In order to determine the differences between metric-affine formulation and metric formulation of disformal effects clearly, we express results as a difference between both formulations, such that

$$\begin{aligned}\Delta\rho_{eff} &= (\rho_{eff})_{MAG} - (\rho_{eff})_{GR}\\ &= a_1\epsilon\Bigg\{ - \Big(\frac{2-\epsilon}{1-\epsilon}\Big)(\dot{\theta} + \theta^2) + \Big(\frac{16 - 14\epsilon + \epsilon^2}{4 - 2\epsilon}\Big)a^\alpha a_\alpha + \Big(\frac{6-5\epsilon}{2-\epsilon}\Big)V^\alpha\,\nabla_\beta\nabla_\alpha V^\beta\\ &- \Big(\frac{2-\epsilon}{2}\Big)\nabla^\beta V^\alpha\,\nabla_\alpha V_\beta - \Big(\frac{\epsilon(6-\epsilon)}{2(2-\epsilon)}\Big)\nabla^\beta V^\alpha\,\nabla_\beta V_\alpha - \nabla_\alpha a^\alpha\Bigg\}, \qquad (3.40)\end{aligned}$$

and similarly for the pressure part we have

$$\begin{aligned}\Delta p_{eff} &= (p_{eff})_{MAG} - (p_{eff})_{GR}\\ &= a_1\epsilon\Bigg\{\Big(\frac{2-5\epsilon}{9(\epsilon-1)}\Big)(\dot{\theta} + \theta^2) + \Big(\frac{16 - 18\epsilon + 3\epsilon^2}{6(2-\epsilon)}\Big)\Big(a^\alpha a_\alpha + \nabla_\beta V_\alpha\,\nabla^\beta V^\alpha\Big)\\ &+ \Big(\frac{2+3\epsilon}{6}\Big)\nabla_\alpha V_\beta\,\nabla^\beta V^\alpha + \Big(\frac{10-7\epsilon}{3(2-\epsilon)}\Big)V^\alpha\,\nabla_\beta\nabla_\alpha V^\beta - \frac{5}{9}\nabla_\alpha a^\alpha\Bigg\}. \qquad (3.41)\end{aligned}$$

In this chapter, we have studied the effects of background 4-vectors on metric-affine and metric framework. Our findings show that, when viewed from the field equations perspective, disformal effects contributes to standard energy-momentum tensor $T_{\mu\nu}$ and can be seen as an effective energy-momentum tensor $T^{eff}_{\mu\nu}$. Some of the contributions involves the $T_{\mu\nu}$ and it's trace T whereas other contributions comes from the derivatives of the 4-velocity. We have seen that the disformal coupling effects differs between the metric and metric-affine formulation and presented the difference in terms of $\Delta\rho_{eff}$ and Δp_{eff}. These differences contains various order of magnitudes in ϵ.

CHAPTER 4

CROSS-CURVATURE METRIC-AFFINE GRAVITY

4.1. Metrical $F(\mathrm{R})$ Theory

Generalized action of the metrical $F(\mathrm{R})$ theory is described by the action [28–30]

$$S[\mathrm{g}] = \frac{M_{pl}^2}{2}\int \mathrm{d}^4x\sqrt{-\mathrm{g}}\; F(\mathrm{R}) + S_M[\mathrm{g},\psi], \tag{4.1}$$

where R is the curvature scalar of the Levi-Civita connection ${}^{\mathrm{g}}\Gamma^{\lambda}_{\mu\nu}$, g is the determinant of the metric $\mathrm{g}_{\mu\nu}$ and the M_{pl} is the reduced planck mass given as

$$M_{pl} \equiv \frac{1}{\sqrt{8\pi G_N}}. \tag{4.2}$$

Standard procedure to obtain field equations is the applying least action principle to action (4.1). Varying the gravitational part of the action with respect to $\mathrm{g}^{\mu\nu}$ gives

