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RELATIVISTIC BURGERS AND NONLINEAR SCHRÖDINGER EQUATIONS

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We construct relativistic complex Burgers–Schrödinger and nonlinear Schrödinger equations. In the nonrelativistic limit, they reduce to the standard Burgers and nonlinear Schrödinger equations and are integrable through all orders of relativistic corrections.

Keywords: semirelativistic nonlinear Schrödinger equation, relativistic Burgers–Schrödinger equation, nonlinear Schrödinger hierarchy, relativistic dispersion, recursion operator

1. General Burgers–Schrödinger hierarchy

The relativistic linear Schrödinger equation was discussed in the early stages of quantum mechanics but was subsequently replaced by the Klein–Gordon and the Dirac equations. Relativistic versions of the Schrödinger equation were recently considered in the study of relativistic quark–antiquark bound states [1] and the gravitational collapse of a boson star [2]. A nonlinear version of the model appeared in the form of the semirelativistic Hartree-Fock equation [3]. But none of those models is known to be integrable. Here, we construct an integrable relativistic nonlinear Schrödinger (NLS) equation, preserving the integrability through all orders of the $1/c$ expansion.

We consider the Schrödinger equation in 1+1 dimensions for a free particle with classical dispersion of the general analytic form $E = E(p)$

$$i\hbar \frac{\partial \Psi}{\partial t} = \mathcal{H}(P_1)\Psi, \tag{1}$$

where $P_0 = i\hbar\partial/\partial t$ and $P_1 = -i\hbar\partial/\partial x$ are the respective operators of time and space translations commuting with the Schrödinger operator $S = i\hbar\partial/\partial t - \mathcal{H}(P_1)$: $[P_\mu, S] = 0$, $\mu = 0, 1$. The general boost operator, defined as $K = x - t\mathcal{H}'(P_1)$, also commutes with S: $[K, S] = 0$. Commuting it with the space and time translations, we obtain the algebra of symmetry operators

$$[P_0, P_1] = 0, \qquad [P_0, K] = -\hbar\mathcal{H}'(P_1), \qquad [P_1, K] = -i\hbar. \tag{2}$$

If Ψ is a solution of (1) and W is an operator from this algebra, i.e., $[W, S] = 0$, then $W\Psi$ is also a solution of (1).

For a given classical dispersion $E = E(p)$, $E_0 \equiv E(0)$, we define E-polynomials $H_n^{(E)}(x,t)$ by the generating function

$$\exp\left[\frac{i}{\hbar}\bigl(px - \bigl(E(p) - E_0\bigr)t\bigr)\right] = \sum_{n=0}^{\infty}\left(\frac{i}{\hbar}\right)^n \frac{p^n}{n!} H_n^{(E)}(x,t) \tag{3}$$

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or, equivalently,

$$H_n^{(E)}(x,t) = \exp\left[-\frac{i}{\hbar}\left(\mathcal{H}\left(-i\hbar\frac{\partial}{\partial x}\right) - E_0\right)t\right]x^n. \tag{4}$$

The polynomial $H_n^{(E)}(x,t)$ is a solution of

$$i\hbar\frac{\partial}{\partial t}H_n^{(E)}(x,t) = (\mathcal{H} - E_0)H_n^{(E)}(x,t) \tag{5}$$

with the initial value $H_n^{(E)}(x,0) = x^n$. From commutativity, $[S,K] = 0$, the time evolution of the operator K satisfies the equation

$$i\hbar\frac{\partial K}{\partial t} = [\mathcal{H}, K] \tag{6}$$

and has the form

$$K(t) = e^{-i\mathcal{H}t/\hbar}K(0)e^{i\mathcal{H}t/\hbar} = e^{-i\mathcal{H}t/\hbar}xe^{i\mathcal{H}t/\hbar}.$$

The operator K consequently generates an infinite hierarchy of polynomials:

$$KH_n(x,t) = K\exp\left[-\frac{i}{\hbar}(\mathcal{H} - E_0)t\right]x^n = \exp\left[-\frac{i}{\hbar}(\mathcal{H} - E_0)t\right]x^{n+1} = H_{n+1}(x,t). \tag{7}$$

1.1. Nonrelativistic Schrödinger equation. The nonrelativistic dispersion law $E(p) = p^2/2m$ implies the Hamiltonian operator

