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Locally isomorphic torsionless modules over domains of finite character

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In a 2002 paper, P. Goeters and B. Olberding compare local, near, and stable isomorphisms of torsionless modules over h-local domains. In this paper, we compare these weaker forms of isomorphisms of torsionless modules over domains of finite character.

Keywords: Krull–Schmidt property; torsion-free modules; finitely generated modules; integral domains of finite character; almost local-global rings.

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1. Introduction

Let R be a commutative ring with identity and C a class of indecomposable R-modules. The *Krull–Schmidt property* holds for C if, whenever $G_1 \oplus \cdots \oplus G_n \cong H_1 \oplus \cdots \oplus H_m$ for $G_i, H_j \in C$, then $n = m$ and, after reindexing, $G_i \cong H_i$ for all $i \leq n$. This property fails broadly for modules over commutative rings, and even the weaker property of cancellation $A \oplus B \cong A \oplus C \Rightarrow B \cong C$, holds only in special situations. In [6], the authors discuss *weaker* forms of isomorphisms to recover properties such as cancellation over *h-local* domains. They extend the notion of near isomorphism from the theory of torsion-free finite rank abelian groups to modules over commutative integral domains, studying its properties for the class of torsionless modules (see below for definitions).

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We say that an integral domain R is of *finite character* if every nonzero element of R is contained in only finitely many maximal ideals of R. An integral domain R is called *h-local* if R is of finite character and each nonzero prime ideal of R is contained in a unique maximal ideal of R. Thus, a domain R of finite character has the property that R/I is a semilocal ring for each nonzero ideal I of R, while an h-local domain R has the property that R/I is a finite direct sum of local rings for each nonzero ideal I of R.

If R is a ring and A is a matrix over R, then the *content* of A is the ideal of R generated by all the entries of A. Let B be a submodule of $R^{(n)}$. Then B is said to be *basic* if the ideal of R generated by the contents of all vectors in B equals R. We say that a commutative ring R has the *BCS-property* if for each finitely generated projective R-module P and each basic submodule B of P, B contains a rank one projective summand of P. The BCS-property is equivalent to the UCS-property of [7].

We say that R is *local-global* if every polynomial over R in finitely many indeterminates which represents units locally, assumes a unit value when evaluated at properly chosen elements of R [4]. Semi-local domains are local–global. A ring is almost local–global if every of its proper factor ring is local–global. We note that domains of finite character are almost local–global.

We call a module *torsionless* if it is isomorphic to a submodule of a finitely generated free module. If R is an integral domain with quotient field Q and G is a torsion-free R-module, then the divisible hull QG of G is $Q \otimes_R G$. We identify G with its image in QG. The rank of G is the dimension of the Q-vector space QG. We write $G^{(n)}$ for a direct sum of n copies of G. Two torsionless R-modules G and H are said to be *nearly isomorphic* if G and H are of the same rank and, for each nonzero ideal I of R, there exists an embedding $f : G \to H$ such that the ideal $\mathrm{Ann}_R(\mathrm{Coker}(f))$ is comaximal with I. The R-modules G and H are called *locally isomorphic* if $G_M \cong H_M$ for all maximal ideals M of R. It is easy to see that, if G and H are torsionless nearly isomorphic R-modules, then G and H are locally isomorphic. The R-modules G and H are *stably isomorphic* if $G \oplus R^{(n)} \cong H \oplus R^{(n)}$ for some $n > 0$. These weaker forms of isomorphisms are equivalence relations. Furthermore, for domains with nonzero Jacobson radical, near isomorphism implies isomorphism. Also, stable isomorphism implies isomorphism for semilocal commutative rings [4, Theorem 2.5]. So, for semilocal domains, near isomorphism and stable isomorphism are equivalent to isomorphism.

In [6], the authors compare local, near, and stable isomorphisms for torsionless modules over an h-local domain. The key to their results is [6, Lemma 2.1], where they show that, over a finite direct sum R of local rings, if F is a finitely generated free module, then an isomorphism $F/G \cong F/H$ (for submodules G and H of F) can be lifted to an automorphism of F. The application to their results is obvious, since a proper homomorphic image of an h-local domain is a finite direct sum of local rings. The key to our results (Lemma 2.1 below) is that one can lift such an isomorphism $F/G \cong F/H$ to an automorphism of F provided the ring R is semilocal. Since a proper homomorphic image of a domain of finite character is

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semilocal, we hope for comparable results for local, near, and stable isomorphism of torsionless modules over a domain of finite character.

We prove a generalization of [6, Lemma 2.6] (Lemma 3.2). In this generalization, we show that $F/G \cong F/H$ (F is a finitely generated free module and G and H are its submodules) when G and H are locally isomorphic under the assumption that G is finitely generated. We do not have an example to show that this restriction is necessary. Most of our main results are based on this lemma, and hence they are limited to finitely generated torsionless modules (over an integral domain of finite character).

This paper is organized as follows. In Sec. 2, we give some preliminary results which will be handy in comparing local, near, and stable isomorphism. In Sec. 3, we prove when local isomorphism implies isomorphism and when local, stable, and near isomorphisms coincide. In Sec. 4, we provide some applications of the results in Sec. 3.