$$\begin{aligned}\delta S_G &= \frac{M_{pl}^2}{2}\int \mathrm{d}^4x\Big\{\delta\sqrt{-\mathrm{g}}\;F + \sqrt{-\mathrm{g}}\;\frac{\partial F}{\partial \mathrm{R}}\;\delta\mathrm{R}\Big\}\\ &= \frac{M_{pl}^2}{2}\int \mathrm{d}^4x\Big\{-\frac{\sqrt{-\mathrm{g}}}{2}\;F\mathrm{g}_{\mu\nu}\delta\mathrm{g}^{\mu\nu} + \sqrt{-\mathrm{g}}\frac{\partial F}{\partial \mathrm{R}}\mathrm{R}_{\mu\nu}\delta\mathrm{g}^{\mu\nu} + \sqrt{-\mathrm{g}}\frac{\partial F}{\partial \mathrm{R}}\mathrm{g}^{\mu\nu}\delta\mathrm{R}_{\mu\nu}\Big\}\\ &= \frac{M_{pl}^2}{2}\int \mathrm{d}^4x\sqrt{-\mathrm{g}}\Big\{\Big(-\frac{1}{2}F\mathrm{g}_{\mu\nu} + \frac{\partial F}{\partial \mathrm{R}}\mathrm{R}_{\mu\nu}\Big)\delta\mathrm{g}^{\mu\nu} + \frac{\partial F}{\partial \mathrm{R}}\mathrm{g}^{\mu\nu}\Big[\nabla_\alpha\delta\Gamma^{\alpha}_{\mu\nu} - \nabla_\nu\delta\Gamma^{\alpha}_{\mu\alpha}\Big]\Big\}.\end{aligned} \tag{4.3}$$

First term in curly braces in Eq.(4.3) is already in the form of expression multiplied by $\mathrm{g}^{\mu\nu}$, integrating by parts the second term and neglecting the surface terms we have

$$\delta S_G = \frac{M_{pl}^2}{2}\int \mathrm{d}^4x\sqrt{-\mathrm{g}}\Big\{\Big(-\frac{1}{2}F\mathrm{g}_{\mu\nu} + \frac{\partial F}{\partial \mathrm{R}}\mathrm{R}_{\mu\nu}\Big)\delta \mathrm{g}^{\mu\nu} + \mathrm{g}^{\mu\nu}\Big[-\nabla_\alpha\Big(\frac{\partial F}{\partial \mathrm{R}}\Big)\,\delta\Gamma^\alpha_{\mu\nu} + \nabla_\nu\Big(\frac{\partial F}{\partial \mathrm{R}}\Big)\,\delta\Gamma^\alpha_{\mu\alpha}\Big]\Big\}. \tag{4.4}$$

Before we go into next step, we need to express $\delta\Gamma^\alpha_{\mu\nu}$ in terms of variation of the inverse metric $\delta\mathrm{g}^{\mu\nu}$, which can be easily found from taking the variation of metric compatibility equation (2.4) such as

$$\delta(\nabla_\alpha \mathrm{g}_{\mu\nu}) = \delta\Big(\partial_\alpha \mathrm{g}_{\mu\nu} - \Gamma^\sigma_{\mu\alpha}\mathrm{g}_{\sigma\nu} - \Gamma^\sigma_{\nu\alpha}\mathrm{g}_{\mu\sigma}\Big) = 0, \tag{4.5}$$

where we have used the open form of covariant differentiation. Expanding the above equation for three different permutations of indices we get

$$\begin{aligned}
\partial_\alpha\delta \mathrm{g}_{\mu\nu} - \Gamma^\sigma_{\mu\alpha}\delta \mathrm{g}_{\sigma\nu} - \Gamma^\sigma_{\nu\alpha}\delta \mathrm{g}_{\mu\sigma} - \delta\Gamma^\sigma_{\mu\alpha}\mathrm{g}_{\sigma\nu} - \delta\Gamma^\sigma_{\nu\alpha}\mathrm{g}_{\mu\sigma} &= 0\\
\partial_\mu\delta \mathrm{g}_{\nu\alpha} - \Gamma^\sigma_{\nu\mu}\delta \mathrm{g}_{\sigma\alpha} - \Gamma^\sigma_{\alpha\mu}\delta \mathrm{g}_{\nu\sigma} - \delta\Gamma^\sigma_{\nu\mu}\mathrm{g}_{\sigma\alpha} - \delta\Gamma^\sigma_{\alpha\mu}\mathrm{g}_{\nu\sigma} &= 0\\
\partial_\nu\delta \mathrm{g}_{\alpha\mu} - \Gamma^\sigma_{\alpha\nu}\delta \mathrm{g}_{\sigma\mu} - \Gamma^\sigma_{\mu\nu}\delta \mathrm{g}_{\alpha\sigma} - \delta\Gamma^\sigma_{\alpha\nu}\mathrm{g}_{\sigma\mu} - \delta\Gamma^\sigma_{\mu\nu}\mathrm{g}_{\alpha\sigma} &= 0.
\end{aligned} \tag{4.6}$$

We add the first and second equation and subtract the third one, and identify the covariant differentiation of metric variation $\delta\mathrm{g}_{\mu\nu}$ to obtain

$$2\delta\Gamma^\sigma_{\mu\alpha}\mathrm{g}_{\sigma\nu} = \nabla_\alpha(\delta \mathrm{g}_{\mu\nu}) + \nabla_\mu(\delta \mathrm{g}_{\nu\alpha}) - \nabla_\nu(\delta \mathrm{g}_{\alpha\mu}). \tag{4.7}$$