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \tag{8}$$

and the Gallilean boost operator

$$K = x + it\frac{\hbar}{m}\frac{\partial}{\partial x}. \tag{9}$$

From the generating function

$$\exp\left[\frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)\right] = \sum_{n=0}^{\infty}\left(\frac{i}{\hbar}\right)^n\frac{p^n}{n!}H_n^{(\mathrm{S})}(x,t), \tag{10}$$

we have the Schrödinger polynomials

$$H_n^{(\mathrm{S})}(x,t) = \exp\left[\frac{i}{\hbar}t\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right]x^n. \tag{11}$$

If $H_n^{(\mathrm{KF})}(x,t) = e^{td^2/dt^2}x^n$ is the Kampe de Feriet polynomial, then $H_n^{(\mathrm{S})}(x,t) = H^{(\mathrm{KF})}(x, i\hbar t/2m)$ or, in terms of the Hermite polynomial,

$$H_n^{(\mathrm{S})}(x,t) = \left(-\frac{i\hbar}{2m}t\right)^{n/2}H_n\left(\frac{x}{\sqrt{-2i\hbar t/m}}\right). \tag{12}$$

1.2. Semirelativistic Schrödinger equation. The relativistic dispersion $E(p) = \sqrt{m^2c^4 + c^2p^2}$ implies the Hamiltonian

$$\mathcal{H} = mc^2\sqrt{1 - \frac{\hbar^2}{m^2c^2}\frac{\partial^2}{\partial x^2}} \tag{13}$$

and the semirelativistic boost operator

$$K = x + \frac{i\hbar}{m}t\frac{\partial/\partial x}{\sqrt{1 - (\hbar^2/m^2c^2)(\partial^2/\partial x^2)}}. \tag{14}$$

In the nonrelativistic limit as $c \to \infty$, it reduces to Galilean boost (9). The generating function in the form of a relativistic plane wave

$$\exp\left[\frac{i}{\hbar}\left(px - \left(\sqrt{m^2c^4 + c^2p^2} - mc^2\right)t\right)\right] = \sum_{n=0}^{\infty}\left(\frac{i}{\hbar}\right)^n \frac{p^n}{n!} H_n^{(\mathrm{SRS})}(x,t) \tag{15}$$

then gives the semirelativistic polynomials

$$H_n^{(\mathrm{SRS})}(x,t) = \exp\left[-\frac{i}{\hbar}mc^2t\left(\sqrt{1 - \frac{\hbar^2}{m^2c^2}\frac{\partial^2}{\partial x^2}} - 1\right)\right]x^n. \tag{16}$$

In the nonrelativistic limit, $H_n^{(\mathrm{SRS})} \to H_n^{(\mathrm{S})}$. The first three polynomials coincide exactly with the Schrödinger polynomials,

$$H_1^{(\mathrm{SRS})}(x,t) = x, \qquad H_2^{(\mathrm{SRS})}(x,t) = x^2 + i\frac{\hbar}{m}t, \qquad H_3^{(\mathrm{SRS})} = x^3 + i\frac{\hbar}{m}3xt,$$

while starting from the fourth one,

$$H_4^{(\mathrm{SRS})}(x,t) = x^4 + i\frac{\hbar}{m}6x^2t - \frac{\hbar^2}{m^2}3t^2 + i\frac{\hbar^3}{m^3c^2}3t, \tag{17}$$

we have relativistic corrections of the order $1/c^2$. For the complex-valued space coordinate x, as in the (2+1)-dimensional Chern–Simons theory [4], the zeros of these polynomials describe a motion of point vortices in the plane. The equations of motion for N vortices are

$$\dot{x}_k = \frac{i}{\hbar}\operatorname{Res}\Big|_{x=x_k}\mathcal{H}\left[\frac{\hbar}{i}\left(\frac{\partial}{\partial x} + \sum_{l=1}^{N}\frac{1}{x - x_l}\right)\right]\cdot 1, \quad k = 1, 2, \ldots, N. \tag{18}$$