2. Preliminary Results

Let R be a commutative ring. If P is a projective R-module, then P_M is a free R_M-module for each maximal ideal M of R. We define the rank of P to be the maximum of the ranks of the free modules P_M and say that P has constant rank n if the rank of P_M, for each M, is n. The following is a generalization of [6, Lemma 2.1].

Lemma 2.1. *Let R be a semilocal ring and F a finitely generated free R-module of finite rank and G and H be submodules of F such that $F/G \cong F/H$. Then there exists an automorphism $\theta : F \to F$ such that $\theta(G) = H$.*

Proof. Let ϕ be the isomorphism between F/G and F/H. Our aim is to show that ϕ can be lifted to an automorphism of F which maps G onto H.

Let J be the Jacobson radical of R, $G' = G \cap JF$, and $H' = H \cap JF$. We claim that the exact sequence

$$0 \to G/G' \to F/G' \to F/G \to 0$$

is split. Let us consider the composition of the inclusion map $\iota : G/G' \to F/G'$ with the natural map $\rho : F/G' \to (F/G')/J(F/G') \cong F/JF$. Let $g + G' \in G/G'$ such that its image is 0. Hence, $g \in JF$, so that $g \in G'$, implying that $g + G' = G'$. Thus, this composition is injective. Furthermore, F/JF is semisimple, and the composition map $\alpha : G/G' \to F/JF$ is injective. So, α is split by a map $\beta : F/JF \to G/G'$, and ι is split by $\beta \circ \rho$, so that the exact sequence is split. Since the sequence is split exact, $F/G' \cong G/G' \oplus F/G$, implying that

$$F/JF \cong (F/G')/(J(F/G')) \cong (G/G')/(J(G/G')) \oplus (F/G)/(J(F/G)).$$

Since $J(G/G') = 0$, $F/JF \cong G/G' \oplus (F/G)/(J(F/G))$. Similarly, the exact sequence for F/H' is split, also, and hence $F/JF \cong H/H' \oplus (F/H)/(J(F/H))$. Since the simple summands of F/JF have local endomorphism rings, we can cancel

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them from the direct sum [5, Corollary V.8.3], and hence $G/G' \cong H/H'$. Since $F/G \cong F/H$, ϕ lifts to an isomorphism $\psi : F/G' \to F/H'$.

Let $\mu : F \twoheadrightarrow F/G'$ and $\nu : F \twoheadrightarrow F/H'$ be the natural maps. Since F is projective, ψ lifts to a map $\theta : F \to F$. Thus, $\psi\circ\mu = \nu\circ\theta$. Now $ker(\nu) = H' \subseteq JF$, the Jacobson radical of the finitely generated module F. So, it follows from Nakayama's Lemma that θ is surjective. Then by comparing ranks, θ must be injective as well. So, $\theta : F \to F$ is an isomorphism that induces the isomorphism $\psi : F/G' \to F/H'$, which induces the isomorphism $\phi : F/G \to F/H$. Thus, $\theta : F \to F$ is an automorphism such that $\theta(G) = H$. □

Next lemma is a generalization of [1, Lemma 3.1]. We recall that a square matrix over a ring R is called a *transvection* if its diagonal entries are all ones and there is at most one nonzero entry off the diagonal. We say that a ring R has *stable range one* if $\alpha R + \beta R = R$ for $\alpha, \beta \in R$ implies $(\alpha + \beta\gamma)R = R$ for some $\gamma \in R$.

Lemma 2.2. *Let R be a semilocal ring. Then any matrix of determinant* 1 *over R is a product of transvections.*

Proof. Suppose that R is a semilocal ring. Then R has stable range one by [5, Lemma V.8.2]. Let A be an $n \times n$ matrix of determinant 1 over R. Then the entries of each row of A generate R, and hence A can be reduced to a matrix whose each row and each column contain a unit. Now, the lemma follows from the proof of [1, Lemma 3.1]. □

Next we give a generalization of [6, Lemma 2.2].

Lemma 2.3. *Let R be an integral domain of finite character, F a finitely generated free R-module and I a nonzero ideal of R. If ϕ is an R/I-automorphism of F/IF such that* $\det \phi = 1$, *then ϕ lifts to an automorphism of F.*

Proof. Since R is of finite character, R/I is a semilocal ring. By assumption ϕ can be represented by a matrix of determinant 1. Since R/I is a semilocal ring, matrices of determinant 1 are products of transvections by Lemma 2.2. So, ϕ is a product of transvections with each transvection lifting to an automorphism of F; the product of these lifted automorphisms is an automorphism. □

Proposition 2.4. *Let R be an almost local–global integral domain and P a finitely generated projective R-module of finite rank. Then P is isomorphic to a finite rank free module direct sum with an invertible ideal.*