Last step to obtain $\delta\Gamma^\sigma_{\mu\nu}$ in terms of $\delta\mathrm{g}^{\alpha\beta}$, we multiply the Eq.(4.7) with inverse metric $\mathrm{g}^{\nu\lambda}$ and switch the variations of metric to variations of inverse metric via the relation $\delta(\mathrm{g}^{\mu\nu}\mathrm{g}_{\mu\alpha}) = 0$. The result is

$$\delta\Gamma^\sigma_{\mu\nu} = -\frac{1}{2}\Big[\mathrm{g}_{\alpha\mu}\nabla_\nu(\delta \mathrm{g}^{\alpha\sigma}) + \mathrm{g}_{\alpha\nu}\nabla_\mu(\delta \mathrm{g}^{\alpha\sigma}) - \mathrm{g}_{\mu\alpha}\mathrm{g}_{\nu\beta}\nabla^\sigma(\delta \mathrm{g}^{\alpha\beta})\Big]. \tag{4.8}$$

Now we can plug in the above equation to (4.4) and get the desired form. After integrating by parts and relabelling some dummy indices we find variation of the gravitational part of action as

$$\delta S_G = \frac{M_{pl}^2}{2}\int \mathrm{d}^4x\sqrt{-\mathrm{g}}\Big\{-\frac{1}{2}F\mathrm{g}_{\mu\nu} + \frac{\partial F}{\partial \mathrm{R}}\mathrm{R}_{\mu\nu} - \Big[\nabla_\mu\nabla_\nu - \mathrm{g}_{\mu\nu}\nabla_\alpha\nabla^\alpha\Big]\frac{\partial F}{\partial \mathrm{R}}\Big\}\delta\mathrm{g}^{\mu\nu}. \tag{4.9}$$

Finally adding the matter part and applying the least action principle $\delta S = 0$ we find the field equations for the $F(\mathrm{R})$ theory in metrical formulation as

$$-\frac{1}{2}F(\mathrm{R})\mathrm{g}_{\mu\nu} + f_\mathrm{R}\mathrm{R}_{\mu\nu} - \Big(\nabla_\nu\nabla_\mu - \mathrm{g}_{\mu\nu}\Box\Big)f_\mathrm{R} = \frac{1}{M_{pl}^2}T_{\mu\nu}, \tag{4.10}$$

where $\Box \equiv \nabla_\mu\nabla^\mu$ and f_R is a shorthand notation for differentiation of the $F(\mathrm{R})$ with respect to its subscript

$$f_\mathrm{R} \equiv \frac{\partial F(\mathrm{R})}{\partial \mathrm{R}}, \tag{4.11}$$

and the energy-momentum tensor $T_{\mu\nu}$ is by definition given as

$$T_{\mu\nu} = \frac{-2}{\sqrt{-\mathrm{g}}}\frac{\delta S_M}{\delta \mathrm{g}^{\mu\nu}}. \tag{4.12}$$

One can go further and construct the Einstein tensor $G_{\mu\nu}$ from Eq.(4.10) and define an effective energy momentum tensor to study its properties [28]. Our plan is to study the cosmological effects of the $F(\mathrm{R})$ theory. Combining computer based simulations with cosmological observations of the cosmic microwave background (CMB) and galaxy distributions indicate that our universe is highly homogeneous and isotropic at the large scale [32–35, 37]. A mathematical representation of our universe can be given by the metric

$$ds^2 = -dt^2 + a(t)^2\Big(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta\, d\phi^2)\Big) \tag{4.13}$$

which is known as the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric in the literature. This is the common form of the FLRW metric where $a(t)$ is the dimensionless scale factor determines how big is the 3-dim space at the given time, k represents the type of geometry of the space-time and can be greater than, equal to, or less than one. We can now plug in the FLRW metric into our field equations to study the cosmological effects of $F(R)$ theory in the metrical framework. Assuming the energy-momentum tensor of the matter field is described by the perfect fluid as in (3.30), we can calculate modified Friedmann equations from the field equations (4.10) as [28]

$$H^2 = \frac{1}{3M_{pl}^2}(f')^{-1}\Big(\rho + \frac{\mathrm{R}f' - F(\mathrm{R})}{2} - 3H\dot{\mathrm{R}}f''\Big), \tag{4.14}$$

$$2\dot{H} + 3H^2 = -\frac{1}{M_{pl}^2}(f')^{-1}\Big(p + (\dot{\mathrm{R}})^2 f''' + 2H\dot{\mathrm{R}}f'' + \ddot{\mathrm{R}}f'' + \frac{1}{2}(F(\mathrm{R}) - \mathrm{R}f')\Big), \tag{4.15}$$