1.3. Relativistic Burgers–Schrödinger equation. Using the Schrödinger $\log\Psi$ transformation [5] $\Psi = e^{\log\Psi}$ and the identity

$$e^{-\log\Psi}\frac{\partial^n}{\partial x^n}e^{\log\Psi} = \left(\frac{\partial}{\partial x} + \frac{\partial\log\Psi}{\partial x}\right)^n \cdot 1, \tag{19}$$

we can rewrite Schrödinger equation (1) in the form

$$i\hbar\frac{\partial}{\partial t}\log\Psi = \mathcal{H}\left[-i\hbar\left(\frac{\partial}{\partial x} + \frac{\partial\log\Psi}{\partial x}\right)\right]\cdot 1. \tag{20}$$

For the complex function $\Psi = e^{iF/\hbar} = e^{R+iS/\hbar}$, we introduce a new complex function

$$V = -i\frac{\hbar}{m}\frac{\partial}{\partial x}\log\Psi = \frac{1}{m}\frac{\partial}{\partial x}F$$

with the dimension of velocity. Then $F = S - i\hbar R$ is a complex potential, whose real and imaginary parts are the classical and quantum velocities, $V = V_{\mathrm{c}} + iV_{\mathrm{q}} = S_x/m - i\hbar R_x/m$. Hence, (20) takes the complex Hamilton–Jacobi form (quantum Hamilton–Jacobi equation)

$$\frac{\partial F}{\partial t} + \mathcal{H}\left(-i\hbar\frac{\partial}{\partial x} + F_x\right)\cdot 1 = 0. \tag{21}$$

In the classical (dispersionless) limit $\hbar \to 0$, the quantum velocity V_{q} vanishes, and the complex potential F reduces to the real velocity potential S, playing the role of Hamilton's principal function. In this case, (21) becomes the classical Hamilton–Jacobi equation $\partial S/\partial t+\mathcal{H}(\partial S/\partial x)=0$. Differentiating (20), we obtain the equation for the complex velocity

$$i\hbar\frac{\partial V}{\partial t}=-i\frac{\hbar}{m}\frac{\partial}{\partial x}\left[\mathcal{H}\left(-i\hbar\frac{\partial}{\partial x}+mV\right)\cdot 1\right]. \tag{22}$$

This equation is the Madelung fluid-type representation of Schrödinger equation (1). In the classical limit, it gives the Newton equation $m\partial V_\mathrm{c}/\partial t=-\partial[\mathcal{H}(mV_\mathrm{c})]/\partial x$ or, in the hydrodynamic-type form,

$$\frac{\partial V_\mathrm{c}}{\partial t}+\mathcal{H}'(mV_\mathrm{c})\frac{\partial V_\mathrm{c}}{\partial x}=0, \tag{23}$$

which is just a differentiation of the classical Hamilton–Jacobi equation. Equation (23) has the implicit general solution $V_\mathrm{c}(x,t)=f\big(x-\mathcal{H}'(mV_\mathrm{c})t\big)$, where f is an arbitrary function, and it develops a singularity at a critical time when the derivative $(V_\mathrm{c})_x$ blows up.

1.3.1. Nonrelativistic Burgers–Schrödinger equation. In this case, Schrödinger equation (1) with nonrelativistic Hamiltonian (8) is equivalent to the nonlinear equation for the complex velocity V

$$i\hbar\frac{\partial V}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2 V}{\partial x^2}-i\hbar V\frac{\partial V}{\partial x}, \tag{24}$$

which we call the Burgers–Schrödinger equation. In terms of the real and imaginary parts, it gives the Madelung fluid with the density $\rho=e^R$ and velocity V_c. In the classical limit, it reduces to one real equation for the classical velocity V_c, namely, the ordinary dispersionless Burgers equation.

1.3.2. Semirelativistic Burgers–Schrödinger equation. For Hamiltonian (13), the "Burgerization" procedure described above (see (22)) gives the semirelativistic Burgers–Schrödinger equation

$$\frac{1}{c}\frac{\partial V}{\partial t}+c\frac{\partial}{\partial x}\left[\sqrt{1+\frac{1}{m^2c^2}\left(-i\hbar\frac{\partial}{\partial x}+mV\right)^2}\cdot 1\right]=0. \tag{25}$$

In the nonrelativistic limit, it reduces to (24) with the relativistic corrections in the first order

$$\begin{aligned}i\hbar\frac{\partial V}{\partial t} = &-\frac{\hbar^2}{2m}\frac{\partial^2 V}{\partial x^2}-i\hbar V\frac{\partial V}{\partial x}+\frac{1}{8m^3c^2}[-\hbar^4 V_{xxxx}]+\\ &+\frac{1}{8m^3c^2}[-im\hbar^3(10V_xV_{xx}+4VV_{xxx})+m^2\hbar^2(12VV_x^2+6V^2V_{xx})+4im^3\hbar V^3V_x].\end{aligned} \tag{26}$$