Proof. Let J and I be invertible ideals of R. Since R is almost local–global, R/IJ is local–global. Since I is invertible, by [5, Proposition V.4.4], I/IJ is cyclically generated. Since J is invertible, by [5, Proposition I.2.1], $[IJ : J] = I$, and hence $\mathrm{Ann}_{R/IJ}(I/IJ) = I/IJ$. Thus, $I/IJ \cong R/I$. By [7, Proposition 1], $I \oplus J \cong R \oplus IJ$. Since R is almost local-global, by [7, Theorem 3], R has the UCS-property, and hence the BCS-property. So, by [11, Lemma 1.6], P is isomorphic to the direct sum of rank one projective R-modules, and so the direct sum of invertible (integral) ideals of R, say $P = I_1 \oplus \cdots \oplus I_n$. If $n = 2$, then $P = I_1 \oplus I_2 \cong R \oplus I_1 I_2$. If $n = 3$, then $P = (I_1 \oplus I_2) \oplus I_3 \cong R \oplus (I_1 I_2 \oplus I_3) \cong R \oplus R \oplus I_1 I_2 I_3$. By continuing this way, we conclude that P is a direct sum of a finite rank free module with an invertible ideal. □

In [6, Lemma 2.3], given two finitely generated free modules of the same rank F_1 and F_2, over an h-local domain R, it is investigated when, for submodules $G \subseteq F_1$ and $H \subseteq F_2$ of full rank, an isomorphism $F_1/G \cong F_2/H$ lifts to an isomorphism $F_1 \oplus R \cong F_2 \oplus R$. Now, we are ready to prove a generalization of [6, Lemma 2.3]: given a finitely generated free module F and a finitely generated projective module P of the same rank, over an integral domain of finite character R, we investigate when, for submodules $G \subseteq F$ and $H \subseteq P$ of full rank, an isomorphism $F/G \cong P/H$ lifts to an isomorphism $F \oplus R \cong P \oplus A$, for some projective rank one R-module A.

Lemma 2.5. *Let R be an integral domain of finite character, F a finitely generated free R-module of rank n, and P a finitely generated projective R-module of rank n. Suppose that G is an R-submodule of F with rank n and H is a rank n R-submodule of P. If $F/G \cong P/H$, then there is an isomorphism $\alpha : F \oplus R \to P \oplus A$ for some rank one projective R-module A such that $\alpha(G \oplus R) = H \oplus A$.*

Proof. Let R be an integral domain of finite character. Then, by Proposition 2.4, $P \cong R^{(n-1)} \oplus J$ for some invertible ideal J of R. Since $J \oplus J^{-1} \cong R \oplus JJ^{-1} \cong R^{(2)}$, $P \oplus J^{-1}$ is a free R-module of rank $n+1$. Let $F' = F \oplus R$, $P' = P \oplus A$, where $A = J^{-1}$. Since both F' and P' have the same rank, there is an isomorphism $\psi : P' \to F'$. Let $I = \mathrm{Ann}_R(F/G)$.

We first claim that P/IP is a free R/I-module of rank n. Since P is a finitely generated projective R-module of rank n, P_M is a free R_M-module of rank n for all maximal ideals M of R containing I, and hence $(P/IP)_M \cong P_M/IP_M$ is a free R_M/I_M-module of rank n. So, P/IP is a projective R/I-module which is locally free of constant rank, and hence, by [9, Theorem IV.30], P/IP is free of rank n. So, there is an isomorphism $\beta : P/IP \to F/IF$.

Let ϕ be the isomorphism from F/G onto P/H. We note that $IP \subseteq H$ and $\beta(H/IP) = H_1/IF$ for some submodule H_1 of F such that $IF \subseteq H_1$, so that β induces an isomorphism $\bar{\beta} : P/H \to F/H_1$. So, there are two exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H/IP \arrow[r] & P/IP \arrow[r] \arrow[d, "\beta"] & P/H \arrow[r] \arrow[d, "\bar{\beta}"] & 0 \\
0 \arrow[r] & H_1/IF \arrow[r] & F/IF \arrow[r] & F/H_1 \arrow[r] & 0
\end{tikzcd}$$

where $\bar{\beta}$ and β are isomorphisms. Hence, $\bar{\beta} \circ \phi : F/G \to F/H_1$ is an isomorphism. By Lemma 2.1, $\bar{\beta} \circ \phi$ lifts to an automorphism λ of F/IF such that $\lambda(G/IF) = H_1/IF$. So, $\theta = \beta^{-1} \circ \lambda : F/IF \to P/IP$ is an isomorphism such that $\theta(G/IF) = H/IP$.

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Since J^{-1}/IJ^{-1} is a free R/I-module, there is an isomorphism $\gamma : R/I \to J^{-1}/IJ^{-1}$. Hence, $\theta \oplus \gamma : F'/IF' \to P'/IP'$ is an isomorphism. Also, the isomorphism $\psi : P' \to F'$ induces an isomorphism $\bar{\psi} : P'/IP' \to F'/IF'$, so that $\bar{\psi} \circ (\theta \oplus \gamma)$ is an automorphism of F'/IF'. Since $u = \det(\bar{\psi} \circ (\theta \oplus \gamma))$ is a unit in R/I, we can define an automorphism $(1 \oplus u^{-1})$ on $F'/IF' = F/IF \oplus R/I$. Hence, $\bar{\psi} \circ (\theta \oplus \gamma) \circ (1 \oplus u^{-1})$ is an automorphism of F'/IF' of determinant 1. By Lemma 2.3, this automorphism lifts to an automorphism δ of F'. Let $\alpha = \psi^{-1} \circ \delta$. Then $\alpha(IF') = IP'$. We note that, modulo I, α induces $\bar{\alpha} = \bar{\psi}^{-1} \circ \bar{\psi} \circ (\theta \oplus \gamma) \circ (1 \oplus u^{-1}) = \theta \oplus (\gamma \circ u^{-1})$ such that $\bar{\alpha}(G/IF \oplus R/I) = H/IP \oplus J^{-1}/IJ^{-1}$. Therefore, $\alpha(G \oplus R) = H \oplus J^{-1}$. □

3. Main Results

The following two lemmas are the main tools which are used widely in a series of results to compare local, stable, and near isomorphism of torsionless modules over integral domains of finite character.