where f' is the differentiation of functional $F(R)$ with respect to its argument, H is known as the Hubble parameter defined as $H = \dot{a}/a$. Hubble parameter is one of the key parameters in cosmological studies and observations, it is related to rate of expansion of the universe at the given time. Current value of the Hubble parameter is called Hubble constant, H_0. Latest value of the Hubble constant H_0 coming from the direct measurements is $H_0 = 74.03 \pm 1.42 km\ s^{-1} Mpc^{-1}$ [38]. Mpc stands for mega-parsec, which is an unit of distance commonly used in cosmology and it is defined as $1Mpc = 3.086 * 10^{19}\ km$. Unit of Hubble constant means that for every $3.086 * 10^{19}\ km$ further away a star or galaxy is from us, it appears to be recedes away from us with the speed of $74.03\ km/s$ because of the expansion of the universe. Indirect measurements of the H_0 involves applying a cosmological model to CMB observations which is the relic radiation from early universe. Latest indirect measurement coming from the planck group is $H_0 = 67.4 \pm 0.5 km\ s^{-1} Mpc^{-1}$ [34]. This increasing discrepancy between direct and indirect (model dependent) measurements known as ”Hubble tension”, and it indicates that we might need a new cosmological model to describe features of our universe [39–41].

4.2. Metric-Affine Formalism of $F(\mathbb{R})$

Action of the generalized metric-affine theory can be written as [31]

$$S[\Gamma, \mathrm{g}] = \tfrac{M_{pl}^2}{2} \int \mathrm{d}^4 x \sqrt{-\mathrm{g}}\, F(\mathbb{R}) + S_M[\mathrm{g}, \psi]. \tag{4.16}$$

where $F(\mathbb{R})$ is the functional of affine curvature scalar $\mathbb{R}(\Gamma)$ and again we examine the case where matter part of the action is independent of the affine connection. Varying the action with respect to $\mathrm{g}_{\mu\nu}$ and applying least action principle as before yields

$$-\tfrac{1}{2}F(\mathbb{R})\mathrm{g}_{\mu\nu} + f_{\mathbb{R}}\mathbb{R}_{\mu\nu} = \frac{1}{M_{pl}^2}T_{\mu\nu}, \tag{4.17}$$

Similarly varying the action with respect to general affine connection $\Gamma^{\lambda}_{\mu\nu}$ gives

$$\nabla_\alpha \left(\sqrt{-\mathrm{g}}\ \mathrm{g}^{\mu\nu} f_{\mathbb{R}}\right) = 0. \tag{4.18}$$

Following the same approach as in section 3, we can define a dynamical metric $h_{\mu\nu}$ as

$$h_{\mu\nu} = f(\mathbb{R})\mathrm{g}_{\mu\nu}. \tag{4.19}$$

Using the covariant conservation of the dynamical metric $h_{\mu\nu}$, we can solve affine connection $\Gamma_{\mu\nu}$ in terms of Levi-Civita connection ${}^{\mathrm{g}}\Gamma_{\mu\nu}$ plus contributions from the derivatives of $F(\mathbb{R})$. The result is

$$\Gamma^{\lambda}_{\mu\nu} = {}^{\mathrm{g}}\Gamma^{\lambda}_{\mu\nu} + \frac{1}{2}(f_{\mathbb{R}})^{-1}\Big(\delta^{\lambda}_{\nu}\partial_\mu f_{\mathbb{R}} + \delta^{\lambda}_{\mu}\partial_\nu f_{\mathbb{R}} - \mathrm{g}^{\lambda\sigma}\mathrm{g}_{\mu\nu}\partial_\sigma f_{\mathbb{R}}\Big), \tag{4.20}$$

with the corresponding affine Ricci tensor $\mathbb{R}_{\mu\nu}(\Gamma)$ can be written in terms of metrical Ricci tensor $\mathrm{R}_{\mu\nu}({}^{\mathrm{g}}\Gamma)$ plus contributions from the derivatives of $F(\mathbb{R})$, such as

$$\begin{aligned}\mathbb{R}_{\mu\nu}(\Gamma) = \mathrm{R}_{\mu\nu}({}^{\mathrm{g}}\Gamma) + \tfrac{3}{2}(f_{\mathbb{R}})^{-2}\,{}^{\mathrm{g}}\nabla_\mu f_{\mathbb{R}}\,{}^{\mathrm{g}}\nabla_\nu f_{\mathbb{R}} \\ - (f_{\mathbb{R}})^{-1}\,{}^{\mathrm{g}}\nabla_\mu\,{}^{\mathrm{g}}\nabla_\nu f_{\mathbb{R}} - \tfrac{1}{2}(f_{\mathbb{R}})^{-1}\,\mathrm{g}_{\mu\nu}\,{}^{\mathrm{g}}\Box f_{\mathbb{R}}.\end{aligned} \tag{4.21}$$