In the classical (dispersionless) limit, (25) becomes the equation of hydrodynamic type

$$(V_\mathrm{c})_t+\frac{V_\mathrm{c}}{\sqrt{1+V_\mathrm{c}^2/c^2}}(V_\mathrm{c})_x=0. \tag{27}$$

In the nonrelativistic limit, it reduces to the dispersionless Burgers equation with the first relativistic correction

$$(V_\mathrm{c})_t+V_\mathrm{c}(V_\mathrm{c})_x-\frac{1}{2c^2}V_\mathrm{c}^3(V_\mathrm{c})_x=0. \tag{28}$$

The general implicit solution of (27) is

$$V_\mathrm{c}(x,t)=f\left(x-\frac{V_\mathrm{c}t}{\sqrt{1+V_\mathrm{c}^2/c^2}}\right), \tag{29}$$

and it develops a singularity at a finite time.

1.4. Bäcklund transformation for the Burgers–Schrödinger equation. Using the boost transformation (see the definition in the text after (1)), from a given solution Ψ_1 of Schrödinger equation (1), we can generate another solution as

$$\Psi_2 = K\Psi_1 = \left[x - t\mathcal{H}'\left(-i\hbar\frac{\partial}{\partial x}\right)\right]\Psi_1. \tag{30}$$

Using the identity

$$\Psi^{-1}G\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi = G\left(-i\hbar\frac{\partial}{\partial x} + mV\right)\cdot 1, \tag{31}$$

for the complex velocities $V_a = -(i\hbar/m)\log\Psi_a$, $a = 1, 2$, for (22), we obtain the Bäcklund transformation

$$V_2 = V_1 - i\frac{\hbar}{m}\frac{\partial}{\partial x}\log\left[x - tH'\left(-i\hbar\frac{\partial}{\partial x} + mV_1\right)\cdot 1\right]. \tag{32}$$

In the nonrelativistic quantum mechanics in the case of (8), this gives the complex Bäcklund transformation

$$V_2 = V_1 - i\frac{\hbar}{m}\frac{1 - (V_1)_x t}{x - V_1 t} \tag{33}$$

for Burgers–Schrödinger equation (24).

For the semirelativistic quantum mechanics with Hamiltonian (13) for (25), we have the Bäcklund transformation of the form

$$V_2 = V_1 - i\frac{\hbar}{m}\frac{\partial}{\partial x}\log\left(x - \frac{1}{\sqrt{1 + 1/(m^2c^2)(-i\hbar\partial/\partial x + mV_1)^2}}\,V_1 t\right). \tag{34}$$

It is noteworthy that in the classical limit as $\hbar \to 0$ and $V \to V_{\mathrm{c}}$, these Bäcklund transformations reduce to the trivial identity $V_{\mathrm{c}1} = V_{\mathrm{c}2}$.

2. Integrable general NLS hierarchy

In the preceding section, we studied the so-called C-integrable relativistic Burgers–Schrödinger equation. We now use the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy for the NLS equation to construct a relativistic NLS hierarchy.

2.1. The NLS hierarchy. We consider the Zakharov–Shabat linear problem

$$\frac{\partial}{\partial x}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -ip/2 & -\kappa^2\bar{\psi} \\ \psi & ip/2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = J_1\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{35}$$

for the spatial evolution, and the generalized AKNS problem [6]

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -iA & -\kappa^2\overline{C} \\ C & iA \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = J_0\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{36}$$

for the temporal evolution, where we substitute $A_N = \sum_{n=0}^{N} A^{(n)}(-p/2)^n$ and $C_N = \sum_{n=0}^{N} C^{(n)}(-p/2)^n$ for the real $A(x,t,p)$ and complex $C(x,t,p)$ functions, determined by the zero-curvature condition. This gives the evolution equation

$$\partial_{t_N}\psi = \partial_x C^{(0)} + 2iA^{(0)}\psi$$

and the conditions $C^{(N)} = 0$ and $A^{(N)} = a_N = \text{const}$. We fix this constant such that $a_N = (-2)^{N-1}$. We then have the recurrence relations

$$C^{(n)} = \frac{1}{2i}\partial_x C^{(n+1)} + A^{(n+1)}\psi, \qquad \partial_x A^{(n)} = i\kappa^2(\overline{C}^{(n)}\psi - C^{(n)}\bar{\psi}), \quad n = 0, 1, 2, \ldots, N-1.$$