Lemma 3.1. *Let R be an integral domain of finite character and G a torsionless R-module. If G_M is a free R_M-module for all maximal ideals M of R, then G is a projective R-module, and if the* $\operatorname{rank}(G) \geq 2$, *then G is isomorphic to a direct sum of a free R-module and an invertible ideal ideal of R.*

Proof. If G is a torsionless R-module, then there is an embedding $\rho : G \to R^{(n)}$. If G_M is free for all maximal ideals M of R, then G_M is finitely generated for all M. Hence, each coordinate of $\rho(G)$, which is an ideal of R, is finitely generated and proper in the localizations of R at only finitely many maximal ideals of R since R is of finite character. Hence, by [5, Lemma V.2.11], each coordinate becomes a finitely generated ideal in R. Thus, $\rho(G)$ is a finitely generated submodule of $R^{(n)}$, and so, G is finitely generated. Since G_M is free of constant rank for all M of R, G is projective. If G has rank one, then G is isomorphic to an invertible ideal of R. If $\operatorname{rank}(G) \geq 2$, then, by Proposition 2.4, G is isomorphic to a direct sum of a free R-module and a projective R-module, that is an invertible ideal of R. □

Lemma 3.2. *Let R be an integral domain of finite character and F a finitely generated free module of rank n. If G is a rank n R-submodule of F and H is a torsionless R-module such that either H is nearly isomorphic to G, or G is finitely generated and H is locally isomorphic to G, then there exists a finitely generated projective R-submodule P of QH such that $H \subseteq P$ and $F/G \cong P/H$.*

Proof. Since F is a free module of rank n, $F \cong R^{(n)}$. Then $aF \subseteq G$ for some nonzero element $a \in R$ because F is finitely generated and $QF = QG$. Since H is torsionless and locally isomorphic to G (because nearly isomorphic implies locally isomorphic), H is of rank n and a submodule of a finitely generated free module. Thus, there exists an injection from H into F, so we can assume that $H \subseteq F$. Since

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$H \cong aH \subseteq aF \subseteq G$, we can further assume that $H \subseteq aF \subseteq G \subseteq F$. Since H has rank n, $QF = QH$. So, there is some nonzero $b \in R$ such that $bF \subseteq H$. Since R is of finite character, b is contained in finitely many maximal ideals of R, say $M_1, \ldots, M_t$. Let $S = R - \bigcup_{i=1}^{t} M_i$.

If G and H are nearly isomorphic, then there is an injection $\psi : G \to H$ such that $\mathrm{Ann}(H/\psi(G))$ and Rb are comaximal. So, localizing these ideals at M_i, for each $i = 1, \ldots, t$, we get $H_{M_i} = (\psi(G))_{M_i}$, and hence ψ_{M_i} is surjective for all i. Since ψ is injective, so is the homomorphism $S^{-1}\psi : S^{-1}G \to S^{-1}H$. Since $S^{-1}M_1, S^{-1}M_2, \ldots, S_{M_t}$'s are precisely the maximal ideals of $S^{-1}R$ and ψ_{M_i} is surjective for each i, $S^{-1}\psi$ is surjective. Therefore, $S^{-1}\psi$ is an isomorphism. Since $\psi(aF) \subseteq aF$ implies $\psi(F) \subseteq F$, $S^{-1}\psi(S^{-1}F) \subseteq S^{-1}F$.

Suppose that G is finitely generated and H is locally isomorphic to G. Let $\phi_i : G_{M_i} \to H_{M_i}$ be an isomorphism for each i. Since G is finitely generated, there exists a $b_i \notin M_i$ such that, taking $\psi_i = b_i\phi_i$, $\psi_i(G) \subseteq H$, and $(\psi_i)_{M_i}$ is an isomorphism from G_{M_i} to H_{M_i}. For each i, there exists $r_i \in M_1 \cdots M_{i-1} \cdot M_{i+1} \cdots M_t - M_i$. Let $\psi = r_1\psi_1 + \cdots + r_t\psi_t$. Note that $(r_i\psi_i)_{M_i}(G_{M_i}) = H_{M_i}$ since r_i is a unit in R_{M_i}. For $j \neq i$, $(r_j\psi_j)_{M_i}(G_{M_i}) \subseteq M_iH_{M_i}$. So, $\psi_{M_i} = (r_1\psi_1)_{M_i} + \cdots + (r_t\psi_t)_{M_i}$ induces a surjective map $\overline{\psi_{M_i}} : G_{M_i}/M_iG_{M_i} \to H_{M_i}/M_iH_{M_i}$. By Nakayama's Lemma, $\psi_{M_i} : G_{M_i} \to H_{M_i}$ is surjective. We note that $S^{-1}\psi : S^{-1}G \to S^{-1}H$ is an R-module homomorphism. Since the maximal ideals of $S^{-1}R$ are precisely $S^{-1}M_1, \ldots, S^{-1}M_t$ and ψ_{M_i} is surjective for each i, $S^{-1}\psi$ is surjective. Now G and H are torsionless R-modules of the same (finite) rank. So tensoring up to Q gives a surjection of vector spaces of the same (finite) rank. Thus, $S^{-1}\psi$ must be injective. As in the case of the previous paragraph, $S^{-1}\psi(S^{-1}F) \subseteq S^{-1}F$. Since we can extend ψ to an automorphism of QF, we denote this automorphism (and all of its restrictions) just by ψ.