All the derivatives in the above equation are with respect to Levi-Civita connection ${}^{\mathrm{g}}\Gamma^{\lambda}_{\mu\nu}$ and the box operator has the same definition as before. As we mentioned in the last section, at this point, it is possible to construct Einstein tensor $G_{\mu\nu}$ with the help of Eq.(4.21) and define an effective energy-momentum tensor. Turning back to cosmological setup, we can calculate $\mathbb{R}_{00}$ and $\mathbb{R}_{ii}$ from (4.21) with the FLRW metric (4.13). The results are

$$\mathbb{R}_{00} = -3\frac{\ddot{a}}{a} + \frac{3}{2}(f_{\mathbb{R}})^{-1}\Big(\frac{(\dot{f}_{\mathbb{R}})^2}{f_{\mathbb{R}}} - \ddot{f}_{\mathbb{R}} - H\dot{f}_{\mathbb{R}}\Big), \tag{4.22}$$

$$\mathbb{R}_{ii} = a^2\Big[\frac{\ddot{a}}{a} + 2H^2 + \frac{1}{2}(f_{\mathbb{R}})^{-1}(5H\dot{f}_{\mathbb{R}} + \ddot{f}_{\mathbb{R}})\Big]. \tag{4.23}$$

Substituting Eq.(4.22) and (4.23) back into the field equations (4.17) and as before assuming the form of perfect fluid for the $T_{\mu\nu}$

$$T_{\mu\nu} = (\rho + p)V_\mu V_\nu + p\ \mathrm{g}_{\mu\nu}, \tag{4.24}$$

we find the modified Friedmann equations as [42]

$$\Big(H+\frac{1}{2}\frac{\dot{f}_{\mathbb{R}}}{f_{\mathbb{R}}}\Big)^2=\frac{1}{6}(f_{\mathbb{R}})^{-1}\Big(F(\mathbb{R})+\frac{(\rho+3p)}{2M_{pl}^2}\Big), \tag{4.25}$$

$$\dot{H}=-\frac{1}{2}(f_{\mathbb{R}})^{-1}\Big(\frac{\rho+p}{2M_{pl}^2}-\frac{3}{2}\frac{(\dot{f}_{\mathbb{R}})^2}{f_{\mathbb{R}}}+\ddot{f}_{\mathbb{R}}-H\dot{f}_{\mathbb{R}}\Big), \tag{4.26}$$

where definitions are same as the previous. Looking at the Eq.(4.25), we notice that Hubble parameter H gets an additive term in the metric-affine formulation which was not present in the metric formulation.

4.3. Metric-Affine Formalism with Cross-Curvature terms $F(\mathrm{R},\mathbb{R})$

Our main goal of this chapter is the study the cosmological effects of cross-curvature $F(\mathrm{R},\mathbb{R})$ theory, where $F(\mathrm{R},\mathbb{R})$ is now the functional of both curvature scalars $\mathrm{R}({}^{\mathrm{g}}\Gamma)$ and $\mathbb{R}(\Gamma)$. Since we are formulating the theory in the metric-affine framework, our priori assumption is, the curvature of our space-time is represented by the affine Riemann tensor $\mathbb{R}^{\lambda}_{\mu\alpha\nu}(\Gamma)$ and the affine connection $\Gamma^{\lambda}_{\mu\nu}$ should be the connection which is responsible for the motion of material particles. If one is to study the motion of particles in space-time, he should use the affine connection $\Gamma^{\lambda}_{\mu\nu}$ in the geodesic equation [12]. Then one might ask, what is the reason for using $\mathrm{R}({}^{\mathrm{g}}\Gamma)$ in the functional at all? We simply approach to problem from the perspective of classical field theory, where both $\Gamma^{\lambda}_{\mu\nu}$ and ${}^{\mathrm{g}}\Gamma^{\lambda}_{\mu\nu}$ are the fields with corresponding field strengths $\mathbb{R}_{\mu\nu}(\Gamma)$ and $\mathrm{R}_{\mu\nu}({}^{\mathrm{g}}\Gamma)$. As in the previous sections, we study the non-fermionic matter fields for simplicity. Most general action including the cross-curvature terms can be written as

$$S[\Gamma,\mathrm{g}]=\frac{M_{pl}^2}{2}\int \mathrm{d}^4x\sqrt{-\mathrm{g}}\,F(\mathrm{R},\mathbb{R})+S_M[\mathrm{g},\psi], \tag{4.27}$$