Integrating the last equation, we obtain

$$A^{(n)} = -i\kappa^2 \int^x (\bar{\psi}C^{(n)} - \psi\overline{C}^{(n)}). \tag{37}$$

Substituting (37) in the recurrence formula, we obtain

$$\begin{pmatrix} C^{(n)} \\ \overline{C}^{(n)} \end{pmatrix} = -\frac{1}{2}\mathcal{R}\begin{pmatrix} C^{(n+1)} \\ \overline{C}^{(n+1)} \end{pmatrix}, \tag{38}$$

where $\mathcal{R}$ is a matrix-valued integro-differential operator, which is just the recursion operator for the NLS hierarchy [6]

$$\mathcal{R} = i\sigma_3 \begin{pmatrix} \partial_x + 2\kappa^2\psi\int^x\bar{\psi} & -2\kappa^2\psi\int^x\psi \\ -2\kappa^2\bar{\psi}\int^x\bar{\psi} & \partial_x + 2\kappa^2\bar{\psi}\int^x\psi \end{pmatrix}, \tag{39}$$

and σ_3 is the third Pauli matrix. We then have

$$i\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}_{t_N} = \mathcal{R}^N \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \tag{40}$$

where t_N, $N = 1, 2, 3, \ldots$, is an infinite time hierarchy. In the linear approximation with $\kappa = 0$, the recursion operator is the momentum operator $\mathcal{R}_0 = i\sigma_3\partial/\partial x$, and NLS hierarchy (40) becomes the linear Schrodinger hierarchy

$$i\psi_{t_n} = i^n\partial_x^n\psi. \tag{41}$$

The Madelung representation for this hierarchy, produced by the complex Cole–Hopf transformation, is given by the complex Burgers hierarchy [4].

Every equation of hierarchy (40) is integrable. The linear problem for the Nth equation is given by Zakharov–Shabat problem (35) for the spatial part and the problem

$$\frac{\partial}{\partial t_N}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -iA_N & -\kappa^2\overline{C}_N \\ C_N & iA_N \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = J_{0_N}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{42}$$

for the temporal part. The coefficient functions C_N can be found in the convenient form

$$\begin{pmatrix} C_N \\ \overline{C}_N \end{pmatrix} = \sum_{k=1}^{N} p^{N-k}\mathcal{R}^{k-1}\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = (p^{N-1} + p^{N-2}\mathcal{R} + \cdots + \mathcal{R}^{N-1})\begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{43}$$

To rewrite this expression in a compact form, we introduce the notation for the operator in terms of the q-number

$$1 + q + q^2 + \cdots + q^{N-1} \equiv [N]_q, \tag{44}$$

where q is a linear operator. Hence, in terms of the operator $q \equiv \mathcal{R}/p$, we have the finite Laurent expansion in the spectral parameter p

$$1 + \frac{\mathcal{R}}{p} + \left(\frac{\mathcal{R}}{p}\right)^2 + \cdots + \left(\frac{\mathcal{R}}{p}\right)^{N-1} \equiv [N]_{\mathcal{R}/p}. \tag{45}$$

We then have

$$\begin{pmatrix} C_N \\ \overline{C}_N \end{pmatrix} = p^{N-1}[N]_{\mathcal{R}/p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{46}$$

Similarly, we obtain

$$A_N = -\frac{p^N}{2} - i\kappa^2 p^{N-1} \left(\int^x \bar{\psi}, \, -\int^x \psi \right) [N]_{\mathcal{R}/p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{47}$$

Equations (42), (46), and (47) give the time part of the linear problem (the Lax representation) for the Nth flow of NLS hierarchy (40) in the q-calculus form.