Let $F' = \psi(S^{-1}F)$ and $P = F' \cap F_b$. For a maximal ideal M not containing b, $(F_b)_M = F_M$, and $F_M \subseteq (F')_M$, so $P_M = F_M$ is free. For each maximal ideal M_i, $i = 1, 2, \ldots, t$, containing b, $F_{M_i} \subseteq (F_b)_{M_i}$, and $(F')_{M_i} = \psi(F_{M_i}) \subseteq F_{M_i}$, so that $P_{M_i} = \psi(F_{M_i})$ is free. So, P is locally free of finite constant rank. Since $P_M \subseteq F_M$ for every maximal ideal M, it follows that P is torsionless, so P is projective by Lemma 3.1. Since $H \subseteq F \subseteq F_b$ and $H \subseteq S^{-1}H = \psi(S^{-1}G) \subseteq F'$, $H \subseteq P$.

Since $bF \subseteq H \subseteq G$, $b(F/G) = 0$. Also, for all $s \in S$, there is no maximal ideal of R that contains both s and b. Thus, $Rb + Rs = R$ for all such $s \in S$. Let $\eta : F/G \to S^{-1}(F/G)$ be the natural map. Then η is an isomorphism. So, $S^{-1}(F/G) \cong (S^{-1}F)/(S^{-1}G) \cong F'/(S^{-1}H)$. Similarly, $P/H \cong (S^{-1}P)/(S^{-1}H) \cong (S^{-1}F' \cap S^{-1}F_b)/(S^{-1}H)$. Now $S^{-1}F' = F' = \psi(S^{-1}F) \subseteq S^{-1}F \subseteq S^{-1}F_b$. Thus, $P/H \cong F'/(S^{-1}H)$, which is isomorphic to F/G. □

Lemma 3.2 is a generalization of [6, Lemma 2.6]. As the proof of [6, Lemma 2.6] notes, over h-local domains, one does not need that G is finitely generated, when G and H are locally isomorphic. At this point, we know of no example to show that

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the assumption of G being finitely generated is necessary, over integral domains of finite character.

In [6, Proposition 2.8], two locally isomorphic torsionless modules over an h-local domain with a trivial Picard group are proven to be isomorphic when one of them contains a direct summand isomorphic to a nonzero ideal. Next we prove this in the more general case in which the domain has finite character and trivial Picard group. Moreover, we see that the finitely generated assumption is not needed if these modules are nearly isomorphic.

Proposition 3.3. *Let R be an integral domain of finite character with trivial Picard group. Suppose that G and H are torsionless R-modules such that either H is nearly isomorphic to G, or G is finitely generated and H is locally isomorphic to G. If G has a direct summand isomorphic to a nonzero ideal of R, then $G \cong H$.*

Proof. Let $G = A \oplus Je$, where A has rank $n-1$ and $J \neq 0$ is an ideal of R. Then let $A \subseteq F' \subseteq QA$ for some free R-module F' of rank $n-1$. Let $F = F' \oplus Re$ be a free R-module with $G \subseteq F$. By Lemma 3.2, there is a projective torsionless R-submodule P of QH such that $F/G \cong P/H$.

Since R has the BCS-property, P is the direct sum of rank one projective modules. So, P has to be free because R has trivial Picard group. Thus, $F \cong P$, and hence we may assume that $H \subseteq F$ and $F/G \cong F/H$. We note that $\mathrm{Ann}_R(F/G) = \mathrm{Ann}_R(F/H)$. Let $I = \mathrm{Ann}_R(F/G)$. Since F/G is finitely generated and torsion, $I \neq 0$, implying that F/G and F/H are isomorphic as R/I-modules. Since R is of finite character, R/I is a semi-local ring. We note that F/IF is a finitely generated free R/I-module with $F/G \cong (F/IF)/(G/IF) \cong F/H \cong (F/IF)/(H/IF)$, so by Lemma 2.1, there exists an automorphism $\phi\colon F/IF \to F/IF$ such that $\phi(G/IF) = H/IF$. Thus, $u = \det\phi$ is a unit in R/I. Since $F = F' \oplus Re$, $F/IF = F'/IF \oplus Re/Ie$. We can define an isomorphism $\theta : F'/IF \oplus Re/Ie \to F'/IF \oplus R/I$ such that $\theta = 1 \oplus u^{-1}$, where 1 is the identity map on F'/IF and u^{-1} is the multiplication by u^{-1} from Re/Ie to R/I. Let $\psi = \phi \circ \theta$, then $\det\psi = u \cdot u^{-1} = 1$. Also, $\psi(G/IG) = \phi(\theta(A/IF' \oplus Je/Ie)) = \phi(A/IF' \oplus Je/Ie) = \phi(G/IG) = H/IH$. So, by Lemma 2.3, ψ lifts to an R-automorphism Ψ of F. Therefore, $\Psi(IF) = I\Psi(F) = IF$, and $\psi(G/IG) = H/IH$, where $\psi(x+IF) = \Psi(x)+IF$ for each $x \in F$, which implies that $\Psi(G) = H$. $\square$