Varying the action with respect to metric and affine connection respectively yields

$$-\frac{1}{2}F(\mathrm{R},\mathbb{R})\mathrm{g}_{\mu\nu}+f_{\mathrm{R}}\mathrm{R}_{\mu\nu}+f_{\mathbb{R}}\mathbb{R}_{\mu\nu}-\Big({}^{\mathrm{g}}\nabla_{\nu}\,{}^{\mathrm{g}}\nabla_{\mu}-\mathrm{g}_{\mu\nu}\,{}^{\mathrm{g}}\Box\Big)f_{\mathrm{R}}=\frac{1}{M_{pl}^2}T_{\mu\nu}, \tag{4.28}$$

$$\nabla_{\alpha}\left(\sqrt{-\mathrm{g}}\,\mathrm{g}^{\mu\nu}f_{\mathbb{R}}\right)=0. \tag{4.29}$$

Now field equations Eqn.(4.28) and (4.29) includes contributions from both curvature scalars through $F(\mathrm{R},\mathbb{R})$ and its derivatives. As always, Eq.(4.29) lets us to define dynamical metric $h_{\mu\nu}$ which will be covariantly conserved by the affine connection, therefore we can write affine connection $\Gamma^{\lambda}_{\mu\nu}$ in terms of Levi-Civita connection ${}^{\mathrm{g}}\Gamma^{\lambda}_{\mu\nu}$ and contributions from the derivatives of $F(\mathrm{R},\mathbb{R})$ such as

$$\Gamma^{\lambda}_{\mu\nu} = {}^{\mathrm{g}}\Gamma^{\lambda}_{\mu\nu} + \tfrac{1}{2}[f_{\mathbb{R}}]^{-1}\Big(\delta^{\lambda}_{\nu}\partial_{\mu}f_{\mathbb{R}} + \delta^{\lambda}_{\mu}\partial_{\nu}f_{\mathbb{R}} - \mathrm{g}^{\lambda\sigma}\mathrm{g}_{\mu\nu}\partial_{\sigma}f_{\mathbb{R}}\Big), \tag{4.30}$$

and the corresponding $\mathbb{R}_{\mu\nu}$ can be calculated by simpliy putting affine connection into definition of $\mathbb{R}_{\mu\nu}$ given by (2.2). Resulting expression is

$$\begin{aligned}\mathbb{R}_{\mu\nu}(\Gamma) = \mathrm{R}_{\mu\nu}({}^{\mathrm{g}}\Gamma) &+ \frac{3}{2}(f_{\mathbb{R}})^{-2}\,{}^{\mathrm{g}}\nabla_{\mu}f_{\mathbb{R}}\,{}^{\mathrm{g}}\nabla_{\nu}f_{\mathbb{R}} \\ &- (f_{\mathbb{R}})^{-1}\,{}^{\mathrm{g}}\nabla_{\mu}\,{}^{\mathrm{g}}\nabla_{\nu}f_{\mathbb{R}} - \frac{1}{2}(f_{\mathbb{R}})^{-1}\,\mathrm{g}_{\mu\nu}\,{}^{\mathrm{g}}\Box f_{\mathbb{R}}\end{aligned} \tag{4.31}$$

Eliminating $\mathbb{R}_{\mu\nu}(\Gamma)$ from (4.28) and (4.31) yields

$$\begin{aligned}\mathrm{R}_{\mu\nu} = (\tilde{f})^{-1}\Big[&- \frac{3}{2}(f_{\mathbb{R}})^{-1}\,{}^{\mathrm{g}}\nabla_{\mu}f_{\mathbb{R}}\,{}^{\mathrm{g}}\nabla_{\nu}f_{\mathbb{R}} + \frac{1}{2}F(\mathrm{R},\mathbb{R})\mathrm{g}_{\mu\nu} \\ &+ {}^{\mathrm{g}}\nabla_{\mu}\,{}^{\mathrm{g}}\nabla_{\nu}\tilde{f} + \mathrm{g}_{\mu\nu}\,{}^{\mathrm{g}}\Box\big(\frac{1}{2}f_{\mathbb{R}} - f_{\mathrm{R}}\big) + \frac{1}{M_{pl}^2}T_{\mu\nu}\Big],\end{aligned} \tag{4.32}$$

where we defined $\tilde{f}$ to slightly simplify equations as

$$\tilde{f} \equiv f_{\mathrm{R}} + f_{\mathbb{R}}. \tag{4.33}$$

Following the same procedure as in section 4.2, we get the following modified Friedmann equations for the cross-curvature theory as