2.2. General NLS hierarchy equation. For the time t determined by the formal series $\partial_t = \sum_{N=0}^{\infty} E_N \partial_{t_N}$, where E_N are arbitrary constants, the general NLS hierarchy equation is

$$i\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}_t = (E_0 + E_1 \mathcal{R} + \cdots + E_N \mathcal{R}^N + \cdots) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{48}$$

The integrability of this equation is associated with Zakharov–Shabat problem (35) and the time evolution

$$J_0 = \sum_{N=0}^{\infty} E_N J_{0_N} = \begin{pmatrix} -iA & -\kappa^2 \overline{C} \\ C & iA \end{pmatrix}, \tag{49}$$

where

$$\begin{pmatrix} C \\ \overline{C} \end{pmatrix} = \sum_{N=0}^{\infty} E_N \begin{pmatrix} C_N \\ \overline{C}_N \end{pmatrix} = \sum_{N=1}^{\infty} E_N p^{N-1} [N]_{\mathcal{R}/p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \tag{50}$$

and we use the fact that $C_0 = 0$ for $N = 0$. We then have

$$A = \sum_{N=0}^{\infty} E_N A_N = -\frac{1}{2} \sum_{N=0}^{\infty} E_N p^N - i\kappa^2 \left(\int^x \bar{\psi}, \, -\int^x \psi \right) \begin{pmatrix} C \\ \overline{C} \end{pmatrix}. \tag{51}$$

Equation (48) gives an integrable nonlinear extension of linear Schrdinger equation (1) with a general analytic dispersion. We consider the classical particle system with the energy–momentum relation $E(p) = E_0 + E_1 p + E_2 p^2 + \ldots$. Then the corresponding time-dependent Schrödinger wave equation is (1), where the Hamiltonian operator results from the standard momentum substitution $p \to -i\hbar \partial/\partial x$ in the dispersion law. Equation (1) together with its complex conjugate can be rewritten as

$$i\hbar\sigma_3 \frac{\partial}{\partial t} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = H\left(-i\hbar\sigma_3 \frac{\partial}{\partial x} \right) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{52}$$

The momentum operator here is the recursion operator in the linear approximation $\mathcal{R}_0 = i\sigma_3 \partial/\partial x$. Hence, (52) is the linear Schrödinger equation with an arbitrary analytic dispersion. The nonlinear integrable extension of (52) appears as (48), which corresponds to the substitution $\mathcal{R}_0 \to \mathcal{R}$ ($\hbar = 1$), and hence

$$i\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}_t = H(\mathcal{R}) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{53}$$

From this standpoint, the standard classical momentum substitution $p \to -i\hbar \partial/\partial x$ or, equivalently, $p \to -i\hbar\sigma_3 \partial/\partial x = \mathcal{R}_0$ for the equation in spinor form gives the quantization in the form of the linear Schrödinger equation, while the substitution $p \to \mathcal{R}$ gives a "nonlinear quantization" and the nonlinear Schrödinger hierarchy equation.

The related Lax representation for (53) is given by (50) and (51). By the definition of the q-derivative

$$D_q^{(\zeta)} f(\zeta) = \frac{f(q\zeta) - f(\zeta)}{(q-1)\zeta},$$

for the operator $q = \mathcal{R}/p$, we have the relation $D^{(p)}_{\mathcal{R}/p}\zeta^N = [N]_{\mathcal{R}/p}p^{N-1}$. We can then rewrite Eq. (50) as

$$\begin{pmatrix} C \\ \overline{C} \end{pmatrix} = \sum_{N=1}^{\infty} E_N p^{N-1}[N]_{\mathcal{R}/p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = \sum_{N=1}^{\infty} E_N D^{(p)}_{\mathcal{R}/p} p^N \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \tag{54}$$

or, using the linearity of the q-derivative and the specific analytic dispersion form,

$$\begin{pmatrix} C \\ \overline{C} \end{pmatrix} = D^{(p)}_{\mathcal{R}/p} \sum_{N=0}^{\infty} E_N p^N \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} = D^{(p)}_{\mathcal{R}/p} E(p) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{55}$$

By the above definition, (55) reduces to the simple formula

$$\begin{pmatrix} C \\ \overline{C} \end{pmatrix} = \frac{E(\mathcal{R}) - E(p)}{\mathcal{R} - p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{56}$$

For A, we then obtain

$$A = -\frac{1}{2}E(p) - i\kappa^2 \left(\int^x \bar{\psi},\ -\int^x \psi \right) \frac{E(\mathcal{R}) - E(p)}{\mathcal{R} - p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{57}$$

Equations (56) and (57) give the Lax representation of general integrable NLS hierarchy model (53) in a simple compact form. It noteworthy that the particular form of the dispersion law $E = E(p)$ is fixed by the physical problem. We consider the relativistic form of this dispersion law and the corresponding semirelativistic NLS equation in Sec. 3.