Lemma 3.4. *Let R be an almost local-global domain. If I is an ideal of R and $n > 0$, then $I^n \cong R$ if and only if $I^{(n)} \cong R^{(n)}$.*

Proof. Suppose that $I^n \cong R$. Then I^n is a principal ideal of R, say $I^n = Ra$ for some $a \in I$. So, $I^n a^{-1} = R$ implies that $I \cdot I^{n-1}a^{-1} = R$. Thus, I is invertible, and hence finitely generated and projective, so is $I^{(n)}$. Since $I^{(n)}$ is a torsionless R-module and locally free at every maximal ideal of R, by Lemma 3.1, $I^{(n)} \cong R^{(n-1)} \oplus J$ for some invertible ideal J. By [8, Lemma 1], $I^n \cong J$, and hence $J \cong I^n \cong R$. The converse follows directly from [8, Lemma 1]. $\square$

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In [6, Proposition 2.10], given two locally isomorphic torsionless modules, G and H, over an h-local domain with a torsion Picard group, it is proven that $G^{(n)} \cong H^{(n)}$ for some $n > 0$. Next we prove this when one of these modules is finitely generated over an integral domain of finite character with a torsion Picard group. Moreover, we see that the finitely generated assumption is not needed if these modules are nearly isomorphic.

Proposition 3.5. *Let R be an integral domain of finite character with torsion Picard group. Suppose that G and H are torsionless R-modules such that either H is nearly isomorphic to G, or G is finitely generated and H is locally isomorphic to G. Then there exists $n > 0$ such that $G^{(n)} \cong H^{(n)}$.*

Proof. By Lemma 3.2, there exist finitely generated projective R-modules P_1 and P_2 such that $P_1/G \cong P_2/H$. By Lemma 3.1, P_1 and P_2 are each isomorphic to a direct sum of a free R-module and an invertible ideal of R. Thus, by Lemma 3.4 and the assumption that $Pic(R)$ is torsion, there exists $k > 0$ such that $P_1^{(k)}$ and $P_2^{(k)}$ are free R-modules of the same rank, and so isomorphic. Let $F = P_1^{(k)}$, $A = G^{(k)}$ and $B = H^{(k)}$.

As in the proof of Lemma 3.2, we may reduce to the case that A and B are submodules of F and $F/A \cong F/B$. Let $I = \mathrm{Ann}_R(F/A)$. By Lemma 2.1, there exists an automorphism $\phi : F/IF \to F/IF$ such that $\phi(A/IA) = B/IB$. Set $u = \det\phi$, and let m be the rank of F/IF as a free R/I-module. Since u is a unit in R/I, we may define an automorphism $\Psi : F^{(m)}/IF^{(m)} \to F^{(m)}/IF^{(m)}$ by $\Psi(x_1, \ldots, x_m) = (u^{-1}\Phi(x_1), \Phi(x_2), \ldots, \Phi(x_m))$ for all $(x_1, \ldots, x_m) \in F^{(m)}/IF^{(m)}$. Then $\det \Psi = u^{-m}(\det \Phi)^m = u^{-m}u^m = 1$, and hence, by Lemma 2.3, Ψ lifts to an automorphism of $F^{(m)}$ with $\Psi(A^{(m)}) = B^{(m)}$. □

Our next result states that two torsionless modules are locally isomorphic if and only if they are stably isomorphic over an integral domain of finite character with a trivial Picard group when one of the modules is finitely generated, and hence, generalizes [6, Theorem 2.11] under this finitely generated assumption.

Theorem 3.6. *Let R be an integral domain of finite character with trivial Picard group. Suppose that G and H are rank n torsionless R-modules. If G is finitely generated, then the following are equivalent.*

(a) *G and H are locally isomorphic.*
(b) *$F_1/G \cong F_2/H$ for some free R-modules F_1 and F_2 with $G \subseteq F_1 \subseteq QG$ and $H \subseteq F_2 \subseteq QH$.*
(c) *G and H are stably isomorphic.*
(d) *$G \oplus A \cong H \oplus A$ for some finitely generated R-module A.*
(e) *$G^{(m)} \cong H^{(m)}$ for some $m > 0$.*

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Proof. (a) $\Rightarrow$ (b): By Lemma 3.1 and the assumption that R has trivial Picard group, every finitely generated projective R-module is free. Hence, by Lemma 3.2, $F_1/G \cong F_2/H$ for some free R-modules F_1 and F_2 with $G \subseteq F_1 \subseteq QG$ and $H \subseteq F_2 \subseteq QH$.