$$\begin{aligned}\Big(H + \frac{1}{2}\frac{\dot{f}_{\mathbb{R}}}{\tilde{f}}\Big)^2 &= \frac{1}{6}(\tilde{f})^{-1}\Big(F(\mathrm{R},\mathbb{R}) + \frac{(\rho+3P)}{2M_{pl}^2}\Big) \\ &- \frac{1}{2}(\tilde{f})^{-1}\Big[\frac{1}{2}(\dot{f}_{\mathbb{R}})^2\Big(\frac{1}{f_{\mathbb{R}}} - \frac{1}{\tilde{f}}\Big) + a^{-1}\Big(a\;\dot{\ } f_{R}\Big)^{\bullet}\Big],\end{aligned} \tag{4.34}$$

$$\dot{H} = -\frac{1}{2}(\tilde{f})^{-1}\left(\frac{\rho + P}{2M_{pl}^2} - \frac{3}{2}\frac{(\dot{f}_{\mathbb{R}})^2}{f_{\mathbb{R}}} + \ddot{\tilde{f}} - H\dot{\tilde{f}}\right). \tag{4.35}$$

Before concluding this chapter, we give a comparison table of the key equations for the both $F(\mathbb{R})$ and $F(\mathrm{R},\mathbb{R})$ theories in the metric-affine framework as follows

Comparison of $F(\mathbb{R})$ and $F(\mathrm{R},\mathbb{R})$		
Grav. part of action	$S_G = \frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}\, F(\mathbb{R})$ $Eq.(4.16)$	$S_G = \frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}\, F(\mathrm{R},\mathbb{R})$ $Eq.(4.27)$
Field Equations	$-\frac{1}{2}F(\mathbb{R})g_{\mu\nu} + f_{\mathbb{R}}\mathbb{R}_{\mu\nu} = \frac{1}{M_{pl}^2}T_{\mu\nu}$ $Eq.(4.17)$ ${}^{\Gamma}\nabla_\alpha\left(\sqrt{-g}\, g^{\mu\nu} f_{\mathbb{R}}\right) = 0$ $Eq.(4.18)$	$-\frac{1}{2}F(\mathrm{R},\mathbb{R})g_{\mu\nu} + f_{\mathrm{R}}\mathrm{R}_{\mu\nu} + f_{\mathbb{R}}\mathbb{R}_{\mu\nu} - \left({}^{g}\nabla_\nu\, {}^{g}\nabla_\mu - g_{\mu\nu}\, {}^{g}\Box\right) f_{\mathrm{R}} = \frac{1}{M_{pl}^2}T_{\mu\nu}$ $Eq.(4.28)$ ${}^{\Gamma}\nabla_\alpha\left(\sqrt{-g}\, g^{\mu\nu} f_{\mathbb{R}}\right) = 0$ $Eq.(4.29)$
Friedmann Equations	$\left(H + \frac{1}{2}\frac{\dot{f}_{\mathbb{R}}}{f_{\mathbb{R}}}\right)^2 = \frac{1}{6}(f_{\mathbb{R}})^{-1}\Big(F + \frac{(\rho+3P)}{2M_{pl}^2}\Big)$ $Eq.(4.25)$ $\dot{H} = -\frac{1}{2}(f_{\mathbb{R}})^{-1}\left(\frac{\rho+P}{2M_{pl}^2} - \frac{3}{2}\frac{(\dot{f}_{\mathbb{R}})^2}{f_{\mathbb{R}}} + \ddot{f}_{\mathbb{R}} - H\dot{f}_{\mathbb{R}}\right)$ $Eq.(4.26)$	$\left(H + \frac{1}{2}\frac{\dot{f}_{\mathbb{R}}}{\tilde{f}}\right)^2 = \frac{1}{6}(\tilde{f})^{-1}\Big(F + \frac{(\rho+3P)}{2M_{pl}^2}\Big) - \frac{1}{2}(\tilde{f})^{-1}\Big[\frac{1}{2}(\dot{f}_{\mathbb{R}})^2\big(\frac{1}{f_{\mathbb{R}}} - \frac{1}{\tilde{f}}\big) + a^{-1}(a\, \dot{f}_R)^{\cdot}\Big]$ $Eq.(4.34)$ $\dot{H} = -\frac{1}{2}(\tilde{f})^{-1}\left(\frac{\rho+P}{2M_{pl}^2} - \frac{3}{2}\frac{(\dot{f}_{\mathbb{R}})^2}{f_{\mathbb{R}}} + \ddot{\tilde{f}} - H\dot{\tilde{f}}\right)$ $Eq.(4.35)$

We have studied the cosmological effects of $F(\mathrm{R})$ and $F(\mathbb{R})$ up to Friedmann equations in metric and metric-affine frameworks respectively and compared the cross-curvature theory $F(\mathrm{R}, \mathbb{R})$ with the latter one. We highlighted differences between $F(\mathbb{R})$ and $F(\mathrm{R}, \mathbb{R})$ in red. Friedmann equations in both theory highly depend on the functional form of F, looking back at the Eq.(4.31), if the functionals dependence on curvatures is linear (as in case of Einstein-Hilbert action) then the both Ricci tensors become equal such as $\mathrm{R}({}^{\mathrm{g}}\Gamma) = \mathbb{R}(\Gamma)$. The difference of the formulation becomes emergent only when quadratic or more power of curvature terms are involved in the action.