3. Semirelativistic NLS

In Sec. 1.2, we considered the relativistic dispersion relation $E(p) = \sqrt{m^2c^4 + p^2c^2}$. This can be used to construct a semirelativistic Schrödinger equation with Hamiltonian (13). Combining the two complex conjugate equations, we obtain

$$i\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}_t = mc^2 \sqrt{1 + \frac{1}{m^2c^2}\left(i\sigma_3 \frac{\partial}{\partial x} \right)^2} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{58}$$

We emphasize that if ψ describes the relativistic particle moving forward in time with positive energy, then $\bar{\psi}$ corresponds to backward time with negative energy. From this standpoint, Eq. (58) is complete because it describes both cases simultaneously.

Following the general procedure described in Sec. 2, we can replace the derivative operator $\mathcal{R}_0 = i\sigma_3 \partial/\partial x$ corresponding to the linear momentum p with the full recursion operator $\mathcal{R}$ given by (39), which eventually leads to the integrable relativistic nonlinear Schrödinger equation

$$i\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}_t = mc^2 \sqrt{1 + \frac{1}{m^2c^2}\mathcal{R}^2} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}, \tag{59}$$

where the square root of the operator expression is understood as the formal power series and hence

$$i\sigma_3 \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}_t = mc^2 \left(1 + \frac{1}{2m^2c^2}\mathcal{R}^2 - \frac{1}{8m^4c^4}\mathcal{R}^4 + \frac{1}{16m^6c^6}\mathcal{R}^6 - \cdots \right) \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}. \tag{60}$$

For the above relativistic dispersion and Eq. (59), we have the linear problem

$$\begin{aligned} \frac{\partial}{\partial x} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} -ip/2 & -\kappa^2\bar{\psi} \\ \psi & ip/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \\ \frac{\partial}{\partial t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} -iA & -\kappa^2\overline{C} \\ C & iA \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \end{aligned} \tag{61}$$

where

$$\begin{pmatrix} C \\ \overline{C} \end{pmatrix} = \frac{\sqrt{m^2c^4 + \mathcal{R}^2c^2} - \sqrt{m^2c^4 + p^2c^2}}{\mathcal{R} - p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix},$$

$$A = -\frac{1}{2}\sqrt{m^2c^4 + p^2c^2} - i\kappa^2 \left(\int^x \bar{\psi},\ -\int^x \psi \right) \frac{\sqrt{m^2c^4 + \mathcal{R}^2c^2} - \sqrt{m^2c^4 + p^2c^2}}{\mathcal{R} - p} \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix},$$

and the spectral parameter p has the meaning of the classical momentum. Model (59) is an integrable nonlinear Schrodinger equation with a relativistic dispersion

$$i\psi_t = mc^2 \sqrt{1 - \frac{1}{m^2c^2}\frac{\partial^2}{\partial x^2}}\psi + F(\psi), \tag{62}$$

where the nonlinearity expanded in $1/c^2$ is the infinite sum

$$F(\psi) = \frac{1}{2m}[-2\kappa^2|\psi|^2\psi] - \frac{1}{8m^3c^2}\left[2\kappa^2\left(2|\psi_x|^2\psi + 4|\psi|^2\psi_{xx} + \bar{\psi}_{xx}\psi^2 + 3\bar{\psi}\psi_x^2\right) + 6\kappa^4|\psi|^4\psi\right] + O\left(\frac{1}{c^4}\right).$$

If we also expand the dispersion part in $1/c^2$, then we obtain an integrable system in every order of $1/c^2$. This means that we obtain integrable relativistic corrections to the NLS equation at all orders of the perturbation theory. Neither of the known relativistic integrable models, the sine-Gordon nor the Liouville equation, has this property. Finally, we note that the nonlinear relativistic equations considered here differ from those obtained in [1]–[3] and the references therein. Our models might be used to analyze relativistic corrections to solitons, Bose–Einstein condensates, or other condensed matter systems described by an effective equation of relativistic form.

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