(b) $\Rightarrow$ (c): Follows from Lemma 2.5.

(c) $\Rightarrow$ (d): Immediate.

(d) $\Rightarrow$ (a): Follows immediately from the fact that finitely generated modules cancel over local rings [4, Theorem 2.5].

(a) $\Rightarrow$ (e): Proposition 3.5.

(e) $\Rightarrow$ (a): Proved in [4, Theorem 2.11]. □

Corollary 3.7. *Let R be a Bézout domain of finite character. Suppose that G and H are torsionless R-modules and that G is finitely generated. G is locally isomorphic to H if and only if there exists a torsion-free finite rank R-module A such that $G \oplus A \cong H \oplus A$.*

Proof. If G and H are locally isomorphic, then, by Theorem 3.6(c), $G \oplus R \cong H \oplus R$. For the converse, since a Bézout domain is a Prüfer domain, it is locally a valuation domain. Since a torsion-free module over a valuation domain has the cancellation property [12, Theorem 5.4], A cancels locally from the isomorphism $G \oplus A \cong H \oplus A$, so that G is locally isomorphic to H. □

Before we prove when local, stable, and near isomorphism coincide over an integral domain of finite character, we show that two torsionless modules are locally isomorphic if and only if they are nearly isomorphic when one of the modules is finitely generated, and hence, generalize [6, Theorem 2.11] under this finitely generated assumption.

Theorem 3.8. *Let R be an integral domain of finite character. Suppose that G and H are torsionless R-modules of rank n. If G is finitely generated, then the following are equivalent.*

(a) *G and H are locally isomorphic.*
(b) *$F/G \cong P/H$ for some finitely generated projective R-modules F and P with $G \subseteq F \subseteq QG$ and $H \subseteq P \subseteq QH$.*
(c) *$G \oplus R \cong H \oplus J$ for some invertible ideal J of R.*
(d) *$G \oplus A \cong H \oplus B$ for some finitely generated projective R-modules A and B.*
(e) *G is nearly isomorphic to H.*

Proof. (a) $\Rightarrow$ (b): This is Lemma 3.2.

(b) $\Rightarrow$ (c): This is Lemma 2.5.

(c) $\Rightarrow$ (d): Clear.

(d) $\Rightarrow$ (a): Since projective modules over a local domain are free and stably isomorphic modules are locally isomorphic [4, Theorem 2.5], (a) follows.

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(e) $\Rightarrow$ (a): If G is nearly isomorphic to H, then for any maximal ideal M there is an embedding $f : G \to H$ such that $M + \mathrm{Ann}_R(H/f(G)) = R$, and hence $f_M : G_M \to H_M$ is an isomorphism.

(a) $\Rightarrow$ (e): Let $I \neq 0$ be an ideal of R. Since R is of finite character, I is contained in finitely many maximal ideals, say $M_1, \ldots, M_t$. Let $S = R - \cup_{i=1}^{t} M_i$, so that $S^{-1}R$ is a semilocal domain. Moreover, $S^{-1}G$ and $S^{-1}H$ are locally isomorphic $S^{-1}R$-modules of the same rank with $S^{-1}G$ finitely generated. By Theorem 3.6, $S^{-1}G$ and $S^{-1}H$ are stably isomorphic $S^{-1}R$-modules. Then, by semi-local cancellation [5, Corollary V.8.3], $\psi : S^{-1}G \cong S^{-1}H$.

Since G is finitely generated, there is an element $s \in S$ such that $s\psi(G) \subseteq H$. Hence, we have a short exact sequence $0 \to G \to H \to \mathrm{Coker}(s\psi) \to 0$. Since $s\psi$ is an isomorphism locally at S, we get $S^{-1}(\mathrm{Coker}(s\psi)) = 0$. Now H is finitely generated, so $\mathrm{Coker}(s\psi)$ is as well, and hence $r \cdot \mathrm{Coker}(s\psi) = 0$ for some $r \in S$. Since $r \notin M_i$ for $i = 1, \ldots, t$, it follows that $\mathrm{Ann}_R(\mathrm{Coker}(s\psi))$ is relatively prime to I. $\square$

Corollary 3.9. *Let R be an integral domain of finite character with trivial Picard group. Suppose that G and H are rank n torsionless R-modules. If G is finitely generated, then the following are equivalent.*

(a) *G and H are locally isomorphic.*
(b) *G and H are nearly isomorphic.*
(c) *G and H are stably isomorphic.*

Proof. Follows immediately from Theorems 3.6 and 3.8. $\square$

4. Applications

In this section, we provide some applications of the results in the previous section.