CHAPTER 5

CONCLUSIONS

The present thesis mainly focuses on two distinct studies. In chapter 3, we have derived the field equations for the general case of functional $F\mathbb{R}_{\mu\nu}, \mathrm{g}^{\mu\nu}$, which is extended form of the known $F(\mathbb{R})$ functionals. Having solved affine Ricci tensor $\mathbb{R}_{\mu\nu}$ in terms of metrical Ricci tensor $\mathrm{R}_{\mu\nu}$ and contributions from dynamical metric $h_{\mu\nu}$, we then specified our functional as (3.15) to examine the effects of disformal gravity. To pinpoint the effects of disformal coupling in a cosmological setting, we constructed the Einstein tensor $G_{\mu\nu}$ defined in (3.26) and identified an effective energy-momentum tensor $T^{eff}_{\mu\nu}$. We have seen that $T^{eff}_{\mu\nu}$ includes many contributions through derivatives of 4-velocity vector as well as contributions from the standard energy-momentum tensors itself weighted by the coupling strength ϵ. Assuming either energy-momentum tensor can be written in the form of a perfect fluid, we identified the effective energy density ρ_{eff} and effective pressure p_{eff}. We then gave a brief derivation of disformal effects in the metrical framework and compared the findings of both formulations. Even though both formulations have the same starting point, namely the action, they lead to different predictions on the effective energy density and pressure. We have shown the differences with the Eqs.(3.40) and (3.41). As we said before, contributions to $T^{eff}_{\mu\nu}$ are highly dependent on the functional form of $F\mathbb{R}_{\mu\nu}, \mathrm{g}^{\mu\nu}$, our model can be studied further with the modifications of (3.15). For example, letting the coupling strength to be a function of space-time such as $\epsilon = \epsilon(x^{\mu})$, of course the equations will be much more complicated in that setting.

Second focus of the present thesis is the study of cross-curvature effects on metric-affine formulation. In chapter 4, for convenience, we have derived the known equations of metric $F(\mathrm{R})$ and metric-affine $F(\mathbb{R})$ up to Friedmann equations and presented the idea of both curvature scalars in the functional such that $F(\mathrm{R}, \mathbb{R})$. Our approach to the idea of cross-curvature terms in the functional $F(\mathrm{R}, \mathbb{R})$ was simple, we considered both connections as classical fields with corresponding field strengths given by the Ricci tensors and derived the both Friedmann equations within the content of the cross-curvature theory. Comparison of $F(\mathbb{R})$ and $F(\mathrm{R}, \mathbb{R})$ is presented in the table. Although there are similar terms in between corresponding Friedmann equations, namely (4.25) to (4.34) and (4.26) to (4.35), we can see the effects of the cross-curvature approach. Once again we should point the importance of the open form the functional $F(\mathrm{R}, \mathbb{R})$. We see from the Friedmann equations, $F(\mathrm{R}, \mathbb{R})$ is the decisive factor for the study of cosmological parameters.

We presented both of our studies as general as possible, so that once the functional forms are specified (functionals in the actions Eq.3.4 and Eq.4.27) , it is straightforward to just plug it in the related expressions. Of course, there are infinite number of choices for the functionals, so one might as "how to choose the right functional?". The open form of the functional should be deduced from careful examinations of observations and measurements. For example, it is well known that the addition of quadratic curvature terms to Einstein-Hilbert action can achieve gravity-driven inflation [43].

Another important point to improve the present study is that involvement of the fermion fields in the actions. In both of our studies, we have approached from the cosmological perspective and assumed that our matter fields are in the form of perfect fluid (3.29) which is naturally independent of the connections. Since fermions play a great role in our universe, a comprehensive study must include them by relaxing both restrictive assumptions that matter action is independent of connection and the affine connection is symmetric. These two assumptions might not be independent of each other after all. As we have said before, the study of fermions demands the involvement of the affine connection in matter action (through spin connection) and studies show that, this dependence on the affine connection of the matter part couples to anti-symmetric part of the connection (which is called torsion). Just as the metric dependent part of the matter action couples to Ricci tensor (2.14), namely Einstein field equations. This clearly indicates that the existence of spin itself induces the torsion in space-time [31, 44].

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