Proposition 4.1. *Let R be an integral domain of finite character such that for almost all maximal ideals M of R, R_M is a valuation domain. Let A, B, and C be torsionless R-module. Suppose A is finitely generated and every homomorphic image of A is finitely presented. If A and $B \oplus C$ are locally isomorphic, then $A = B' \oplus C'$ for some submodules B' and C' locally isomorphic to B and C, respectively.*

Proof. Since A is a torsionless R-module, we may assume that there exists a free F such that $A \subseteq F \subseteq QA$. Since R is of finite character, $\mathrm{Ann}_R(F/A)$ is contained in only finitely many maximal ideals of R. We note that A_M, B_M and C_M are free R_M-modules for all but finitely many maximal ideals of R. Let $\mathcal{N}$ be the set of maximal ideals N of R such that A_N is a free R_N-module and R_N is a valuation domain. Let $\mathcal{M}$ be the set of maximal ideals of R not contained in $\mathcal{N}$. If $\mathcal{M}$ is empty, then by Lemma 3.1, the result follows. Otherwise, if $\mathcal{M}$ is not empty, we write $\mathcal{M} = \{M_1, \ldots, M_t\}$ and let $J = M_1 \cdots M_t$. Then, by Theorem 3.8, there is an embedding $f : A \to B \oplus C$ such that J and $\mathrm{Ann}_R(\mathrm{Coker}(f))$ are

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relatively prime. Let $g = \pi \circ f$, where π is the projection map $\pi : B \oplus C \to B$, and $I = \mathrm{Im}(g)$ and $K = \mathrm{Ker}(g)$. We claim that I and K are locally isomorphic to B and C, respectively. We note that f_{M_i} is surjective for each index i because J and $\mathrm{Ann}_R(\mathrm{Coker}(f))$ are relatively prime. Then $I_{M_i} = B_{M_i}$, for each i, since $I_{M_i} = g(A)_{M_i} = g_{M_i}(A_{M_i}) = (\pi \circ f)_{M_i}(A_{M_i}) = \pi_{M_i}(B_{M_i} \oplus C_{M_i}) = B_{M_i}$. Thus, $\mathrm{rank}(I) = \mathrm{rank}(B)$. Since R_N is a valuation domain for all $N \in \mathcal{N}$ and I_N and B_N are finitely generated torsion-free, hence free, R_N-modules of the same rank, it follows that I and B are locally isomorphic.

Next, we claim that the exact sequence $0 \to K \to A \to I \to 0$ splits. Since R_N is a valuation domain for all maximal ideals $N \in \mathcal{N}$ and I_N is finitely generated torsion-free, I_N is projective, so the map g_N splits. On the other hand, for each $M_1, \ldots, M_t$, $g_{M_i} = \pi_{M_i} o f_{M_i}$, where f_{M_i} is an isomorphism, while π_{M_i} is a split surjection, so that the map g_{M_i} splits. By assumption, $I = \mathrm{Im}(g)$ is finitely presented, so the map g splits [2, Exercise 2.13]. Thus, $A \cong I \oplus K$, where I is locally isomorphic to B. By [3, Theorem 2], locally at each maximal ideal of R, we can cancel those summands from both sides of the isomorphism, so K is locally isomorphic to C. □

We note that an example of a ring satisfying the hypotheses of the previous proposition is a Prüfer domain of finite character. Since R is a Prüfer domain, R is coherent. So, by [5, Lemma IV.2.5], any torsionless module over R is coherent, and hence any homomorphic image of a finitely presented module is finitely presented.

Corollary 4.2. *Let R be as in Proposition* 4.1. *If G is a finitely generated torsionless R-module, and every homomorphic image of G is finitely presented then G is indecomposable if and only if G is locally isomorphic to an indecomposable torsionless R-module.*

Proof. Immediately follows from Proposition 4.1. We just note that, following the proof of Proposition 4.1, $I = G$, which is finitely presented. □

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References

[1] S. U. Chase, Torsion-free modules over $K[X, Y]$, *Pacific J. Math.* **12** (1962) 437–447.
[2] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, (Springer, NY, 1994).
[3] E. G Evans, Krull-Schmidt and cancellation over local rings, *Pacific J. Math.* **46** (1973) 115–121.
[4] D. R. Estes and R. M. Guralnick, Module Equvalences: Local to global when primitive polynomials represent units, *J. Algebra* **77**(1) (1982) 138–157.

[5] L. Fuchs and L. Salce, *Modules Over Non-Noetherian Domains*, Math. Surveys Monographs, Vol. 84 (Amer. Math. Soc., New York, 2001).

[6] B. Olberding and P. Goeters, On locally isomorphic torsion-free modules, *Commutative Rings* (Nova Sci. Publ., Hauppauge, NY, 2002), pp. 83–94.

[7] J. Brewer and L. Klingler, Pole assignability and the invariant factor theorem in Prüfer domains, *J. Algebra* **111**(2) (1987) 536–545.

[8] I. Kaplansky, Modules over Dedekind rings and valuation rings, *Trans. AMS* **72** (1952) 327–340.

[9] B. R. McDonald, *Linear Algebra Over Commutative Rings* (Marcel Dekker, NY, 1984).

[10] W. V. Vasconcelos, On local and stable cancellation, *An. Acad. Brasil. Ci.* **37** (1965) 389–393.

[11] W. V. Vasconcelos and C. A. Weibel, BSC rings, *J. Pure Appl. Algebra* **52** (1988) 173–185.

[12] R. B Jr Warfield, Cancellation of modules and groups and stable range of endomorphism rings, *Pacific J. Math.* **91** (1980) 457–485.

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